Novembre 1965 LRP 18-65

LABORATOIRE DE RECHERCHES SUR LA PHYSIQUE DES PLASMAS
FINANCÉ PAR LE FONDS NATIONAL SUISSE DE LA RECHERCHE SCIENTIFIQUE

THE ANOMALOUS SKIN EFFECT IN A PLASMA.

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I Introduction and Results

Any material which satisfies an Ohm's law

$$j = \sigma(\omega) E \qquad (1)$$

for sinusoidally varying fields exhibits a skin effect. (Natural units are used except where MKS units are explicitely indicated.) As the field progresses from the surface towards the interior, it decreases exponentially, the depth of penetration δ being given by

$$\delta^{-1} = \mathcal{Re}\,(i\omega\sigma)^{1/2} \qquad (2)$$

Consider now a plasma in which the ion motion is neglected. Let u be the mean thermal speed of the electrons, ν their collision frequency and $\ell = u/\nu$ their mean free path. Further denote by d = u/ω the distance travelled by the average electron in 1/2π period of the applied field, and define the quantities ω_p, s, ε, δ_o by the relations

$$\omega_p^2 = \frac{e^2 n}{m}$$

$$\varepsilon = tg^{-1}\frac{\nu}{\omega}$$

$$s = \frac{i\omega + \nu}{|i\omega + \nu|} = i\,e^{-i\varepsilon}$$

$$\delta_o = \left(\frac{|i\omega + \nu|}{\omega_p^2\,\omega}\right)^{1/2}$$

If either the distance d or the mean free path is small, compared to the scale of spacial variation, that is, if

$$\ell \ll \delta_o \quad \text{or} \quad d \ll \delta_o$$

then (1) applies with

$$\sigma = \frac{e^2 n}{m(i\omega + \nu)} = \frac{\omega_p^2}{i\omega + \nu}$$

and the normal skin effect, given by (2) obtains :

$$\delta = \delta_0 / \cos(\varepsilon/2)$$

On the other hand, if both

$$\ell \gg \delta_0 \quad \text{and} \quad d \gg \delta_0$$

then the relation between current and electric field is no longer local. "If the mean free path of the electrons is comparable with the depth of penetration of the field, an electron during one free path will be moving through regions of varying field and the drift velocity which it acquires will be related to the field strength at all points along its path" (2). Hence the current at any one point in the plasma is determined by all the field values within a certain distance.

We shall consider a semi-infinite plasma with a sharp boundary at x = 0 and extending to $+\infty$. For the propagation of a transverse wave in the x-direction the relation between current and field has the form

$$j(x) = \omega_p^2 \int_0^\infty K(\omega, x-x') \, E(x') \, dx'$$

The form of the kernel depends on the type of reflection which the electrons suffer at the surface, and on the unperturbed velocity distribution.

In this paper we assume that the electrons are specularly reflected at x = 0*. An arbitrary velocity distribution will be assumed for the derivation of the principal equations. The problem will be solved explicitely for a Maxwellian distribution using the Boltzmann equation with a relaxation term and Maxwell's equations without displacement current. The solution E (x) will be obtained in the form

$$E(x) = E(\lambda, s; z)$$

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* The much more difficult case of diffuse reflection will be treated in a subsequent report.

where z is a normalised distance

$$z = \frac{|i\omega + \nu|}{u}\, x$$

The parameters s and

$$\lambda = \frac{u^2 \omega_p^2 \omega}{|i\omega + \nu|^3} \qquad (3)$$

determine the shape of the curve. The parameter λ can also be expressed as

$$\frac{1}{\lambda} = \delta_o^2 \left[d^{-2} + \ell^{-2} \right]$$

Only if $\lambda \ll 1$ does the well known formula (2) apply.

Taking for the collision frequency

$$\nu = \nu_o n T^{-3/2} \qquad \text{(MKS)}$$

with

$$\nu_o = \frac{1}{8} \frac{e^4}{\varepsilon_o^2 m^{1/2} k^{3/2}} \qquad \text{(MKS)}$$

one can express (3) as follows :

$$\omega\lambda = b \frac{T(n/\omega)}{\left[1 + \nu_o^2 (n/\omega)^2 T^{-3}\right]^{3/2}} \qquad \text{(MKS)}$$

with

$$b = \frac{2 k e^2}{\varepsilon_o m_e^2 c^2} \qquad \text{(MKS)}$$

This relation lends itself to the graphic representation of Fig. 1. The curves of constant ωλ and those of constant ν/ω allow a rapid estimate of λ and s for a given temperature T, density n and angular frequency ω.

For $\lambda \geqslant .3$ the classical formula fails. It will be shown that if $\lambda \gg 1$ the following approximate formulas hold for the skin depth δ and the surface impedance Z :

$$\delta = \frac{8}{9\pi^{1/6}} \left(\frac{c^2 u}{\omega_p^2 \omega} \right)^{1/3} \qquad \text{(MKS)}$$

$$Z = \frac{4}{3\sqrt{3}\,\pi^{1/6}} \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \left(\frac{\mu_0^2 \omega^2 u}{\varepsilon_0 \omega_p^2} \right)^{1/3} \qquad \text{(MKS)}$$

The formula for Z agrees up to a constant factor with the one found by Reuter and Sondheimer (5) for a degenerate Fermi distribution. If one takes for u the speed at the Fermi surface then their value of Z is 12.8 times larger than ours. We have not found the reason for this difference.

In the extreme anomalous case $\lambda \gg 1$ neither skin depth δ nor the surface impedance Z depend on the collision frequency. The dissipation of energy is independent of ν and present even if $\nu = 0$
A better approximation to δ and Z is given in section 8.

The analysis was completed by numerical calculations carried out on the IBM 1040 computer of the Ecole Polytechnique de Lausanne. Figs. 8 and 9 show E as a function of distance from the surface, for various values of λ and s. The curves for the absolute value of E are not monotonic, which is quite unexpected. The increase in amplitude which occurs at a certain depth in all cases where $\lambda > 1$ is a property of the model which has been investigated, and not due to any error of computation*. Physically this effect can be explained by electrons which carry momentum acquired near the surface into deeper layers where they are out of phase with the field. Mathematically it can be traced to interference of the term due to the boundary and the terms that are also present in the infinite medium. While the ratio of a minimum to a maximum can be quite large, the effect is nearly negligible in terms of the amplitude at the surface and probably unobservable.

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* Several curves were computed by two entirely independent methods with identical results.

II Prior Work

In his famous paper on "the vibrations of an electron plasma" L. Landau (1) treats also the problem of the penetration of a longitudinal field into a semi infinite plasma, with $\nu = 0$ and for specular reflection. In this case E satisfies an integral equation with a kernel that is different from ours, but which has the same asymptotic behaviour. Only this asymptotic form of E (x) is given, which is similar to the behaviour of the transverse field, but not identical. From a practical point of view it is useless, since it applies only at distances where E has decreased to a tiny fraction of its value at the surface. The case of ω near ω_p is treated in detail. From a formal point of view the theory is marred by the use of a macroscopic concept, the dielectric constant of the plasma, where as it ought to be developped entirely from Boltzmann's equation.

The following papers discuss the skin effect in metals, and for a degenerate fermi distribution.

A.B. Pippard (2) observed disagreement of measured surface impedance in metals at low temperature with classical skin effect formulae. He gives an intuitive solution by assuming that only those electrons which traverse a whole mean free path within the depth of field penetration contribute to the impedance. These "effective electrons make angle with the surface of equal or less than arcsin (δ/ℓ), l being the mean free path and δ the yet undetermined skin depth". Using this effectiveness concept he obtains expressions for the skin depth and surface impedance which are qualitatively correct. It does not appear to be easy to obtain a justification of Pippard's concept on the basis of the exact treatment. In later papers (3), (4) the "ineffectivness" concept is further developed.

Reuter and Sondheimer (5) treat the anomalous skin effect for metals with a degenerate Fermi distribution and including a collision frequency. Propagation of the transverse wave is normal to the metal surface. Both diffuse and specular reflection of electrons are treated. The analytic behaviour of the kernel and of E(x) is quite different from our case of a plasma with a Maxwellian velocity distribution. Expressions for the surface impedance are given, but not for the skin depth.

