Mean-Field Theory Revives in Self-Oscillatory Fields with Non-Local Coupling

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A simple mean-field idea is applicable to the pattern dynamics of large assemblies of limit-cycle oscillators with non-local coupling. This is demonstrated by developing a mathematical theory for the following two specific examples of pattern dynamics. Firstly, we discuss propagation of phase waves in noisy oscillatory media, with particular concern with the existence of a critical condition for persistent propagation of the waves throughout the medium, and also with the possibility of noise-induced turbulence. Secondly, we discuss the existence of an exotic class of patterns peculiar to non-local coupling called *chimera* where the system is composed of two distinct domains, one coherent and the other incoherent, separated from each other with sharp boundaries.

§1. Introduction

The mean-field idea is one of the most elementary ideas in statistical mechanics of many-body systems with strong interaction. The usefulness of this idea seems to have been realized recently through the fact, e.g., that its advanced form turned out successful in dealing with frustrated system like spin glasses, providing also a useful tool for solving various problems of probabilistic information processing.[1)] Apart from such direction of application, the simple original idea of the mean field seems still useful for a better understanding of collective behavior of a variety of cooperative systems composed of nonlinear non-equilibrium elements such as periodic or non-periodic self-sustained oscillators. It has in fact been realized, particularly through the study of synchronization phase transition over the last few decades, that the same idea works well for some statistical problems of large populations of limit-cycle oscillators when the coupling is global.[2)–4)]

The goal of the present paper is to show how the classical mean-field idea finds application to oscillator systems when the coupling range becomes *finite* and even far smaller than the linear dimension of the system. In such cases, the spatial degrees of freedom involved become so large as to enable us to discuss pattern dynamics. Our argument will be demonstrated through the following two topics. The first one is relevant to the general and practically important problem as to how and under what conditions phase waves can propagate without decay through random oscillatory media. We will sketch our previous theory[5)] with some generalization on the onset of long-range coherence, i.e., the onset of persistent propagation of waves, in noisy oscillatory media. This theory may be regarded as a non-local generalization of the theory on the onset of collective oscillation in globally coupled phase oscillators with noise proposed earlier.[6)] Similarly to the case of globally coupled oscillators, exact critical condition and an asymptotic theory valid near the critical point can be formulated by virtue of the mean-field idea, which would be hard when the coupling were local. Unlike the case of global coupling, however, the critical condition now depends on the wavelength of the phase wave concerned. We shall also argue that under suitable conditions *noise-induced turbulence* occurs through the mechanism that noise changes the effective dynamics of the system in such a way that the stability of spatially uniform oscillations may become violated.

In the second topic, we will discuss exotic patterns characterized by a coexistence of coherent and incoherent domains separated from each other with sharp boundaries. The non-local nature of the coupling is crucial for giving rise to this type of patterns. The existence of such pattern was first confirmed and explained by Kuramoto and Battogtokh [7)] for a one-dimensional array of phase oscillators, and studied in further detail by Abrams and Strogatz who also called such composite patterns *chimera states*.[8)] It was found later that chimera states also appear in two-dimensional spiral waves in reaction-diffusion systems when the systems involves some diffusion-free components.[9)] In the present paper, we will review the one- and two-dimensional chimera states a little more informally and also from a little broader perspective than before.

§2. Hierarchy of mathematical models for non-locally coupled oscillators

We will start with globally coupled phase oscillators with sine coupling, namely, the model proposed by one of the present authors in 1975 as a solvable model for studying synchronization phase transition.[2)] Then we will deform the model step by step in such a manner that a much broader class of oscillator dynamics involving spatial degrees of freedom may be discussed and analyzed mathematically at the same level of precision as for the original solvable model. The original model takes the form

$$\frac{d\phi_i}{dt} = \omega_i - \frac{k}{N}\sum_{j=1}^{N}\sin(\phi_i - \phi_j), \quad i = 1, 2, \cdots N. \tag{2·1}$$

In the arguments which follow, we will mainly be concerned with the oscillators with identical nature, and therefore drop the suffix i from ω_i. Let us first relax the condition of global coupling by modifying the above equation in the form

$$\frac{d\phi_i}{dt} = \omega - k\sum_{j=1}^{N} G_{ij}\sin(\phi_i - \phi_j + \alpha). \tag{2·2}$$

Here the radius of the coupling G_{ij} is assumed finite and even much shorter than the linear size of the system. With this generalization we would have sufficiently large spatial degrees of freedom to discuss pattern dynamics, not only the collective motion of the system as a whole. Also, the original sine coupling has been slightly generalized by inserting a constant α. The system may further be generalized to include external noise, and this will separately be discussed later.

We will now take a continuum limit by replacing the sum with a spatial integral:

$$\frac{\partial}{\partial t}\phi(\boldsymbol{r},t) = \omega - k\int d\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}')\sin(\phi(\boldsymbol{r},t)-\phi(\boldsymbol{r}',t)+\alpha). \tag{2·3}$$

In taking this limit, we keep the coupling range finite. The last point is crucial, because in this way the number of oscillators which fall within the coupling range becomes infinite, precisely as for globally coupled systems. And this enables us to apply the mean-field theory as an exact theory. The sine-coupled non-local phase oscillator model thus obtained will be used later to illustrate our theory. Regarding the spatial integral in the equation above, our understanding is that it does not necessarily imply that the system is a true continuum, but that it may only mean that the spatial distribution of the oscillators is sufficiently dense.

