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INFLUENCE OF LIMITERS ON THE PENETRATION OF AN OSCILLATING AXIAL CURRENT IN A STRAIGHT DISCHARGE

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A b s t r a c t

Schemes of dynamic stabilization of pinches which rely on oscillating magnetic fields transversely to the main static field encounter severe difficulties caused by the screening of the field by residual plasma. It is shown that insulating limiters, regularly placed along a column, suppress the screening effect by exciting stationary Alfvèn waves, and allow the axial oscillating current to flow on the central plasma core.

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1. Introduction

In some experiments aimed at obtaining dynamic equilibrium and/or dynamic stabilization of high-β plasma columns, it is necessary to apply a high frequency B_θ (cylindrical geometry being considered) magnetic field which is perpendicular (or nearly perpendicular)to the main confining field[1-7]. In general, the pinched plasma column is surrounded by a low-density plasma. The origin of this residual plasma is not well established but it can be supposed to arise from, amongst other causes, incomplete collection during the implosion phase of the pinch or from ionization of gas liberated from the discharge tube wall. It is to be expected that any applied alternating axial current will flow in this tenuous plasma thus confining the high-frequency B_θ field to a skin layer near the wall of the discharge tube. This indeed turns out to be the case in some of the experiments referred to above (see Refs. 5 and 6).

In other experiments[1-3], limiters have been incorporated into the apparatus with the stated purpose of restricting the diameter of the plasma column. The role that these limiters play in determining the distribution of the high-frequency axial current has not yet been fully elucidated, either theoretically or experimentally. In one experiment[5], the use of quartz limiters shifted the current distribution only slightly inwards from the wall region. However, in other experiments[1,7], the use of limiters clearly allowed the alternating axial current to flow on the plasma column.

For some time, van der Laan has stressed that the B_θ field distribution in linear combined pinches (combined Z-and θ-pinches) depends greatly on the manner of feeding in the axial Z-current. He has outlined a mechanism[8] (which does not invoke the use of limiters) which can allow a vacuum field to be formed in the region between the plasma column and the wall of the discharge tube. This mechanism fails if breakdown occurs at the discharge tube wall and is inoperative in a toroidal system. It is because of the possible occurrence of this mechanism that van der Laan has counseled caution in basing the design of toroidal combined pinches on results obtained in linear or toroidal sector experiments.

The purpose of this paper is to point out that if certain experimental requirements are met, then limiters, by enforcing the mechanism described by van der Laan, alter the current distribution in such a way as to allow most of the current to flow on the plasma column. The effect of the limiters on the current distribution is described below and the necessary experimental conditions for enhanced penetration are established. It should be noted that the use of limiters in toroidal systems is not precluded and would indeed allow alternating axial currents to be induced on toroidal pinches.

2. Description of the system

Consider a straight cylindrical plasma column of radius a confined by an axial steady magnetic field B_o and surrounded by a low density plasma. Insulating limiters are placed regularly with a spacing 2L inbetween (Fig. 1). The inside radius is equal to the plasma column radius a. The limiters extend to the insulating wall of the tube of radius b.

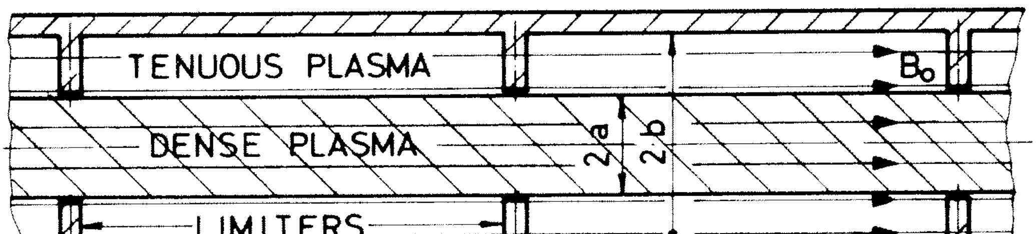

Fig.1 Schematic of the system

An alternating current $I\cos\omega t$ flows in the axial direction. At the tube radius b the oscillating magnetic field $B_\theta(r = b)$ is then given by

$$B_\theta(r=b) = \frac{\mu_0 I}{2\pi b}\cos\omega t = H\cos\omega t$$

We want to find the oscillating field profile in this system.