Smith (6) gives a comprehensive list of references on the anomalous skin effect and surface impedance in anisotropic metals.

Demirkhanov, Kadysh and Khodyrev (7) have measured the anomalous skin effect in a plasma at 9, 5.6 and 4.6 Mc and densities of $n_e = 10^{11}$ to $10^{13}\,cm^{-3}$ and electron temperatures of 2 to 10 x 10^4 ^oK.

R. Keller (8) observed the anomalous skin effect in a plasma at 4.8 Mc, $n_e = .5$ to 3 $10^{15}\,cm^{-3}$ and Te = 5 to 20 10^4 ^oK.

III Analysis

1. The perturbation of the velocity distribution

The plasma occupies the half space $x > 0$. The equilibrium velocity distribution is assumed to have the form

$$f_0(\underline{v}) = F(v_x^2,\ v_y^2 + v_z^2)$$

An electromagnetic wave travels in the x-direction such that B and E are parallel to the y and z axes respectively. Let all fields depend on time as exp (iωt). Then they obey the equations

$$\frac{d^2E}{dx^2} + \omega^2 E = i\omega j$$
$$B = \frac{1}{i\omega}\frac{dE}{dx} \qquad (4)$$

The distribution function shall be governed by the Boltzmann equation with a collision term of the form $-\nu f$ where f is the perturbation. Thus

$$(i\omega+\nu)f + v_x \frac{\partial f}{\partial x} = -\frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right)\frac{\partial f_0}{\partial \underset{\sim}{v}}$$

$$= -\frac{2e}{m}\left[v_z F_\perp E + \frac{v_x v_z}{i\omega}\left(F_\perp - F_\parallel\right)\frac{dE}{dx}\right] \qquad (5)$$

$$= v_x R(x, \underset{\sim}{v})$$

Here $F_\parallel$ and $F_\perp$ denote the derivatives $\partial F(\xi,\eta)/\partial\xi$ and $\partial F(\xi,\eta)/\partial\eta$ respectively. The general solution of (5) is

$$f = \exp\left(-\frac{i\omega+\nu}{v_x}x\right)\left[\int_0^x \exp\left(\frac{i\omega+\nu}{v_x}x'\right)R(x', \underset{\sim}{v})\,dx' + \varphi(\underset{\sim}{v})\right]$$

The function $\varphi(v)$ is determined by the boundary conditions. For $x \to \infty$ f must remain finite. At x = 0 the particles are reflected. Two extrem cases are specular reflection and diffuse reflection.

In the latter case the electrons are reemitted from the boundary into the unperturbed distribution. The boundary condition has the form

$$f(x=0, \underset{\sim}{v}) = \varepsilon f(x=0, \underset{\sim}{v}^*) \quad , \quad v_x > 0$$

where v^* is obtained form v by changing the sign of v_x.
For specular reflection $\varepsilon = 1$, for diffuse reflection $\varepsilon = 0$.

Thus

$$f = \int_{-\varepsilon\infty}^{x} \exp\left(\frac{i\omega+\nu}{v_x}(x'-x)\right) R(x', \underset{\sim}{v})\, dx' \qquad v_x > 0$$

$$f = \int_{\infty}^{x} \exp\left(\frac{i\omega+\nu}{v_x}(x'-x)\right) R(x', \underset{\sim}{v})\, dx' \qquad v_x < 0$$

2. The current density

The current density is

$$j(x) = en \int v_z\, f(x, \underset{\sim}{v})\, dv_x\, dv_y\, dv_z \qquad (6)$$

It is convenient to express F as

$$F(\tau, \xi) = \partial^2 \phi(\tau, \xi)/\partial \xi^2 = \phi_{\xi\xi}(\tau, \xi)$$

with

$$\phi(\tau, \infty) = \phi_\xi(\tau, \infty) = 0$$

The field E(x) is of interest only for $x > 0$. It can be continued at will into $x < 0$. The most useful definition is

$$E(-x) = E(x) \qquad (7)$$

Performing the integrations of (6) one obtains

$-j\ (x)/\ \pi\omega_p^{\ 2}$

$$= \int_{-\varepsilon\infty}^{x} dx' \left[I_1((i\omega+\nu)(x-x'))\, E(x') + \frac{1}{i\omega} I_2((i\omega+\nu)(x-x'))\, E(x') \right]$$

$$+ \int_{+\infty}^{x} dx' \left[-I_1(-(i\omega+\nu)(x-x'))\, E(x') + \frac{1}{i\omega} I_2(-(i\omega+\nu)(x-x')\, E(x') \right]$$

Where

$$I_1(\alpha) = \int_0^\infty e^{-\alpha/v}\, \phi_\xi(v^2, 0)\, dv/v$$

$$I_2(\alpha) = \int_0^\infty e^{-\alpha/v} \left[\phi_\xi(v^2, 0) - \phi_\tau(v^2, 0) \right] dv$$

If one whishes to study the skin effect for anisotropic distributions (for beams of particls in the x direction, for instance) one must take into acount I_2, which arises from the term $(\underset{\sim}{v} \times \underset{\sim}{B})\, \partial f_0 / \partial \underset{\sim}{v}$.

For isotropic distributions one has

$$\phi(\tau, \xi) = \phi(\tau + \xi)$$

$$I_2 = 0 \qquad I_1(\alpha) = -(1/\pi)\ H(\alpha)$$

$$H(\alpha) = -\pi \int_0^\infty \phi'(v^2)\, e^{-\alpha/v}\, dv/v$$

and

$$j(x) = \omega_p^2 \int_{-\varepsilon\infty}^{\infty} H\,((i\omega+\nu)|x-x'|)\, E(x')\, dx' \qquad (8)$$

For a completely degenerate Fermi distribution one must take

$$\phi'(\tau) = \begin{cases} \frac{3}{4\pi u^2}(\tau - u^2) & \tau < u^2 \\ 0 & \tau > u^2 \end{cases}$$

from which one obtains the kernel of Sondheimer and Reuter (5)

$$H(\alpha) = \pi u^2 \int_1^\infty \left(\frac{1}{\xi} - \frac{1}{\xi^3}\right) e^{-\frac{\alpha\xi}{u}} d\xi .$$

For a Maxwellian distribution one must take

$$\phi'(\xi) = - \frac{\exp(-\xi / u^2)}{\pi^{3/2}\, u}$$

and one obtains

$$H_u(\alpha) = \frac{1}{\sqrt{\pi}} u^{-1} \int_0^\infty e^{-\frac{\alpha}{u\xi} - \xi^2} d\xi / \xi \qquad (9)$$

One notices that

$$\int_0^\infty H_u(\alpha)\, d\alpha = \frac{1}{2}$$

and that

$$\int_0^\infty \varphi(\alpha)\, H(\alpha)\, d\alpha = \frac{1}{2}\varphi(0) \qquad (10)$$

so that $H_u(\alpha)$ tends to a Dirac delta function as u tends to zero.

3. The integrodifferential equation for E

Combining equations (4), (8) and (9) and neglecting the displacement current, one obtains the integrodifferential equation for E :

$$\frac{d^2E}{dx^2} = i\omega\omega_p^2 \int_{-\infty}^{\infty} H_u((i\omega+\nu)|x-x'|)\, E(x')\, dx' \qquad (11)$$

Letting $u \to 0$ or $\nu \to \infty$ (11) simplifies to

$$\frac{d^2E}{dx^2} = \frac{i\omega\omega_p^2}{i\omega+\nu} E$$

by virtue of (10). This equation describes the ordinary skin effect as discussed in the introduction.

