Data-driven Precompensator Tuning for Linear Parameter Varying Systems

Mark Butcher, Alireza Karimi and Roland Longchamp

***Abstract*— Methods for direct data-driven tuning of the parameters of precompensators for LPV systems are developed. Since the commutativity property is not always satisfied for LPV systems, previously proposed methods for LTI systems that use this property cannot be directly adapted. When the ideal precompensator giving perfect mean tracking exists in the proposed parameterisation of the precompensator, the LPV transfer operators do commute and an algorithm using only two experiments on the real system is proposed. It is shown that this algorithm gives consistent estimates of the ideal parameters despite the presence of stochastic disturbances. For the more general case, when the ideal precompensator does not belong to the set of parameterised precompensators, another technique is developed. This technique requires a number of experiments equal to twice the number of precompensator parameters and it is shown that the calculated parameters minimise the mean squared tracking error.**

I. Introduction

It is commonplace to use precompensators, based on the inverse of the closed-loop system, in order to improve the tracking performance of linear time-invariant (LTI) systems. This technique typically uses the inverse of a model of the closed-loop system for the precompensator. However, the model will be subject to uncertainty and when this is above a certain level, the tracking performance of the system can be adversely affected [1]. In [2] a data-driven method is proposed for direct tuning of the parameters of precompensators for LTI systems. This method minimises the tracking control criterion directly using measured data, rather than passing through a system modelling step and then minimising a criterion based on the uncertain model. This approach means that the achieved tracking is not affected by system model uncertainty and leads to high performance tracking. The method is based on parameter estimation algorithms using instrumental variables and takes advantage of the commutativity of LTI transfer operators. However, in many applications the LTI assumption is not satisfied e.g. certain mechatronic systems such as x-y positioning tables where the dynamics change as a function of position and consequently the method proposed in [2] cannot be applied.

A class of systems whose dynamics change as a function of the operating point are linear parameter varying (LPV) systems. For LPV systems, methods have been proposed ([3], [4]) to tune precompensators and feedforward controllers whose parameters vary also as a function of the operating point. These methods, however, are based on uncertain identified LPV models and thus, unlike direct data-driven methods, suffer from model uncertainty.

No data-driven precompensator, or feedforward controller, tuning methods for LPV systems have been proposed to the authors' knowledge. But, as is the case for LTI systems, system identification techniques for LPV systems should be adaptable to the tuning of these controllers.

Research into the problem of identifying LPV systems has been active in recent years (see e.g. [5], [6], [7], [8]). In [9] a method is proposed for the identification of the parameters of Single Input Single Output (SISO) LPV systems in input-output form. Each parameter of the system transfer operator is a linear combination of predefined, operating point dependent functions. The identification procedure is then one of identifying the coefficients multiplying these functions, which is a linear regression problem, and so can be computed using the standard least squares technique. However, as occurs in the LTI case, the least squares technique generally gives biased parameter estimates. Consistent estimates can be obtained in the general case using instrumental variables [10].

In this paper the application of instrumental variables to the problem of direct, data-driven tuning of precompensators for LPV systems is considered. It is shown that if the ideal precompensator giving zero mean tracking exists in the proposed precompensator parameterisation, the LPV transfer operators commute and a tuning technique is proposed which gives consistent estimates using measurements from just two experiments. For the more general case, where the LPV transfer operators do not commute, another algorithm is proposed requiring a number of experiments equal to twice the number of precompensator parameters. The algorithm leads to parameter estimates that converge to those that minimise the desirable mean squares criterion.

The paper is organized as follows. Notation and preliminaries are given in Section II. The tuning scheme when the ideal precompensator exists in the precompensator parameterisation is presented in Section III. In Section IV, the tuning method for the general case is explained. Finally, some concluding remarks are made in Section V.

II. Preliminaries

The output of a SISO Linear Parameter Varying (LPV) system $G(\sigma(t), q^{-1})$ is given by:

$$\begin{aligned} y(t) &= G(\sigma(t), q^{-1})u(t) + H(\sigma(t), q^{-1})e(t) \\ &= G(\sigma(t), q^{-1})u(t) + v(t) \end{aligned} \tag{1}$$

The authors are with the Automatic Control Laboratory, Ecole Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland.