It would be interesting to take a few more steps of generalization of our model. Firstly, we replace the sine function of the phase coupling with a general coupling function Γ which is a 2π-periodic function of the phase difference between the pair of interacting oscillators:

$$\frac{\partial}{\partial t}\phi(\boldsymbol{r},t) = \omega + \int d\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}')\Gamma(\phi(\boldsymbol{r},t)-\phi(\boldsymbol{r}',t)). \tag{2·4}$$

Furthermore, it is generally known that the form of non-locally coupled phase oscillator model obtained in this way can be derived under suitable conditions from a vector dynamical system model for non-locally coupled oscillators of the form[9)]

$$\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{F}(\boldsymbol{A}(\boldsymbol{r},t)) + \boldsymbol{k}S(\boldsymbol{r},t), \tag{2·5}$$

$$S(\boldsymbol{r},t) = \int d\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}')h(\boldsymbol{A}(\boldsymbol{r}',t)). \tag{2·6}$$

The part $\dot{\boldsymbol{A}} = \boldsymbol{F}$ appearing in the first equation represents a local limit-cycle oscillator. The local oscillators are commonly under the influence of a field S multiplied by a coupling parameter $\boldsymbol{k}$. The S-field is generally space-time dependent, and its value at a given point $\boldsymbol{r}$ in space actually represents the net force due to the non-local coupling experienced by the oscillator situated there. Thus, S may be called the mean field. The method of reduction to be used here for deriving the phase equation (2·4) is the so-called phase reduction, and the condition under which this method is applicable is that the coupling between the oscillators and the mean field, represented by the parameter $\boldsymbol{k}$, is sufficiently weak. If some external noises are introduced, which will be done later, they must be weak as well.

We may further argue that the vector model given by Eqs. (2·5) and (2·6) itself is a consequence of reduction from a class of reaction-diffusion systems of the form

$$\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{F}(\boldsymbol{A}(\boldsymbol{r},t)) + \boldsymbol{k}S(\boldsymbol{r},t), \tag{2·7}$$

$$\tau\frac{\partial}{\partial t}S(\boldsymbol{r},t) = -S + D\nabla^2 S + h(\boldsymbol{A}(\boldsymbol{r},t)). \tag{2·8}$$

While the first equation Eq. (2·7) is identical with Eq. (2·5), S is no longer a simple mean field but it represents an independent dynamical field variable obeying the second equation Eq. (2·8). The characteristic time scale of S, denoted by τ, has been inserted for the sake of convenience. It is clear that the model given by Eqs. (2·5) and (2·6) results from adiabatic elimination of S in the last model by assuming that τ is sufficiently small. The quantity G in Eq. (2·6) is then the Green's function of the Helmholtz differential equation obtained by equating the right-hand side of Eq. (2·8) to zero. Thus, G decays exponentially at long distances, and the characteristic length of decay is given by $\sqrt{D}$. This means that if S diffuses infinitely fast, then the oscillators in the reduced system become globally coupled, while if D is sufficiently small, then the effective coupling stays practically local. In what follows, a few more remarks will be given on the last reaction-diffusion system, because this model seems to have some interesting physical implications.[10),11)]

A possible interpretation of the model given by Eqs. (2·7) and (2·8) would be that the system represents a continuum model of a hypothetical biological cellular assembly, where each cell is exhibiting oscillatory activity. The cells are not in direct contact with each other, but their interaction is mediated by an extra chemical with concentration S which is diffusive. As before, the local dynamics is influenced by the local concentration of this chemical which itself is produced from the individual cells.

It is clear that the above reaction-diffusion model must involve at least three components, because Eq. (2·7) which represents oscillator dynamics already involves two components or more. A simple example of this model is given by

$$\frac{dX}{dt} = f(X,Y) + kS, \tag{2·9}$$

$$\frac{dY}{dt} = g(X,Y), \tag{2·10}$$

$$\tau\frac{\partial S}{\partial t} = -S + D\nabla^2 S + X, \tag{2·11}$$

where

$$f = \epsilon^{-1}(X - X^3 - Y), \tag{2·12}$$

$$g = aX + b. \tag{2·13}$$

Here the local dynamics is given by the FitzHugh-Nagumo type equation, and the local coupling between the oscillators and the diffusive field S is simplified to a linear form. In later sections, we will work with this specific model numerically to confirm our theory.

Up to now, we have discussed a hierarchy of mathematical models of coupled oscillators. These models may appear rather different from each other, but they may all be describing one identical system; what is different may only be the level of description. In studying a specific problem of pattern dynamics, it seems often useful to jump flexibly from one level of description to another, because this may often consolidate the validity of our arguments. In the remaining part of this article, this attitude will be taken in approaching the two problems announced before.

