Tangent Spaces to Metric Spaces: Overview and Motivations

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Abstract. This is the text of an introductory lecture I made at the Séminaire Borel 2003 (IIIème Cycle Romand de Mathématiques). The general theme of that seminaire was *Tangent Spaces of Metric Spaces*.

The goal of the Borel 03 seminar is to study the notion of tangent spaces in various geometric contexts. This notion has its origin in the early developements of differential geometry and plays a central role in the whole theory. In fact, we can define, in a modern language, differential geometry as *the study of smooth tensors living on smooth manifolds*.

Recall that, by definition, a manifold M of dimension n is a space which locally looks like Euclidean space $\mathbb{R}^n$ in a small neighborhood of any of its point $p \in M$. A blowing up of M near the point p provides a better and better approximation of the geometry of M near p with the geometry of $\mathbb{R}^n$, and a limiting procedure gives an identification of an infinitesimally small neighborhood of p with a copy of $\mathbb{R}^n$; which is called the *tangent space* of M at p, and denoted T_pM. Using standards results from calculus (the implicit function theorem), one may turn this heuristic idea into a rigourous definition. It can also be shown that the union of all tangent spaces to a given manifold M can be given the structure of a new manifold, which is called the tangent bundle of M and denoted by TM.

Various chapters of differential geometry can then be defined as the study of some specific tensors in the tangent bundle. For instance symplectic geometry is the study of a certain non degenerate skew-symmetric bilinear form ω on the tangent bundle and Riemannian geometry is the study of a certain non degenerate symmetric bilinear form g.

There is of course life for a geometer outside differential geometry; and subjects like affine or projective geometry, discrete geometry or algebraic geometry do not fit in the differential geometry framework.

Metric geometry is a particularly important part of contemporary, non differential, geometry and it is an important question to understand what type of metric spaces admit a notion of tangent spaces as described above. Note that the tools from classic calculus are no longer available, but, on the other hand, a new mathematical area known as *Analysis on Metric Spaces* has seen important developpments the last decade. This new analysis will obviously play an important role in the study of tangent spaces to metric spaces.

There are several motivations, even for a differential geometer, to study the field of metric geometry.

The first motivation is that several important subjects such as Riemannian geometry, Sub-riemannian geometry or Finsler geometry belong to the intersection of differential geometry and metric geometry; and the metric viewpoint can sometimes bring unification and a new insight in these subjects.

A second motivation is that various type of non Riemannian metric spaces naturally appear as "boundary at infinity" or as "limits" (in a sense to be made precise) of Riemannian manifolds. [1]

A third motivation is that some problems in Riemannian geometry might be solved by studying a corresponding problem in some other class of metric spaces approximating the considered Riemannian manifold, and then using some limiting argument.

1 Metric Spaces

Recall that a *metric space* is a set X together with a distance function $d : X \times X \to \mathbb{R}$ satisfying for any $x, y, z \in X$

i) $d(x, y) = d(y, x)$;

ii) $d(x, y) = 0$ iff $x = y$;

iii) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangular inequality).

Observe that we always have $d(x, y) \geq 0$ since $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$.

Metric spaces are everywhere in mathematics and the definition is so basic, that it seems that they always where around. In fact they are only one century old :

[1] Think of the following analogy: in functionnal analysis, one considers natural topologies on the space of smooth functions and the completion of the function spaces thus defined contains non smooth functions (such as Sobolev functions) which the analyst needs to understand.

the definition has been given by Frechet in 1906 in the context of function spaces and it is only in the 1920's that Menger proposed to consider them as a subject for geometric investigations.

Let us recall a few basic definitions and facts :

1 Definitions :

1) The *diameter* of a non empty subset $A \subset X$ is
$$\operatorname{diam}(A) := \sup\{d(x,y) \mid x,y \in A\}.$$

2) The *codiameter* of $A \subset X$ is
$$\operatorname{codiam}_X(A) := \sup\{d(x,y) \mid x \in A \text{ and } y \in X \setminus A\}.$$

3) A sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ is a *Cauchy sequence* if
$$\lim_{k\to\infty} \operatorname{diam}\left(\{x_i\}_{i\geq k}\right) = 0.$$

4) The metric space X is said to be *complete* if every Cauchy sequences converges.