Corresponding author: alireza.karimi@epfl.ch

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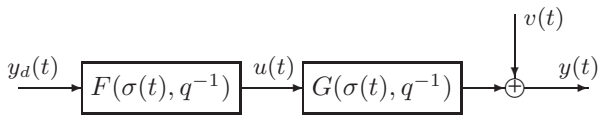

Fig. 1. System with precompensator

where $\sigma(t) \in \mathbb{R}^{n_\sigma}$ is a measurable scheduling parameter vector at time t, $H(\sigma(t), q^{-1})$ a, possibly LPV, transfer operator filtering the sequence of zero-mean, independent random variables $e(t)$ to give $v(t)$, and q^{-1} the backward-shift time operator. The scheduling parameter vector contains the measurable signal(s) which correspond to the system's current operating point. It should be noted that $G(\sigma(t), q^{-1})$ and $H(\sigma(t), q^{-1})$ may be the transfer operators of either an open or closed-loop system, under the condition that they are uniformly stable for all $\sigma(t)$ in the operating zone.

Definition: An LPV transfer operator $P(\sigma(t), q^{-1})$, $\sigma(t) \in \mathcal{A}$:

$$P(\sigma(t), q^{-1}) = \sum_{k=0}^{\infty} p_k(\sigma(t)) q^{-k}, \quad \sigma(t) \in \mathcal{A} \tag{2}$$

is said to be *uniformly stable* if

$$|p_k(\sigma(t))| \leq p_k, \quad \text{for } t = 0, 1, \ldots, \quad \sum_{k=0}^{\infty} p_k < \infty. \tag{3}$$

The output of the system, with an LPV precompensator $F(\sigma(t), q^{-1})$, is given by (see Fig. 1):

$$y(t) = G(\sigma(t), q^{-1}) F(\sigma(t), q^{-1}) y_d(t) + v(t). \tag{4}$$

The objective is to calculate the parameters of the precompensator $F(\sigma(t), q^{-1})$ such that the tracking error:

$$\epsilon(t) = y_d(t) - y(t) \tag{5}$$

is reduced, where $y_d(t)$ is the desired system output, which is defined over the duration $t = 0, \ldots, N-1$.

In this paper we consider the ideal precompensator to be that which gives zero mean tracking error. It can clearly be seen from (4) and (5) that the ideal precompensator is the precompensator for which:

$$G(\sigma(t), q^{-1}) F(\sigma(t), q^{-1}) = 1. \tag{6}$$

A fact which should be noted is that due to the time-varying nature of the transfer operators, commutativity does not apply to them, in general. In fact the backward-shift operator should obey a non-commutative multiplicative operation '$\circ$' defined as [11]:

$$q^{-i} \circ q^{-j} = q^{-(i+j)}, \quad q^{-i} \circ x(t) = x(t-i) q^{-i}. \tag{7}$$

A. Precompensator Parameterisation

The precompensator is parameterised such that $F(\sigma(t), q^{-1})$ is linear in its parameters and can be expressed as:

$$F(\rho, \sigma(t), q^{-1}) = \beta^T(\sigma(t), q^{-1}) \rho \tag{8}$$

where

$$\begin{aligned} \beta^T(\sigma(t), q^{-1}) = [&\beta_0^1(q^{-1}) \circ \sigma(t)^T \circ \beta_0^2(q^{-1}), \\ &\beta_1^1(q^{-1}) \circ \sigma(t)^T \circ \beta_1^2(q^{-1}), \ldots, \\ &\beta_{n_\rho}^1(q^{-1}) \circ \sigma(t)^T \circ \beta_{n_\rho}^2(q^{-1})], \end{aligned} \tag{9}$$

and $\rho \in \mathbb{R}^{(n_\sigma+1)(n_\rho+1)}$ is the vector of controller parameters:

$$\begin{aligned} \rho = [&\rho_0^0, \rho_0^1, \ldots, \rho_0^{n_\sigma}, \rho_1^0, \rho_1^1, \ldots, \rho_1^{n_\sigma}, \ldots, \rho_{n_\rho}^0, \\ &\rho_{n_\rho}^1, \ldots, \rho_{n_\rho}^{n_\sigma}]^T. \end{aligned} \tag{10}$$

For ease of notation the number of precompensator parameters $(n_\sigma + 1)(n_\rho + 1)$ will be denoted by n.

