Octobre 1970 LRP 45/70

CENTRE DE RECHERCHES EN PHYSIQUE DES PLASMAS
FINANCÉ PAR LE FONDS NATIONAL SUISSE DE LA RECHERCHE SCIENTIFIQUE

# THEORY OF THE DYNAMIC STABILIZATION OF THE RAYLEIGH-TAYLOR INSTABILITY

F. Troyon and R. Gruber

LAUSANNE

# Erratum LRP 45/70

p. 7 : replace ...used by Wold... by ...used by Wolf...

p. 8 Fig. 2: Interchange the indexes 1 and 2

p. 8 Fig. 3: replace ...$\mu_2$=4 P... by ...$\mu_1$=4 P

p. 9 Fig. 9: replace ...$\mu_2$=4 P... by ...$\mu_1$=4 P

p. 10 : replace ...$X \approx - X_o + (ah)^2 p_o$... by ...$X \approx - X_o + ah^2 p_o$

Octobre 1970 LRP 45/70

# THEORY OF THE DYNAMIC STABILIZATION OF THE RAYLEIGH-TAYLOR INSTABILITY

F. Troyon and R. Gruber

A b s t r a c t

Recent experiments by G.H. Wolf have demonstrated the possibility of dynamically stabilizing the Rayleigh-Taylor instability of the interface between two heavy fluids. This problem is considered here. It is shown quite generally that both viscosity and interfacial tension are necessary to stabilize the short wavelength perturbations. As an example the experimental results on the liquid-air interface are reproduced. Dynamic stabilization of plasma confinement can be conceived by analogy with the fluid case. The differences between a plasma and a fluid are examined to show the limitations of the analogy.

Lausanne

## Introduction

The equilibrium of two immiscible superposed fluids of density $\rho_1$ and $\rho_2$ with a horizontal boundary is unstable if the upper fluid (1) is heavier than the lower fluid (2) . [1,2] The growthrate is given by

$$\Omega = \sqrt{\frac{gh\,(\rho_1 - \rho_2)}{\rho_1 + \rho_2}} \qquad (1)$$

where h is the perturbation wavenumber.

Recently G.H. Wolf [3,4] has shown experimentally that, if certain conditions are satisfied, this equilibrium could be dynamically stabilized by enforcing a harmonic oscillation in the vertical direction to the two fluids. The harmonic oscillation causes a periodic acceleration of the fluids $G = g + \Delta g \cos\omega t$, perpendicular to the horizontal interface. If $h_m$ designates the minimum value of h which can exist on the surface, Wolf found stability conditions of the form

$$F\,(\mu_1,\mu_2,\rho_1,\rho_2,\,\Delta g) < \omega < \frac{\Delta g}{\sqrt{2g}}\sqrt{h_m\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}}\,,\quad \Delta g \gg g \qquad (2)$$

for the case where the depth of the fluids is much larger than their horizontal extension. $h_m$ is given by the vessel size; for a rectangular cross-section, for example, $h_m = \pi/L$, where L is the length of the rectangle. $\mu_1$ and $\mu_2$ are the viscosities of the fluids and they have been found to be important parameters.

Wolf has derived the right-hand side of the inequality (2) by means of the quasi-potential approximation and he has verified it experimentally. The left-hand side has been determined experimentally only in the case of a viscous liquid supported by a gas[4]. When the left-hand inequality is not satisfied parametric instabilities are observed.

This paper presents the calculation of the stability conditions for such a configuration . It is shown that stability is impossible to achieve if there is no interfacial tension . Viscosity and interfacial tension are both crucial in the stabilization of the short wavelength parametric instabilities.

As an example, the stability conditions for the liquid-air system are explicitely computed and compared with the experimental results[4]. The agreement is good and shows that non-linear stabilizing effects do not limit the growth of the short wavelengths instabilities.

The relevance of the results to proposed schemes of dynamic stabilization of plasma confinement is discussed.