§3. Onset of long-range coherence in noisy oscillatory media

The first issue to be discussed is the onset of long-range coherence in random fields of oscillators. The randomness will be introduced below in the form of external noise rather than frozen randomness, simply because the theory would be far easier for noise. Consider a noisy version of Eqs. (2·7) and (2·8) obtained by adding a noise term to Eq. (2·7):

$$\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{F}(\boldsymbol{A}(\boldsymbol{r},t)) + \boldsymbol{k}S(\boldsymbol{r},t) + \boldsymbol{\zeta}(\boldsymbol{r},t). \tag{3·1}$$

The noise $\boldsymbol{\zeta}$ has been assumed to be white Gaussian and applied independently from site to site. Throughout the present paper, the time constant τ is assumed to be relatively small. This means that adiabatic elimination of S from our system is allowed, and thus the notion of *effective coupling range* is clear. Under these conditions, one would like to know under what condition phase waves can propagate through such noisy reaction-diffusion medium. It is naturally expected that waves would propagate without decay when the noise is sufficiently weak, while this would be impossible when the noise is too strong. Thus, there should be a critical strength of noise somewhere in between, and if we want this could be located through numerical simulation.[12] Note, however, that such critical condition would depend on the wavelength of the phase wave under consideration. In particular, we expect that the critical noise strength would be larger for longer wavelengths because their stability is usually higher and therefore should be more robust against noise also. The most stable waves should therefore be uniform waves, which corresponds to uniform oscillations of the system as a whole. After all, the onset of persistent propagation of waves in noisy oscillatory media is closely related to the onset of collective oscillation in the same media. Thus, if we could formulate a corresponding bifurcation theory, then we could find a wavenumber-dependent critical condition for wave propagation. In what follows, a brief sketch will be given on a theory on this type of phase transition making full use of the mean-field idea.

It would be difficult, however, to apply the mean-field theory directly to our three-component reaction-diffusion system. Therefore, we will go back to the spirit of reduction described in the previous section, and take the two reduction steps by which Eqs. (2·7) and (2·8) are contracted to the form of Eq. (2·4) via Eqs. (2·5) and (2·6). Since the external noise can easily be incorporated into these reduction steps, we obtain the following non-locally coupled phase model with noise.

$$\frac{\partial}{\partial t}\phi(\boldsymbol{r},t) = \omega + \int d\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}')\Gamma(\phi(\boldsymbol{r},t)-\phi(\boldsymbol{r}',t)) + \xi(\boldsymbol{r},t), \tag{3·2}$$

where the noise ξ is again Gaussian and delta-correlated in space and time, i.e.,

$$\langle \xi(\boldsymbol{r},t)\rangle = 0, \quad \langle \xi(\boldsymbol{r},t)\xi(\boldsymbol{r}',t)\rangle = 2\gamma\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'). \tag{3·3}$$

For the limiting case of global coupling, for which the function G is a constant, the above stochastic phase equation was analyzed long ago.[6] Its details are not repeated here, but the main idea is the following. Since statistical correlation between the oscillators is absent due to the mean-field nature of the system, our stochastic phase equation can be transformed to a nonlinear Fokker-Planck equation obeyed by the phase distribution for a single oscillator. Since this equation involves a mean-field parameter which itself is a function of one-oscillator phase distributions, what we are working with is a *nonlinear* Fokker-Planck equation. This equation of course represents a deterministic and dissipative dynamical system, and therefore a standard stability analysis and the center-manifold reduction can be applied to it. In particular, the trivial constant solution which corresponds to the absence of collective oscillation becomes unstable and gives way to an oscillating solution when the noise intensity decreases across a certain threshold. Near the onset of collective oscillation, a small-amplitude equation is obtained in the conventional form

$$\frac{dA}{dt} = (\gamma_c - \gamma + i\Omega)A - g|A|^2 A, \tag{3·4}$$

where A is a complex amplitude whose modulus is proportional to the amplitude of the collective oscillation and is also identical with a suitably defined mean field. The frequency at criticality has been denoted by Ω. The critical value γ_c of the noise intensity equals the imaginary part of the first Fourier component of the phase coupling function Γ, where the Fourier components Γ_l have been defined by

$$\Gamma(\psi) = \sum_{l=-\infty}^{\infty} \Gamma_l e^{il\psi}. \tag{3·5}$$

γ_c must be positive for the existence of a transition. This is equivalent to the condition that the coupling is the in-phase type or $d\Gamma(\psi)/d\psi|_{\psi=0} < 0$. In terms of Γ_l the complex constant g is given by

$$g = \frac{\Gamma_1(\Gamma_2 + \Gamma_{-1})}{2\Im\Gamma_1 - i\Re\Gamma_1 + i\Gamma_2}. \tag{3·6}$$

The sign of $\Re g$ depends on the form of the phase-coupling function $\Gamma(\psi)$. In what follows, we will restrict ourselves to the case of positive $\Re g$, i.e., the case of supercritical Hopf bifurcation.

Coming back to the case of nonlocal coupling, the idea remains almost the same as for the case of global coupling. The small-amplitude equation obtained is also very similar,[5)] except for the appearance of a diffusion term. Thus, we obtain a complex Ginzburg-Landau equation

$$\frac{dA}{dt} = (\gamma_c - \gamma + i\Omega)A - g|A|^2 A + d\nabla^2 A, \tag{3·7}$$

where the complex diffusion constant d is determined, similarly to g, only by the phase-coupling function through

$$d = -i\Gamma_1. \tag{3·8}$$

Note that the original non-locality in coupling has disappeared. This is because the characteristic wavelength becomes sufficiently longer than the coupling radius near the critical point, thus allowing to use the diffusion-coupling approximation to the non-local coupling. As is well known, Eq. (3·7) admits a family of plane wave solutions, and the conditions for their existence and stability are easy to find. In this way, our original purpose of finding a wavelength-dependent critical condition for coherent wave propagation has been achieved.