The $\beta_i^j(q^{-1})$ are linear discrete-time transfer operators, which can be any orthonormal basis functions, such as Laguerre or Kautz. In the sequel, however, for clarity of presentation, we suppose that $\beta_i^1(q^{-1}) = 1$ and $\beta_i^2(q^{-1}) = q^{-i}$. This choice means $\beta^T(\sigma(t), q^{-1})$ is given by:

$$\begin{aligned} \beta^T(\sigma(t), q^{-1}) = [&\sigma_0(t), \sigma_1(t), \ldots, \sigma_{n_\sigma}(t), \\ &\sigma_0(t) q^{-1}, \sigma_1(t) q^{-1}, \ldots, \sigma_{n_\sigma}(t) q^{-1}, \ldots, \\ &\sigma_0(t) q^{-n_\rho}, \sigma_1(t) q^{-n_\rho}, \ldots, \sigma_{n_\sigma}(t) q^{-n_\rho}] \end{aligned} \tag{11}$$

where $\sigma_j(t)$ represents the jth element of $\sigma(t)$. This parameterisation allows a wide range of dependence on the scheduling parameter to be described. For example each $\sigma_j(t)$ could represent a function of a different scheduling parameter. Alternatively the $\sigma_j(t)$ could be a set of orthogonal basis functions of a single scheduling parameter e.g. polynomials:

$$\sigma_j(t) = \bar{\sigma}^j(t), \tag{12}$$

where $\bar{\sigma}(t)$ is the single scheduling parameter.

These choices lead to the following expression for F:

$$\begin{aligned} F(\rho, \sigma(t), q^{-1}) = &\left[\rho_0^0 \sigma_0(t) + \rho_0^1 \sigma_1(t) + \cdots + \rho_0^{n_\sigma} \sigma_{n_\sigma}(t)\right] \\ &+ \left[\rho_1^0 \sigma_0(t) + \rho_1^1 \sigma_1(t) + \cdots + \rho_1^{n_\sigma} \sigma_{n_\sigma}(t)\right] q^{-1} + \ldots \\ &+ \left[\rho_{n_\rho}^0 \sigma_0(t) + \rho_{n_\rho}^1 \sigma_1(t) + \cdots + \rho_{n_\rho}^{n_\sigma} \sigma_{n_\sigma}(t)\right] q^{-n_\rho}. \end{aligned} \tag{13}$$

Remark: In the special case that the desired output $y_d(t)$ and scheduling parameter $\sigma(t)$ are known a priori, they can be used to improve the tracking of systems with a time delay. This improvement is achieved by setting $\beta_i^1(q^{-1}) = q^\delta$, where δ equals the system's time delay. This fact can be illustrated via the following example. Consider the noise-free system with a time delay m:

$$y(t) = -a_1(\sigma(t)) y(t-1) - a_2(\sigma(t)) y(t-2) + u(t-m). \tag{14}$$

We want $y(t) = y_d(t)$, so substituting this equality into the above equation gives:

$$u(t-m) = y_d(t) + a_1(\sigma(t)) y_d(t-1) + a_2(\sigma(t)) y_d(t-2) \tag{15}$$

or

$$\begin{aligned} u(t) = y_d(t+m) &+ a_1(\sigma(t+m)) y_d(t+m-1) \\ &+ a_2(\sigma(t+m)) y_d(t+m-2), \end{aligned} \tag{16}$$

which shows that the structure required for perfect tracking is achieved by choosing $\delta = m$. This implies that values of $\sigma(t)$ and $y_d(t)$ at $t+\delta$ should be used at time t, which is possible if they are known in advance. Unfortunately, in many applications, i.e. those where $\sigma(t)$ is measured in real time, advanced knowledge of $\sigma(t)$ will not be available.