## Equation of motion

The z axis is chosen vertical, pointing down and the x axis lies on the interface. Consider a perturbation of the interface $z = o$ of the form $z = \varepsilon(t)e^{ihx}$. We consider firstly the case of incompressible fluids, in which case the equation of motion reads

$$(X + \Delta g(\rho_2 - \rho_1)\cos\omega t)\,\varepsilon(t) + \int_0^t R(t-t')\dot{\varepsilon}(t')dt' = o \qquad (3)$$

$$X = g\,(\rho_2 - \rho_1) + Th^2$$

where T is the interfacial tension. $R(t)$ is a response function given by its Laplace transform

$$\tilde{R}(s) = \int_0^\infty e^{-st} R(t)\, dt \tag{4}$$

$$\tilde{R}(s) = (\rho_1+\rho_2)\frac{s}{h} + \frac{4\rho_1\rho_2 s}{N} + \frac{4h(\mu_1-\mu_2)(\rho_1(k_2-h)-\rho_2(k_1-h))}{N} - \frac{4h^2(\mu_1-\mu_2)^2\ (k_1-h)(k_2-h)}{sN}$$

where

$$k_i = \sqrt{h^2 + \frac{\rho_i s}{\mu_i}}\ , \quad N = \rho_1\ (k_2-h) + \rho_2\ (k_1-h)$$

$\tilde{R}(s)$ has the following properties: holomorphic in Re $s > o$ ; Re $\tilde{R}(s) > o$ for Re $s \geqslant o$ ; $\tilde{R}(s) \to (\rho_1 + \rho_2)\frac{s}{h}$ as $|s| \to \infty$ .

## S t a b i l i t y

When $\mu_1 = \mu_2 = o$, $\tilde{R}(s) = (\rho_1 + \rho_2)s/h$ and $R(t) = (\rho_1+\rho_2)/h\delta'(t)$ and (3) degenerates into a Mathieu equation.

$$(\rho_1 + \rho_2)\ \ddot{\varepsilon} + h\ (X + \Delta g\ (\rho_2 - \rho_1)\ \cos\omega t)\ \varepsilon = o$$

We shall always assume $\Delta g > g$.

The first stability region of this equation is given by

$$\sqrt{h}\, F_1(X, \Delta g) < \omega < \sqrt{h}\, F_2(X, \Delta g) \quad \text{when} \quad X < o.$$

$$\sqrt{h}\, F_1(X, \Delta g) < \omega \quad \text{when} \quad X > o. \tag{5}$$

There are other stability regions at lower $\omega$ but they are too narrow to be of any interest. These conditions are obviously impossible to satisfy for all $h > h_m$ (where $h_m$ is given by the size of the vessel).

When $\mu_1, \mu_2 \neq o$, the stability regions can be found by the method of the determinant[5]. We form the function

$$D(s) = 1 - \sum_{\ell=o}^{\infty} (-1)^{\ell} \sum_{k=-\infty}^{+\infty} G(s+ik\omega)G(s+ik\omega + 2i\omega)\ldots.G(s+ik\omega+2i\ell\omega) \tag{6}$$

where

$$G(s) = g(s)\, g(s + i\omega)$$

$$g(s) = \frac{(\rho_2-\rho_1)\, \Delta g/2}{X + s\, \tilde{R}(s)} .$$

The necessary and sufficient condition for stability is that $D(s) \neq o$ in $\mathrm{Re}\, s > o$ . $D(s)$ has the properties

$$D(s^*) = D^*(s) \; , \; D(s + i\omega) = D(s)$$

It is thus possible to determine if there are zeroes in the half-plane Re $s > o$, just by looking at $D(iy)$, $o < y < \frac{\omega}{2}$. This method is described in more detail in another paper[6].

We can summarize the important results of the numerical calculations in the following way:

The stability conditions (5) become

$$\sqrt{h}\, F_1(X,\Delta g,\mu_i,h) < \omega < \sqrt{h}\, F_2(X,\Delta g,\mu_i,h) \quad \text{for} \quad X < o,\ h < H_1(X,\Delta g,\mu_i)$$

$$\sqrt{h}\, F_1(X,\Delta g,\mu_i,h) < \omega \quad \text{for} \quad X > o,\ h < H_2(X,\Delta g,\mu_i) \tag{7}$$

$$\omega > o \qquad \text{for} \quad X > o,\ h > H_2$$

For $h > H_1$ there is no longer a stable region. The stability region collapses at $h = H_1$; this limit $H_1$ is an increasing function of $\mu_i^{-1}$, $X$ and $\Delta g$. Viscosity has thus a destabilizing influence on the short wavelengths in the region $X < o$. When there is no interfacial tension, $T = o$, $X = -(\rho_1 - \rho_2)\, g < o$ and the short wavelengths will always be unstable. When $T \neq o$, $X = -(\rho_1 - \rho_2)\, g + Th^2$ and if $T$ is sufficiently large $X$ will become positive at a value of $h < H_1$, making it possible to achieve stability of the short wavelengths.