Such results may not sound very exciting; it may seem that there is little progress beyond the theory on globally coupled oscillators with noise developed 20 or more years ago. However, the above space-dependent generalization tells one non-trivial thing at least. This is the fact associated with the so-called Benjamin-Feir instability, namely, the instability of the uniform oscillation leading to phase turbulence.[6)] The condition for this instability depends on the two complex coefficients d and g, and is given by

$$1 + \frac{\Im d \Im g}{\Re d \Re g} < 0. \tag{3·9}$$

It is clear that the Benjamin-Feir instability condition depends entirely on the nature of the phase-coupling function. When Eq. (3·9) is satisfied, the uniform oscillation becomes unstable, and the mean field becomes turbulent. It would therefore be interesting to examine whether or not this instability condition is satisfied for some representative forms of the coupling function Γ. For example, one may ask how about the case of the sine coupling including parameter α. Remember that our theory is meaningful only for the case of in-phase type coupling, because otherwise the system would be unable to behave coherently even in the absence of noise. The condition for the coupling to be in-phase type is given by

$$|\alpha| < \pi/2. \tag{3·10}$$

Under this restriction, it is found that the system still becomes Benjamin-Feir unstable provided the condition

$$\tan^2 \alpha > 2 \tag{3·11}$$

is satisfied. Thus, there is a finite range of α given by

$$0.9553 \cdots < |\alpha| < \frac{\pi}{2}, \tag{3·12}$$

satisfying these two inequalities simultaneously. This type of instability and the resulting turbulence for the mean field could never occur in the original phase equation if there is no noise. Therefore, the turbulence of this sort may be called *noise-induced turbulence.* Noise-induced turbulence is something quite independent of the small fluctuations in the mean field directly caused by the random noise. In fact, such fluctuations should be completely negligible due to the cancellation among a large number of oscillators within the coupling range, and this is the very basic fact which the mean-field theory relies upon. Thus, the only possible interpretation for our noise-induced turbulence is that the noise changes effective deterministic dynamics in such a way that the condition for the occurrence of spatio-temporal chaos becomes satisfied.

Incidentally, we will make a Benjamin-Feir stability test for another form of the phase coupling function of some practical interest, namely, the coupling function corresponding to the so-called leaky integrate-and-fire neural oscillators with pulsatile coupling. In this model,[13)] each oscillator is described with only one variable u. This quantity is governed by the equation $\dot{u} = a - u \ (a > 1)$ supplemented with an extra dynamical rule that each time u attains the level $u = 1$, then it is immediately reset to the zero value where its evolution is restarted. Thus, the oscillation pattern of u is simply given by a repetition of exponential decays toward the level $u = a$ within the interval $0 \le u \le 1$. Note that the natural frequency ω of this oscillator is given by

$$\omega = \frac{2\pi}{|\ln(1 - a^{-1})|}. \tag{3·13}$$

Let such oscillators be coupled mutually. Consider a coupled pair of such oscillators and name them oscillator A and oscillator B. In a simplest model, oscillator A is assumed to receive a signal from oscillator B in the form of a delta pulse which occurs precisely at the moment of resetting of oscillator B. By omitting the details, it is known that, under the condition of sufficiently weak coupling, the dynamics of u can be transformed to the standard form of the phase dynamics[14)]

$$\frac{d\phi}{dt} = \omega + \Gamma(\phi - \phi'), \tag{3·14}$$

where the phase variable ϕ is defined by a certain nonlinear transformation of u. The phase-coupling function $\Gamma(\phi)$ is simply given by an exponential function in the interval $0 \le \phi < 2\pi$. Since Γ is a 2π-periodic function, it has a discontinuity at $\phi = 0$ or 2π. Many of such oscillators are now coupled non-locally and symmetrically, and weak noise is added. We then apply the mean-field theory and the center-manifold reduction such as described above, and examine the Benjamin-Feir criterion. The condition for the Benjamin-Feir instability in this particular case turns out extremely simple, and is determined only by the natural frequency:

$$\omega < \sqrt{2}. \tag{3·15}$$

In terms of a, this condition is expressed as

$$1 < a < (1 - e^{-\sqrt{2}\pi})^{-1} = 1.0119\cdots. \tag{3·16}$$

Thus the possibility of turbulence is limited to an extremely small region of a. However, the above argument assumes the coupling through delta pulses. More generally,

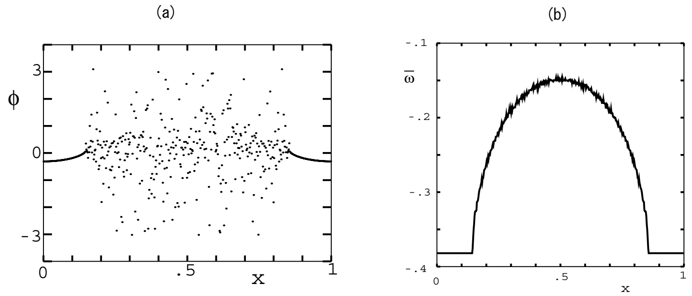

Fig. 1. Numerical results obtained from Eq. (4·1) with 512 oscillators over the unit length with periodic boundary conditions. Parameter values are $\alpha = 1.457$ and $\kappa = 4.0$. (a) Instantaneous spatial distribution of the phases. (b) Distribution of the actual frequencies $\bar{\omega}$ of the oscillators, where $\bar{\omega}$ is defined by a long-time average.

the coupling is often modeled with the so-called α-function with a finite width which gives rise to an effect similar to introducing time delay in coupling. Although no details are shown here, the corresponding phase coupling function is so changed as to make the stability of the complete phase synchronization of the mean field lowered. This conclusion seems consistent with our knowledge that introducing delay in coupling generally favors the instability of phase-synchronized states in noise-free systems.