B. Single Realisation Behaviour and Ergodicity

It is often useful to be able to equate the time average properties of a signal over a single realisation with the ensemble average taken over many realisations. Signals with this property are called ergodic and Theorem 2B.1 in [12] indicates when certain types of nonstationary signals can be ergodic in the correlation.

Theorem 2B.1 [p. 55 in [12]] Let $\{P_\theta(q^{-1}), \theta \in D_\theta\}$ and $\{M_\theta(q^{-1}), \theta \in D_\theta\}$ be uniformly stable families of filters, and assume that the deterministic signal $w(t)$, $t = 1, 2, \ldots,$ is subject to

$$|w(t)| \leq C_w, \quad \forall t. \tag{17}$$

Let the signal $s_\theta(t)$ be defined, for each $\theta \in D_\theta$, by

$$s_\theta(t) = P_\theta(q^{-1})v_l(t) + M_\theta(q^{-1})w(t) \tag{18}$$

where

$$v_l(t) = \sum_{k=0}^{\infty} l_k(t)e_l(t-k) = L(t, q^{-1})e_l(t) \tag{19}$$

and $e_l(t)$ is a sequence of independent random vectors with zero mean values, $E\{e_l(t)e_l^T(t)\} = \Lambda_t$ and bounded fourth moments, and $\{L(t, q^{-1}), t = 1, 2, \ldots\}$ is a uniformly stable family of filters. $E\{\cdot\}$ denotes the mathematical expectation. Then:

$$\sup_{\theta \in D_\theta} \left\| \frac{1}{N} \sum_{t=0}^{N-1} \left[s_\theta(t)s_\theta^T(t) - E\{s_\theta(t)s_\theta^T(t)\} \right] \right\| \to 0 \tag{20}$$
$$\text{w.p. 1, as } N \to \infty,$$

where $\|\cdot\|$ is the Frobenius norm.

III. Tuning when LPV transfer operators commute

As mentioned previously, in general, time-varying operators do not commute. One case, however, where they do is when the two operators considered are reciprocal. Thus, in the case that the precompensator's parameterisation and parameters are such that (6) is satisfied then $F(\rho_0, \sigma(t))G(\sigma(t)) = G(\sigma(t))F(\rho_0, \sigma(t)) = 1$, where ρ_0 are the parameters satisfying (6). This fact gives an idea for a tuning scheme for the precompensator's parameters.

A. Tuning scheme

We have that the tracking error of the system, with a precompensator, is given by:

$$\epsilon(t) = y_d(t) - G(\sigma(t))F(\rho, \sigma(t))y_d(t) - v(t), \tag{21}$$

where the dependence of the transfer operators on the backward-shift time operator has been left out for notational clarity.

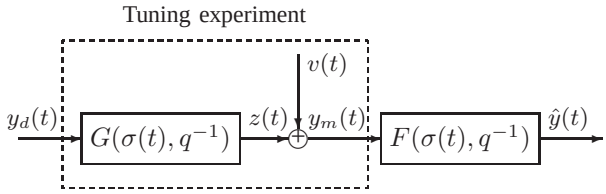

Fig. 2. Precompensator tuning scheme

In the absence of noise, and when $G(\sigma(t))$ and $F(\rho, \sigma(t))$ are commutative, the same tracking error would be obtained if the positions of the system and the precompensator were swapped so that F acts as a post-compensator. Using this idea, it is possible to estimate $\epsilon(t)$ from one set of data obtained from the system without a precompensator as:

$$\begin{aligned}\hat{\epsilon}(t) &= y_d(t) - \hat{y}(t) = y_d(t) - F(\rho, \sigma(t))y_m(t) \\ &= y_d(t) - F(\rho, \sigma(t))G(\sigma(t))y_d(t) - F(\rho, \sigma(t))v(t) \\ &= y_d(t) - F(\rho, \sigma(t))z(t) - F(\rho, \sigma(t))v(t)\end{aligned} \tag{22}$$

where $z(t)$ and $y_m(t)$ are the noise-free and noisy outputs, respectively, of the system when $y_d(t)$ is applied as the input (see Fig. 2).