By a simple change of scale

$$x = \frac{u}{|i\omega+\nu|} z$$

one can cast equation (11) into the standard form

$$\frac{d^2E}{dx^2} = i\lambda \int_{-\varepsilon\infty}^{\infty} H(|z-z'|)\, E(z')\, dz' \tag{12}$$

where $H(\alpha)$ is the function $H_u(\alpha)$ for u = 1

$$H(\alpha) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\alpha/\xi - \xi^2} d\xi / \xi$$

From now on we specialise further to the case of specular reflection : $\varepsilon = 1$. In this case equation (12) can be solved formally by means of Fourier transformation. It is important to keep in mind that the function continued symmetrically on to $x<0$ by (7) has discontinuous derivatives. The second derivative contains a delta function $2E'(0)\,\delta(z)$ which must be subtracted :

$$\hat{E}(k) = \int_{-\infty}^{\infty} e^{-ikz} E(z)\, dz$$

$$\hat{E''}(k) = -k^2 \hat{E}(k) - 2E'(0)$$

Defining

$$h(k) = \int_{-\infty}^{\infty} e^{-ikz}\, H(|z|)\, dz$$

equation (12) becomes

$$\hat{E}(k) = -\frac{2E'(0)}{k^2 + i\lambda h(k)}$$

and the inverse transform gives

$$E(z) = -\frac{E'(0)}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikz}\, dk}{k^2 + i\lambda h(k)} \qquad (13)$$

This is the formal solution of the problem.

To extract information about the behaviour of E(z) from (13) one must first investigate h(k).

4. The Fourier transform h(k)

Let
$$H(z) = \frac{1}{\sqrt{\pi}}\frac{dK}{dz},$$

$$K = \int_{0}^{\infty} e^{-z/t - t^2}\, dt$$

The function K(z) satisfies the differential equation

$$2K + K'' + zK''' = 0$$

and for $|z| \gg 1$, $\mathcal{Re}\, z > 0$ behaves like

$$K \approx \exp -3(z/2)^{2/3}$$

Hence the Fourier transform h(k) is, at first only defined for real values of k :

$$h(k) = \frac{1}{\sqrt{\pi}} \frac{1}{ik} \int_{-\infty}^{\infty} \frac{e^{-\xi^2} d\xi}{\xi - is/k} = \begin{cases} \frac{1}{ik} Z\left(\frac{is}{k}\right) \\ -\frac{1}{ik} Z\left(-\frac{is}{k}\right) \end{cases}$$

Here $Z(\xi)$ is the plasma dispersion function as defined and tabulated by Fried and Conte (9). Obviously there is no analytic function representing h(k) on the whole real axis. But it is possible to extend each representation analytically into the upper half plane. It will be convenient to divide the upper half plane into two regions separated by the straight line passing from 0 through is to infinity. The domain adjoining the positive axis will be called domain I, the other domain II. Thus one can define

$$h(k) = \frac{1}{ik} Z\left(\frac{is}{k}\right) \qquad \text{Domain I}$$

$$h(k) = -\frac{1}{ik} Z\left(-\frac{is}{k}\right) \qquad \text{Domain II}$$

5. Deformation of path of integration

Moving the path of integration from the real axis into the upper half plane allows to split the integral (13) into three contributions : a) the residues from domain I, b) the residues from domain II, and c) the integral along the boundary of I and II (= the "cut"). It will be shown that there is at most one residue in each domain :

$$E(z) = -i2\, E'(0) \sum_{1,2} \frac{A_j \exp(ik_j z)}{dD_j(k_j)/dk_j}$$

$$+ i \frac{2}{\sqrt{\pi}} E'(0)\, \lambda \int_0^{\infty} \frac{\exp(-\tau^{-2} - \tau z)\, d\tau}{\tau\, D_1\, D_2} \tag{14}$$

where A_i are either zero or one and

$$D_{\substack{1\\2}} = k^2 \pm \frac{\lambda}{ik} Z\left(\pm \frac{is}{k}\right) \quad , \quad \tau = k/is \tag{15}$$

Use was made of the relation

$$Z(\xi) + Z(-\xi) = i2\sqrt{\pi}\, \exp(-\xi^2)$$

Equation (14) shows that the spectrum of the Vlasov operator (5) with the present boundary conditions has a discrete and a continuous part.

6. The residues

One must first establish the number of zeros of the functions $D_1(k)$ and $D_2(k)$. The simplest method is provided by Nyquist. The functions are analytic every where except at $k = 0$. Thus the number of zeros of D_j in the domain j equals the number of times $D_j(k)$ turns around zero as k moves once completely around the domain. Instead of constructing the Nyquist diagrams for each value of λ and s we draw those of

$$\pm \frac{1}{ik} Z\left(\pm \frac{is}{k}\right)$$

by using the tables of Fried and Conte (9) for a few values of s as shown in Figs 2 to 6. The effect of adding the term k^2/λ can be inferred by inspection. In this manner one convinces oneself that there is one root or none of (14) in each domain.

Actually there is always one root for each function D_1 and D_2 in the second quadrant, and these are the only ones that may fall into the respective domains. Using the expansions of $Z(\xi)$ for small and large argument one obtains approximate values for the roots of (14)

$$\lambda \gg 1 \qquad k_1 = i\,\pi^{1/6}\,\lambda^{1/3} - \frac{2s}{3\sqrt{\pi}}$$

$$k_2 = \exp\left(i\,\frac{2\pi}{3}\right)\pi^{1/6}\,\lambda^{1/3} + \frac{2s}{3\sqrt{\pi}}$$

$$\lambda \ll 1 \qquad k_1 = k_2 = \exp\left(i\,\frac{3\pi}{4}\right)(\lambda/s)^{1/2} = i\exp(i\,\varepsilon/2)\sqrt{\lambda}$$

Depending on whether the cut passes above, between or below the two roots, one has the three cases A, B, C :

$$\begin{aligned} &A : A_1 = 0 \qquad && A_2 = 1 \\ &B : A_1 = 1 && A_2 = 1 \qquad\qquad (16) \\ &C : A_1 = 1 && A_2 = 0 \end{aligned}$$

A_j being the number of roots in the domain j. For $\lambda \ll 1$ there is a root in I for $0 \leq \varepsilon < \pi/6$ and none for $\pi/6 < \varepsilon \leq \pi/2$ and the reverse applies for domain II. For larger values of λ the situation depends in detail on the values of λ and s. If $\lambda \gg 1$ and if ε increases from zero to $\pi/2$ then the cut crosses over k_1 for $\varepsilon_1 \cong 2/(3\pi^{2/3}\lambda^{1/3})$, going thereby from case A to case B. At $\varepsilon = \pi/3 - \varepsilon_1$ the cut crosses over the root k_2 and excludes it from domain II, going thereby from case B to case C. Fig. 7 shows the cut for $\varepsilon = \pi/4$ and the roots for various values of λ.

For the residues one can give simple approximate expressions in the limiting cases $\lambda \ll 1$ and $\lambda \gg 1$

$$\lambda \ll 1 \qquad \frac{dD_j}{dk_j} = 2k_j$$

$\lambda \gg 1$:

$$\frac{dD_1}{dk_1} = i\,3\pi^{1/6}\lambda^{1/3} - \frac{4}{\sqrt{\pi}}s \qquad (17)$$

$$\frac{dD_2}{dk_2} = \exp\left(i\frac{5\pi}{6}\right) 3\pi^{1/6}\lambda^{1/3} + \frac{4}{\sqrt{\pi}}s$$

7. The integral along the cut.

For small values of λ the integral along the cut is negligible and the residue (there is exactly one) yields the well known formula of the ordinary skin effect.

For $\lambda \gg 1$ the integral along the cut can be approximatly evaluated : if z is very large, the method of the saddle point can be used. If z is very small we calculate the integral and its first few derivatives for z = 0. Introducing :

$$\alpha = \pi^{1/6}\lambda^{1/3} \qquad\qquad A = i\,\frac{4s}{\sqrt{\pi}}$$

one may approximate the integral for

$$\alpha \gg 1 \,,\quad z \gg 1 \qquad I = -\frac{s^2}{\lambda^2}\int_0^{\infty} e^{-1/\tau^2 - s\tau z}\, d\tau/\tau \qquad (18)$$

and for

$$\alpha \gg 1 \,,\quad z \ll 1 \qquad I = \int_0^{\sigma\infty} \frac{\exp(-z\xi)\,\xi\, d\xi}{\xi^6 - \alpha^6 - A\alpha^3\xi^2} \qquad (19)$$

Here the expansion for large and for small arguments of $Z(\xi)$ have been used.

