WEAK METRICS ON EUCLIDEAN DOMAINS

ATHANASE PAPADOPOULOS AND MARC TROYANOV

Abstract. A *weak metric* on a set is a function that satisfies the axioms of a metric except the symmetry and the separation axioms. The aim of this paper is to present some interesting weak metrics and to study some of their properties. In particular, we introduce a weak metric, called the *Apollonian weak metric*, on any subset of a Euclidean space which is either bounded or whose boundary is unbounded. We relate this weak metric to some familiar metrics such as the Poincaré metric, the Klein-Hilbert metric, the Funk metric and the part metric which all play important roles in classical and in recent work on geometric function theory.

This paper will appear in the JP Jour. Geometry & Topology

AMS Mathematics Subject Classification: 30F45, 51M15, 51K99.
Keywords: Weak metric, Apollonian weak metric, hyperbolic geometry, Poincaré model, Klein model, Klein-Hilbert metric, Funk weak metric, part metric, Poincaré metric, Apollonian semi-metric, half-Apollonian semi-metric.

1. introduction

The axioms for a metric space were formulated 100 years ago, in a famous paper by Maurice Fréchet, see [8]. Since then, several important generalizations of the notion of metric space appeared. In the present paper, we shall consider weak metrics and semi-metrics. We first recall the definitions.

Definition 1.1 (Weak metric and semi-metric)**.** A *weak metric* on a set X is a function $\delta : X \times X \to [0, \infty)$ satisfying

i) $\delta(x, x) = 0$ for all x in X;
ii) $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ for all x, y and z in X.

A *semi-metric* is a symmetric weak metric, that is, a weak metric satisfying

iii) $\delta(x, y) = \delta(y, x)$ for all x and y in X.

H. Busemann studied extensively functions satisfying some of the axioms of a metric (see [6, 7]), and he called them "general metrics" or simply "metrics". The name "weak metric" is due to H. Ribeiro ([15]). There are two distinct notions of separation for a weak metric, namely:

Definition 1.2. A weak metric δ is said to be *strongly separating* if we have

$$\min\{\delta(x, y), \delta(y, x)\} = 0 \quad \Leftrightarrow \quad x = y,$$

and it is said to be *weakly separating* if

$$\max\{\delta(x, y), \delta(y, x)\} = 0 \quad \Leftrightarrow \quad x = y,$$

for all x and y in X.

Date: February 21, 2007.

In the case of a semi-metric, these two notions clearly coincide. A *metric* on X, in the usual sense, is a separating semi-metric.

Given a weak metric $\delta : X \times X \to [0,\infty)$, it can be interesting to consider a *symmetrization* of it, and to try and compare this symmetrization with other known metrics or semi-metrics. In fact, there exist several notions of symmetrization, none of them being more natural than the others. We shall deal in this paper with two of these, defined as follows.

Definition 1.3 (Symmetrizations)**.** Let $\delta : X \times X \to [0,\infty)$ be a weak metric. A *symmetrization* of δ is one of the following functions $\sigma\delta : X \times X \to [0,\infty)$ and $S\delta : X \times X \to [0,\infty)$, defined for x and y in X by

$$\sigma\delta(x,y) = \max\{\delta(x,y), \delta(y,x)\}$$

and

$$S\delta(x,y) = \frac{1}{2}\left(\delta(x,y) + \delta(y,x)\right).$$

Both symmetrizations are semi-metrics. The semi-metric $\sigma\delta$ is sometimes called the *max-symmetrization* and $S\delta$ the *mean-value-symmetrization* of δ.

Observe the following inequalities:

$$S\delta(x,y) \leq \sigma\delta(x,y) \leq 2S\delta(x,y).$$

Furthermore, the equality $S\delta = \sigma\delta$ holds if and only if δ is itself symmetric. In that case, both symmetrizations coincide with δ.
It is also clear from the definitions that both symmetrizations of a weakly separating weak metric give rise to a metric.

It is easy to make definitions, but it is important to have interesting examples. We start right away with a concrete example of a weak metric.
Let $\mathbb{E}^n$ be the n-dimensional Euclidean space. We shall denote by $|x-y|$ the Euclidean distance between the points x and y in that space.
Let A be an open subset of $\mathbb{E}^n$, with $A \neq \mathbb{E}^n$. We introduce the function i_A on $A \times A$ defined by

$$i_A(x,y) = \log\left(1 + \frac{|x-y|}{d(x,\partial A)}\right) = \sup_{a \in \partial A} \log\left(1 + \frac{|x-y|}{|x-a|}\right)$$

for all $x, y \in A$.

Proposition 1.4. *The function i_A is a weak metric on A.*

Proof. We prove the triangle inequality. For x, y and z in A, we have

$$|y-z| \geq |x-z| - |x-y|$$

and

$$|x-y| + d(x,\partial A) \geq d(y,\partial A).$$

Multiplying the two inequalities, we obtain

$$|y-z|\left(|x-y| + d(x,\partial A)\right) \geq \left(|x-z| - |x-y|\right) d(y,\partial A)$$

or, equivalently,

$$\left(d(x,\partial A) + |x-y|\right)\left(d(y,\partial A) + |y-z|\right) \geq \left(d(x,\partial A) + |x-z|\right) d(y,\partial A).$$

The last inequality is equivalent to

$$\left(1+\frac{|x-y|}{d(x,\partial A)}\right)\left(1+\frac{|y-z|}{d(y,\partial A)}\right)\geq\left(1+\frac{|x-z|}{d(x,\partial A)}\right).$$

Taking logarithms, we obtain the triangle inequality for i_A. □

In their study of uniform domains in Euclidean spaces [11], Gehring and Osgood considered a metric which is the symmetrization Si_A of our weak metric i_A. It is usually denoted by $\tilde{j}_A$, and it is therefore defined by

$$\tilde{j}_A(x,y)=S\mathrm{i}_A(x,y)=\frac{1}{2}\left\{\log\left(1+\frac{|x-y|}{d(x,\partial A)}\right)+\log\left(1+\frac{|x-y|}{d(x,\partial A)}\right)\right\}.$$

The symmetrization σi_A of this weak metric i_A is a metric that has been considered by M. Vuorinen in his study of conformal invariants and moduli of families of curves, see [18]. Vuorinen's metric is usually written as

$$j_A(x,y)=\log\left(1+\frac{|x-y|}{\min\{\delta(x),\delta(y)\}}\right)$$

where $\delta(z)=d(z,\partial A)$.