The various possibilities are shown schematically in Fig. 1 . Stability of all modes is shown in Fig. 1 d, while Fig. 1 f, 1 e show what happens when $\mu$ is too large or too small compared to T.

These results conform with the results of other calculations[6,7], which show that damping is always destabilizing if $X < o$ and stabilizing if $X > o$.

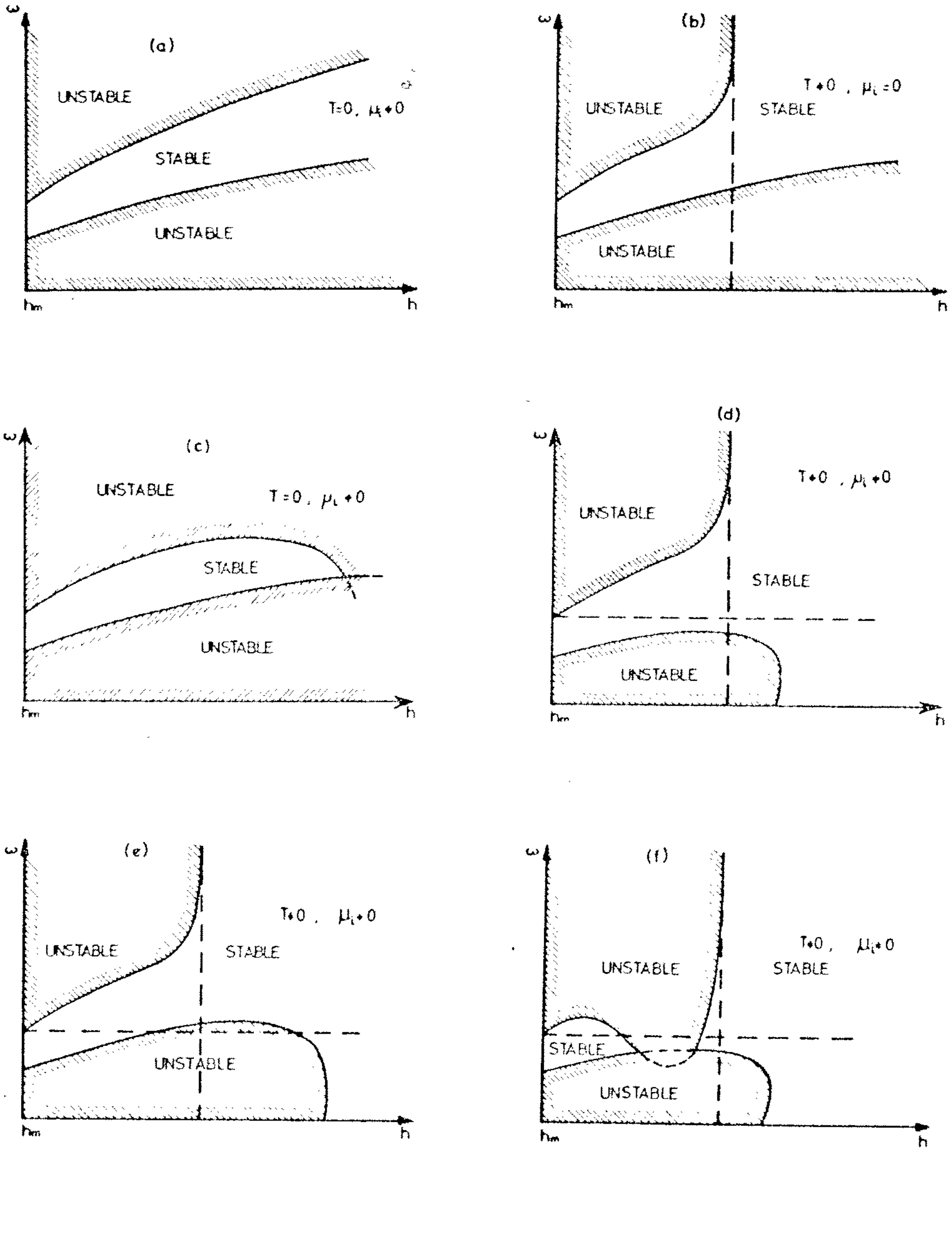


Fig. 1: Schematic representation of the first stability region ω(h) for different values of $\mu_i$, T. The diagram 1d shows the only stable case.The lower boundary delimits the parametric instability bands. There is an infinite number of bands below the first one, near the axis ω = o, which have all be englobed with the first band.