For the first integral (18) one finds a saddle point at $\tau_0 = (2/sz)^{1/3}$ in the 4th quadrant. The contribution to E from this integral becomes

$$E_{out} = -i\frac{2}{\sqrt{3}}\,\frac{s^2}{\lambda}\left[\left(\frac{2}{sz}\right)^{1/3} - \frac{1}{12}\left(\frac{2}{sz}\right)\right] e^{-3\left(\frac{sz}{2}\right)^{2/3}}$$

This asymptotic expression is the same in all three cases A, B, C. At large distance it dominates the contribution of the residues, since it decays more slowly than exponentially. From a practical point of view this expression is quite useless as it becomes accurate enough only at a very large distance. (e.g. for $\lambda = 100$ the error is still 20% at 100 skin depths where the field amplitude is down to 10^{-9} of its surface value).

From (19) one finds the derivatives of I(z) at z = 0 :

$$\left.\frac{d^m I}{dz^m}\right|_{z=0} = I_m = \frac{(-)^{m+1}}{\alpha^{4-m}} \sum_1^6 \frac{t_j^m \lg t_j}{6t_j^4 - 2A/\alpha}\,, \quad m \leq 3 \qquad (20)$$

where the tj are the six roots of the equation

$$t^6 - 1 - \frac{A}{\alpha}t^2 = 0$$

For large values of $\alpha = \pi^{1/6}\lambda^{1/3}$ tj approximates exp $(i\pi j/3)$. Taking the term A/α to first order into account one obtains

$$t_j = \exp\left(i\,\frac{\pi j}{3}\right) + (-)^j \frac{A}{6\alpha}$$

Substituting these values into (20) for m = 0 yields, after laborious algebra, the values

$$I_A = -\frac{\pi}{6\alpha^4}\left\{\frac{1}{\sqrt{3}} - i + \frac{A}{\alpha}\left(\frac{1}{3\sqrt{3}} + \frac{i}{3}\right)\right\}$$

$$I_B = -\frac{\pi}{6\alpha^4}\left\{\frac{1}{\sqrt{3}} + i + \frac{A}{\alpha}\left(\frac{1}{3\sqrt{3}} - \frac{i}{3}\right)\right\} \qquad (21)$$

$$I_C = \frac{\pi}{3\sqrt{3}\,\alpha^4}\left(1 + \frac{A}{3\alpha}\right)$$

The subscripts A,B,C designate the three possible cases in which the cut passes above, between or below the two roots of $D_1 \; D_2 = 0$.

By varying s one passes from case A to B and to C. One might think at first, that E would be discontinuous in s. However the discontinuities of the integral are just compensated by the residues which are present or not according to (15). This must be so since E is analytic as a function of $(\nu + i\omega)$.

Combining (14), (16), (17) and (21) gives the following expression for E (0) :

$$E(0) = E'(0)\left\{\frac{2}{3\alpha}\left(-1 + i\frac{1}{\sqrt{3}}\right) + \frac{2A}{9\alpha^2}\left(1 + i\frac{1}{\sqrt{3}}\right)\right\}$$

It is possible in principle to obtain the second and third derivatives of E from (20). Calculating the first derivative provides a check of the formula (20).

8. The surface impedance and depth of penetration

Defining the surface impedance

$$Z = E(0)\Big/\int_0^\infty j(x)\,dx = -\,i\omega\,E(0)/E'(0)$$

gives at once

$$Z = \frac{4e^{i\pi/3}}{3\sqrt{3}\,\pi^{1/6}}\left(\frac{\omega^2 u}{\omega_p^2}\right)^{1/3} + \frac{16\,e^{i\pi/6}}{9\sqrt{3}\,\pi^{5/6}}(i\omega+\nu)\left(\frac{\omega}{u\,\omega_p^4}\right)^{1/3}$$

or

$$Z = \frac{4e^{i\pi/3}}{3\sqrt{3}\,\pi^{1/6}}\left(\frac{\mu_0^2\omega^2 u}{\varepsilon_0\,\omega_p^2}\right)^{1/3} + \frac{16\,e^{i\pi/6}}{9\sqrt{3}\,\pi^{5/6}}(i\omega+\nu)\left(\frac{\mu_0\omega}{\varepsilon_0^2\,\omega_p^4\,u}\right)^{1/3} \quad \text{(MKS)}$$

Since the electric field does not fall off exponentially, it is not practical to define the skin depth as the depth at which it assumes 1/e of its surface value. Instead we define it simply as

$$\delta = \frac{2\,dx}{d\,\lg(EE^*)}\bigg|_{x=0} = \frac{8}{9\pi^{1/6}}\left(\frac{u}{\omega_p^2\omega}\right)^{1/3}$$

or

$$\delta = \frac{8}{9\pi^{1/6}}\left(\frac{c^2 u}{\omega_p^2\omega}\right)^{1/3}$$

9. Numerical computations

The IBM 7040 computer of the Ecole Polytechnique de Lausanne, was used to evaluate E(z) numerically using the formula (14). As a check, the original integral was also used for a few cases with identical results. In particular the values of E(z) near the "bumps" were all verified by direct integration of (13). For the plasma dispersion function Z (ζ), a subprogram was used which is due to F.Troyon. The results of these calculations are shown in Figs 8, 9. From Fig. 9 it is clear that the contributions of the residues and of the integral along the cut are of about equal importance near the surface.

Acknowledgement : It is a pleasure to acknowledge many stimulating discussions with F. Troyon.

Thanks are also due to Mr. Roux for hand calculations.

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Figures :

1. λ and $s = i \exp(-i\varepsilon)$ as functions of density n, Temperature T, and angular frequency ω. $\varepsilon = tg^{-1}(\nu/\omega)$

2. Map of domain I for the function $Z(is/k)/ik$, $s = 1$

3. Map of domain II for the function $-Z(-is/k)/ik$, $s = 1$

4. Map of domain I for the function $Z(is/K)/ik$, $s = \sqrt{i}$

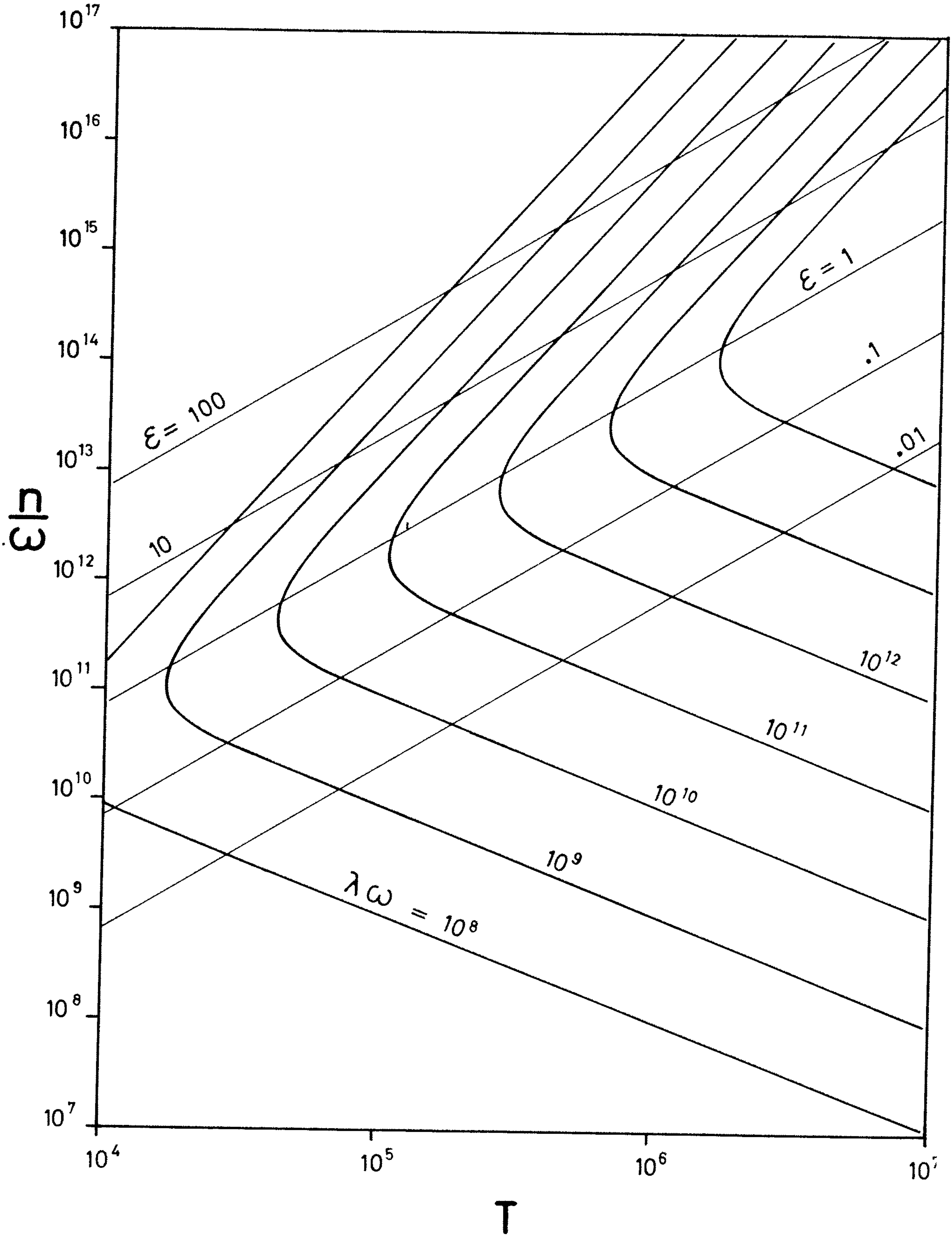
5. Map of domain II for the function $-Z(-is/K)/ik$, $s = \sqrt{i}$