B. Algorithm

It is possible to express $\hat{\epsilon}(t)$ in linear regression form as:

$$\hat{\epsilon}(t) = y_d(t) - F(\rho, \sigma(t))y_m(t) = y_d(t) - \phi_m^T(t)\rho \tag{23}$$

where:

$$\begin{aligned}\phi_m^T(t) = [&\sigma_0(t)y_m(t), \sigma_1(t)y_m(t), \ldots, \sigma_{n_\sigma}(t)y_m(t), \\ &\sigma_0(t)y_m(t-1), \sigma_1(t)y_m(t-1), \ldots, \sigma_{n_\sigma}(t)y_m(t-1), \ldots, \\ &\sigma_0(t)y_m(t-n_\rho), \sigma_1(t)y_m(t-n_\rho), \ldots, \sigma_{n_\sigma}(t)y_m(t-n_\rho)].\end{aligned} \tag{24}$$

The precompensator parameters can then be found by the minimisation of a quadratic cost function:

$$J_m^N(\rho) = \frac{1}{2N} \sum_{t=0}^{N-1} \hat{\epsilon}^2(t). \tag{25}$$

The minimiser of this criterion is given by:

$$\rho_{mLS}^N = \left[\frac{1}{N} \sum_{t=0}^{N-1} \phi_m(t)\phi_m^T(t) \right]^{-1} \frac{1}{N} \sum_{t=0}^{N-1} \phi_m(t)y_d(t). \tag{26}$$

Unfortunately when $F(\rho, \sigma(t))$ is placed as a post-compensator, it filters the noise $v(t)$ also, as seen in Fig. 2. Therefore the parameters which minimise the variance of the tracking error estimate will not be the same as those which minimise the variance of the true tracking error.

The Instrumental Variables (IV) method can be used, nonetheless, to give consistent estimates of the true minimising parameters ρ_0. For the IV estimates to converge to the true values, the IV vector must be correlated with the non-noisy component of $y_m(t)$, but not with the noise $v(t)$. Many choices of IV vector satisfy these conditions, such as a vector of time shifted versions of $y_d(t)$. The choice considered in

this paper is to use a vector similar to $\phi_m(t)$, but with $y_m(t)$ obtained from a second experiment, performed in the same way as the first. The second experiment will, however, be affected by a different, independent noise realisation. This choice has been made as it leads to an IV vector that is strongly correlated with the non-noisy component of $y_m(t)$, and so leads to parameter estimates with low variances. The IV estimate is thus given by:

$$\rho_{mIV}^N = \mathcal{M}^{-1}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)y_d(t), \tag{27}$$

with

$$\mathcal{M} = \left[\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_{m2}^T(t)\right]$$

where $\phi_{m1}(t)$ and $\phi_{m2}(t)$ are the $\phi_m(t)$ from the two experiments.

The consistency of the IV estimates is not directly obvious as, unlike the standard LTI case, the signals considered contain nonstationary stochastic components. The applicability of ergodicity type results typically used in consistency analysis is not, therefore, immediately evident. An analysis is thus performed in the next subsection which demonstrates that IV does indeed lead to consistent estimates, despite the presence of these types of disturbances.

C. Consistency of IV Estimates

To see that the IV method gives consistent estimates we begin by rewriting (27) as:

$$\begin{aligned}\rho_{mIV}^N &= \mathcal{M}^{-1}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_z^T(t)\rho_0\\ &= \mathcal{M}^{-1}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\left(\phi_{m2}^T(t)-\phi_{v2}^T(t)\right)\rho_0\\ &= \rho_0 - \mathcal{M}^{-1}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_{v2}^T(t)\rho_0,\end{aligned} \tag{28}$$

where $\phi_z(t)$ and $\phi_{vi}(t)$ are similar to $\phi_{mi}(t)$, but $y_{mi}(t)$ is replaced by $z(t)$ and $v_i(t)$ respectively. Additionally $\phi_{mi}(t) = \phi_z(t) + \phi_{vi}(t)$.

In order for the parameter estimates to be consistent i.e. that ρ_{mIV}^N converges almost surely to ρ_0 as $N \to \infty$, it is necessary that:

i) $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_{m2}^T(t)$ be nonsingular. (29)

ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_{v2}^T(t)\rho_0 = \mathbf{0}$, (30)

where $\mathbf{0}$ is the zero vector.