3. Field profile for a perfectly conducting plasma

We assume that in the region $a \leq r \leq b$:

1) The tenuous plasma has a low uniform density ρ and is pressureless, which implies that the field B_o is constant in this region.
2) The oscillating field B_θ is small compared to B_o : $|B_\theta| \ll B_o$.
3) The plasma is perfectly conducting.
4) The system is in a stationary regime.

The basic equations are:

$$\begin{aligned}
\frac{\partial B_\theta}{\partial t} &= -\frac{\partial E_r}{\partial z} = B_o\frac{\partial v_\theta}{\partial z} \\
\rho\frac{\partial v_\theta}{\partial t} &= -J_r B_o \\
\mu_0 J_r &= -\frac{\partial B_\theta}{\partial z} \\
\mu_0 J_z &= \frac{1}{r}\,\frac{\partial (rB_\theta)}{\partial r}
\end{aligned} \tag{1}$$

where E, v and J are respectively the electric field, the flow velocity and the current density. By elimination, it results the equation of propagation of torsional waves

$$\frac{\partial^2 B_\theta}{\partial t^2} = V_A^2 \frac{\partial^2 B_\theta}{\partial z^2} \tag{2}$$

where

$$V_A = B_o / \sqrt{\mu_o \rho}$$

In the region between 2 limiters the stationary solutions have the form

$$B_\theta = B_\theta^{(1)}(r) \cos\omega t . \cos\frac{\omega z}{V_A} + B_\theta^{(2)}(r) \cos\omega t . \sin\frac{\omega z}{V_A} \tag{3}$$

Choosing the origin of z equidistant from the two limiters, we have $B^{(2)}(r) = 0$. $B_\theta^{(1)}(r)$ is determined by the boundary condition on the limiters, namely

$$J_z(\pm L) = 0,$$

which is equivalent to the condition

$$B_\theta(r, \pm L) = \frac{Hb}{r}$$

expressing the continuity of B_θ on the limiters. This gives

$$B_\theta^{(1)}(r) = \frac{Hb}{r \cos \varepsilon}$$

where we have introduced the crucial dimensionless parameter ε .

$$\varepsilon = \frac{\omega L}{V_A}$$

For convenience, let us normalize the distance z by

$$z = Ly$$

The two limiters are now in $y = \pm 1$. The solution is then

$$B_\theta(r,y,t) = \frac{Hb}{r}\,\frac{\cos\varepsilon y}{\cos\varepsilon}\cos\omega t$$

$$J_r(r,y,t) = \varepsilon\,\frac{Hb}{L\mu_o r}\,\frac{\sin\varepsilon y}{\cos\varepsilon}\cos\omega t \tag{4}$$

$$J_z(r,y,t) = \frac{Hb}{\mu_o}\left[\frac{\delta(r-a)}{a}\,\frac{\cos\varepsilon y}{\cos\varepsilon} - \frac{\delta(r-b)}{b}\left(\frac{\cos\varepsilon y}{\cos\varepsilon} - 1\right)\right]\cos\omega t$$

In the limit $\varepsilon \longrightarrow 0$, $B_\theta = \frac{Hb}{r}\cos\omega t$, as if there was vacuum instead of the tenuous plasma. All the current is flowing at r = a, namely on the central column. For $\varepsilon \ll 1$ the equations (4) become

$$B_\theta \approx \frac{Hb}{r}\left[1 + \frac{\varepsilon^2}{2}(1-y^2)\right]\cos\omega t$$

$$J_r \approx \varepsilon^2\,\frac{Hb}{L\mu_o r}\,y\cos\omega t \tag{5}$$

$$J_z \approx \frac{Hb}{\mu_o a}\,\delta(r-a)\cos\omega t + \frac{\varepsilon^2}{2}(1-y^2)\frac{Hb}{\mu_o}\left[\frac{\delta(r-a)}{a} - \frac{\delta(r-b)}{b}\right]\cos\omega t$$

We see that the corrections to the vacuum solution are of order ε^2. The field profile is shown in Fig. 2.