6. Map of domain I for the function $Z(is/K)/ik$, $s = i$

7. Roots of $D_1(K)\,D_2(K) = 0$ for $s = \sqrt{i}$ and for λ = .1, .3, 1, 10, 100.

8. The field amplitude E as a function of normalized depth $z = \frac{|i\omega+\nu|}{u}x$ for $s = \sqrt{i}$ and λ = .3, 1, 10, 100.

9. Real and imaginary parts of the integral along the cut, C_r, C_i and of the residues 1_r, 1_i and 2_r, 2_i for $\lambda = 100$, $s = \sqrt{i}$



10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
$\frac{n}{\omega}$
$\varepsilon = 100$
10
$\varepsilon = 1$
.1
.01
10^{12}
10^{11}
10^{10}
10^{9}
$\lambda\omega = 10^{8}$
10^{4}
10^{5}
10^{6}
10^{7}
T

Fig. 1

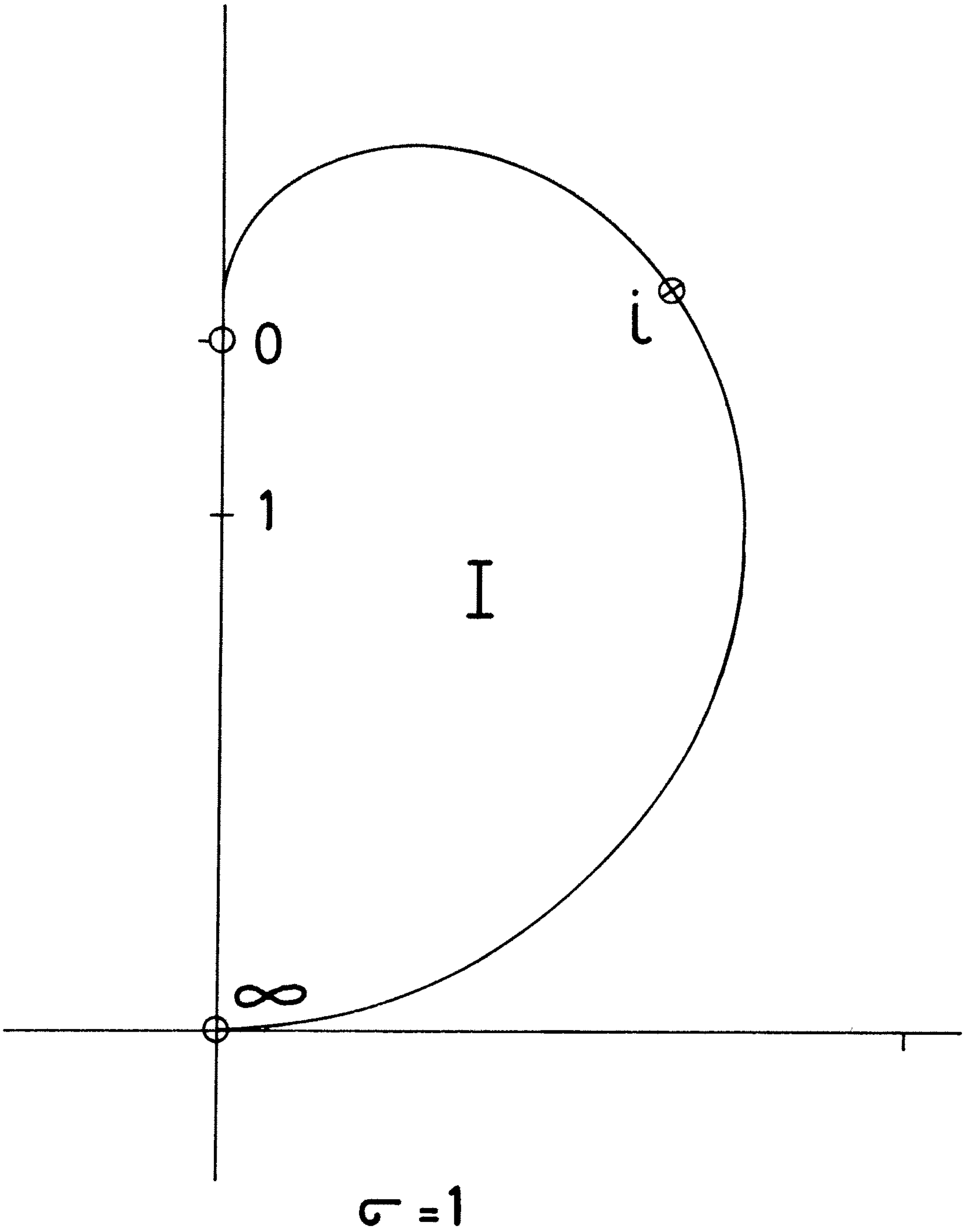


Fig. 2

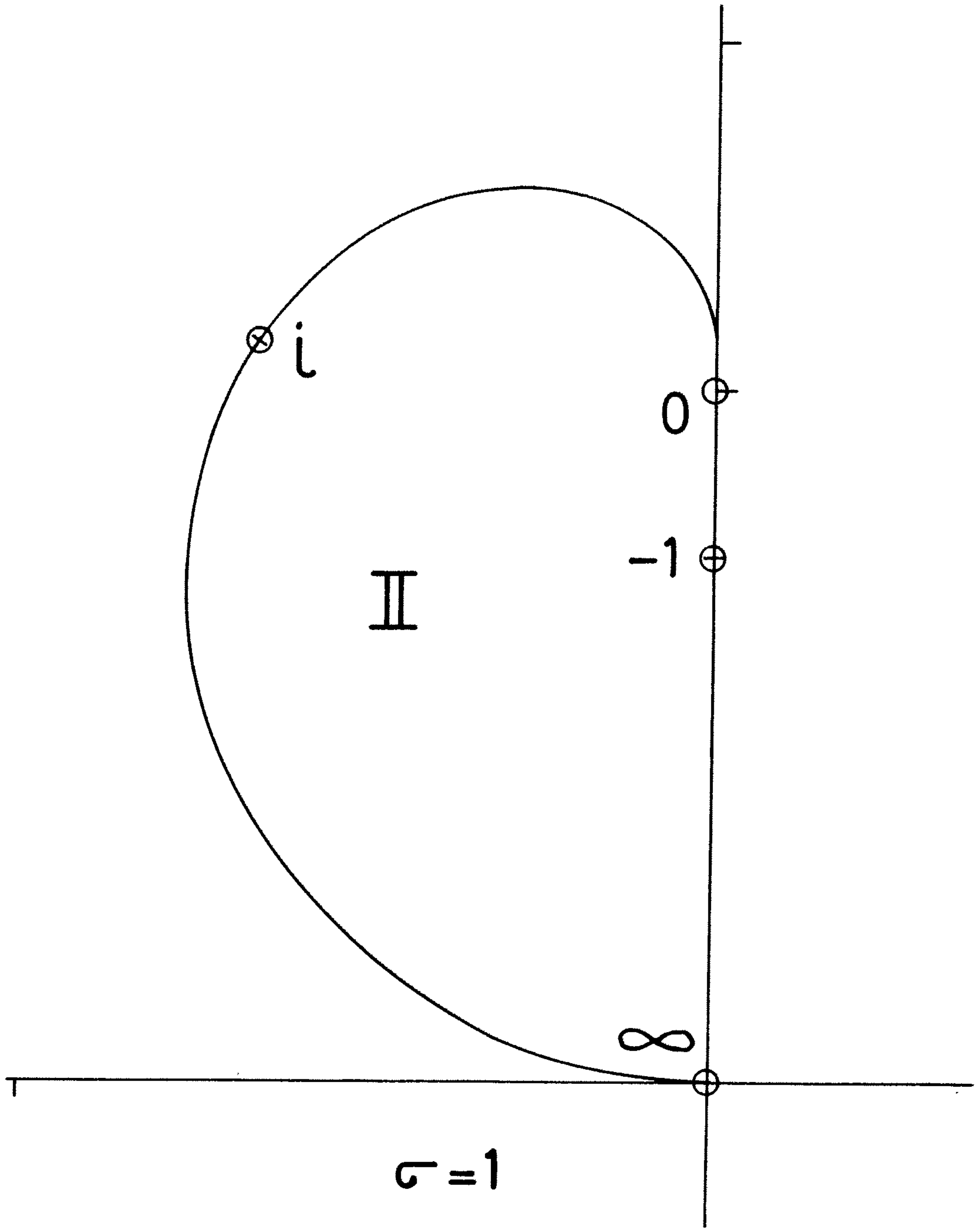
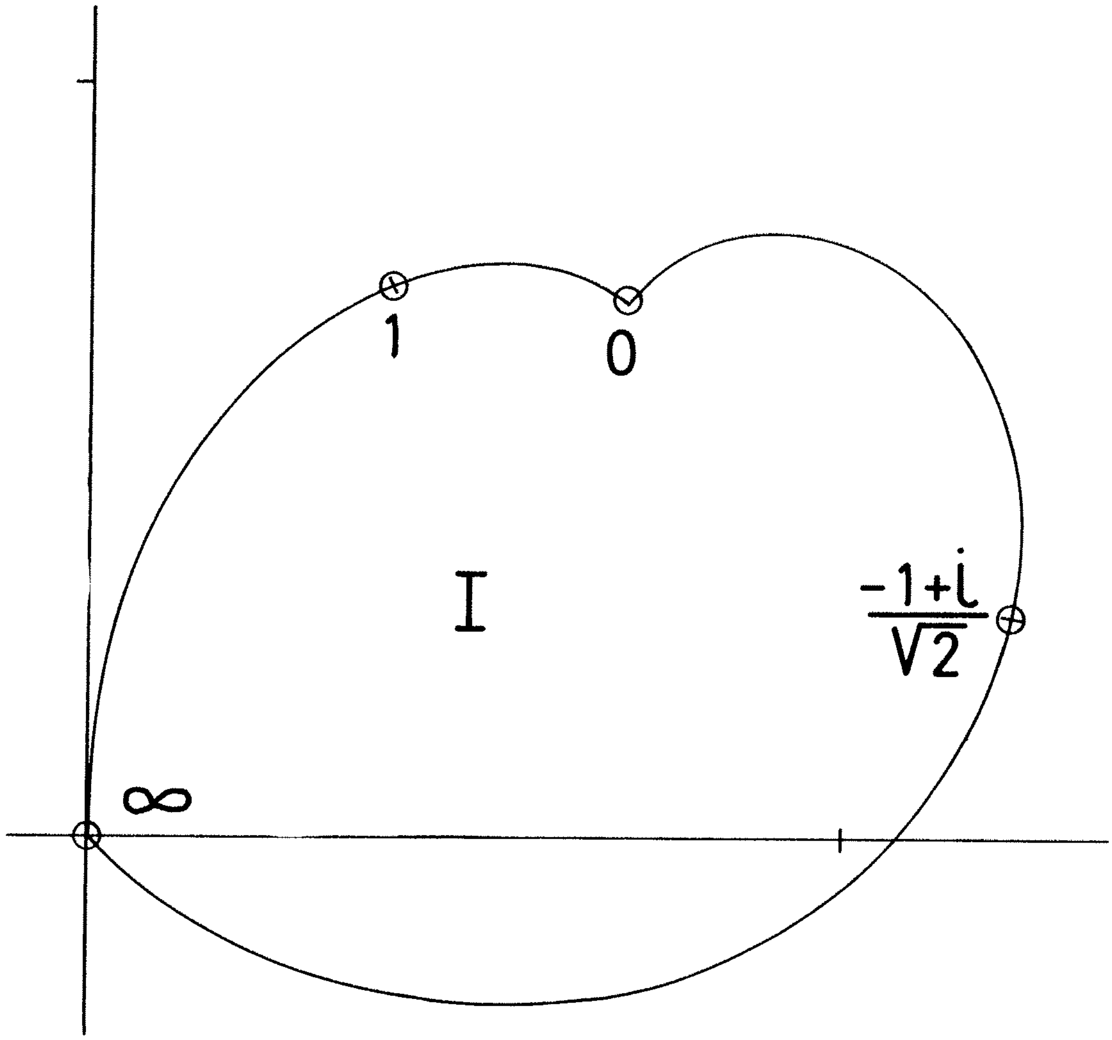