Condition i) is a persistency of excitation condition. It is similar to the persistency of excitation condition found in methods for the identification of input-ouput form LPV models, as similar signals are involved i.e. noisy output signals multiplied by functions of the scheduling parameter. Consistency of excitation for LPV identification was first considered in [9]. There sufficient conditions for polynomial type coefficient dependence on the scheduling parameter are given. It is shown that if the system input signal is 'sufficiently rich' then persistency of excitation is ensured if the scheduling parameter 'visits' $n_\sigma + 1$ distinct points infinitely many times, where n_σ is the order of the polynomial dependence. More recently [13] produced more general sufficient conditions for other types of coefficient dependence.

To show Condition ii) further analysis is required. We have that:

$$\begin{aligned}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{m1}(t)\phi_{v2}^T(t) &= \frac{1}{N}\sum_{t=0}^{N-1}\phi_z(t)\phi_{v2}^T(t)\\ &\quad + \frac{1}{N}\sum_{t=0}^{N-1}\phi_{v1}(t)\phi_{v2}^T(t).\end{aligned} \tag{31}$$

Considering the first matrix on the right hand side of (31), each of its elements is the sum over time of products of terms such as $\sigma_j(t)z(t-n)$ and $\sigma_i(t)v_2(t-p)$. Then, referring to Theorem 2B.1 given in Subsection II-B, we can define:

$$\begin{aligned}s(t) &= \begin{bmatrix}\sigma_j(t)z(t-n)\\ \sigma_i(t)v_2(t-p)\end{bmatrix}\\ &= \begin{bmatrix}0\\ \sigma_i(t)H(\sigma(t-p),q^{-1})e_2(t-p)\end{bmatrix}\\ &\quad + \begin{bmatrix}\sigma_j(t)G(\sigma(t-n),q^{-1})y_d(t-n)\\ 0\end{bmatrix}\\ &= \begin{bmatrix}v_1^1(t)\\ v_2^1(t)\end{bmatrix} + \begin{bmatrix}w_1^1(t)\\ w_2^1(t)\end{bmatrix}\end{aligned} \tag{32}$$

where $w_1^1(t)$, $w_2^1(t)$, $v_1^1(t)$ and $v_2^1(t)$ have their obvious definitions. The signals $w_1^1(t)$ and $w_2^1(t)$ satisfy (17), due to the assumed uniform stability of $G(\sigma(t),q^{-1})$ and the boundedness of $\sigma_j(t)$ and $y_d(t)$. Also $v_1^1(t)$ and $v_2^1(t)$ fit in with the desired form of (19) due to the assumed uniform stability of $H(\sigma(t),q^{-1})$. The components of $s(t)s^T(t)$ give, amongst others, $\sigma_j(t)z(t-n)\sigma_i(t)v_2(t-p)$. So by applying Theorem 2B.1 we can state that:

$$\left\|\frac{1}{N}\sum_{t=0}^{N-1}[\phi_z(t)\phi_{v2}^T(t) - E\{\phi_z(t)\phi_{v2}^T(t)\}]\right\| \to 0$$

w.p. 1, as $N \to \infty$,

and, as $z(t)$ and $v(t)$ are not correlated, $E\{\phi_z(t)\phi_{v2}^T(t)\} = \mathbf{0}$ implying:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\phi_z(t)\phi_{v2}^T(t) = \mathbf{0} \text{ w.p. } 1.$$

A similar result for the second matrix on the right hand side of (31) can be derived using Theorem 2B.1 i.e. that:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\phi_{v1}(t)\phi_{v2}^T(t) = \mathbf{0} \text{ w.p. } 1$$

and combining these two satisfies Condition ii), showing the parameters estimates obtained with the IV method to be consistent.

Remark: The values of the scheduling parameter $\sigma_j(t)$ used in the calculation of ρ^N_{mIV} will normally also be measured. They are, therefore, susceptible to measurement noise as well. In the case that the noise-to-signal ratio is very low, the effect of this noise can be neglected. However, if the ratio is not negligible the measurement noise may degrade the precision of the parameter estimates. It can be shown, however, by an analysis similar to that previously presented and also to that found in [10], that when the scheduling parameter is uncorrelated with the output signal and the dependency on it is affine the use of an IV vector like $\phi_{m2}(t)$, but using $\sigma_j(t)$ measured during the second experiment, leads to consistent estimates.