There is a large literature on the metrics $\tilde{j}_A$ and j_A, see for instance [16] and [12].

As the above example illustrates, we shall see in this paper that a certain number of important metrics are naturally obtained as a symmetrization of some weak metric. The weak metric appears then as a kind of primitive structure on which the actual metric is built. It is then an interesting question to investigate the geometric properties of the weak metric and to compare them with those of the associated symmetric metric (or semi-metric). To our knowledge, this question has not been really studied so far. Let us formulate it as the following general:

Problems. (1) *Given a metric space (X,d), find a natural weak metric δ (which is not a metric) on X such that d is the symmetrization of δ, i.e. $d=S\delta$ or $d=\sigma\delta$.* (2) *Describe the geometry of (X,δ) and compare it to the geometry of (X,d).*

By the word geometry, we mean here the study of geodesics, of isometries, of curvature and so on.

These problems are not precisely formulated, in particular we should not expect question (1) to have a unique answer. However, we believe that these questions are worth investigating, at least in the case of some important metric spaces.

In this paper, we shall in particular address these problems in the case of the hyperbolic plane. We shall see that the hyperbolic metric can be obtained as a symmetrization of at least three natural weak metrics. The first weak metric is the so called *Funk weak metric* and it is related to the projective (Klein) model of hyperbolic geometry. The other two are the *Apollonian weak metrics*, and they are related to the conformal (Poincaré) model of the unit disk and the upper-half plane respectively. We shall give explicit formulas for the Apollonian weak metrics of the upper-half plane and the unit disk in Theorem 1 and Theorem 2 respectively. We shall observe that the isometry group of the Apollonian weak metric is quite different from the isometry group of the hyperbolic metric. On the other hand, we shall show in Theorem 3 that the hyperbolic lines in the unit disk are geodesics for the Apollonian weak metric.

The weak metrics that we consider here are defined for a wide class of domains in Euclidean space. The Funk weak metric is classical although not so popular. The Apollonian weak metric is a new notion which we define in section 4 below. The name "Apollonian" was chosen because a symmetrization of this weak metric is the Apollonian semi-metric discussed in the paper [4] by Beardon.

The outline of the rest of this paper is as follows. In Section 2 and 3 , we recall the definition of the Funk weak metric, and we identify its symmetrizations. In Section 4, we define the Apollonian weak metric. and we relate it to some other known metrics. In Section 5, we give an explicit formula for the Apollonian weak metric of the unit disk in $\mathbb{C}$ and we draw some consequences of that formula. The Poincaré metric of the unit disk is a symmetrization of the Apollonian weak metric in the same way as the Klein-Hilbert metric of that disk is a symmetrization of the Funk metric. Section 6 is concerned with the notion of geodesic associated to a weak metric. We study in particular the geodesics of the Apollonian weak metric of the unit disk in $\mathbb{C}$.

2. The Funk weak metric

The *Funk weak metric* is a weak metric defined on open bounded convex subsets of $\mathbb{E}^n$. It was discovered by P. Funk [9] and is discussed in [7, 19]. To recall its definition, let A be a nonempty bounded open convex subset of $\mathbb{E}^n$. For every x in A, we set $\mathcal{F}(x,x) = 0$. For y distinct from x, we consider the Euclidean ray starting at x and passing through y, we let a be the intersection point of that ray with the boundary of A, and we set

$$\mathcal{F}(x,y) = \log \frac{|x-a|}{|y-a|}. \tag{1}$$

The function $\mathcal{F}$ is a strongly separating weak metric on A, called the Funk weak metric. The proof of the triangle inequality follows from the theorem of Menelaus (see [19, Appendix I]).

If we need to indicate the dependence of the Funk weak metric on the domain A, we denote it by $\mathcal{F}_A$.

We recall that a *similarity* defined on a domain in $\mathbb{E}^n$ is a map ϕ satisfying $|\phi(x) - \phi(y)| = \mu|x-y|$ for some $\mu > 0$ and for all x, y in this domain. A similarity always extends as a global affine map $\phi : \mathbb{E}^n \to \mathbb{E}^n$. Similarities form a subgroup of the affine group. Observe that any similarity $\phi : A \to B$ between bounded convex open subsets $A, B \subset \mathbb{E}^n$ is an isometry from $(A, \mathcal{F}_A)$ to $(B, \mathcal{F}_B)$. This is obvious since similarities preserve ratios of Euclidean distances.