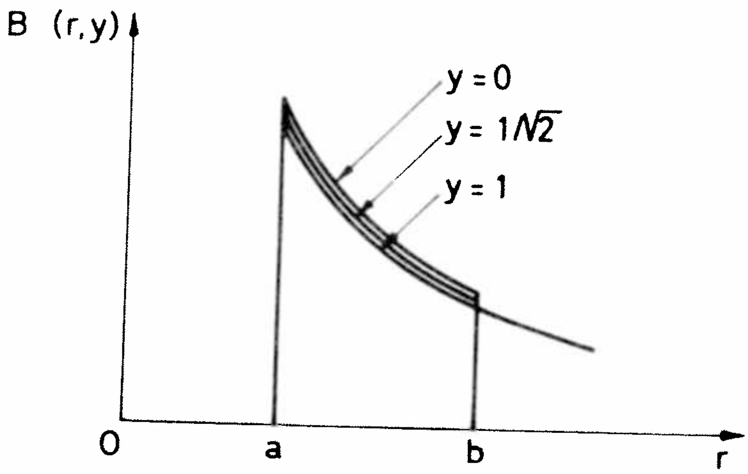

Fig. 2 Profiles of the B_θ field

The solution (4) can be thought of as a superposition of a vacuum solution and a vortex structure as shown in Fig. 3. It is this vortex which creates the surface current at the tube wall.

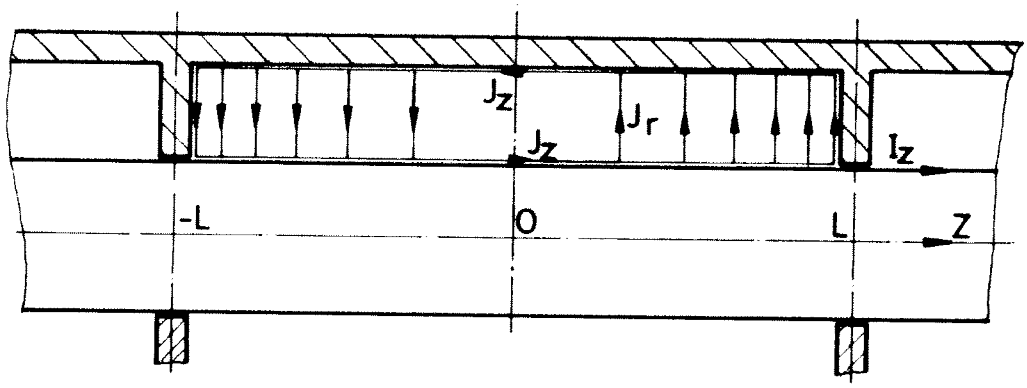

Fig. 3 Distribution of the currents between two limiters

We want to minimize the current in the vortex in order to reduce as much as possible ohmic dissipation in the residual plasma. This can be done by choosing $\varepsilon \ll 1$, which we shall always assume in the following.

This result can be extended to the case where ρ and B_o are functions of r only. The equations (1), (2) and (3) still hold and B_θ and J_r are still given by (4) with ε being now also a function of r. J_z has also a distributed term proportional to $\frac{\partial \varepsilon}{\partial r}$. If $\varepsilon \ll 1$ in the tenuous plasma, the field will still be nearly equal to the vacuum field.

4. Influence of resistivity

The field discontinuities in r = a and r = b will disappear if one takes into account resistivity. The surface currents will be smeared. The width of the transition layers should be, at worst, of the order of the classical skin depth $\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$ (where σ is the electrical conductivity) which we assume to be much smaller than b-a (otherwise limiters are not needed). The situation should look like this: away from the wall and from the central column $a < r < b$ the solution (4) should be valid with two thin transition layers at r = a and r = b.

To verify this conjecture, let us repeat the previous calculation with resistivity included. We assume the conductivity σ to be constant and isotropic (scalar).

The equations (1) become:

$$\begin{aligned}
\frac{\partial B_\theta}{\partial t} &= -\frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} \\
\rho \frac{\partial V_\theta}{\partial t} &= -J_r B_0 \\
\mu_0 J_r &= -\frac{\partial B_\theta}{\partial z} \\
\mu_0 J_z &= \frac{1}{r} \frac{\partial (r B_\theta)}{\partial r} \\
J_z &= \sigma E_z \\
J_r &= \sigma (E_r + V_\theta B_0)
\end{aligned} \tag{6}$$

By elimination we obtain the wave equation

$$\frac{\partial^2 B_\theta}{\partial t^2} = \frac{\partial^2}{\partial t \partial r}\left[\frac{1}{\mu_0 \sigma r} \frac{\partial (r B_\theta)}{\partial r}\right] + \left(V_A^2 + \frac{1}{\mu_0 \sigma} \frac{\partial}{\partial t}\right) \frac{\partial^2 B_\theta}{\partial z^2} \tag{7}$$