IV. Tuning when LPV transfer operators do not commute

In general, unfortunately, (6) is unlikely to be satisfied by a precompensator with the linear parameterisation proposed. In this case, the order of the precompensator can be increased until the condition is approximately satisfied and the method of the previous section can be used. If, however, the required order is too large to be implemented, and the approximation achieved with a reduced order controller is not sufficiently good, the precompensator and system will no longer commute. A method that can be used in this situation is developed below.

The signal $u(t) = F(\rho, \sigma(t))y_d(t)$ can be expressed as:

$$u(t) = F(\rho, \sigma(t))y_d(t) = \phi^T(t)\rho \tag{33}$$

where

$$\begin{aligned}\phi^T(t) = [&\sigma_0(t)y_d(t), \sigma_1(t)y_d(t), \ldots, \sigma_{n_\sigma}(t)y_d(t),\\ &\sigma_0(t)y_d(t-1), \sigma_1(t)y_d(t-1), \ldots, \sigma_{n_\sigma}(t)y_d(t-1), \ldots,\\ &\sigma_0(t)y_d(t-n_\rho), \sigma_1(t)y_d(t-n_\rho), \ldots, \sigma_{n_\sigma}(t)y_d(t-n_\rho)].\end{aligned} \tag{34}$$

It is now possible to write the output as:

$$\begin{aligned}y(t) &= G(\sigma(t))F(\rho, \sigma(t))y_d(t) + v(t)\\ &= G(\sigma(t))\phi^T(t)\rho + v(t)\\ &= x^T(t)\rho + v(t).\end{aligned} \tag{35}$$

where $x(t) = G(\sigma(t))\phi(t)$.

The tracking error (5) can then be expressed as:

$$\epsilon(t) = y_d(t) - x^T(t)\rho - v(t) \tag{36}$$

or in vector form as:

$$\epsilon = y_d - X\rho - v, \tag{37}$$

where the vector ϵ is given by:

$$\epsilon = [\epsilon(0), \epsilon(1), \ldots, \epsilon(N-1)]^T, \tag{38}$$

y_d and v are defined similarly, and the matrix X is:

$$X = [x(0), x(1), \ldots, x(N-1)]^T. \tag{39}$$

The aim is to find ρ such that the average tracking error is small, therefore a logical objective is to minimise its mean squared value i.e. to find the ρ that minimises:

$$J^N(\rho) = \frac{1}{2N}E\left\{\epsilon(\rho)^T\epsilon(\rho)\right\}, \tag{40}$$

where the dependence of ϵ on ρ has been shown explicitly.

The minimiser of (40) is given by:

$$\rho^N_{LMS} = \left[\frac{1}{N}X^TX\right]^{-1}\frac{1}{N}X^Ty_d. \tag{41}$$

In the case that $G(\sigma(t))$ were known exactly, X could be calculated, followed by ρ^N_{LMS}. $G(\sigma(t))$ is never known exactly, however, and the model uncertainty leads to a non optimal ρ.

It is possible to obtain an estimate $\hat{X}$ of the matrix X without the use of a model through a series of experiments on the real system. This can be seen to be the case by noting that the element $X_{t,j}$ of the matrix X is the output of the system $G(\sigma(t))$ when the jth element of $\phi(t)$ is applied as an input. Thus for each column of X an experiment can be carried out on the real system i.e. n experiments in total. In reality an estimate, rather than the exact value, of X will be found as each experiment will have its own noise realisation $v_j(t)$. The estimate of X is:

$$\hat{X} = X + V, \tag{42}$$

where V is a matrix whose t, jth element is $v_j(t)$. Substituting $\hat{X}$ in for X in (41) we have:

$$\begin{aligned}\hat{\rho}^N_{LMS} &= \left[\frac{1}{N}(X+V)^T(X+V)\right]^{-1}\frac{1}{N}(X+V)^Ty_d\\ &= \left[\frac{1}{N}(X^TX + V^TX + X^TV + V^TV)\right]^{-1}\\ &\quad\times\frac{1}{N}(X+V)^Ty_d.\end{aligned} \tag{43}$$

Therefore, unfortunately, when $\hat{X}$ is used in place of X in (41) the presence of the noise in the experiments performed to find $\hat{X}$ will mean that the minimising value ρ^N_{LMS} cannot be calculated i.e. $\hat{\rho}^N_{LMS} \neq \rho^N_{LMS}$.

