Wave Propagation in Nonlocally Coupled Oscillators with Noise

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The onset of undamped wave propagation in noisy self-oscillatory media is identified with a Hopf bifurcation of the corresponding effective dynamical system obtained by properly renormalizing the effects of noise. We illustrate this fact on a dense array of nonlocally coupled phase oscillators for which a mean-field idea works exactly in deriving such effective dynamical equations.

Unlike conservative oscillatory media, dissipative self-oscillatory media admit traveling waves without decay even in the presence of noise or other sources of randomness. This ability of wave transmission in random self-oscillatory media should be functionally relevant in a variety of living organisms for which randomness is unavoidable. For this type of systems, a critical strength of randomness is generally expected to exist such that below which the system is capable of sustaining undamped traveling waves. This critical point should be identical with the Hopf bifurcation point of an effective dynamical system obtained by properly renormalizing the effects of noise. It is reasonable to expect that a theory could be developed unambiguously on this issue in the particular case when each oscillator couples with sufficiently many oscillators, because a mean-field idea should be applicable then. In the present article, we carry out this program for nonlocally coupled phase oscillators with noise, and show how an effective dynamical equation can be derived, and how it is reduced to a small-amplitude equation near the bifurcation point. Our theory may be regarded as a natural extension of a previous theory [1] on the onset of collective oscillation for globally coupled phase oscillators with noise.

Imagine an infinitely long array of nonlocally coupled oscillators which are distributed densely and subject to additive noise. By taking the continuum limit, the phases $\phi(x,t)$ of the oscillators are assumed to obey the following Langevin-type equation.

$$\dot{\phi}(x,t) = \omega + \int_{-\infty}^{\infty} dx' G(x-x') \Gamma(\phi(x,t) - \phi(x',t')) + \xi(x,t). \tag{1}$$

Here ω is the natural frequency, the second term represents the nonlocal coupling, and the last term gives additive noise. The phase coupling function Γ depends only on the phase difference and satisfies the in-phase condition $\Gamma'(0) < 0$, [1] while its strength G depends on the distance. Specifically, we assume the form $G(x) = \gamma \exp(-\gamma|x|)/2$ whose integral is normalized. The noise is assumed to be white Gaussian with vanishing mean, i.e., $\langle \xi(x,t) \rangle = 0$ and $\langle \xi(x,t)\xi(x',t') \rangle = 2D\delta(t-t')\delta(x-x')$. Since we are working with an oscillator continuum, the coupling radius γ^{-1} is so large that infinitely many local oscillators may fall within it. This allows us to apply a mean field idea similar to the one applied successfully to globally coupled

oscillators involving randomness. We also remark that the model given by Eq. (1) is not merely our invention but the one derived under suitable conditions from a certain class of reaction-diffusion systems after eliminating adiabatically a highly diffusive chemical component.[2)]

Periodic traveling waves in our oscillatory field correspond to a family of solutions of Eq. (1) having the form $\phi = kx + \Omega t$ possibly disturbed by the noise. In the absence of noise, it is easy to confirm their existence and analyze their stability for general phase-coupling function. In the simple case given by

$$\Gamma(\phi) = -\sin(\phi + \alpha), \quad (|\alpha| < \pi/2) \tag{2}$$

for instance, the traveling waves of sufficiently small k turn out always stable. The additive noise is now switched on. Numerical simulation for $|\alpha|$ not too close to $\pi/2$ shows that as the noise becomes stronger, traveling waves of shorter wavelengths (still satisfying the stability condition for the noiseless case) successively become unstable. For sufficiently strong noise, no traveling waves can exist, implying that the medium is no longer self-oscillatory because the traveling wave with infinite wavelength corresponds to a uniform oscillation. In order to explain these results, we will now convert Eq. (1) to a nonlinear Fokker-Planck equation and apply the center-manifold reduction to it.