§4. Chimera states

We will proceed to the second topic which is the chimera state in no-locally coupled systems. The existence of this peculiar state was first noticed by Battogtokh[15) while doing a numerical analysis of a simple non-locally coupled phase oscillator model of the form

$$\frac{\partial}{\partial t}\phi(x,t) = \omega - K \int dx' G(x-x') \sin(\phi(x,t) - \phi(x',t) + \alpha) \tag{4·1}$$

with $G(x) = \exp(-\kappa|x|)$. He found the following. Under a certain condition, the system produces a set of numerical data as shown in Fig. 1. In the numerical simulation, the above continuum model was replaced with a densely distributed array of oscillators over an interval of unit length with periodic boundary conditions, and the coupling range κ^{-1} was set to 0.25. The system is Benjamin-Feir stable, that is, a uniform phase pattern which is running upward with a constant velocity is stable. However, under different initial conditions, a phase pattern as displayed in Fig. 1(a) appears, and once created such a pattern persists indefinitely. The pattern looks almost stationary except for a steady drift. Here the term *stationary* has been used in a statistical sense, because in the central domain of the system the actual frequencies of the oscillators are distributed, and therefore they are running independently with their respective velocities. How the actual frequencies are distributed is shown in Fig. 1(b). The distribution of the frequencies itself is quite systematic. The quantity $\bar{\omega}$ represents a long-time average of the instantaneous frequencies of the individual

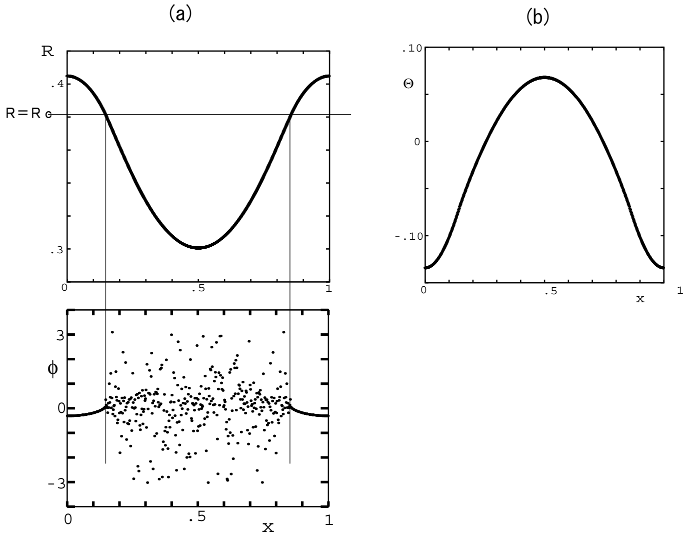

Fig. 2. (a) Spatial profile of a long-time of the order-parameter amplitude $R(r)$, showing that the inner domain satisfying $R < R_c$ corresponds to phase randomized oscillators. (b) Spatial profile of a long-time average of the order-parameter phase. Parameter conditions are the same as in Fig. 1.

oscillators. In contrast to the large central domain, the oscillators near the boundaries have identical frequencies, which means that they are mutually synchronized there.

At first, we were puzzled with such patterns, and could not figure out their origin. Some time after, however, the following was noticed. Similarly to what we usually do for globally coupled oscillators, one may define a complex order parameter with modulus R and phase Θ through

$$R(x,t)e^{i\Theta(x,t)} \equiv \int dx' G(x-x')e^{i\phi(x',t)}, \tag{4·2}$$

that is, the order parameter is given by a spatial average of the quantity $\exp(i\phi)$ weighted by the coupling strength G in the convolution form. Thus, this complex order parameter may also be called the mean field, the one experienced at the spatial point x. In terms of R and Θ, our phase equation can be re-expressed in the form

$$\frac{\partial}{\partial t}\phi(x,t) = \omega - R(x,t)\sin(\phi(x,t) - \Theta(x,t) + \alpha). \tag{4·3}$$

Note that the equation above has a form of a single oscillator dynamics under external forcing, although the forcing comes actually from the mean field. Depending on the amplitude R of the forcing, the oscillator may or may not be synchronized with it. Thus, what we want to know next is how the spatial pattern of R looks. This is easily computed numerically and the result is shown in Fig. 2(a). We see that R is given by a symmetric curve with a single minimum at the center. Since the dot pattern of the phases of the oscillators is stationary (though only statistically as noted above), the spatial profile of R should also be stationary. The spatial profile of the mean field phase Θ, which is shown in Fig. 2(b), is also stationary except that it is rigidly drifting upward at a constant speed Ω or the frequency of the collective oscillation. It would therefore be convenient to separate Θ into two parts, i.e., the stationary part $\Theta_0(x)$ and Ωt as

$$\Theta(x,t) = \Theta_0(x) + \Omega t. \tag{4·4}$$

Coming back to the pattern of R, we see that it is smaller toward the center of the system, which means that the oscillators there are likely to fail to synchronize with the mean-field motion. This implies at the same time that mutual synchronization may also be broken there. In contrast, near the boundaries, the forcing amplitude seems larger enough for the oscillators to be synchronous and behave coherently. There should be a critical forcing amplitude R_c which defines the boundary between coherence and incoherence. Thus, the next question is how such qualitative interpretation can be formulated in a mathematical form and how it is possible to determine the mean field pattern in a self-consistent manner. As we see below, this can be answered almost in parallel with Kuramoto's 1975 theory of synchronization transition where also the whole population consisted of two subpopulations, namely, a coherent subpopulation and an incoherent subpopulation.