The mean-value symmetrization (in the sense of Definition 1.3) of the Funk weak metric gives rise to the *Klein-Hilbert metric*. Let us describe this metric $\mathcal{H}$. . Let A be again a nonempty open bounded convex subset of $\mathbb{E}^n$. For $x = y \in A$, we set $\mathcal{H}(x,y) = 0$, and for $x \neq y$, we consider the Euclidean line containing x and y. It intersects the boundary of A in two points, a and b, these names chosen such that b, x, y, a follow each other in that order on that line. We then set

$$\mathcal{H}(x,y) = \frac{1}{2} \log \left(\frac{|x-a|}{|y-a|} \frac{|y-b|}{|x-b|} \right). \tag{2}$$

Note that the quantity $\dfrac{|x-a|}{|y-a|}\dfrac{|y-b|}{|x-b|}$ is equal to the cross ratio $[b,x,y,a]$ of the four points. Since the notions of Euclidean line and of cross ratio of points on a line are invariant under projective transformations, the Klein-Hilbert metric is also invariant under projective transformations (which include the similarities).
This metric was first defined by F. Klein on the unit disk $\mathbb{D}^2 \subset \mathbb{E}^2$. It defines what is usually called the Klein model of hyperbolic geometry on $\mathbb{D}^2$. This metric was later defined by D. Hilbert, using the same formula, on an arbitrary bounded open covex subset of $\mathbb{E}^n$. Formula (2) shows that the Klein-Hilbert metric is a symmetrization of the Funk semi-metric. More precisely, we have

$$\mathcal{H} = S\mathcal{F}. \tag{3}$$

3. The part metric

This notion was introduced by H. S. Bear in his study of complex function algebras, [2, 3]. It can be defined in the following abstract setting. Consider a set X and a class $\mathcal{B}$ of positive real-valued functions on X. Introduce an equivalence relation (which we call "part-equivalence") on X as follows: two points x and y in X are equivalent if and only if there exists a constant $c > 0$ such that the following Harnack-type inequality

$$\frac{1}{c} \leq \frac{u(y)}{u(x)} \leq c$$

holds for all functions in $\mathcal{B}$. A subset of X consisiting of points in the same equivalent class is called a *part* of X. The collection of parts of X form a partition of X which is associated to $\mathcal{B}$. Such partitions were first considered by A. M. Gleason.

On each part of X, we have the following natural metric:

$$p(x,y) = p_{\mathcal{B}}(x,y) = \sup\left\{\left|\log\left(\frac{u(x)}{u(y)}\right)\right| \;\Big|\; u \in \mathcal{B}\right\}, \tag{4}$$

which is called the *part metric* induced from $(X,\mathcal{B})$.

Proposition 3.1. *Let A be an open bounded convex subset of $\mathbb{E}^n$ and let $\mathcal{B}$ be the class of positive functions on A that are restrictions of affine functions $u : \mathbb{E}^n \to \mathbb{R}$. Then, all the elements of A are part-equivalent for the relation induced by $\mathcal{B}$ and the corresponding part metric on A coincides with the max-symmetrization of the Funk weak metric:*

$$p_{\mathcal{B}}(x,y) = \sigma\mathcal{F}(x,y) = \max\left\{\log\frac{|x-a|}{|y-a|}, \log\frac{|y-b|}{|x-b|}\right\}.$$

Proof. That all the points of A are part-equivalent will follow from the fact that $p(x,y)$ is finite for all x and y in A, which follows from what we prove now. Given $x,y \in A$ we denote by a and b the two points lying on the intersection of the boundary ∂A and the Euclidean line passing through x and y, assuming that b,x,y,a follow each other in that order on the line. We prove that $\mathcal{F}(x,y) \leq p_{\mathcal{B}}(x,y)$. We consider an affine function u such that $u > 0$ on A and $u(a) = 0$. For $t > 0$, set $z(t) = ty + (1-t)a$. Then $y = z(1)$ and $x = z(s)$ for $s = \frac{|x-a|}{|y-a|}$. Furthermore, there exists $\lambda > 0$ such that for $t > 0$ we have $u(z(t)) = \lambda t$, since $u(z(0)) = u(a) = 0$ and $u(x) > 0$. Thus

$$p(x,y) \geq \log \frac{u(x)}{u(y)} = \log \frac{\lambda s}{\lambda} = \log \frac{|x-a|}{|y-a|} = \mathcal{F}(x,y).$$

A similar argument shows that $p(x,y) \geq \mathcal{F}(y,x)$ and thus $p \geq \sigma\mathcal{F}$.
To prove the converse inequality, we consider an arbitrary affine function v such that $v > 0$ on A. We parametrize the segment $[a,b]$ by $z(t) = tb + (1-t)a$ so that $v(z(t)) = \lambda t + \mu$ for some λ, μ. For $0 < t \leq s$, we have

$$\frac{v(z(t))}{v(z(s))} = \frac{\lambda t + \mu}{\lambda s + \mu} = \frac{t + \mu/\lambda}{s + \mu/\lambda} \leq \max\left\{\frac{s}{t}, \frac{t}{s}\right\}.$$

It easily follows from this inequality that $p(x,y) \leq \max\{\mathcal{F}(y,x), \mathcal{F}(x,y)\}$.

□

Let us now consider an open subset A of the complex plane $\mathbb{C}$ on which there exists a non-constant positive harmonic function (for this it suffices that $\mathrm{Card}(\mathbb{C} \setminus A) \geq 2$) and let $\mathcal{B}$ be the set of harmonic functions on A. The corresponding part metric on A is thus given by

$$p(x,y) = \sup\left\{\left|\log\left(\frac{u(x)}{u(y)}\right)\right| \ : \ u > 0, \text{harmonic in } A\right\}, \tag{5}$$

Since the composition of a positive harmonic function with a conformal transformation is again a positive harmonic function, the part metric is a conformal invariant of domains.