## Liquid-air interface

To illustrate the general statements presented above we choose to examine the stability of the liquid-air interface studied experimentally by Wolf. The frequency $\omega$ was sufficiently low for the compressibility of the gas to be neglected. The liquids used were mixtures of mineral oils and their surface tension were not known and could vary with $\mu$. The vessel was a tube of diameter D = 1.6 cm. The height of the liquid in the tube was large enough for it to be considered as infinite. Fig. 2 shows a comparison of the measurements with the calculated curves assuming T = 10 dynes/cm. We assumed the equivalent $h_m$ for a square vessel used by Wold namely $h_m = 2.315\,\mathrm{cm}^{-1}$. The agreement is very good particularly at larger values of $\mu$, bearing in mind the uncertainty on T. The simple equation

$$\omega = \frac{\Delta g}{\sqrt{2}g}\sqrt{\frac{gh_m(\rho_1-\rho_2) - Th_m^3}{\rho_1+\rho_2}} \qquad (8)$$

obtained with the quasi-potential approximation, reproduces quite well the upper stability boundary for all the values of $\mu$ considered.

Fig. 3 shows the stability diagram for fixed values of $\mu$ and T as a function of h. When X > o the lower boundary, which corresponds to the first parametric instability, has a turning point which moves towards $\Delta g = \infty$ as $h \to \infty$. The stability boundary shown in Fig. 2 is the envelope of all these curves. The other possible parametric instability regions, which are crowded near the axis $\omega = o$, are not shown since they seem experimentally inaccessible.

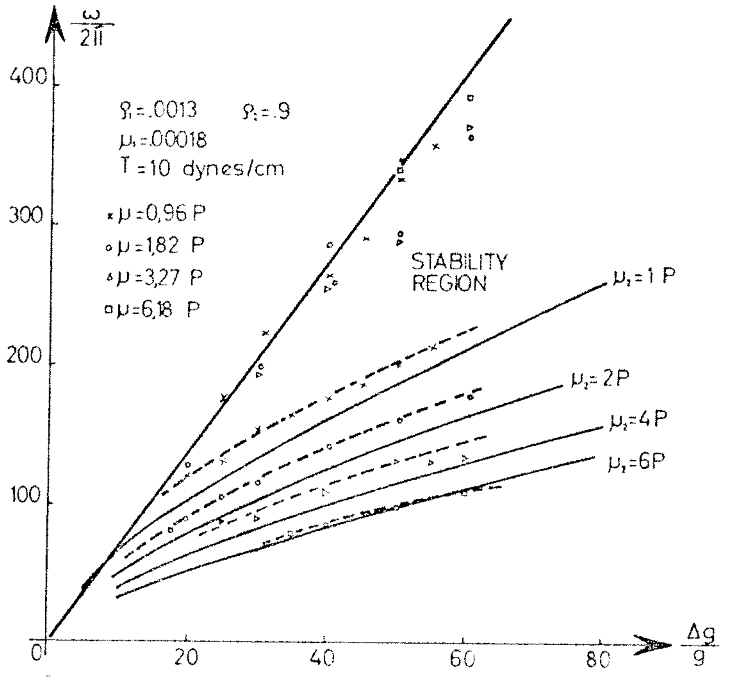


Fig. 2: Air-Oil System. Comparison of the computed stability domain with some experimental results. The continuous curves are the calculated ones. The dotted lines are drawn through the experimental points. The region of stability is between the upper continuous curve and one of the lower curves.