Let us introduce the constant $k^2 = \omega\mu_0\sigma$ which relates to the skin-depth by $\delta = \sqrt{2}/k$. We write

$$B_\theta(r,z,t) = \Re e\left\{B(r,y)e^{i\omega t}\right\} \tag{8}$$

where y is the normalized variable already introduced: equation (7) becomes:

$$\frac{i}{k^2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rB)}{\partial r}\right] + \frac{1}{\varepsilon^2}\left(1+\frac{i\varepsilon^2}{k^2L^2}\right)\frac{\partial^2 B}{\partial y^2} + B = 0 \tag{9}$$

The term $\frac{\varepsilon^2}{k^2L^2} = \frac{\omega^2\delta^2}{2V_A^2} = \left(\frac{\omega}{\omega_{ci}}\right)\left(\frac{\nu}{\omega_{ce}}\right)$, where ν is the usual collision frequency defined by $\sigma = \frac{ne^2}{m_e\nu}$, is always very small and can be neglected in all cases of interest. We shall neglect it.

The general solution of equ. (9) which satisfies the boundary conditions at r = b and on the limiters can be written (Appendix)

$$B(r,y) = \frac{Hb}{r} + \frac{Hb}{r}2\varepsilon^2\sum_{n=0}^{\infty}\frac{(-1)^n\cos Ny}{N(N^2-\varepsilon^2)}\left[1-\frac{r}{b}\frac{H_1^{(1)}(\beta_{2n+1}r)}{H_1^{(1)}(\beta_{2n+1}b)}\right]$$

$$+ H\sum_{n=1}^{\infty}A_2^{(n)}\left[\frac{H_1^{(2)}(\beta_n r)}{H_1^{(2)}(\beta_n a)} - \frac{H_1^{(2)}(\beta_n b)H_1^{(1)}(\beta_n r)}{H_1^{(2)}(\beta_n a)H_1^{(1)}(\beta_n b)}\right]\cos\frac{n\pi}{2}y \tag{10}$$

$$N \equiv (2n+1)\frac{\pi}{2}, \quad \beta_n = (1+i)\frac{k\sqrt{\left(\frac{n\pi}{2}\right)^2-\varepsilon^2}}{\sqrt{2}\,\varepsilon} = \frac{1+i}{\varepsilon\delta}\sqrt{\left(\frac{n\pi}{2}\right)^2-\varepsilon^2}$$

$A_2^{(n)}$ are constants to be determined from the boundary conditions at r = a, namely the continuity of B and E_z.

Instead of the developments (10), which are limited to the range $-1 \leq y \leq 1$, we expand B and E_z as

$$B = \sum_{\ell=0}^{\infty} B_\ell(r) \cos \ell\pi y$$
$$E_z = \sum_{\ell=0}^{\infty} E_\ell(r) \cos \ell\pi y \tag{11}$$

Note that the B_ℓ in (11) are different from the B_n in (A4), and that the thickness of the limiters is neglected.

Designate by Z_ℓ the complex quantities

$$Z_\ell = \frac{E_\ell(a)}{B_\ell(a)} \tag{12}$$

They have the dimension of a velocity and they depend only on the properties of the central plasma column. The $A_2^{(n)}$ can be expressed in terms of the Z_ℓ.

For $\frac{a}{\varepsilon\delta} \gg 1$ equation (10) gives

$$B(a,y) \approx \frac{Hb}{a} \frac{\cos \varepsilon y}{\cos \varepsilon} + H \sum_{n=1}^{\infty} A_2^{(n)} \cos \frac{n\pi}{2} y$$
$$E_z(a,y) \approx \frac{1}{\mu_0 \sigma r} \frac{\partial(rB)}{\partial r} \approx \frac{i\omega}{k^2} H \sum_{n=1}^{\infty} \beta_n A_2^{(n)} \cos \frac{n\pi}{2} y \tag{13}$$

By comparison with (11) we find

$$H A_2^{(n)} = -\frac{ik}{\omega\beta_n} \sum_{\ell=0}^{\infty} c_{n\ell} E_\ell(a) = -\frac{ik^2}{\omega\beta_n} \sum_{\ell=0}^{\infty} c_{n\ell} Z_\ell B_\ell(a) \tag{14}$$

$$c_{n\ell} = 0 \text{ if n even ; } c_{no} = \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}}$$

$$c_{n\ell} = \frac{(-1)^\ell n\pi \sin \frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2 - (\ell\pi)^2}, \ \ell \neq 0$$

