SOME UNIVERSAL ESTIMATES ON EIGENVALUES OF ORIENTABLE RIEMANNIAN SURFACES

Marc Troyanov, December 1990

Abstract *We give some upper bounds for eigenvalues on complete orientable Riemannian Surfaces depending only on a lower bound on the curvature.* (This unpublished paper has been written in 1990 at the University of Utah.)

1. Introduction. Let (M, g) be a connected Riemannian manifold with or without boundary. For a compact domain $\Omega \subset\subset M$, we denote by $\lambda_j(\Omega)$ the jth eigenvalue of the Laplace operator for the Dirichlet problem on Ω. We then define

$$\lambda_j(M, g) := \inf \lambda_j(\Omega)\,,$$

where the infimum is taken over all compact domains of M [(1)].

An old result of Weyl tells us that for a smooth domain $M \subset \mathbf{R}^n$ with finite volume V, we have

$$\lambda_k(M) \leq c \left(\frac{k}{V}\right)^{2/n}, \tag{1}$$

where c is a constant depending on M (see [B, p. 70]).

Recentely (working with Neumann eigenvalues), Nick Korevaar obtained a similar inequality for every manifold (M, g) whose metric g is conformal to a metric $g_0 = f \cdot g$ with nonnegative Ricci curvature (see [K]).

Using the fact that every closed Riemann surface is a branched cover over S^2, it is then possible to show that for every compact orientable Riemannian surfaces (S, g), we have

$$\lambda_k(S, g) \leq \gamma(p)\frac{k}{A}\,,$$

where A is the area of (S, g) and $\gamma(p)$ is a constant depending only on the genus p of S. Thus answering positively to a question of S.T. Yau (see [Y2, p.19]).

If the curvature K of (S, g) satisfies $K \geq -a^2$, then $a^2 A \geq 2\pi(2p - 2)$ (by Gauss-Bonnet) and we thus have

$$\lambda_k(S, g) \leq (\tilde{\gamma}(p)a^2)\, k\,,$$

[1] We follow the convention in [B], in particular, on a compact manifold, $\lambda_1 = 0$ and the first non zero eigenvalue is λ_2.

where $\tilde{\gamma}(p) := \dfrac{\gamma(p)}{2\pi(2p-2)}$.

The aim of this paper is to show that, given a lower bound on the curvature, the constant $\tilde{\gamma}(p)$ in the above inequality is in fact independant on the genus of S :

Theorem 2. *There exists a universal constant $\bar{\mu}$ such that the inequality*

$$\lambda_k(S,g) \leq (\bar{\mu}\, a^2)\, k$$

holds for every compact orientable Riemannian surface with curvature $K \geq -a^2$ and genus ≥ 2.

We also extend this result to non compact surfaces, see theorem 5.

2. Background on eigenvalues. We denote by $H_0^1(M,g)$ the completion of the space $C_0^\infty(M)$ of smooth functions on M with compact support and vanishing on ∂M, with respect to the Sobolev norm :

$$\|u\|_1^2 = \int_S (u^2 + |\nabla u|^2) dvol_g \,.$$

The Rayleigh quotient of a function $u \in H_0^1(M)$ is defined by

$$\mathcal{R}(u) := \frac{D(u)}{\|u\|^2_{L^2(M)}}\,,$$

where $D(u) := \displaystyle\int_S |\nabla u|^2 dvol_g$ is the Dirichlet integral of u.

For a finite dimensional subspace $E \subset H_0^1(M)$, we set

$$\mathcal{R}(E) := \sup\{\mathcal{R}(u) :\, u \in E \setminus \{0\}\}\,.$$

It is then well known (see e.g. [B], p.61) that $\lambda_j(M,g)$ satisfies :

$$\lambda_j(M,g) = \inf_E \mathcal{R}(E)\,,$$

where E ranges through all $j-$dimensional linear subspaces of $H_0^1(M)$.

It follows from the above caracterisation of λ_j that these numbers can be estimated from local geometry :

Lemma 3. *If $\Omega_1, \Omega_2, \dots \Omega_m$ are m disjoint domains in (M,g), then*

$$\lambda_{km}(M,g) \leq \max_{1\leq j\leq m} \lambda_k(\Omega_j, g)\,.$$

Proof. Given $\epsilon > 0$, we may find for each $1 \leq j \leq m$ a $k-$dimensional vector space $E_j \subset H_0^1(\Omega_j)$ such that $\mathcal{R}(E_j) \leq \lambda_k(\Omega_j, g) + \epsilon$. Let $E = \oplus_j E_j \subset H_0^1(M)$. This is a $km-$dimensional vector space. We have then

$$\lambda_{km}(M,g) \leq \mathcal{R}(E) \leq \max_{1\leq j\leq k} \mathcal{R}(E_j) \leq \max_{1\leq j\leq k} \lambda_k(\Omega_j, g) + \epsilon$$

□

Lemma 4. *Let g and h be two conformal metrics on a surface S, and assume $h = f^2 \cdot g$ for some function $f : M \to]0, c]$. Then $\lambda_j(g) \leq c^2\, \lambda_j(h)$.*

The proof is elementary. □

3 The Main Result. Before stating the main result of this paper, let us recall some facts from surface theory.

An *end* in a surface of finite topological type S is a closed subset $E \subset S$ homeomorphic to $S^1 \times [0, \infty)$ (2).