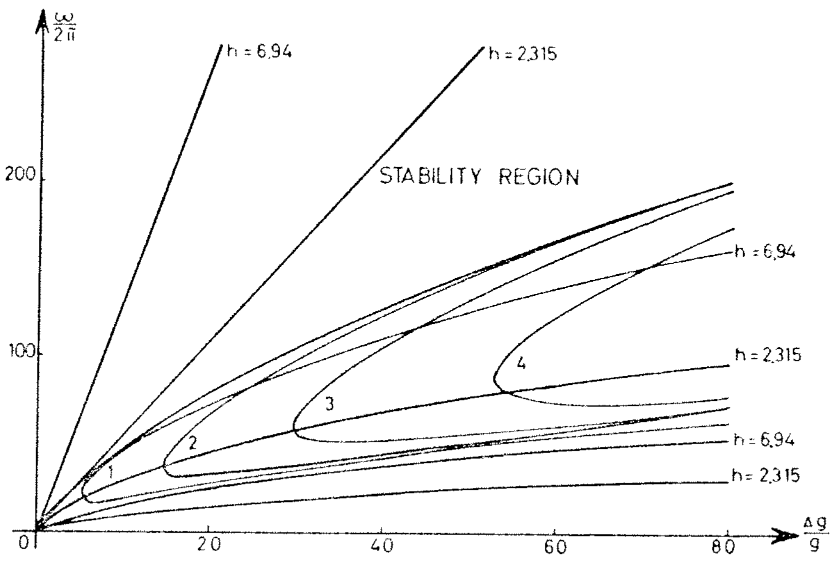


Fig. 3: Build-up of the $\mu_2$ = 4 P, T = 10 dynes/cm stability diagram from the individual stability diagrams for values of h = n $h_m$. The curves markes 1,2,3 and 4 correspond to h = 11.6, 16.7, 20.8 and 25.4 $cm^{-1}$ respectively. For these 4 modes X > 0 and there is no upper stability boundary. Only the first parametric instability band is shown.

Fig. 4 shows what happens when T = o. The "quasi-potential" boundary

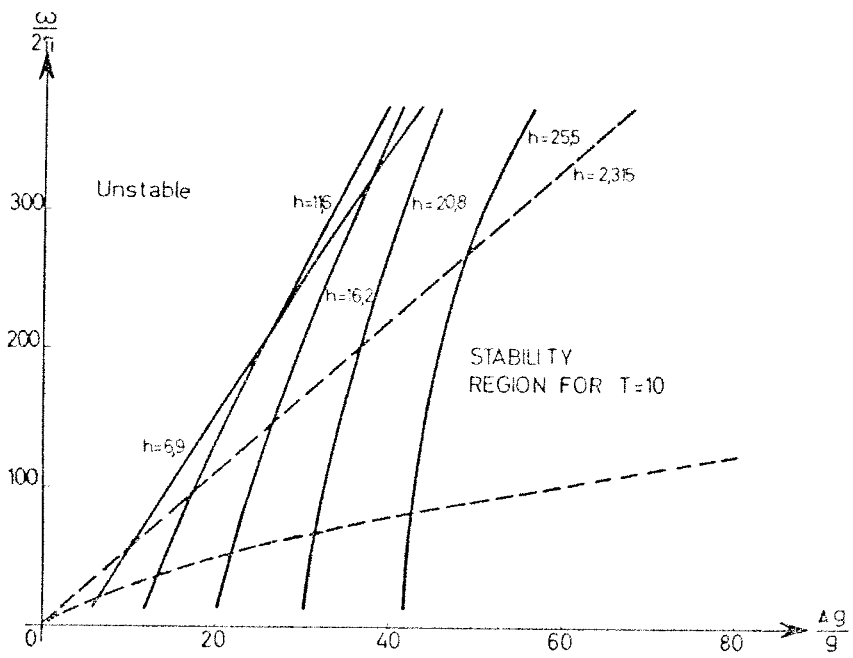


Fig. 4: Short wavelength instabilities for T = o, $\mu_2$ = 4P. The dotted lines delimit the stability region for T = 10, shown as a reference . Each solid line corresponds to a fixed value of h, with stability to the right of the line. The parametric instability regions are not shown.

for large h crosses the boundary for $h = h_m$. As $h \to \infty$ the whole stability region, shown for T = 10 as reference, disappears. This corresponds to the case shown in Fig. 1c. For T ≠ o, but very small, the same situation would arise, corresponding to Fig. 1f. This may have been already observed by Wolf who has noticed that for very large $\mu_2$ he could not obtain stability.

## Dynamic Stabilization of Plasma Confinement

The frequently used analogy between fluid and plasma suggests that analogous schemes should work for dynamic stabilization of plasma confinement. Two such schemes seem promising for high $\beta$ plasma confinement:

The first scheme consists of modulating the confining magnetic field. The modulated magnetic pressure will then produce a periodic motion of the plasma which should be the equivalent of the shaking of the vessel in Wolf's experiment. A low $\beta$ scheme of this type has been investigated by J. Wesson[9].