We see that $A_2^{(2n)} \equiv 0$. From (13), by reexpanding in y, we find

$$B_{\ell=0}(a) = \frac{Hb}{a}\frac{tg\,\varepsilon}{\varepsilon} + H\sum_{n=0}^{\infty} c_{2n+1,0}\,A_2^{(2n+1)}$$

$$B_{\ell\neq 0}(a) = \frac{Hb}{a}\frac{(-1)^{\ell}\,2\,\varepsilon\, tg\,\varepsilon}{\varepsilon^2-(\ell\pi)^2} + H\sum_{n=0}^{\infty} c_{2n+1,\ell}\,A_2^{(2n+1)}$$

Substituting into (14) we find the desired equation for the $A_2^{(n)}$

$$A_2^{(2n+1)} = -\frac{ib k^2}{a\omega\beta_{2n+1}}\left[\frac{tg\,\varepsilon}{\varepsilon}\,c_{2n+1,0}\,Z_0 - 2\,\varepsilon\, tg\,\varepsilon\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}c_{2n+1,\ell}}{\varepsilon^2-(\ell\pi)^2}\,Z_\ell\right]$$

$$-\frac{ik^2}{\omega\beta_{2n+1}}\sum_{k=0}^{\infty} A_2^{(2k+1)}\sum_{\ell=0}^{\infty} c_{2n+1,\ell}\,c_{2n+1,k}\,Z_\ell \qquad (15)$$

The Z_ℓ depend on the properties of the central column, but some general statements are possible. Z_ℓ can be written

$$Z_\ell = \frac{\omega\,\delta_\ell}{\sqrt{2}}\,e^{i\varphi_\ell}$$

δ_ℓ has the dimension of a length. δ_0 is independent of ε while the other δ_ℓ depend on it. The requirement that the Poynting vector points towards r = o implies that $0 \leq \varphi_\ell \leq \frac{\pi}{2}$. If equ. (7) applies in the central column, with σ constant, δ_0 becomes the usual skin depth ($\delta_0 \ll a$) and δ_ℓ is a decreasing function of ℓ. This last property remains correct for more general conditions. Since the Z_ℓ are at most of the order of Z_0, which is independent of ε, we can expand in equ. (15) in powers of ε. We are interested in the leading terms only. It gives, at the lowest order,

$$A_2^{(2n+1)} \approx -\left(\frac{\delta_0}{\delta}\right) e^{i\left(\varphi_0+\frac{\pi}{4}\right)}\frac{b}{a}\,\frac{\varepsilon\,(-1)^n}{\left[\frac{(2n+1)\pi}{2}\right]^2} \qquad (16)$$

It is pleasant that the leading term depends only on δ_0 and φ_0. Putting this result in (10), we find for $\varepsilon \ll 1$.

$$B(r,y) \approx \frac{Hb}{r} + 2\varepsilon^2 H \sum_{n=0}^{\infty} \frac{(-1)^n \cos Ny}{N^3} \left[\frac{b}{r} - e^{(-1+i)N\frac{b-r}{\varepsilon\delta}} \right]$$

$$- \frac{\varepsilon H \delta_0 b}{\delta a} e^{i(\varphi_0 + \frac{\pi}{4})} \sum_{n=0}^{\infty} \frac{(-1)^n \cos Ny}{N^2} e^{(-1+i)N\frac{r-a}{\varepsilon\delta}} \quad (17)$$

This result has the form expected. B(r,y) is equal to the vacuum field except in narrow transition regions. The transition regions in r = b and r = a have the same thickness $\delta_A \sim \varepsilon\delta$, which is much smaller than δ .

This is qualitatively understood as a balance between the influence of the limiter which tends to create the discontinuity and the resistive diffusion which tends to smear it. The profiles of the amplitude of B and E_z are shown schematically in Fig. 4.