The outline of the theory is the following.[7] The set of equations to start with are Eqs. (4·3) and (4·2) or, with the use of Eq. (4·4),

$$\frac{\partial}{\partial t}\phi(x,t) = \omega - R(x,t)\sin(\phi\,(x,t) - \Theta_0(x) - \Omega t + \alpha)\,, \tag{4·5}$$

$$R(x,t)e^{i(\Theta_0(x)+\Omega t)} \equiv \int dx' G(x-x')e^{i\phi(x',t)}. \tag{4·6}$$

It would be more convenient to eliminate the explicit time-dependence appearing in the above equations through Ωt. Thus, we work with a new phase variable ψ defined in a frame of reference co-moving with the mean field phase, i.e.,

$$\psi(x,t) = \phi(x,t) - \Omega t, \tag{4·7}$$

and rewrite Eqs. (4·5) and (4·6) as

$$\frac{\partial}{\partial t}\psi(x,t) = \omega - \Omega - R(x,t)\sin(\psi(x,t) - \Theta_0(x) + \alpha), \tag{4·8}$$

$$R(x,t)e^{i\Theta_0(x)} \equiv \int dx' G(x-x')e^{i\psi(x',t)}. \tag{4·9}$$

In the above representation, if the solution ψ turns out constant in time, then it is implied that the corresponding oscillator is synchronized with the mean field motion, while if ψ drifts, then the oscillator fails to synchronize. Thus, what should be done next is to solve Eq. (4·8) for the individual oscillators as a function of the mean field, which is easy, and then substitute these solutions into Eq. (4·9). In this way, the mean field should be expressed in terms of the mean field itself.

It is clear that the solution of Eq. (4·8) may differ qualitatively depending on the value of R. If $R(x)$ is greater than the critical value $|\omega - \Omega|$ at a given spatial point x, then we have a stable fixed point, implying that the oscillator there is in a synchronized state. In contrast, if $R(x)$ is below this critical value, then we have a drifting solution, implying that the corresponding oscillator is unable to synchronize. In deriving a self-consistency equation for the mean field, it is therefore convenient to restrict the unknown function $R(x)$ to a class of symmetric functions with a single minimum, by taking advantage of the facts suggested from the numerical simulation. It is also assumed that the curve of $R(x)$ crosses the level of its critical value at $x = \pm x_c$, by which the boundaries between coherence and incoherence are defined. Correspondingly, the spatial integral in Eq. (4·9) defining the mean field is divided into two domains. For the outer domain, where the oscillators are synchronized, we substitute the fixed-point solutions $\psi_{equil}(x)$, while for the inner domain we substitute the drifting solutions $\psi_{drift}(x,t)$. Thus,

$$R(x,t)e^{i\Theta_0(x)} = \int_{|x'|>x_c} dx' G(x-x') e^{i\psi_{equil}(x')} + \int_{|x'|\leq x_c} dx' G(x-x') e^{i\psi_{drift}(x',t)}. \tag{4·10}$$

In this way, as was desired, the right-hand side of Eq. (4·9) becomes a functional of the mean field.

However, there is still a final step which is crucial. This step is necessary to resolve a seeming contradiction. What looks contradictory is that the drifting solutions have an explicit time-dependence by definition, which are inserted into the right-hand side of Eq. (4·9), while the left-hand side of the same equation has been assumed time-independent. How to resolve this seeming dilemma is the following. The time-dependence of the individual drifting solutions does not necessarily imply time-dependence of an integral of the effects over all drifting solutions. In fact, the time-dependence is expected to cancel statistically after integration over infinitely many drifting solutions. This suggests that we may replace the exponential factor appearing in the first integral on the right-hand side of Eq. (4·10) with its statistical average. In calculating the statistical average, we of course need to know the one-oscillator probability distribution function $p(\psi, x)$. It is clear, almost by definition, that for a given drifting oscillator, the probability density is inversely proportional to the drift velocity. Thus, we can use the following expression for $p(\psi, x)$.

$$p(\psi, x) = C[\omega - \Omega - R(x)\sin(\psi - \Theta_0(x) + \alpha)]^{-1}, \tag{4·11}$$

where the right-hand side is proportional to the inverse of the right-hand side of the phase equation Eq. (4·8), and C is the normalization constant. In this way, we finally arrive at a functional self-consistency equation

$$R(x)e^{i\Theta_0(x)} = H[R(x'), \Theta_0(x'), \alpha], \tag{4·12}$$

which however could only be solved numerically. This equation still involves an unknown parameter Ω, i.e., the collective frequency, and the solution of this equation exists only for a particular value of Ω. Thus, we are working with a non-linear eigenvalue problem. This problem can be solved numerically, and the results agree perfectly with our numerical simulation.[7)]

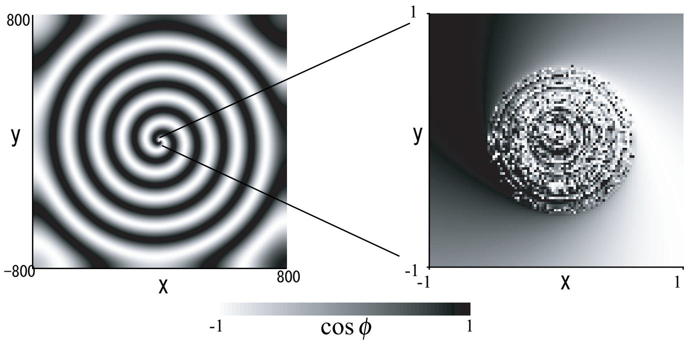

Fig. 3. Spiral pattern obtained from a two-dimensional version of Eq. (4·5) with $\alpha = 0.3$, and a blowup of the structure near the core. For details, see Ref. 9).