Fig. 3



$\sigma = \sqrt{i}$

Fig. 4

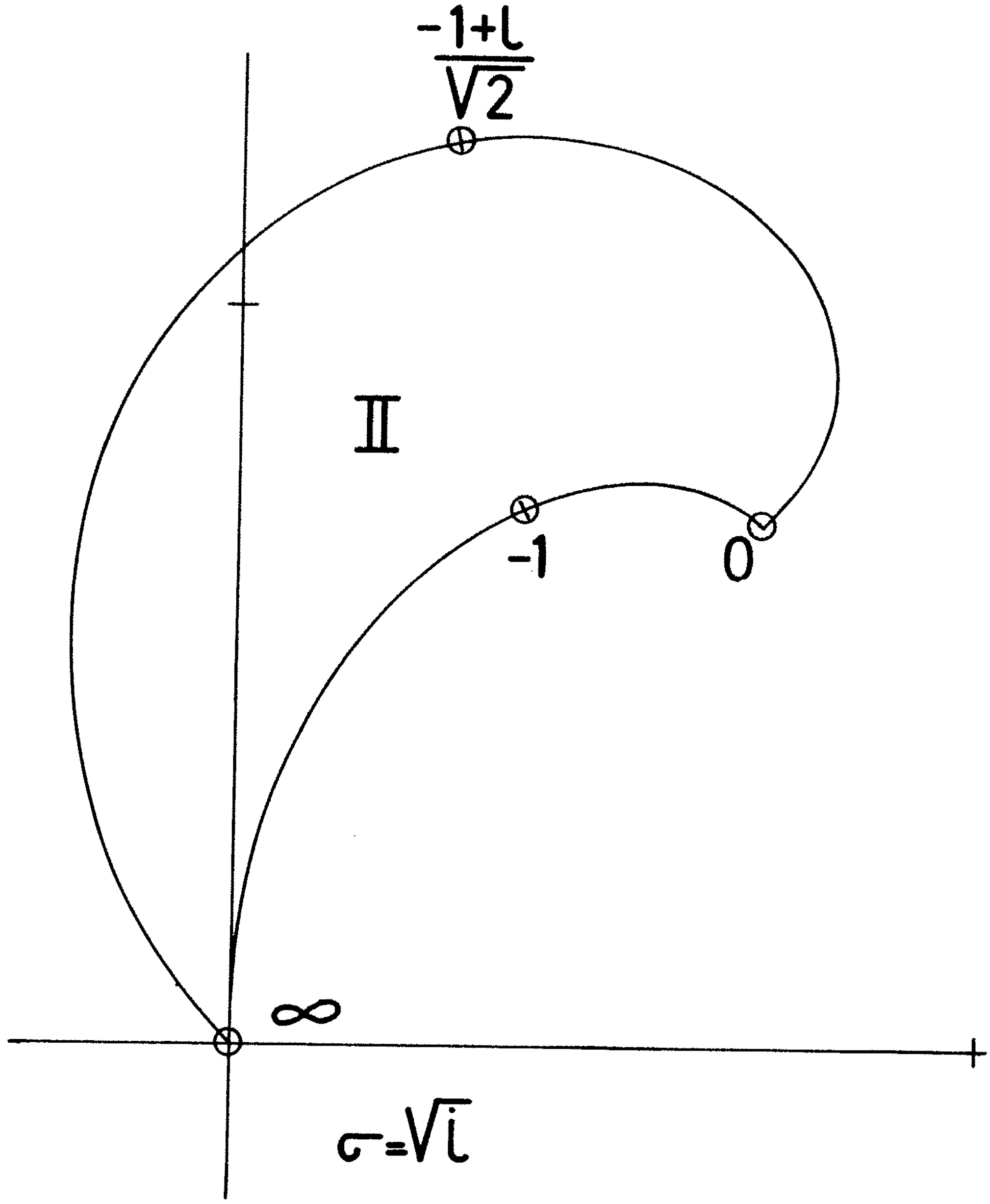


Fig. 5

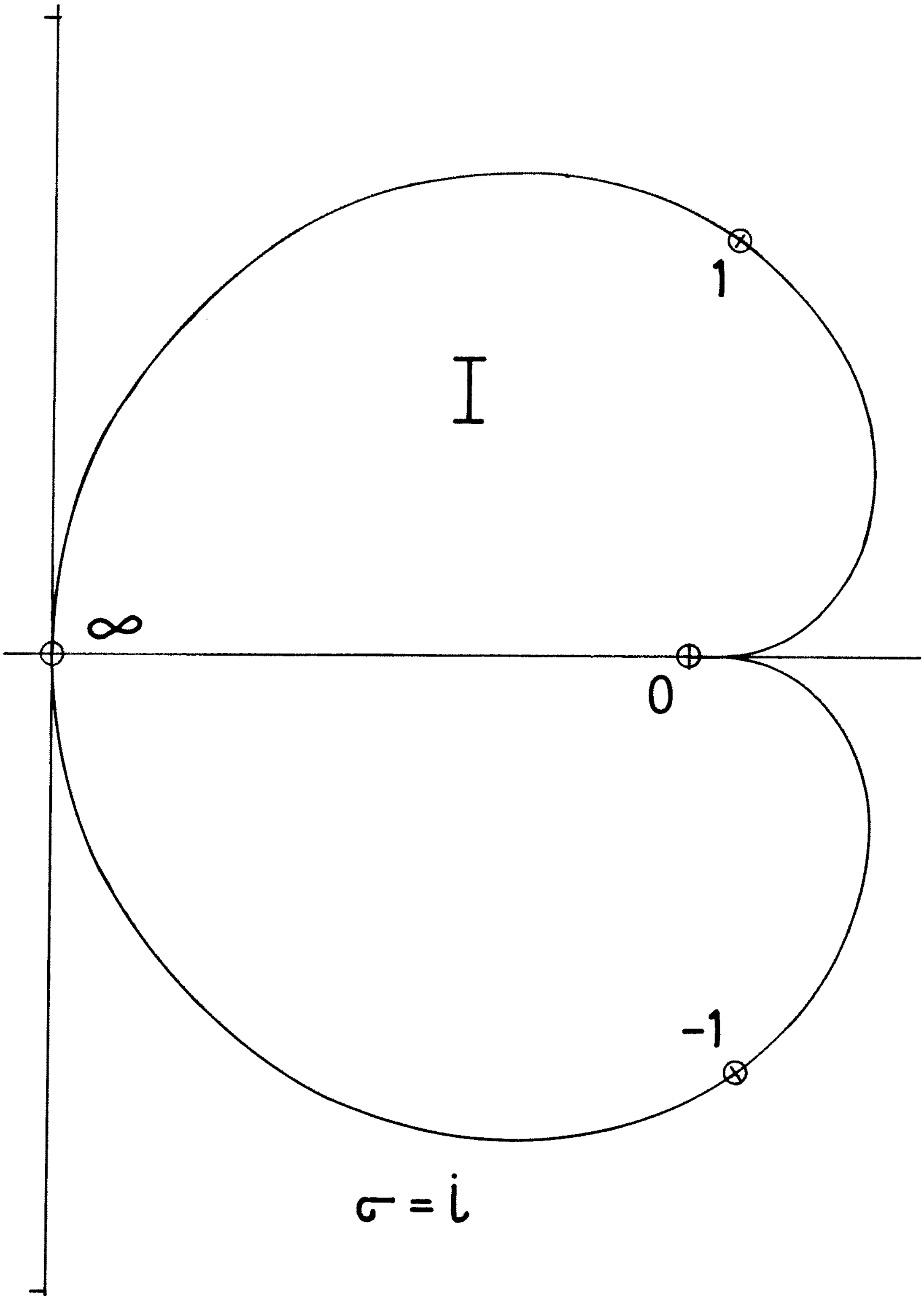


Fig. 6

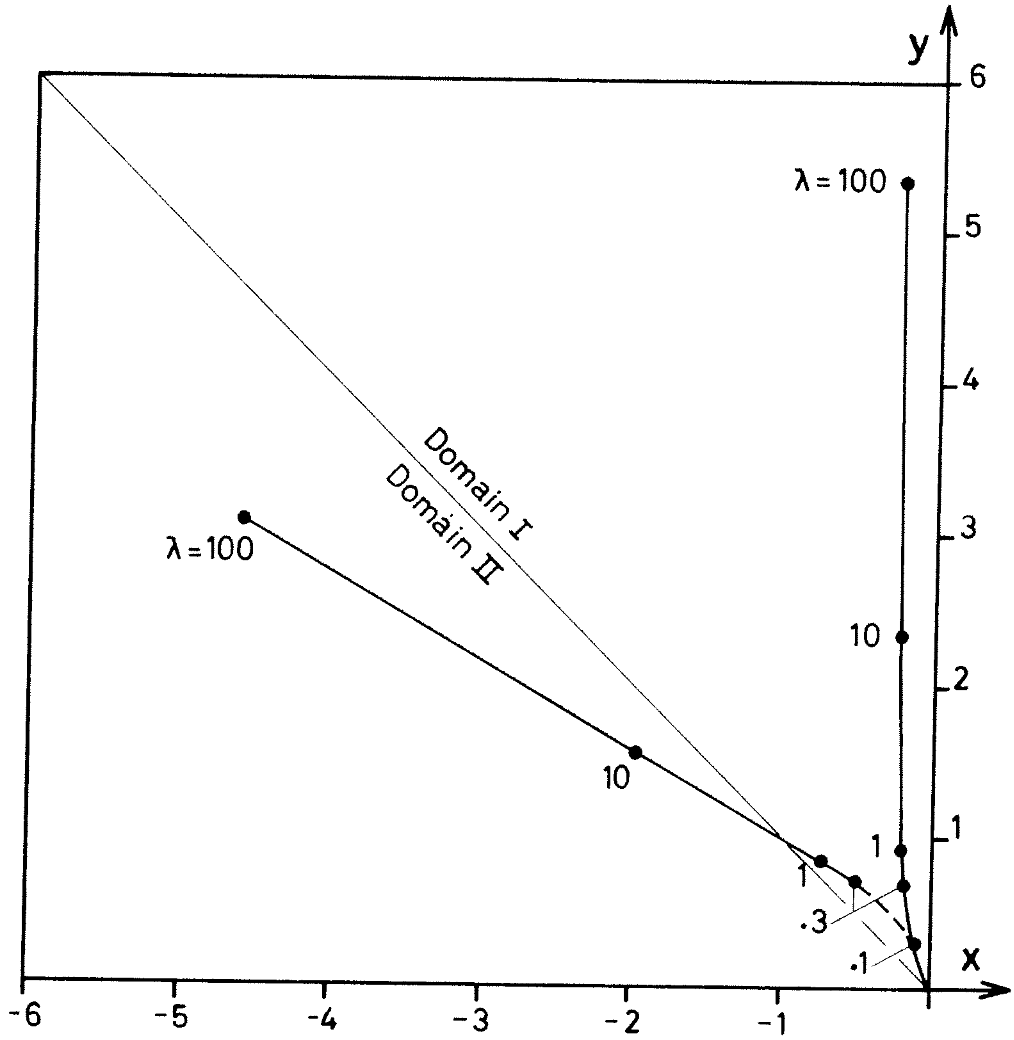
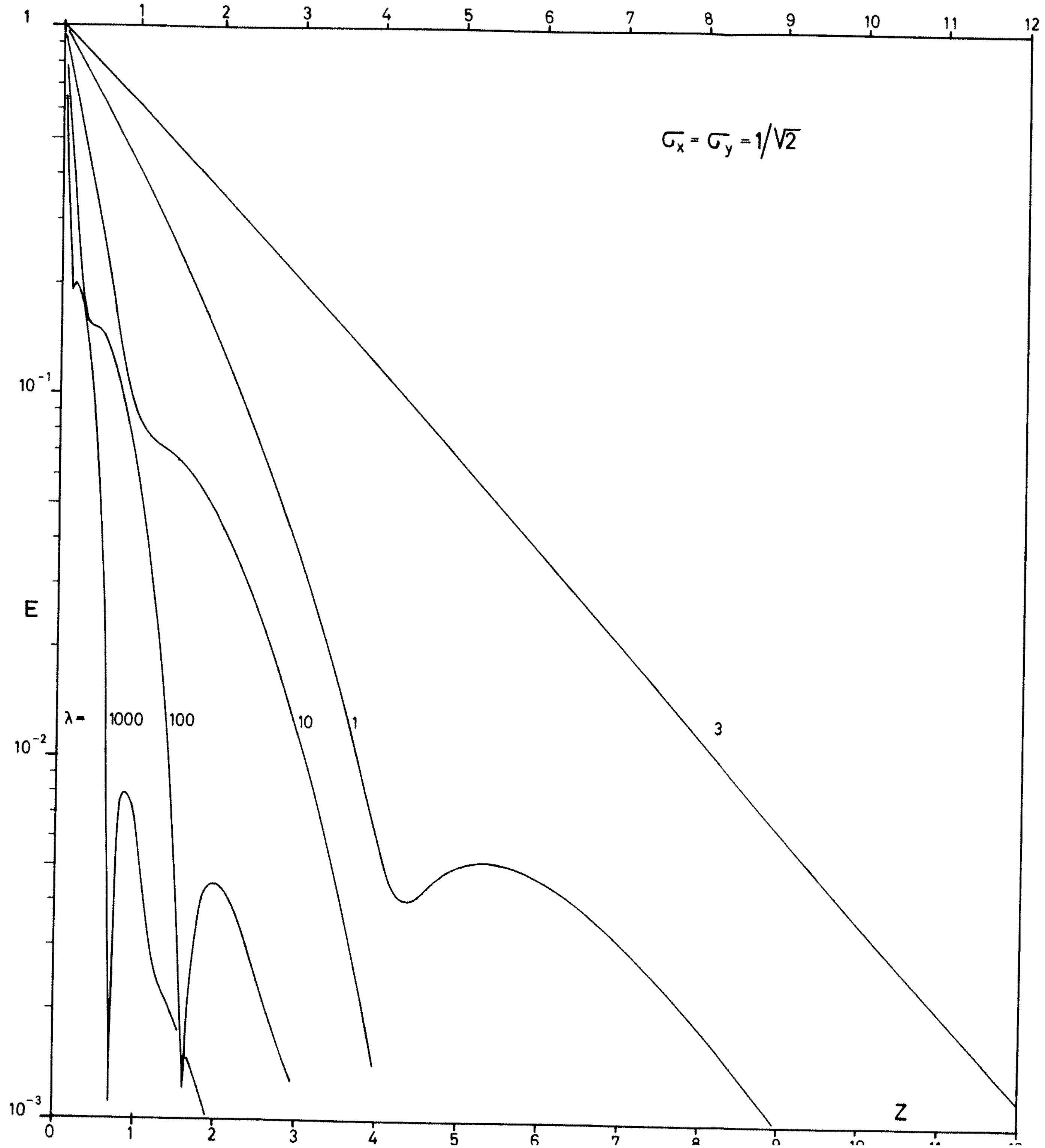