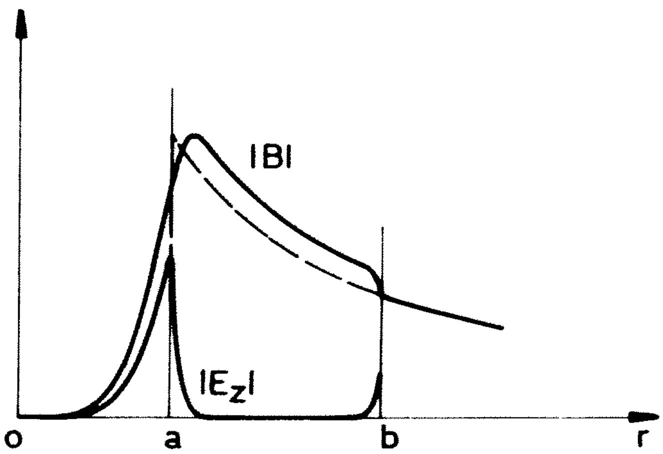

Fig. 4 Profiles of |B| and $|E_z|$ with resistivity

Note that if $\delta_o << \delta$, there is no transition layer at r = a.

The inclusion of other terms, like viscous damping or a variable resistivity should not affect the main result. It should only affect the boundary layers which should still maintain their dependence on ε . Away from the boundaries the plasma rotates almost as a rigid rotor, which makes viscosity irrelevant.

5. The transient

We have assumed until now a steady state. It is important for the experiment to know the transient phase in order to determine its duration and its effect on the field profile. We assume the oscillating current $I_z \sin\omega t$ is switched at t = o on the configuration of Fig. 1. To determine the duration of the transient we include resistive damping, but we forget about the transition layers. The equation for B_θ is then

$$\frac{\partial^2 B_\theta}{\partial t^2} = \left(V_A^2 + \frac{1}{\mu_o \sigma}\frac{\partial}{\partial t}\right)\frac{\partial^2 B_\theta}{\partial z^2} \tag{18}$$

Laplace transforming in time (variable s) and denoting by a wiggle the transformed functions, this equation reads

$$s^2 \tilde{B}_\theta = \frac{\omega^2}{\varepsilon^2}\frac{\partial^2 \tilde{B}_\theta}{\partial y^2} + \frac{\omega s}{k^2 L^2}\frac{\partial^2 \tilde{B}_\theta}{\partial y^2}$$

which has the solution

$$\tilde{B}_\theta (r,y,s) = \frac{Hb}{r}\frac{\omega}{s^2+\omega^2}\,\frac{\cosh\left[\dfrac{\varepsilon s y}{\omega\sqrt{1+\dfrac{\varepsilon^2 s}{k^2 L^2 \omega}}}\right]}{\cosh\left[\dfrac{\varepsilon s}{\omega\sqrt{1+\dfrac{\varepsilon^2 s}{k^2 L^2 \omega}}}\right]} \tag{19}$$

The inverse transform is

$$B_\theta(r,y,t) \cong \frac{Hb}{r}\left\{\frac{\cos \varepsilon y}{\cos \varepsilon}\sin\omega t + 2\varepsilon \sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)\frac{\pi}{2}y}{\varepsilon^2-\left[(2n+1)\frac{\pi}{2}\right]^2}\, e^{-\frac{\left[(2n+1)\frac{\pi}{2}\right]^2}{2k^2L^2}\omega t} \cdot \sin\frac{(2n+1)\pi}{2\varepsilon}\,\omega t\right\} \qquad (20)$$

where we have neglected damping in the first term, which is obviously the steady state solution and where we have made the small damping approximation in the terms of the sum. This last approximation fails for large n but this is irrelevant since the sum is very strongly convergent. The solution (20) shows that the amplitude of the transient is of the order of $\frac{8\varepsilon}{\pi^2}$ for $\varepsilon \ll 1$ and this transient dies out in a time of the order of

$$t_c = \frac{8k^2L^2}{\omega\pi^2}$$

Since we always have $kL \gg 1$ this transient will disappear slowly. This is another reason to choose $\varepsilon \ll 1$ in experiments to minimize the influence of the transient. It should be noted that the dependence in r is the same for the transient than for the steady state (except for the transition layers where large currents flow).