Recently, Abram and Strogatz carried out a deeper analysis of the same chimera state.[8)] While our study was limited to the case of fixed values of the parameters, chimera states may disappear and reappear when some parameters are changed. This can be studied through a global bifurcation analysis, which was what these authors did.

The chimera state we have discussed above seems rather special. Its relevance to the real-world phenomena may seem questionable, particularly because it is a one-dimensional pattern and the boundary effects are crucial there. Therefore, it would be interesting to seek the possibility of chimera states in a more realistic setting, e.g., in two-dimensional space with large spatial extension. Within the same model of non-locally coupled phase oscillators, two-dimensional chimera do exist at least in the form of rotating spiral waves.[9)] We know well a possible objection against speaking of spiral waves in terms of a phase oscillator model. The objection is that spiral patterns must involve a phase singularity, whereas the phase model presupposes a well-defined phase everywhere in the system. Our claim is that the phase description of spiral waves does not imply contradiction once spatial continuity has been lost. Loss of spatial continuity can actually occur in reaction-diffusion systems if diffusion is absent for some chemical components involved. We will come back to this issue later.

In Fig. 3 we see a chimera spiral obtained from the same non-local phase model as used above except that the spatial dimension is now two.

The picture shows the pattern of the phases, or more precisely, the pattern of $\cos\phi$. There seems nothing unusual as long as we look at the overall pattern. However, if we look closely into the central part of the pattern, there is something very strange. We find there a circular domain with a sharp boundary in which the oscillators are completely randomized in phase. The size of this incoherent domain is comparable with the size of the spiral core, and their linear dimension is comparable with the coupling radius. If we look at the pattern in terms of the mean field, and not in terms of the states of the individual oscillators, then there is nothing anomalous. This is clear from Fig. 4 where the radial profile $R(r)$ of the mean-field amplitude R about the center of rotation is shown. There is definitely a phase singularity in the

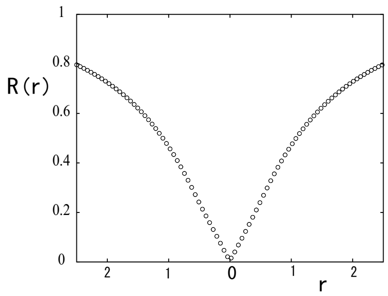

Fig. 4. Instantaneous radial profile of the mean field amplitude R about the center of rotation for the spiral pattern shown in Fig. 3.

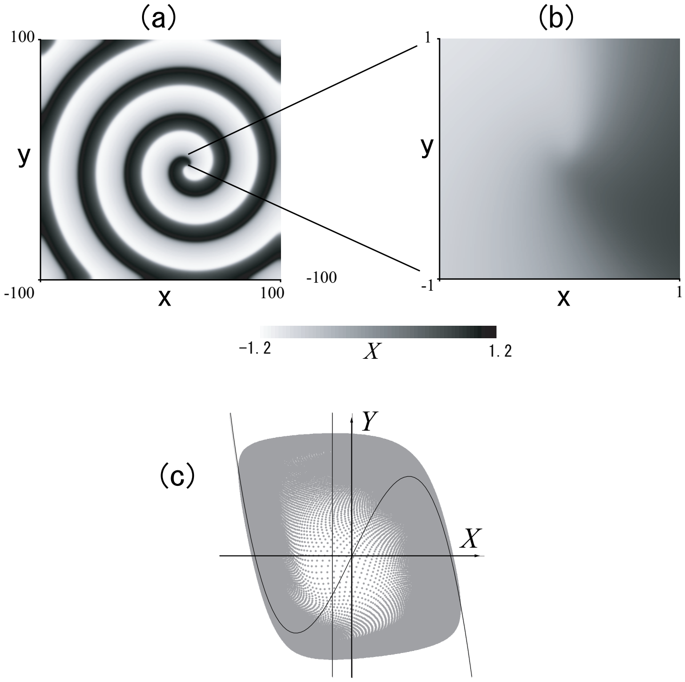

Fig. 5. Spiral pattern obtained from the reaction-diffusion model given by Eqs. (2·9)–(2·13) (left), a blowup of the structure near the center of rotation (right), and the corresponding phase portrait projected onto the X-Y plane (bottom). Parameter values are $k = 10.0$, $a = 1.0$, $b = 0.2$, $\epsilon = 0.1$ and $D = 1.0$.

mean field. We confirmed that $R(r)$ is isotropic except for a very small anisotropic distortions due to the boundary effects. The picture of $R(r)$ also implies that near the center of rotation the mean-field forcing would be too weak to entrain the individual oscillators, while in the outer domain, where the mean-field forcing is strong, the condition for entrainment could be satisfied. The situation is thus quite similar to that of the one-dimensional chimera.