Let (S, g) be a complete Riemannian surface, then $E \subset (S, g)$ is a *parabolic end* if it is isometric to $\{z \in \mathbf{C} : |z| \geq 1\}$ equipped with some conformal complete metric $ds^2 = \rho(z)|dz|^2$. A parabolic may have finite or infinite area.

A *hyperbolic end* is a subset $E \subset (S, g)$ isometric to $\{z \in \mathbf{C} : 1 \leq |z| < r_0\}$ equipped with some conformal complete metric $ds^2 = \rho(z)|dz|^2$. A result of A. Huber tells us that If the curvature of (S, g) has a lower bound, then all hyperbolic ends have infinite area (this follows e.g. from the Schwarz lemma [Tr] or from [H, theorem 15]).

It follows from Uniformization theory that a complete Riemannian surface of finite topological type is a union of a compact surface and finitely many hyperbolic or parabolic ends (see e.g. Appendix B in [HT2]). If (S, g) is a complete surface with finite total curvature, then it is of finite topological type and has only parabolic ends (see [H] or [HT2]).

Let us define $\mu(k)$ to be the k^{th} eigenvalue $\mu(k) = \lambda_k(T)$ of an ideal triangle T in the hyperbolic plane (recall that an ideal triangle T in H^2 is a triangle with all its vertices at infinity; all ideal triangles are congruent).

Theorem 5. *Let (S, g) be a complete connected orientable Riemannian surface of finite topological type without boundary.*

Assume either that S has negative Euler characteristic or that S has a hyperbolic end.

Assume also that the curvature K_g of g is bounded below by a negative constant : $K_g \geq -a^2$, we then have :

(A) If (S, g) has finite area, we have

$$\lambda_{km}(S, g) \leq a^2 \mu(k)$$

where $m = m(S) = 2|\chi(S)|$.

(B) If (S, g) has a hyperbolic end, then $\lambda_j(S, g) \leq a^2/4$ for all j.

(C) If (S, g) has a parabolic end of infinite area, then $\lambda_j(S, g) = 0$ for all j.

Comments. (i) We trivially have $\lambda_1 \leq \lambda_2 \leq \dots$; so the theorem actually gives estimates on all eigenvalues. In particular, theorem 1 is a consequence of (A) with $\bar{\mu}$ defined as :

$$\bar{\mu} := \sup_k \frac{\mu(k)}{k}\,.$$

[2] Two ends E' and E'' are *equivalent* if their intersection contains an end. We do not distinguish between two equivalent ends.

(It follows e.g. from lemma 4 and the Weyl inequality (1) that $\bar{\mu}$ is finite.)

(ii) Using the work of R. Schoen, S. Wolpert and S.T. Yau, we see that our theorem also implies an upper bound on the length of small geodesics on compact hyperbolic surfaces (i.e. surfaces with $K \equiv -1$) depending only on the genus. These authors consider the set of all system of disjoint curves Γ_n in a compact oriented hyperbolic surface disconnecting the surface in $n+1$ parts. They showed that for $n \leq 2p-3$,

$$\inf_{\Gamma_n} \text{Length}(\Gamma_n) \leq c(p)\, \lambda_{n+1}(S)$$

where $c(p)$ is a constant depending only on the genus p of S (see [SWY]). Combining this result with theorem 5A, we find

$$\inf_{\Gamma_n} \text{Length}(\Gamma_n) \leq c(p)\lambda_{4(p-1)}(S) \leq c(p)\mu(1)\,,$$

for all $n \leq 2p-3$.

4 Geometry of surfaces. To prove our main theorem, we will need some facts from the theory of surfaces. The first one is the existence of the Poincaré metric :

Theorem 6. *Let (S, g) be a complete Riemannian surface of finite topological type. If $\chi(S) < 0$ or if (S, g) has a hyperbolic end, then there is a unique complete conformal metric $h = f^2 \cdot g$ of constant curvature $K_h \equiv -1$ (called the Poincaré metric).*

Furthermore, if $K_g \geq -1$, then $f \leq 1$.

See for instance [HT1], [HT2] or any text on Uniformization for a proof of the first statement and [Y] or [Tr] for the second one. □

A complete connected Riemannian surface (S, h) with constant curvature $K_h \equiv -1$ will be called a *hyperbolic surface*. A hyperbolic surface has no hyperbolic end if and only if its area is finite. The next result is a structure theorem for hyperbolic surfaces.