In terms of the number density $n(\psi, x, t) \equiv \delta(\phi(x,t) - \psi)$, Eq. (1) can be rewritten as

$$\begin{aligned}\frac{\partial \phi(x,t)}{\partial t} &= \omega + \int_{-\infty}^{\infty} dx' G(x - x') \int_0^{2\pi} d\psi' \Gamma(\phi(x,t) - \psi') n(\psi', x', t) + \xi(x,t) \\ &\equiv V(\phi(x,t), x, t) + \xi(x,t).\end{aligned} \tag{3}$$

The drift velocity V introduced above is expressed as

$$V(\psi, x, t) = \omega + \int_0^{2\pi} d\psi' Z(\psi', x, t) \Gamma(\psi - \psi') d\psi', \tag{4}$$

where

$$Z(\psi', x, t) = \int_{-\infty}^{\infty} dx' G(x - x') n(\psi', x', t). \tag{5}$$

The quantity Z may be interpreted as a space-time dependent mean field. If we regard Z as an externally given quantity, Eq. (3) represents a non-autonomous one-oscillator equation. Thus, the normalized probability density $f(\psi, x, t)$ such that $\phi(x,t)$ takes value ψ is governed by the Fokker-Planck equation[3)]

$$\frac{\partial f(\psi, x, t)}{\partial t} = -\frac{\partial}{\partial \psi}\Big[V(\psi, x, t) f(\psi, x, t)\Big] + D \frac{\partial^2 f(\psi, x, t)}{\partial \psi^2}. \tag{6}$$

We now take a statistical average of the dynamical variable $n(\psi', x', t)$ appearing in the mean field Z. This is allowed because Z is given by a weighted sum over infinitely many oscillators which are individually subject to random forcing so that the statistical fluctuation of Z may safely be neglected. Noting that the average

$\langle n(\psi', x', t)\rangle$ is identical with $f(\psi', x', t)$ by definition, we finally find a nonlinear evolution equation for f in the form

$$\frac{\partial f(\psi, x, t)}{\partial t} = -\frac{\partial}{\partial \psi}\Big[\Big\{\omega + \int_{-\infty}^{\infty} dx' G(x - x') \int_0^{2\pi} d\psi' \Gamma(\psi - \psi') f(\psi', x', t)\Big\} f(\psi, x, t)\Big] + D\frac{\partial^2 f(\psi, x, t)}{\partial \psi^2}. \tag{7}$$

Wave propagation and collective dynamics could be discussed on the basis of this deterministic kinetic equation. In particular, as is shown below, one may apply the center-manifold reduction near the critical noise strength $D = D_c$ at which the trivial constant solution $f_0 = (2\pi)^{-1}$ loses stability. For this purpose, it is convenient to rewrite Eq. (7) in terms of the fluctuation $\rho(\psi, x, t)$ defined by $f(\psi, x, t) = f_0 + \rho(\psi, x, t)$. The 2π-periodic functions $\rho(\psi)$ and $\Gamma(\psi)$ are further developed into Fourier series like $\rho(\psi, x, t) = 2\pi^{-1}\sum_{l\neq 0}\rho_l(x, t)\exp(il\psi)$ and $\Gamma(\psi) = \sum_l \Gamma_l \exp(il\psi)$. Using these expressions, Eq. (7) becomes

$$\frac{\partial \rho_l(x, t)}{\partial t} = -[l^2 D + il(\omega + \Gamma_0)]\rho_l(x, t) - il\Gamma_l \int_{-\infty}^{\infty} dx' G(x - x')\rho_l(x', t) - il\int_{-\infty}^{\infty} dx' G(x - x') \sum_{m\neq 0, l} \Gamma_m \rho_m(x, t)\rho_{l-m}(x, t). \tag{8}$$

Regarding the loss of stability of the constant solution f_0, spatially uniform perturbation is the most dangerous. This is because by definition the uniform Fourier component of G is the largest of all its Fourier components, thus making the linear growth rate of the uniform perturbation of ρ_l the largest. Therefore, the sign of this growth rate σ_l defined by

$$\sigma_l = -l^2 D + l\Im(\Gamma_l) \tag{9}$$

determines the stability of f_0 with respect to the fluctuation ρ_l. Note that $\Im(\Gamma_{-l}) = -\Im(\Gamma_l)$, so that σ_l is independent of the sign of l. Suppose that σ_l as a function of l takes the largest value at $l = \pm\lambda$ ($\lambda > 0$). As the noise becomes weaker, a Hopf bifurcation occurs at $D = D_c = \Im(\Gamma_\lambda)/\lambda$, the corresponding frequency being given by $\Omega = -\lambda(\omega + \Re(\Gamma_\lambda) + \Gamma_0)$.