It is worthwile to note that in the case where A is the unit disk $\mathbb{D}^2$, the corresponding metric space is essentially isometric to the hyperbolic plane. More precisely, H. Bear proved in [2, Corollary 1] the following

Proposition 3.2. *In the unit disk $\mathbb{D}^2$, the part metric (5) associated to the class of harmonic functions coincides with twice the Poincaré metric of that disk.*

□

4. The Apollonian weak metric

Let $A \subset \mathbb{E}^n$ be an open subset and let $\partial A = \overline{A} \setminus A$ be its boundary. In this section, we suppose that either A is bounded or ∂A is unbounded. Note that any nonempty convex subset A of $\mathbb{E}^n$ with $A \neq \mathbb{E}^n$ satisfies this hypothesis.
We define a function δ_A on $A \times A$ by the formula

$$\delta_A(x,y) = \sup_{a \in \partial A} \log \frac{|x-a|}{|y-a|}. \tag{6}$$

Proposition 4.1. *The function δ_A is a weak metric.*

Proof. The proof is straightforward. We only say a few words on the fact that δ_A is nonnegative. First, suppose that A is bounded. For any distinct points x and y in A, consider the Euclidean ray starting at x and passing through y and let a be an intersection point of that ray with ∂A. We have $\dfrac{|x-a|}{|y-a|} > 1$, which implies $\delta_A(x,y) \geq 0$. Now suppose that ∂A is unbounded. Let (a_n) be a sequence of points in ∂A such that $|x - a_n| \to \infty$ for some (or equivalently, for any) x in A. Then, for any x and y in A, we have $\dfrac{|x-a_n|}{|y-a_n|} \to 1$ as $n \to \infty$, which shows $\delta_A(x,y) \geq 0$.

□

Definition 4.2 (The Apollonian weak metric)**.** For any open subset $A \subset \mathbb{E}^n$ which is either bounded or whose boundary ∂A is unbounded, the weak metric provided by Proposition 4.1 is called the *Apollonian weak* metric of A. (The name is chosen because of Definitions 4.7 and 4.8, and Proposition 4.9 below.)

The following invariance property is straightforward.

Proposition 4.3. *For any similarity ϕ of $\mathbb{E}^n$, we have, for every x and y in A,*

$$\delta_A(x,y) = \delta_{\phi(A)}(\phi(x), \phi(y)).$$

□

We have the following relation between the Apollonian weak metric δ_A and the weak metric i_A that we defined in §1:

Proposition 4.4. *For every x and y in A, we have $\delta_A(x,y) \leq i_A(y,x)$.*

Remark 4.5. Observe that in this statement, the last term is $i_A(y,x)$ and not $i_A(x,y)$. We note that for any given weak metric δ the function δ' defined by $\delta'(y,x) = \delta(x,y)$ is also a weak metric, which we call the weak metric *dual* to δ. Therefore, Proposition 4.4 gives a comparison between the weak metric δ_A and the weak metric dual to i_A.

Proof. For x and y in A and for z in ∂A, we can write

$$\log \frac{|x-z|}{|y-z|} \leq \log \frac{|x-y| + |y-z|}{|y-z|} = \log \left(1 + \frac{|y-x|}{|y-z|}\right).$$

Taking the supremum over $z \in \partial A$, we obtain the desired result. □

We shall see examples of weak metrics on sets A satisfying both kinds of hypotheses of Definition 4.2. The first example is the following:

Example 4.6 (The Apollonian weak metric on the upper half-plane)**.** In this example, A is the upper half-plane $\mathbb{H}^2 = \{\, z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. The associated Apollonian weak metric $\delta_{\mathbb{H}^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to [0, \infty)$ is given by

$$\delta_{\mathbb{H}^2}(x,y) = \sup_{a \in \mathbb{R}} \log \frac{|x-a|}{|y-a|}.$$

It is easy to see that the restriction of δ_A to the vertical half-line $\{z = is\}$, $s > 0$ is given by

$$\delta_{\mathbb{H}^2}(x,y) = \max\left\{0, \log \frac{t}{s}\right\} = \begin{cases} \log \frac{t}{s} & \text{if} \quad s \leq t \\ 0 & \text{if} \quad s \geq t \end{cases} \tag{7}$$

for $x = is$ and $y = it$. From this, we can see that the weak metric δ_A is not symmetric and not strongly separating. It is weakly separating. This Apollonian weak metric has been studied in the paper [5], in relation with a weak metric introduced by Thurston on the Teichmüller space of a hyperbolic surface [17] and it was shown in [5] that δ_A coincides with Thurston's geometrically defined weak metric, if we interpret the upper half-plane as the Teichmüller space of Euclidean metrics on the torus. The following result was obtained in [5]:

Theorem 1. *The Apollonian weak metric of the upper-half plane is given by*

$$\delta_{\mathbb{H}^2} = \log\left(\frac{|y-\overline{x}|+|y-x|}{|x-\overline{x}|}\right) \tag{8}$$

for every x and y in $\mathbb{H}^2$.

□

It follows that the symmetrization $S\delta_{\mathbb{H}^2}$ coincides with the Poincaré metric $h_{\mathbb{H}^2}$ on $\mathbb{H}^2$. In other words, for all x and y in $\mathbb{H}^2$, we have

$$S\delta_{\mathbb{H}^2}(x,y) = h_{\mathbb{H}^2}(x,y) = \frac{1}{2}\log\left(\frac{|x-\bar{y}|+|x-y|}{|x-\bar{y}|-|x-y|}\right). \tag{9}$$

Next, we want to relate the Apollonian weak metric to some semi-metrics that appear in recent works of Beardon and others. We first recall these semi-metrics.