5) X is *totally bounded* if for every $\varepsilon > 0$, there exists a finite subset $F \subset X$ such that $\operatorname{codiam}_X(F) \leq \varepsilon$ (Hausdorff 1927).

6) X is *compact* if every bounded sequence contains a convergent subsequence.

7) X is *proper* if every closed bounded subset is compact.

8) X is *separable* if it contains a countable dense subset.

2 Propositions *1) A metric space is compact if and only if it is both complete and totally bounded.*

2) Every proper metric space is complete and separable.

3) The Euclidean space $\mathbb{R}^n$ is a proper metric space (Heine-Borel).

4) A metric space X is separable if and only if evry open cover admits a countable subcover (Lindelöf property).

□

There are several notions of morphisms associated to metric spaces, the most important are the following ones:

3 Definitions : Let X and Y be metric spaces, then a map $f : X \to Y$ is said to be

1) A *Hölder map* with coefficient $\alpha > 0$ if
$$d(fx,fy) \leq C(d(x,y))^\alpha$$
for some constant C and all $x,y \in X$.

2) A *Lipschitz map* if it is Hölder with coefficient $\alpha = 1$. The *Lipschitz constant* of f is then defined as
$$Lip(f) = \sup\left\{\frac{d(fx,fy)}{d(x,y)} \mid x,y \in X, x \neq y\right\}.$$

3) A *contracting map* if it is Lipschitz with $Lip(f) \leq 1$ and *strictly contracting* if $Lip(f) < 1$.

4) A *bilipschitz map* if it is bijective and both f and f^{-1} are Lipschitz maps.

5) An *isometry* if it is bijective and $d(fx,fy) = d(x,y)$ for all $x,y \in X$.

6) A *quasi-symetric map* if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that
$$\frac{d(fx,fz)}{d(fx,fy)} \leq \eta\left(\frac{d(x,z)}{d(x,y)}\right)$$
for and all distincts $x,y,z \in X$.

(We denote by the same letter d the distance in any metric spaces, unless there is a risk of confusion).

2 Isometric embeddings

Recall that a *Banach space* is complete normed real vector space; an example is the space $\ell^\infty(\mathbb{N})$ of bounded sequences with the sup norm.

4 Theorem *1) Every metric space can be isometrically embedded in a Banach space (Kuratowski).*

2) Every separable metric space can be isometrically embedded in $\ell^\infty(\mathbb{N})$.

3) There exists a universal metric space $\mathbb{U}$ such that

a) $\mathbb{U}$ is complete and separable;

b) Every separable metric space can be isometrically embedded in $\mathbb{U}$;

c) For any pair $F, F' \subset \mathbb{U}$ of finite isometric subsets, there exists a global isometry $\phi : \mathbb{U} \to \mathbb{U}$ which maps F onto F'.

This space $\mathbb{U}$ is unique up to isometry (Urysohn, 1927).

□

It follows at once from (1) that every metric space X has a completion $\overline{X}$.

3 Hausdorff Distance and Gromov-Hausdorff Distance

The *ε-neighbourhood* of a non empty subset A of a metric space X is the set

$$U_\varepsilon(A) = \{x \in X \,|\, \exists a \in A \text{ such that } d(x,a) < \varepsilon\}.$$

The *Hausdorff distance* between the non empty subsets $A, B \in X$ is then defined as

$$d_H(A,B) = \inf\{\varepsilon > 0 \,|\, A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}.$$

The Hausdorff distance is non negative and satisfies the triangular inequality; yet it is not exactly a distance since it can take infinite value and $d_H(A,B) = 0$ does not imply $A = B$ (for instance the Hausdorff distance between a set and any of its dense subset is zero).