The second scheme consists of modulating the applied magnetic field in time and space such that the field line curvature at a given point oscillates in time. Such a scheme has been proposed by G. Berge and G.H. Wolf[8]. In this last scheme there is also an acceleration of the plasma but, at low frequency, it is less important than the change in curvature.*

In the fluid calculation we have assumed that the fluid is incompressible, viscous and possesses a surface tension. In a plasma, for frequencies $\omega \ll uh_m$, where u is the sound speed in the plasma, the plasma behaves as an incompressible fluid for long wavelengths perturbations. The plasma is always dissipative. There is no surface tension in a plasma but there are two mechanisms which are equivalent. In the case of a sharp boundary with no field inside the plasma the magnetic field acts as an anisotropic surface tention $T = \frac{p_o}{h} (\hat{h} \cdot \hat{B})^2$, where $p_o$ is the average magnetic pressure. For $\hat{h} \cdot \hat{B} = o$ there is no surface tension and there will be unstable short wavelengths instabilities. Taking into

---

*In ref. 8 the expressions do not seem to contain the effect of the acceleration of the plasma, but only the effect of the change in curvature.

consideration the finite magnetic field penetration into the plasma and finite ion Larmor radius effects the modes $\hat{h} \cdot \hat{B} \approx o$ can probably be stabilized[9,10]. The situation looks indeed promising with one restriction: the size of the amplitude of the oscillation has to be small to avoid the parametric instabilities of the modes $\hat{h} \cdot \hat{B} \approx 1$, which are weakly damped in the range of frequency considered. This means a domain of application limited to weakly unstable static configurations.

Until now we have considered the same configurations for the plasma as we have used for the fluids. In a realistic configuration the depth of the plasma is much smaller than its length. Consider, as an example, an axisymmetric plasma of radius a with no azimuthal magnetic field. Consider successively the various m values. The behaviour of the modes $m = o$, $h \neq o$ is not at all incompressible. For these modes the function $\tilde{R}(s)$ is such that there are no resonances which can be excited at low frequency ($\omega << u/a$). For a small amplitude of oscillation the wall stabilizing effect can be sufficient to make X sufficiently large for all modes and thus ensure stability of all modes.

The modes $m = 1$ are the most likely to bring instabilities. For $\omega << u/a$ the plasma moves as a string, with almost no compression or flow in the plasma. This implies very little dissipation. For $ah << 1$ we can thus write $\tilde{R}(s) \approx \rho as$ and $X \sim - X_o + (ah)^2 \, p_o$, where $X_o$ simulates the unstable force to be stabilized (curvature of the lines of force). We are thus in the situation depicted in Fig. 1b. Considering the small amount of dissipation expected, the situation shown in Fig. 1e could very well arise.*

Only detailed calculations with an accurate plasma model could give a definite answer. The second scheme where curvature changes in space

---

*In the case where Re $\tilde{R}(0) = 0$, it appears that the whole stability region for $X < 0$ disappears. (see for example in Ref. 6 the stability diagram for $H = 0$. (A) ). The importance and generality of this phenomenon is not known.

may help the long wavelengths by providing a second space averaging process.

For $m \geq 2$ geometrical effects should not be too important and qualitatively the preceeding results for the infinite geometry should apply.

In conclusion the analogy between a plasma and an incompressible fluid is not so complete as to guarantee that schemes of dynamic stabilization based upon these principles will work for plasma confinement. Long wavelength kink modes seem to be the most dangerous since in these modes the plasma moves as a whole and is thus undamped.

## Acknowledgments

The authors gratefully acknowledge the interest and encouragement of E.S. Weibel. One of the authors would like to thank G.H. Wolf for frequent and valuable discussions. This work was performed under the auspices of the Swiss National Science Foundation.

## References


1. Lord RAYLEIGH: Scientific Papers (Cambridge University, England, 1900) Vol. 2, pp. 200 - 207

2. G. TAYLOR: Proc. Roy. Soc. (London) Ser A 201, 192 (1950)

3. G.H. WOLF: Z. Physik 227, 291 (1969)

4. G.H. WOLF: Phys. Rev. Letters 24, 444 (1970)

5. F. TROYON: Phys. Fluids 10, 2660 (1967)

6. F. TROYON: (to be published)

7. E.S. WEIBEL: Proceedings Fourth European Conference on Controlled Fusion and Plasma Physics, Rome 1970

8. G. BERGE and G.H. WOLF: Phys. Rev. Letters 22, 1096 (1969)

9. J. WESSON: Phys. Fluids 13, 761 (1970)

10. G.H. WOLF: Proceedings of the "Symposium on Feedback and Dynamic Control of Plasmas", Princeton (1970)