Since the theory of chimera spiral can be formulated completely in parallel with the case of one-dimensional chimera, we will not give its detailed account. See Ref. 9) for details. We obtain again a functional self-consistency equation for the mean field,

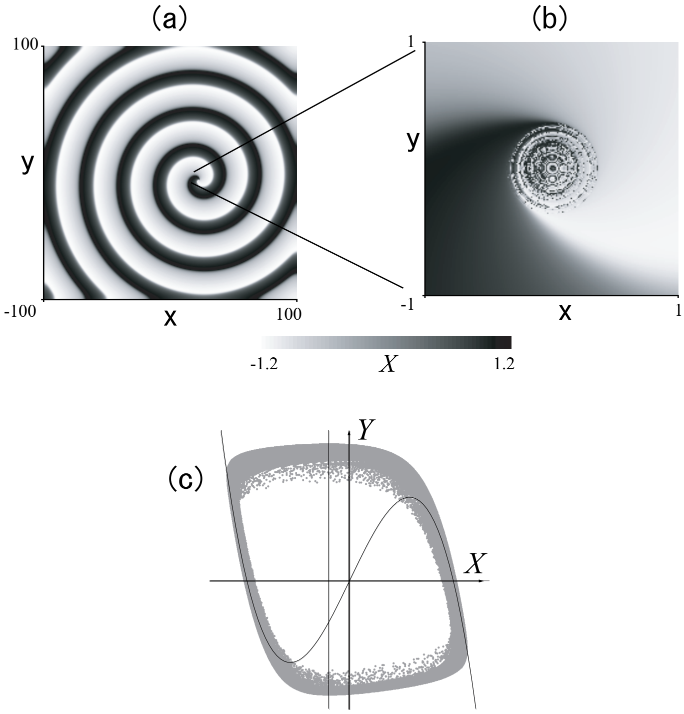

Fig. 6. The same as Fig. 5 except that $k = 2.0$.

and its numerical solution reproduces perfectly the simulation results.

We now come back to the previous question as to whether or not the phase model can be used in describing spiral waves. Without answering this question directly, we first show some results of numerical simulation which was carried out for the three-component reaction-diffusion model given by Eqs. (2·9)–(2·13). Note that diffusion is absent for the first two components X and Y, and therefore spatial continuity of their pattern is not guaranteed. The important parameter is the coupling constant k. For sufficiently strong coupling, e.g., $k = 10$, spiral pattern is completely normal, as we see from Fig. 5. There is no sign of incoherence even near the center of rotation. The corresponding phase portrait is shown in the same figure. This object is actually suspended in a three-dimensional phase space, but we are looking at its projection onto the XY plane. The phase portrait shows a well-known feature characteristic to spiral patterns. That is, it is given by a simply connected object containing a phase singularity somewhere in it.

In contrast, when the coupling constant becomes smaller, perfect coherence becomes impossible. The most fragile part of the pattern in this respect is the center of rotation from which the oscillators start to behave independently. It seems that precisely at the onset of individual motion of such oscillators, the corresponding phase portrait starts to lose its simple connectedness. This is signified by the appearance of a small hole somewhere in the phase portrait. The size of the hole grows as the coupling becomes weaker. Creation of a hole implies that the mapping between the two-dimensional physical space and the phase plane is no longer homeomorphic, namely, a neighborhood in the physical space does not always correspond to a neighborhood in the phase space. This is an unmistakable sign of spatial discontinuity of the pattern.

When the coupling becomes even smaller, the hole in the phase portrait becomes so large that the latter may look rather like a closed loop (see Fig. 6). The ribbon forming this loop becomes thinner and thinner, and the shape of the loop comes to trace approximately the limit-cycle orbit of an isolated oscillator. The situation here is such that the individual oscillators oscillate with almost full amplitude, implying that the only relevant variable is the phase. Thus, we should be allowed to use the phase description, and this fact is consistent with the expected fact that the phase reduction becomes exact in the weak-coupling limit. Application of the phase reduction method to our reaction-diffusion model leads, as usual, to a non-locally coupled phase oscillator model. In this way, we come back to the beginning of the story. Of course, the phase-coupling function appropriate for the present reaction-diffusion model is no longer a simple sine function but its form is much more complex, but this difference would only be secondary to our immediate concern.

§5. Concluding remarks

In relation to the two topics discussed in this paper, there are still a number of questions to be answered. Two of them would be the following. As for the first topic, what we studied were noisy systems. It may be wondered whether or not the case of frozen randomness, especially when the natural frequencies are distributed, can be treated analytically. The answer is not clear, but we suspect that such a theory might be possible, e.g., by extending the sophisticated center-manifold reduction of Crawford[16)] developed for globally coupled phase oscillators to non-local cases. Regarding the second topic, we hope to collect more examples of chimera patterns. The two examples discussed in this paper represent an ideal case in the sense that the collective dynamics is simply time-periodic. There should exist more general chimera states exhibiting complex collective dynamics for which the boundaries between coherence and incoherence would be unsteady and therefore more or less obscured.

Although analytically solvable examples may be limited in each topic, the mean-field picture itself, namely the one-oscillator picture under the control of the internal field in a self-consistent manner, will remain useful for qualitative understanding of complex dynamics of large assemblies of coupled oscillators.

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