The center-manifold reduction can be achieved almost in parallel with the case of globally coupled phase oscillators with noise.[1)] The only difference is the existence of spatial integrals in the governing equation. Since the characteristic spatial scale is expected to be very large near the bifurcation point, which can actually be confirmed from the results obtained, one may develop $\rho(x')$ into a Taylor series like $\rho(x') = \rho(x) + (x' - x)\rho'(x) + (x' - x)^2\rho''(x)/2 + \cdots$ in Eq. (8), and neglect higher order effects. It turns out that the replacement $\rho(x')$ with $\rho(x)$ is allowed in the nonlinear terms, while the second order derivative of $\rho(x)$ must be included in the linear term. In this way, Eq. (8) takes the form of a reaction-diffusion system, and its reduced form near D_c is the complex Ginzburg-Laudau equation. As usual, a complex amplitude $A(x, t)$ related to $\rho(\psi, x, t)$ is introduced as

$$\rho(\psi, x, t) = \frac{1}{2\pi}\Big(A(x, t)\exp[i(\lambda\psi + \Omega t)] + \text{c.c.}\Big). \tag{10}$$

By applying the standard method, the reduced equation near $D = D_c$ becomes

$$\frac{\partial A}{\partial t} = \lambda^2 (D_c - D) A + d \frac{\partial^2 A}{\partial x^2} - g|A|^2 A, \tag{11}$$

where d and g are complex coefficients given by

$$d = -\frac{i\lambda \Gamma_\lambda}{\gamma^2}, \quad g = \frac{\lambda \Gamma_\lambda (\Gamma_{2\lambda} + \Gamma_{-\lambda})}{2\Im(\Gamma_\lambda) - i\Re(\Gamma_\lambda) + i\Gamma_{2\lambda}}. \tag{12}$$

We now consider the simple coupling function given by Eq. (2), and show some results for this particular case. Obviously, we have $\lambda = 1$, $\Gamma_{\pm 1} = \pm i \exp(i\alpha)/2$ and all Γ_l with $|l| \geq 2$ vanish. The quantities d and g are simplified as

$$d = -\frac{i\Gamma_1}{\gamma^2} = \frac{1}{2\gamma^2} \exp(i\alpha), \tag{13}$$

$$g = \frac{|\Gamma_1|^2}{2\Im(\Gamma_1) - i\Re(\Gamma_1)} = \frac{1}{4\cos\alpha + 2i\sin\alpha}. \tag{14}$$

In this particular case, the bifurcation is supercritical because $\Re(g) > 0$.

From the well-known properties of the complex Ginzburg-Landau equation, Eq. (11) admits a family of traveling plane waves $A_k(x,t)$ in the form $A_k = R_k \exp[i(kx + \Omega_k t)]$. The stability condition of the uniform solution A_0 is given by $1 + c_1 c_2 > 0$, where $c_1 = \Im(d)/\Re(d)$ and $c_2 = \Im(g)/\Re(g)$. For the sine-coupling given by Eq. (2), the corresponding condition becomes $\tan^2\alpha < 2$ or $|\alpha| < 0.9553\cdots$. Under this condition, plane waves with k up to some critical value of order $(D_c - D)^{1/2}$ can propagate through the medium stably. This is consistent with the aforementioned results of our numerical simulation on the stochastic system Eq. (1).

Note that if $|\alpha|$ exceeds the above critical value, the uniform oscillation becomes unstable and phase turbulence sets in. In terms of the original model given in Eq. (1), this means that the noise can strongly be amplified to a turbulent level. Note also that Eq. (11) admits the Nozaki-Bekki hole solutions[4] for which the existence of amplitude degree of freedom is crucial. If the phase model Eq. (1) which apparently lacks the amplitude variables can also exhibit hole structures, this should solely be due to a strong phase scattering caused by the noise in a local group of phase oscillators by which the effective amplitude there becomes vanishingly small. Detailed numerical study associated with these remarks will be reported elsewhere.

References

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