6. Feasibility

The parameter ε can be written as

$$\varepsilon = 2.9 \times 10^{-8}\,\frac{f(\mathrm{MHz})\,L(\mathrm{cm})\sqrt{M_A}\sqrt{n_i(\mathrm{cm}^{-3})}}{B(\mathrm{kG})}$$

where M_A is the atomic weight of the ion, n_i the ion density and f the applied frequency. A hot θ-pinch has a residual plasma of density unknown but certainly not more than 10^{14} cm^{-3}. For a field amplitude of 40 kG and a plasma of hydrogen, it gives

$$\varepsilon \approx 0.0075 \, fL.$$

$\varepsilon = 0.3$ corresponds to a maximum axial non-homogeneity of the oscillating field of 5 %. We take this as the maximum value of ε , which then gives for the distance 2L between 2 limiters

$$2L_{max} \approx \frac{80}{f\,(\mathrm{MHz})} \quad \mathrm{cm}.$$

The big unknown is the frequency to use, but it appears that dynamic stabilization will only be interesting if the frequency required is well below 1 MHz. This means limiters could be spaced meters apart. The situation would be even better in low-β configurations with fields in the range of 60 kG and above, and densities in the range 10^{12} - 10^{13} cm^{-3}.

Limiters can be used in the same manner for other configurations. The essential point is that, whenever $\left(\frac{\omega L}{V_A}\right)^2 \ll 1$, where 2L is the distance between two limiters measured along the field lines, the current can only flow along the field lines. Examples of interesting geometries where such a scheme would be useful are screw-pinches, high-β stellarators. Note that the toroidicity does not invalidate the results. It only changes the spatial field distribution of the residual currents which are still of order ε^2.

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Appendix

Let us introduce the new variable $x = \frac{kr}{\varepsilon}$. The equation (9) now reads:

$$i \frac{\partial}{\partial x}\left[\frac{1}{x} \frac{\partial (xB)}{\partial x}\right] + \frac{\partial^2 B}{\partial y^2} + \varepsilon^2 B = 0 \tag{A1}$$

At the surface, $x_s = \frac{kb}{\varepsilon}$, the boundary condition is

$$B(x_s, y) = H$$

and at the limiters, $y = \pm 1$,

$$B(x, \pm 1) = \frac{Hkb}{\varepsilon x}$$

Since we are only interested in the range between two limiters, namely $-1 \leqslant y \leqslant +1$, B is not defined outside this range. We can use this to extend the definition of B to the range $-2 \leqslant y \leqslant +2$, by the following definitions:

$$\begin{aligned} -2 \leq y < -1 \; ; \quad & \tilde{B}(y) = -B(y+2) \\ -1 < y < +1 \; ; \quad & \tilde{B}(y) = B(y) \\ +1 < y \leq +2 \; ; \quad & \tilde{B}(y) = -B(y-2) \end{aligned} \tag{A2}$$

B(y) satisfies the equation (A1) on the enlarged domain. It has two discontinuities in $y = \pm 1$ with the jump relations

$$\tilde{B}(\pm 1 + 0) - \tilde{B}(\pm 1 - 0) = \mp \frac{2Hkb}{\varepsilon x} \tag{A3}$$

and $\tilde{B}(+2) = \tilde{B}(-2)$. Since the solution is symmetrical in y, $\frac{\partial \tilde{B}}{\partial y}$ is continuous and $\frac{\partial \tilde{B}}{\partial y}(+2) = \frac{\partial \tilde{B}}{\partial y}(-2) = 0$.

We expand $\tilde{B}$ in a Fourier series

$$\tilde{B}(x,y) = \sum_{n=0}^{\infty} B_n(x) \cos \frac{n\pi y}{2} \tag{A4}$$

We also need the Fourier expansion of $\frac{\partial^2 \tilde{B}}{\partial y^2}$, given by

$$\frac{\partial^2 \tilde{B}}{\partial y^2} = \sum_{n=0}^{\infty} C_n \cos \frac{n\pi y}{2}$$

with

$$C_n = \frac{1}{2} \int_{-2}^{+2} dy \frac{\partial^2 \tilde{B}}{\partial y^2} \cos \frac{n\pi y}{2} = \frac{n\pi}{4} \int_{-2}^{+2} dy \frac{\partial \tilde{B}}{\partial y} \sin \frac{n\pi y}{2} =$$

$$= \frac{n\pi}{4} \sin \frac{n\pi}{2} \left[\tilde{B}(-1+0) - \tilde{B}(-1-0) - \tilde{B}(1+0) + \tilde{B}(1-0)\right] - \left(\frac{n\pi}{2}\right)^2 \frac{1}{2} \int_{-2}^{+2} dy\, \tilde{B} \cos \frac{n\pi y}{2}$$

$$= n\pi \sin \frac{n\pi}{2} \frac{Hkb}{\varepsilon x} - \left(\frac{n\pi}{2}\right)^2 B_n$$

for $n \neq 0$ and $C_o = 0$.