Definition 4.7 (The Apollonian semi-metric)**.** Let A be any open subset of $\mathbb{E}^n$. The *Apollonian semi-metric* α_A on A is defined by

$$\alpha_A(x,y) = \sup_{a\in\partial A}\log\frac{|x-a|}{|y-a|} + \sup_{b\in\partial A}\log\frac{|y-b|}{|x-b|} = \sup_{a,b\in\partial A}[b,x,y,a].$$

The name "Apollonian" was given by A. Beardon who studied that semi-metric in [4]. (As mentionned in [10], this metric has been earlier introduced by D. Barbilian [1], and Beardon rediscovered it independently). This semi-metric is also discussed in several later papers, for example [10]. The Apollonian semi-metric is a genuine metric if ∂A is not contained in an $(n-1)$-dimensional sphere or an $(n-1)$-dimensional hyperplane (see [4, Theorem 1.1]). The Apollonian semi-metric is invariant under Möbius transformations. If A is the upper-half plane or the unit ball of $\mathbb{E}^n$, the Apollonian semi-metric coincides with the Poincaré metric of these spaces.

Definition 4.8 (The half-Apollonian semi-metric)**.** Let A be an open subset of $\mathbb{E}^n$. The *half-Apollonian semi-metric* η_A on A is defined by

$$\eta_A(x,y) = \sup_{a\in\partial A}\left|\log\frac{|x-a|}{|y-a|}\right|.$$

The half-Apollonian semi-metric was introduced by P. Hästö and H. Lindén in [13]. It is invariant under similarities (cf. [13, Theorem 1.2]) and it is a genuine metric whenever $\mathbb{E}\setminus A$ is not contained in a hyperplane of $\mathbb{E}^n$.

The following proposition, whose proof is immediate from the definitions, shows the relation between the Apollonian and the half-Apollonian semi-metrics and the Apollonian weak metric δ_A:

Proposition 4.9. *for any open subset A of $\mathbb{E}^n$, we have*

$$\sigma\delta_A = \eta_A \qquad \textit{and} \qquad S\delta_A = \alpha_A.$$

□

5. The Apollonian weak metric of the unit disk

The rest of the paper is mainly devoted to a discussion of the Apollonian weak metric of the unit disk $\mathbb{D}^2 \subset \mathbb{C}$. In this section, we give an explicit formula for that weak metric.

Theorem 2. *The Apollonian weak metric $\delta_{\mathbb{D}^2}$ is given by the following formula:*

$$\delta_{\mathbb{D}^2}(x,y) = \log\left(\frac{|x-y| + |x\overline{y}-1|}{\left|1-|y|^2\right|}\right). \tag{10}$$

Proof. The result follows directly from the first statement of Proposition 5.4 below. □

Before stating the needed Proposition, we first draw a few consequences of formula (10).

Corollary 5.1. *The symmetrization $S\delta_{\mathbb{D}^2}$ of the weak metric $\delta_{\mathbb{D}^2}$ on the unit disk $\mathbb{D}^2$ coincides with the Poincaré metric $h_{\mathbb{D}^2}$ of that disk:*

$$S\delta_{\mathbb{D}^2}(x,y) = \frac{1}{2}\left(\delta_{\mathbb{D}^2}(x,y) + \delta_{\mathbb{D}^2}(y,x)\right) = h_{\mathbb{D}^2} = \frac{1}{2}\log\left(\frac{|1-x\overline{y}| + |x-y|}{|1-x\overline{y}| - |x-y|}\right).$$

Proof. The proof is a direct calculation from Theorem 2. Observe that the result also follows from Proposition 4.9 and the result of Beardon stating that the Apollonian semi-metric of the unit disk is the Poincaré metric. □

Corollary 5.2. *The Apollonian weak metric $\delta_{\mathbb{D}^2}$ is nonsymmetric, unbounded and weakly separating.*

Proof. Using Formula (10), we obtain the following special values:

$$\delta_{\mathbb{D}^2}(x,0) = \log|1+|x||\,, \qquad \delta_{\mathbb{D}^2}(0,x) = -\log|1-|x||$$

This shows that $\delta_{\mathbb{D}^2}$ is non-symmetric, and it is unbounded since $\delta_{\mathbb{D}^2}(0,x) \to \infty$ as $|x| \to 1$. The fact that it is weakly separating follows from Corollary 5.1. □

Corollary 5.3. *The Apollonian weak metric $\delta_{\mathbb{D}^2}$ is not invariant under the group of Möbius transformation preserving the unit disk.*

This property is in contrast with a property of the hyperbolic metric.

Proof. Given an arbitrary pair of points $x, y \in \mathbb{D}^2$, there exists a Möbius transformation g preserving the disk and exchanging x and y (g is the 180^0-hyperbolic rotation around the mid-point of the hyperbolic segment $[x,y]$). If $\delta_{\mathbb{D}^2}$ were invariant under the Möbius group, then we would have $\delta_{\mathbb{D}^2}(y,x) = \delta_{\mathbb{D}^2}(g(x), g(y)) = \delta_{\mathbb{D}^2}(x,y)$, which contradicts Corollary 5.2. □

The following Proposition was used in the proof of Theorem 2. For later use, we formulate a more complete statement than what is needed in that proof.