However we have the following result :

5 Theorem *Let us denote by $\mathcal{K}_X$ the set of all non empty compact subsets of the metric space X. Then*

1) $(\mathcal{K}_X, d_H)$ is a metric space;

2) If X is separable, so is $\mathcal{K}_X$;

3) If X is complete, so is $\mathcal{K}_X$;

4) If X is totally bounded, so is $\mathcal{K}_X$;

5) If X is compact, so is $\mathcal{K}_X$;

6) If X is proper, so is $\mathcal{K}_X$.

Proof (1) Is a nice exercice. (2) If X contains a countable dense subset $S \subset X$, then the collection of all non empty finite subsets of S is easyly seen to be dense in $\mathcal{K}_X$.
(3) Suppose X is complete and let $\{K_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{K}_X$, then $L_k := \overline{\cup_{i\geq k} K_i}$ belongs to $\mathcal{K}_X$ and it is not difficult to check that

$$K := \bigcap_{k\in\mathbb{N}} L_k = \lim_{i\to\infty} K_i.$$

(4) Suppose that X is totally bounded and fix $\varepsilon > 0$. Then there exists a finite subset $F \subset X$ such that $\operatorname{codiam}_X(F) \leq \varepsilon$ and the collection $\mathcal{P}^*(F) \subset \mathcal{K}_X$ of all non empty subsets of F is itself of codiameter at most ε.
Finally (5) and (6) are easy consequences of (3) and (4).

□

6 Remark If $\{K_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}_X$, then its limit $K = \lim_{i\to\infty} K_i$ can also be described as the set of all points $x \in X$ such that there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ with $x_i \in K_i$ and $\lim_{i\to\infty} x_i = x$.

Around 1980, M. Gromov extended the notion of Hausdorff distance to the case of abstract metric spaces.

7 Definition The *Gromov-Hausdorff distance* between two metric spaces X and Y is defined by considering triples (Z, φ, ψ), where Z is a metric space and $\varphi : X \to Z$, $\psi : Y \to Z$ are isometric embeddings, and then setting

$$d_{GH}(X,Y) := \inf_{(Z,\varphi,\psi)} d(\varphi(X), \psi(Y)).$$

8 Proposition *The collection $\mathcal{M}$ of all isometry classes of compact metric spaces is a metric space for the Gromov-Hausdorff distance. The space $\mathcal{M}$ is complete and separable.*

The space $\mathcal{M}$ is not compact and the following result is a basic tool in compactness arguments:

9 Gromov compactness theorem *A subset $\mathcal{D} \subset \mathcal{M}$ is totally bounded (i.e. precompact) if and only if the following two conditions are satisfied:*

i) $\sup_{X\in\mathcal{D}} \operatorname{diam}(X) < \infty$;

ii) For every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that every $X \in \mathcal{D}$ contains a finite subset $F \subset X$ with $\operatorname{codiam}_X(F) \leq \varepsilon$ *and* $\operatorname{card}(F) \leq N$.

To deal with unbounded spaces, we need to consider pointed metric spaces.

10 Definition The *pointed Gromov-Hausdorff distance* between X, x_0 and Y, y_0 is defined as

$$d^*_{GH}(X, x_0; Y, y_0) := \inf_{(Z,\varphi,\psi)} \left(d(\varphi(X), \psi(Y)) + d(\varphi(x_0), \psi(y_0))\right)$$

where (Z, φ, ψ) is as above.

11 Definition A sequence $\{(X_n, x_n)\}$ of pointed metric spaces is then said to converge to (X, x) in the *pointed Gromov-Hausdorff sense* if for every $R, \varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for any $n \geq m$ the pointed Gromov-Hausdorff between the ball of center x and radius R in X and the ball of center x_n and radius R in X_n is less than ε

$$d^*_{GH}(B_X(x,R); B_{X_n}(x_n,R)) \leq \varepsilon.$$

A *tangent cone* of a space X at a point $p \in X$ is then a pointed metric space (T, o) such that there exists a sequence of scale factors $\lambda_n \to \infty$ for which

$$(\lambda_n X, p) \to (T, o)$$

in the pointed Gromov-Hausdorff sense as $n \to \infty$.

A tangent cone does not always exist; and if it does, it may not be unique !