Fig. 7



$\sigma_x = \sigma_y = 1/\sqrt{2}$
E
Z
1
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
6
7
8
9
10
11
12
λ =
1000
100
10
1
3

Fig. 8

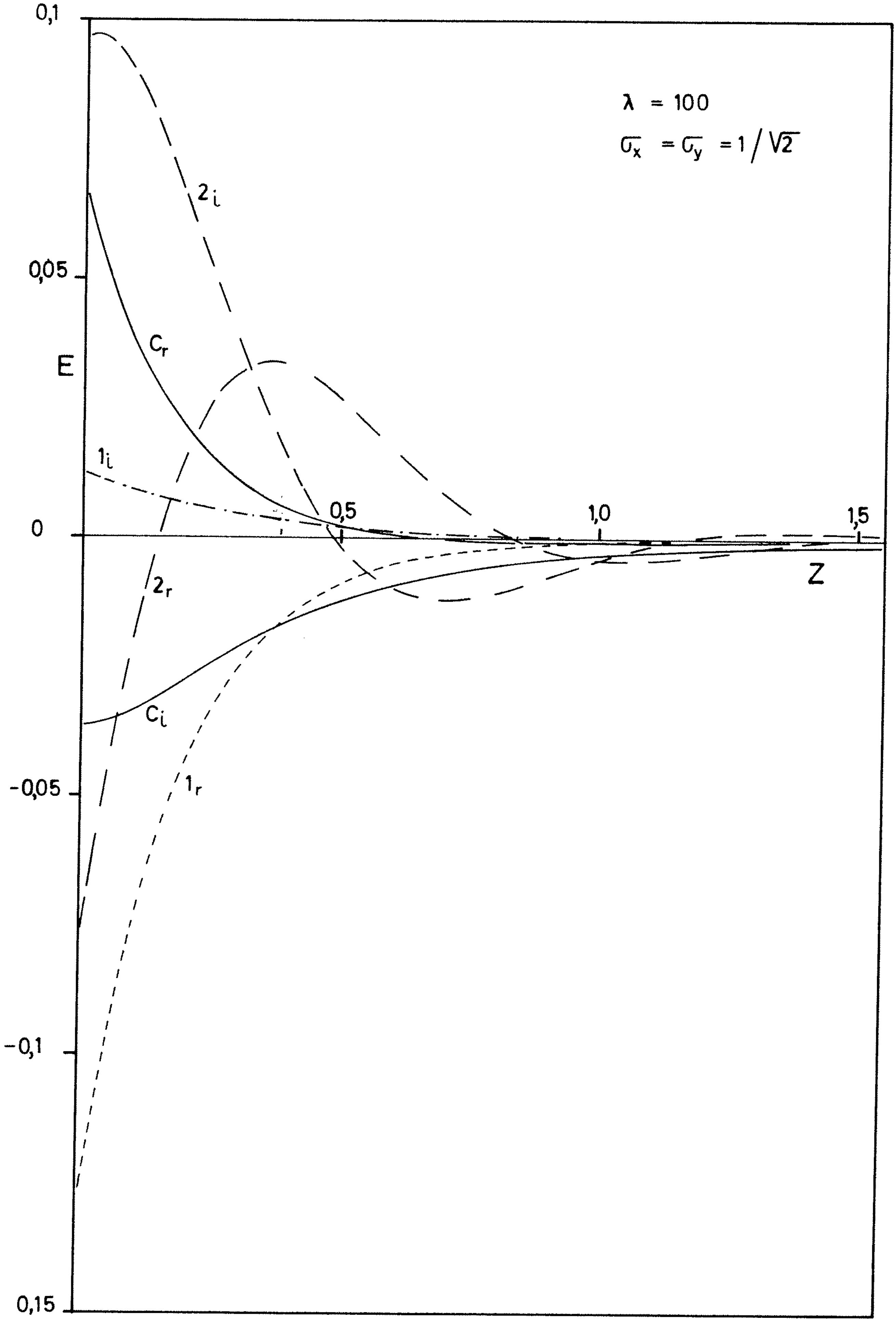


Fig. 9