Proposition 5.4. *Let us fix two distinct points x and y in $\mathbb{C}$, and consider the function $f : \mathbb{S}^1 \to [0,\infty)$ defined by*

$$f(a) = \left| \frac{x-a}{y-a} \right|.$$

The maximum value of this function on the circle $\mathbb{S}^1$ is given by

$$\max_{|a|=1} f(a) = \frac{|x-y| + |x\overline{y}-1|}{\left| |y|^2 - 1 \right|}$$

and this maximum is achieved at a unique point $a^+(x,y) \in \mathbb{S}^1$ given by

$$a^+(x,y) = \frac{|x-y|\,(x\overline{y}-1)y + (x-y)\,|x\overline{y}-1|}{|x-y|\,(x\overline{y}-1) + (x-y)\,|x\overline{y}-1|\,\overline{y}}.$$

The minimum value of f on $\mathbb{S}^1$ is given by

$$\min_{|a|=1} f(a) = \left| \frac{|x-y| - |x\overline{y}-1|}{|y|^2 - 1} \right|$$

and it is achieved at a unique point

$$a^-(x,y) = \frac{|x-y|\,(x\overline{y}-1)y - (x-y)\,|x\overline{y}-1|}{|x-y|\,(x\overline{y}-1) - (x-y)\,|x\overline{y}-1|\,\overline{y}}.$$

To prove the proposition, we shall use the following lemma:

Lemma 5.5. *Let $g : \mathbb{C} \to \mathbb{C}$ be the function given by*

$$g(z) = \lambda(\mu z + 1),$$

where $\lambda, \mu \in \mathbb{C}$ and $\mu \neq 0$. Then

$$\max_{|z|=1} |g| = |\lambda|(|\mu| + 1)$$

and this maximum is achieved at the unique point $z^+ = |\mu|/\mu$. Likewise

$$\min_{|z|=1} |g| = |\lambda|\,||\mu| - 1|$$

and this minimum is achieved at the unique point $z^- = -|\mu|/\mu$.

Proof. Without loss of generality, we can assume $\lambda = 1$. Then, for $|z| = 1$, the point $g(z) = \mu z + 1$ in the complex plane describes, as z varies, a circle of centre 1 and radius $|\mu|$. The function $|g(z)|$ is the distance from that point to the origin, and therefore it has a unique maximum, which is equal to $1 + |\mu|$. The rest of the proof follows by an analogous reasoning.

$\square$

Proof of Proposition 5.4. Let us set $q = \dfrac{\overline{y}a - 1}{y - a}$. Then, we have $a = \dfrac{qy+1}{q+\overline{y}}$. We claim that $|a| = 1 \iff |q| = 1$. Indeed, if $|a| = 1$, we can write $a = e^{i\theta}$ with $\theta \in \mathbb{R}$. Then, $q = \dfrac{\overline{y}e^{i\theta} - 1}{y - e^{i\theta}} = -e^{i\theta}\dfrac{e^{-i\theta} - \overline{y}}{e^{i\theta} - y}$ which shows that $|q| = 1$. In the same way, we can see that if $|q| = 1$ then $|a| = 1$.

Now set

$$\begin{aligned} g(q) = \frac{x-a}{y-a} &= \left(x - \left(\frac{qy+1}{q+\overline{y}}\right)\right) \cdot \left(y - \left(\frac{qy+1}{q+\overline{y}}\right)\right)^{-1} \\ &= \left(\frac{x\overline{y}-1}{|y|^2-1}\right) \cdot \left(\left(\frac{x-y}{x\overline{y}-1}\right) q + 1\right) \cdot \end{aligned}$$

Applying Lemma 5.5 with $\lambda = \dfrac{x\overline{y}-1}{|y|^2-1}$ and $\mu = \dfrac{x-y}{x\overline{y}-1}$, we see that

$$\begin{aligned} \max_{|a|=1} f(a) = \max_{|q|=1} |g(q)| &= \left|\frac{x\overline{y}-1}{|y|^2-1}\right| \cdot \left(\left|\frac{x-y}{x\overline{y}-1}\right| + 1\right) \\ &= \frac{|x-y| + |x\overline{y}-1|}{\left||y|^2-1\right|}. \end{aligned}$$

This maximum is achieved at the unique point

$$q^+ = \left|\frac{x-y}{x\overline{y}-1}\right| \cdot \left(\frac{x-y}{x\overline{y}-1}\right)^{-1} = \frac{(x\overline{y}-1)|x-y|}{|x\overline{y}-1|(x-y)},$$

which corresponds to

$$a^+ = \frac{q^+ y + 1}{q^+ + \overline{y}} = \frac{|x-y|\,(x\overline{y}-1)y + |x\overline{y}-1|\,(x-y)}{|x-y|\,(x\overline{y}-1) + |x\overline{y}-1|\,(x-y)\overline{y}}.$$

Likewise, we have

$$\begin{aligned} \min_{|a|=1} f(a) = \min_{|q|=1} |g(q)| &= \left|\frac{x\overline{y}-1}{|y|^2-1}\right| \cdot \left|\left|\frac{x-y}{x\overline{y}-1}\right| - 1\right| \\ &= \left|\frac{|x-y| - |x\overline{y}-1|}{|y|^2-1}\right| \end{aligned}$$

and this minimum is achieved at the unique point

$$q^- = \left|\frac{x-y}{x\overline{y}-1}\right| \cdot \left(\frac{x-y}{x\overline{y}-1}\right)^{-1} = \frac{(x\overline{y}-1)|x-y|}{|x\overline{y}-1|(x-y)},$$

which corresponds to

$$a^- = \frac{q^- y + 1}{q^- + \overline{y}} = \frac{|x-y|\,(x\overline{y}-1)y - (x-y)\,|x\overline{y}-1|}{|x-y|\,(x\overline{y}-1) - (x-y)\,|x\overline{y}-1|\,\overline{y}}.$$

□

Remark 5.6. It follows from this proof that

$$\begin{aligned} \frac{x-a^+}{y-a^+} &= g(q^+) = \left(\frac{x\overline{y}-1}{|y|^2-1}\right) \cdot \left(\left(\frac{x-y}{x\overline{y}-1}\right) q^+ + 1\right) \\ &= \left(\frac{x\overline{y}-1}{|y|^2-1}\right) \left(\left(\frac{x-y}{x\overline{y}-1}\right) \frac{(x\overline{y}-1)|x-y|}{|x\overline{y}-1|(x-y)} + 1\right) \\ &= (x\overline{y}-1) \left(\frac{|x-y| + |x\overline{y}-1|}{(|y|^2-1)|x\overline{y}-1|}\right). \end{aligned}$$

This observation will be used later.

