Piecewise flat Metrics on Surfaces and the Moduli Space

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The role of piecewise flat surfaces in Teichmuller theory has been studied by a number of authors the last 20 years. See in particular B. H. Bowditch [1], D.B.A. Epstein and R.C. Penner [2], F. Fillastre [3], W. Thurston [4], and W.A. [7]. In this presentation, we will show how the theory of deformations of geometric structures (development and holonomy) applied to the case of piecewise flat surfaces leads to some interesting geometric structures on the moduli space of a punctured surface. The details and proofs are given in the paper [6].

We define a *punctured surface* $\Sigma_{g,n}$ to be an oriented, closed connected surface Σ of genus g together with a distinguished set of n pairwise distinct points $p_1, p_2, ..., p_n \in \Sigma_{g,n}$, and we denote by $\Sigma'_{g,n} := \Sigma_{g,n} \setminus \{p_1, p_2, ..., p_n\}$ the same surface with the points p'_j removed. The fundamental group $\pi_{g,n} = \pi_1(\Sigma'_{g,n})$ is a free group on $2g + n - 1$ generators.

A flat metrics with conical singularities on $\Sigma_{g,n}$ of is a flat metric m on $\Sigma'_{g,n}$ such that in the neighbourhood of a p_j, we can introduce polar coordinates (r, φ), where $r \geq 0$ is the distance to p and $\varphi \in \mathbb{R}/(\theta_j\mathbb{Z})$ is the angular variable (defined modulo θ_j). The number θ_j is the *total angle* at the singular point p_k and $\beta_j = \theta_j/(2\pi) - 1$ is called the *order* of the singularity. These metrics have been classified by the author in 1986 [5]:

Theorem 1. *Let $\Sigma_{g,n}$ be a punctured surface with punctures $p_1, p_2, \dots, p_n$. Fix n real numbers $\beta_1, \beta_2, \dots, \beta_n \in (-1, \infty)$ satisfying the Gauss-Bonnet condition:*

$$\chi(\Sigma) + \sum_i \beta_i = 0\,,$$

For each conformal structure μ on $\Sigma_{g,n}$, there exists a metric m such that

i) m is a flat metric on Σ having a conical singularity of order β_j at p_j ($j = 1, \dots, n$);

ii) m belongs to the conformal class μ.

This metric is unique up to a dilation (homothety).

Associated to any flat metrics with conical singularities on $\Sigma_{g,n}$, we have a *developing map* and a *holonomy representation*. These invariant are defined as follow: Consider the punctured surface $\Sigma_{g,n}$ with a fixed flat metric m with conical singularity of order β_j at p_j ($j = 1, \dots, n$). We conformally have $\Sigma'_{g,n} \simeq \mathbb{U}/\Gamma$ where $\mathbb{U}$ is the unit disk and $\Gamma \subset \mathrm{Aut}(\mathbb{U})$ is a Fuchsian group isomorphic to the fundamental group $\pi_{g,n}$. Thus, the unit disk $\mathbb{U}$ inherits a (incomplete) conformal flat metric $\tilde{m}$. If f_0 is a germ of an isometry near a point $\tilde{z}_0$, to the euclidean plane (identified with $\mathbb{C}$), then we obtain a map $f : \mathbb{U} \to \mathbb{C}$ by analytic continuation from f_0. This map is called *the developing map*, it is a local isometry for the metric $\tilde{m}$ on $\mathbb{U}$ and the canonical metric on $\mathbb{C}$. The corresponding *holonomy* is the unique homomorphism $\varphi_m : \Gamma \to \mathrm{SE}(2)$ such that

$$f(\gamma u) = \varphi_m(\gamma) f(u),$$

here, SE(2) is *special Euclidean group*, i.e. the group of orientation preserving isometries of the Euclidean plane.

Thus, to each flat metric m with conical singularities and germ of isometry f_0, we have associated an element

$$\varphi_m \in \mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2)).$$

Changing the developing map (i.e. the germ f_0) does not affect the conjugacy class of φ_m. Hence to each flat metric, the element

$$[\varphi_m] \in \mathcal{R}(\pi_{g,n}, \mathrm{SE}(2)) = \mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2))/\,\mathrm{SE}(2)$$

is well defined.

Remarks. *A)* If h is a diffeomorphism of Σ preserving the punctures and the orientation, and $m' = h^*m$, then $\varphi_{m'}$ is conjugate to φ_m.

B) If $m'' = \lambda m$ is a dilation of m, then $\varphi_{m''} = \lambda\varphi_m$. Thus

$$[\varphi_m] \in \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2)) = \mathcal{R}(\pi_{g,n}, \mathrm{SE}(2))/\mathbb{R}_+$$

is a well defined invariant of the similarity class of the metric m invariant under any isotopy of Σ preserving the punctures.

C) If m has a conical singularity of order β_j at the puncture p_j, then $\varphi_m(c_j)$ is a rotation of angle $\theta_j = 2\pi(\beta_j + 1)$.

Let us denote by $\Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2))$ the set of equivalent classes of representations $\varphi : \pi_{g,n} \to \mathrm{SE}(2)$ such that $\varphi_m(c_j)$ is a rotation of angle $\theta_j = 2\pi(\beta_j + 1)$ for $j = 1, 2, .., n$. We have associated to each flat metric m on Σ with a conical singularity of order β_j at the puncture p_j a well defined element $[\varphi_m] \in \Xi$.

This element is invariant under any dilation of the metric m and any isotopy. Combining this construction with the previous theorem about the existence of flat singular metrics in each conformal class, we obtain a well defined map :

$$\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2)).$$

Theorem 2. *Ξ has a natural structure of real algebraic variety. If $\beta_j \notin \mathbb{Z}$, then we have*

$$\Xi \simeq \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}$$

Theorem 3. *The map* $\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi$ *is a local homeomorphism.*

Any automorphism of $\pi_{g,n}$ acts on $\mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2))$ by twisting the representation. This leads to a natural action of the pure mapping class group $\mathrm{PMod}_{g,n}$ on $\Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2))$, i.e. we have constructed a natural homomorphism

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\Xi) = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}\,\mathbb{C}$$

Theorem 4. *The map* $\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi$ *is* Φ*-equivariant.*

The previous results taken together give the following

Theorem 5. *Given a punctured surface $\Sigma_{g,n}$ such that $2g+n-2>0$ and $\beta_j>-1$ satisfying the Gauss-Bonnet condition and such that no β_i is an integer, there is a well defined group homomorphism*

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}(\mathbb{C}),$$

and a Φ-equivariant local homeomorphism

$$\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}.$$

In other words, the theorem says that

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \mathrm{PMod}_{g,n}$$

is a good orbifold with a $(\mathcal{G}, \Xi)$-structure.

In the special case of the punctured sphere, a stronger form of this theorem has been obtained by Deligne and Mostow [8] using some techniques from algebraic geometry and by Thurston [4] using an approach closer to ours.

References

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