Theorem 7. *Let (S, h) be an orientable hyperbolic surface of finite type.*

(a) If S has no hyperbolic end, then S contains $m = 2|\chi(S)|$ domains $T_i \subset S$ such that :

- *i) Each T_i is isometric to an ideal triangle T of the hyperbolic plane;*
- *ii) $S = \cup_{i=1}^m \bar{T}_i$;*
- *iii) $T_i^0 \cap T_j^0 = \emptyset$ if $i \neq j$.*

(b) If S has a hyperbolic end, then there is a complete hyperbolic sector V isometrically embedded in (S, h).

Definition. A (non degenerate) *complete hyperbolic sector V* is an open subset in the hyperbolic plane H^2 bounded by two distinct rays with same origin, (i.e. $V := \{(r, \theta) : r > 0,\, 0 < \theta < \theta_0\}$ with metric $ds^2 = dr^2 + \sinh^2(r)d\theta^2$).

The proof of (a) consists in showing that S can be decomposed in $|\chi(S)|$ pair of pants, and then, each pair of pants in 2 ideal triangles. See [Th]. (A "pair of pants" is a hyperbolic surface with geodesic boundary and/or cusps and whose interior is homeomorphic to a thrice punctured sphere).

To prove (b), we represent (S,h) as H^2/Γ. If D is a fundamental polygon in H^2 for the action of Γ, then D has infinite area and contains a nondegenerate sector. $\square$

Proposition 8. *Let (V,h) be a non degenerate complete hyperbolic sector. Then*

$$\lambda_j(V,h) \leq \frac{1}{4} \qquad \forall j\,.$$

Proof. It is known that $\lambda_1(H^2) = \frac{1}{4}$ (see [McK]), thus we may find for every $\epsilon > 0$ a hyperbolic disk $D_r(H^2)$ such that $\lambda_1(D_r(H^2)) \leq \frac{1}{4} + \epsilon$. We may now fit j disjoint copies of $D_r(H^2)$ in the complete sector V and conclude by lemma 2. $\square$

Finally, we will need a result on the eigenvalues of a (parabolic) end of a Riemannian surface.

Proposition 9. *Let $\Omega = \{z \in \mathbf{C} : |z| \geq 1\}$ be equipped with a conformal metric $g = e^{2\varphi}|dz|^2$ such that the area of (Ω, g) is infinite. Then*

$$\lambda_j(\Omega,g) = 0 \qquad \forall j\,.$$

Proof. For a disk $D = \{z \in \mathbf{C} : |z - z_0| \leq r\}$ and a number $\alpha > 1$, it will be convenient to denote by αD the disk $\alpha D = \{z \in \mathbf{C} : |z - z_0| \leq \alpha r\}$.

Now let $D = \{z \in \mathbf{C} : |z - z_0| \leq r\} \subset \Omega$ be any disk such that $\text{Area}(D,g) \geq 1$, and consider the function u on αD given by $u \equiv 1$ on D, $u \equiv 0$ on $\partial(\alpha D)$ and

$$u(z) = \cos\left(\frac{\pi \log(|z - z_0|/r)}{2\log(\alpha)}\right)$$

on the annulus $\alpha D - D$.

Since the Dirichlet integral on surfaces is conformally invariant, we may compute $D(u)$ using the euclidean metric. An explicit calculation gives us the following estimate :

$$D(u) \leq \frac{\pi^3}{2\log(\alpha)}\,.$$

From $\text{Area}(D,g) \geq 1$, we have $\|u\|_{L^2(g)} \geq 1$, hence

$$\lambda_1(D,g) \leq \mathcal{R}(u) \leq \frac{\pi^3}{2\log(\alpha)}\,.$$

Now since $\text{Area}(\Omega,g)$ is infinite, it is possible, given any number $\alpha > 1$, to find an infinite sequence of disks $\{D_j\}_{j=1}^{\infty}$ such that

i) $\alpha D_j \subset \Omega$;
ii) $\alpha D_j \cap \alpha D_k = \emptyset$ if $j \neq k$;
iii) $\text{Area}(D_j, g) \geq 1$.

So lemma 3 implies that

$$\lambda_m(\Omega,g) \leq \max_{1\leq j\leq m} \lambda_1(\alpha D_j, g) \leq \frac{\pi^3}{2\log(\alpha)}\,.$$

Proposition 9 now follows, since α is arbitrarly large. $\square$

5. Proof of Theorem 5.

(A) By lemma 4, we may assume that $K_g \geq -1$. Using now theorem 6, and lemma 4, we see that it is enough to prove the theorem for complete hyperbolic surfaces (S, h) of finite topological type without hyperbolic ends. For such surfaces, however, the theorem follows immediatelly from lemma 3 and theorem 7 a).

(B) As in (A), the proof reduces to the case of hyperbolic surfaces. Now (S, h) has a hyperbolic end and the theorem follows from theorem 7 b) and proposition 8.

(C) There exists a subset Ω of S such that (Ω, g) is conformally equivalent to $\{z \in \mathbf{C} : |z| \geq 1\}$ and has infinite area. We conclude from proposition 9. $\square$

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