This gives the expansion

$$\frac{\partial^2 \tilde{B}}{\partial y^2} = \sum_{n=0}^{\infty} \left[n\pi \sin \frac{n\pi}{2} \frac{Hkb}{\varepsilon x} - \left(\frac{n\pi}{2}\right)^2 B_n\right] \cos \frac{n\pi y}{2} \tag{A5}$$

Substituting into (A1)

$$i \frac{d}{dx}\left[\frac{1}{x} \frac{d(xB_n)}{dx}\right] + \left[\varepsilon^2 - \left(\frac{n\pi}{2}\right)^2\right] B_n = -n\pi \sin \frac{n\pi}{2} \frac{Hkb}{\varepsilon x} \tag{A6}$$

The general solution of equation (A6) is

$$B_n(x) = \frac{n\pi \sin\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2 - \varepsilon} \; \frac{Hkb}{\varepsilon x} + H Z_1^{(n)}(\alpha_n x) \tag{A7}$$

where $\alpha_n = \frac{(1+i)k}{\sqrt{2}\,\varepsilon}\sqrt{\left(\frac{n\pi}{2}\right)^2 - \varepsilon^2}$ and $Z_1^{(n)}(x)$ is a general solution of the Bessel equation of first order.

We choose to express $Z_1^{(n)}(\alpha_n x)$ in the form

$$Z_1^{(n)}(\alpha_n x) = A_1^{(n)} \frac{H_1^{(1)}(\alpha_n x)}{H_1^{(1)}(\alpha_n x_s)} + A_2^{(n)} \frac{H_1^{(2)}(\alpha_n x)}{H_1^{(2)}(\alpha_n x_c)}$$

in terms of the Hankel functions $H^{(1)}(x)$ and $H^{(2)}(x)$ which behave asymptotically as $\frac{e^{-ix}}{\sqrt{x}}$ and $\frac{e^{ix}}{\sqrt{x}}$ respectively for large x ($x_c = \frac{ka}{\varepsilon}$). The constants A_1 and A_2 are fixed for each n by the boundary conditions.

The function $\theta(y)$ defined by

$$\theta(y) = 1 \quad , \quad |y| < 1$$
$$\theta(y) = -1 \quad , \quad 1 < |y| \leq 2$$

has the expansion

$$\theta(y) = 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} \cos\frac{n\pi y}{2} \tag{A8}$$

Using this relation the boundary condition at $x = x_s$, extended by (A2) , gives

$$B_0(x_s) = 0$$
$$B_n(x_s) = 2H \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} \quad , \quad n \neq 0$$

which imply

$$\mathcal{B}_0(x) = 0$$

$$A_1^{(n)} = -\frac{2\,\varepsilon^2 \sin\frac{n\pi}{2}}{\frac{n\pi}{2}\left[\left(\frac{n\pi}{2}\right)^2 - \varepsilon^2\right]} - A_2^{(n)}\,\frac{H_1^{(2)}(\alpha_n x_s)}{H_1^{(2)}(\alpha_n x_c)} \tag{A9}$$

Replacing into (A4), B(x,y) becomes

$$\tilde{B}(x,y) = \frac{Hkb}{\varepsilon x}\,\Theta(y) + 2\,\frac{Hkb}{\varepsilon x}\,\varepsilon^2 \sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)\frac{\pi}{2}y}{(2n+1)\frac{\pi}{2}\left\{\left[(2n+1)\frac{\pi}{2}\right]^2 - \varepsilon^2\right\}} \left[1 - \frac{\varepsilon x H_1^{(1)}(\alpha_{2n+1}x)}{kb\,H_1^{(1)}(\alpha_{2n+1}x_s)}\right]$$

$$+ H \sum_{n=1}^{\infty} A_2^{(n)} \left[\frac{H_1^{(2)}(\alpha_n x)}{H_1^{(2)}(\alpha_n x_c)} - \frac{H_1^{(2)}(\alpha_n x_s)}{H_1^{(2)}(\alpha_n x_c)}\,\frac{H_1^{(1)}(\alpha_n x)}{H_1^{(1)}(\alpha_n x_s)}\right] \tag{A10}$$

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