One way of dealing with this problem is to use instrumental variables again. This time two estimates of X, $\hat{X}_1$ and $\hat{X}_2$, are used. They are obtained from two sets of the n experiments previously described. The IV minimiser estimate is then given by:

$$\hat{\rho}^N_{IV} = \left[\frac{1}{N}\hat{X}_1^T\hat{X}_2\right]^{-1}\frac{1}{N}\hat{X}_1^Ty_d. \tag{44}$$

Remark: The same $G(\sigma(t), q^{-1})$ should be used at each t for all the experiments i.e. when different signals $u(t)$ are applied. It is therefore necessary that the scheduling parameter $\sigma(t)$ be independent of $u(t)$, which may not be the case for certain quasi-LPV systems where the scheduling parameter can be an input-dependent, internal variable. Nonetheless, a number of LPV systems found in practice do satisfy this requirement, such as x-y positioning tables where the dynamics of one stage depend on the position of the other and not their own.

A. Consistency of estimates

To demonstrate that the IV estimate (44) converges to the true minimiser of (40) when $N \to \infty$ we first write:

$$\begin{aligned}\hat{\rho}_{IV}^N &= \left[\frac{1}{N}(X+V_1)^T(X+V_2)\right]^{-1}\frac{1}{N}(X+V_1)^T y_d \\ &= \left[\frac{1}{N}(X^TX + V_1^TX + X^TV_2 + V_1^TV_2)\right]^{-1} \\ &\times \frac{1}{N}(X+V_1)^T y_d \end{aligned} \tag{45}$$

where V_1 and V_2 denote matrices of noise realisations associated with the first and second set of experiments. For $\hat{\rho}_{IV}^N$ to converge to ρ_{LMS}^N we require $V_1^TX + X^TV_2 + V_1^TV_2$ and $V_1^Ty_d$ to converge to zero as $N \to \infty$. This can be shown to be the case by noting that the i,jth element of the matrix V_1^TX is given by:

$$\begin{aligned}[V_1^TX]_{i,j} &= \sum_{t=0}^{N-1} v1_i(t)X_{t,j} \\ &= \sum_{t=0}^{N-1}[H(\sigma(t),q^{-1})e1_i(t)][G(\sigma(t),q^{-1})\phi_j(t)],\end{aligned} \tag{46}$$

where $v1_i(t)$ corresponds to the t,i element of V_1, and $v1_i(t) = H(\sigma(t),q^{-1})e1_i(t)$. Theorem 2B.1 is applicable to this time sum and can be used to show that:

$$\left\|\frac{1}{N}\sum_{t=0}^{N-1}[v1_i(t)X_{t,j} - E\{v1_i(t)X_{t,j}\}]\right\| \to 0$$
w.p. 1, as $N \to \infty$,

and since $E\{v1_{t,i}X_{t,j}\} = 0$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} v1_{t,i}X_{t,j} = 0 \text{ w.p. } 1$$

This result can be applied to each element of V_1^TX, and similar results hold for X^TV_2, $V_1^TV_2$ and $V_1^Ty_d$. Therefore, $\hat{\rho}_{IV}^N \to \rho_{LMS}^N$ as $N \to \infty$ almost surely.

V. Conclusions

Direct data-driven tuning methods of precompensators for LPV systems are developed in this paper. Two techniques are proposed. The first one, applicable when the precompensator and the system commute, only requires two experiments in order to obtain consistent estimates of the parameters leading to perfect mean tracking. The second one, which has no restriction on the precompensator parameterisation, requires a number of experiments equal to twice the number of precompensator parameters. It is demonstrated that the computed parameters converge to those minimising the mean squared tracking error in the presence of noise.

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