6. Geodesics of weak metrics

Working in weak metric spaces, it turns out that rather than defining a geodesic as a distance-preserving path (as one usually does in the case of metric spaces), it is more convenient to define it as a path $\gamma : I \to X$ preserving aligned triples (where $I \subset \mathbb{R}$ is some interval). We make the following precise definitions.

Definition 6.1 (Aligned triple)**.** Let (X, δ) be a weak metric space and x, y and z be three points in X. We say that the three points x, y, z (in that order) are *aligned* if $\delta(x, z) = \delta(x, y) + \delta(y, z)$.

We note that the fact that x, y, z are aligned does not imply that z, y, x are aligned.

Definition 6.2 (Geodesic)**.** A $\delta-$*geodesic* (or, simply, a *geodesic*) in X is a path $\gamma : I \to X$, where I is an interval of $\mathbb{R}$, such that for any t_1, t_2 and t_3 in I satisfying $t_1 \leq t_2 \leq t_3$, the points $\gamma(t_1)$, $\gamma(t_2)$, $\gamma(t_3)$ are aligned.

As a first example, a Euclidean segment is a geodesic for the Funk weak metric. Observe that in general, if a path $\gamma : I \to X$ is a geodesic, then the same path traversed in the opposite direction is not necessarily a geodesic.

In this section, we discuss geodesics of Apollonian weak metrics. Let A be a subset of $\mathbb{E}^n$ satisfying the hypothesis stated at the beginning of Section 5 and let δ_A be the Apollonian weak metric on A. For any x and y in A, we consider the following subset of ∂A:

$$M_{x,y} = \left\{ a_0 \in \partial A \ \text{ such that } \frac{|x - a_0|}{|y - a_0|} = \delta_A(x, y) \right\}. \tag{11}$$

We note that in the case where A is bounded, ∂A is compact and nonempty, and therefore, for every x and y, $M_{x,y}$ is nonempty.

Lemma 6.3. *With the above notations, if x, y and z are elements in A satisfying $M_{x,y} \cap M_{y,z} \cap M_{x,z} \neq \emptyset$, then the three points x, y and z are aligned.*

Proof. This follows from the fact that for any a_0 in $M_{x,z} \cap M_{x,y} \cap M_{y,z}$, we have

$$\delta_A(x, y) + \delta_A(y, z) = \log \frac{|x - a_0|}{|y - a_0|} + \log \frac{|y - a_0|}{|z - a_0|} = \log \frac{|x - a_0|}{|z - a_0|} = \delta_A(x, z).$$

□

Conversely, we have the following

Lemma 6.4. *Suppose that A is bounded. If x, y and z are points in A satisfying $\delta_A(x, z) = \delta_A(x, y) + \delta_A(y, z)$, then, there exists a point $a_0 \in \partial A$ such that*

$$\delta_A(x, y) = \frac{|x - a_0|}{|y - a_0|}, \ \delta_A(y, z) = \frac{|y - a_0|}{|z - a_0|} \quad \textit{and} \quad \delta_A(x, z) = \frac{|x - a_0|}{|z - a_0|}.$$

Proof. Since ∂A is compact and since $z \notin \partial A$, we can find a point a_0 in ∂A satisfying $\delta_A(x, z) = \log \frac{|x-a_0|}{|z-a_0|}$. This gives

$$\begin{aligned} \log \frac{|x - a_0|}{|z - a_0|} &= \log \frac{|x - a_0|}{|y - a_0|} + \log \frac{|y - a_0|}{|z - a_0|} \\ &= \sup_{a \in \partial A} \log \frac{|x - a|}{|y - a|} + \sup_{a \in \partial A} \log \frac{|y - a|}{|z - a|}, \end{aligned}$$

which implies that

$$\delta_A(x,y) = \sup_{a\in\partial A} \log \frac{|x-a|}{|y-a|} = \log \frac{|x-a_0|}{|y-a_0|}$$

and

$$\delta_A(y,z) = \sup_{a\in\partial A} \log \frac{|y-a|}{|z-a|} = \log \frac{|y-a_0|}{|z-a_0|}.$$

□

Finally, we apply these results to the case where $A = \mathbb{D}^2$. First, we need a lemma about generalized circles in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (a generalized circle being as usual a Euclidean circle or a Euclidean straight line compactified by the point $\{\infty\}$). Recall that any generalized circle is the image of the unit circle in $\mathbb{C}$ under a Möbius transformation of $\overline{\mathbb{C}}$.

Lemma 6.5. *Four pairwise distinct points $x, y, a, b \in \mathbb{C}$ lie on a generalized circle if and only if the complex cross-ratio*

$$(x,y,a,b) = \frac{x-a}{y-a} \cdot \frac{y-b}{x-b}$$

is a real number. Furthermore these points appear on the circle in the order x, y, a, b if and only $(x,y,a,b) \in (1,\infty)$.

Proof. This lemma is well known. We recall the proof for the convenience of the reader. Let us consider the Möbius transformation ϕ defined by

$$\phi(z) = \frac{y-b}{y-a} \cdot \frac{z-a}{z-b}.$$

The point x belongs to the generalized circle through a, b, y if and only if $\phi(x)$ belongs to the generalized circle through $\phi(a) = 0$, $\phi(b) = \infty$ and $\phi(y) = 1$, which is the extended real line $\mathbb{R}\cup\{\infty\}$. Furthermore, these four points appear on that circle in the order x, y, a, b if and only if $\phi(x)$ belongs to the interval $(\phi(y), \phi(b)) = (1,\infty)$. The proof of the lemma follows once we observe that $\phi(x) = (x, y, a, b)$.

□

Collecting all this information about geodesics of Apollonian weak metrics, we are now ready to prove the following

Theorem 3. *Let x and y be two distinct points in $\mathbb{D}^2$. Then, the arc of generalized circle starting at x, containing y and orthogonal to the unit circle $\mathbb{S}^1$ is a $\delta_{\mathbb{D}^2}$-geodesic starting at x and passing through y.*

Proof. Denote by Γ the generalized circle through x and y and orthogonal to $\mathbb{S}^1$. Observe that Γ is invariant under the inversion $z \mapsto 1/\overline{z}$, and therefore Γ is the generalized circle passing through $x, y, 1/\overline{y}$ and orthogonal to $\mathbb{S}^1$.
The point $a^+(x,y)$ belongs to $\mathbb{S}^1$ (see Proposition 5.4), and Lemma 6.5 implies that the points x, y, a^+ appear in that order on the circle, because $(x, y, a^+, \frac{1}{\overline{y}}) \in (1,\infty)$. Indeed, we have from Remark 5.6

$$\frac{x-a^+}{y-a^+} = (x\overline{y}-1)\left(\frac{|x-y| + |x\overline{y}-1|}{(|y|^2-1)|x\overline{y}-1|}\right),$$

hence

$$
\begin{aligned}
(x, y, a^+, \frac{1}{\overline{y}}) &= \frac{x - a^+}{y - a^+} \cdot \frac{y - 1/\overline{y}}{x - 1/\overline{y}} \\
&= (x\overline{y} - 1) \left(\frac{|x-y| + |x\overline{y} - 1|}{(|y|^2 - 1)|x\overline{y} - 1|} \right) \left(\frac{|y|^2 - 1}{x\overline{y} - 1} \right) \\
&= 1 + \frac{|x-y|}{|x\overline{y} - 1|}.
\end{aligned}
$$

Thus, we have proved that for any pair of points $x, y \in \Gamma$ such that $x, y, a^+(x,y)$ appear in that order, we have $a^+(x,y) \in M_{x,y}$. The Theorem follows now from Lemma 6.3.

□

References

[1] D. Barbilian, *Einordnung von Lobatschewskys Massbestimmung in gewisse allgemeine Metrik der Jordanschen Bereiche*. Casopis Praha **64**, 182-183 (1935)

[2] H. S. Bear, *Part metric and hyperbolic metric*, Am. Math. Monthly, **98** (1991) 109-123.

[3] H. Bauer & H. S. Bear, *The part metric in convex sets*, Pac. J. Math., **30**, No. 1 (1969) 15-33.

[4] A. Beardon, *The Apollonian metric of a domain in* $\mathbf{R}^n$, P. Duren et al. (ed.), Quasiconformal mappings and analysis, Proceedings of an international symposium, Ann Arbor, 1995, Springer, 91-108 (1998).

[5] A. Belkhirat, A. Papadopoulos & M. Troyanov, *Thurston's weak metric on the Teichmüller space of the torus*, Trans. Am. Math. Soc. **357**, No.8, 3311-3324 (2005).

[6] H. Busemann, *Local metric geometry*, Trans. Amer. Math. Soc. **56**, (1944) 200–274.

[7] H. Busemann, *Recent synthetic differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **54**, Springer-Verlag, New York-Berlin, 1970.

[8] M. Fréchet, *Sur quelques points de calcul fonctionnel*, Rend. Circ. Mat. Palermo, 22 (1906), 1-74.

[9] P. Funk, *Über Geometrien, bei denen die Geraden die Kürzesten sind*, Math. Ann. **101** (1929), 226-237.

[10] F. W. Gehring & K. Hag, *The Apollonian metric and quasiconformal mappings*, in: In the tradition of Ahlfors and Bers, Proceedings of the first Ahlfors-Bers colloquium (ed. I. Kra et al.), Contemp. Math. 256, 143-163 (2000), American Mathematical Society.

[11] F. W. Gehring & B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** 50–74 (1979).

[12] P. Hästö, Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics, Proc. Amer. Math. Soc. 134 (2006), 1137-1142.

[13] P. Hästö, & H. Lindén, *Isometries of the half-Apollonian metric*, Complex Var. Theory Appl. **49** 405-415 (2004).

[14] D. Hilbert, *Grundlagen der Geometrie*, B. G. Teubner, Stuttgart 1899, several later editions revised by the author, and several translations.

[15] H. Ribeiro, *Sur les espaces à métrique faible*, Portugaliae Math., **4** (1943) 21–40.

[16] P. Seittenranta, Möbius-invariant metrics, Math. Proc. Cambridge Philos. Soc. 125 (1999), **3**, 511–533.

[17] W. P. Thurston, *Minimal Stretch maps between hyperbolic surfaces*, Preprint (1985) available at http://arxiv.org/abs/math.GT/9801039.

[18] M. Vuorinen *Conformal invariants and quasiregular mappings*, J. Anal. Math. **45** 69-115 (1985).

[19] E. M. Zaustinsky, Spaces with nonsymmetric distance, Mem. Amer. Math. Soc. **34**, 1959.

A. Papadopoulos, Institut de Recherche Mathématique Avancée, Université Louis Pasteur and CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex - France
E-mail address: papadopoulos@math.u-strasbg.fr

M. Troyanov, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, 1015 Lausanne - Switzerland
E-mail address: marc.troyanov@epfl.ch