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**Geometrical derivation of Lagrange's equations for a system of rigid bodies. (English summary)**
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In this paper, based on the assumption that Euler's two balance laws for a rigid body hold, Lagrange's equations for a system of rigid bodies subject to general holonomic and non-holonomic constraints are derived. One advantage of this assumption is that it extends directly to rigid continua (i.e. infinite-dimensional systems [J. Casey, Z. Angew. Math. Phys. **46** (1995), Special Issue, S805–S847; MR1359345 (96g:70024)]).

The authors introduce a representation of the rigid bodies in a well-suited coordinate system which consists of the center of mass $\overline{\mathbf{x}}_i$ (expressed in the current configuration) of each rigid body $i = 1, \ldots, N$, and 9 quantities for each rotation tensor $\mathbf{Q}^i$ that is associated with a rigid body $i = 1, \ldots, N$. This leads to a $12N$-dimensional vector space $\mathcal{C}^{2N}$ representing the system. From the formulation of the two Euler balance laws in tensor form, and using the aforementioned coordinates, a simple formula $\mathbf{\Phi} = m\dot{\mathbf{v}}$ is obtained representing the dynamics. Of course, although $\mathbf{v} = \left(\dot{\overline{\mathbf{x}}}_1, \ldots, \dot{\overline{\mathbf{x}}}_N, \dot{\mathbf{Q}}_1, \ldots, \dot{\mathbf{Q}}_N\right)$ is of a simple structure, the term $\mathbf{\Phi}$ is quite complicated:

$$\mathbf{\Phi} = m\Big(\frac{\mathbf{F}^1}{m^1}, \ldots, \frac{\mathbf{F}^N}{m^N}, \frac{1}{2}\left(\widehat{\mathbf{M}}^1 + \mathbf{W}^1_{\text{sym}}\right)\mathbf{Q}^1(\mathbf{E}^1_0)^{-1}, \ldots, \frac{1}{2}\left(\widehat{\mathbf{M}}^N + \mathbf{W}^N_{\text{sym}}\right)\mathbf{Q}^N(\mathbf{E}^N_0)^{-1}\Big),$$

where $\mathbf{F}^i$ are the external forces, $\widehat{\mathbf{M}}^i$ are the external torques and $\mathbf{W}^i_{\text{sym}}$ is the symmetric part of the tensor

$$\mathbf{W}^i = 2\ddot{\mathbf{Q}}^i\mathbf{E}^i_0(\mathbf{Q}^i)^{\mathsf{T}},$$

with

$$\mathbf{E}^i_0 = \int \rho^i_0\mathbf{\Pi}^i \otimes \mathbf{\Pi}^i dV^i$$

the Euler tensors ($\mathbf{\Pi}^i = \mathbf{X}_i - \overline{\mathbf{X}}_i$ is the relative position to the center of mass $\overline{\mathbf{X}}_i$ expressed in a fixed occupiable configuration, as opposed to $\overline{\mathbf{x}}_i$). Based on this Newton-like formula $\mathbf{\Phi} = m\dot{\mathbf{v}}$, and using some kinematical arguments concerning the holonomic and nonholonomic constraints, the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\gamma}\right) - \frac{\partial T}{\partial q^\gamma} = Q_\gamma$$

with

$$Q_\gamma = [\mathbf{\Phi}, \mathbf{a}_\gamma] \quad \text{and} \quad \mathbf{a}_\gamma = \frac{\partial \mathbf{v}}{q^\gamma}$$

are derived. The coordinate space is embedded in $\mathcal{C}^{12N}$ and is $6N$-dimensional. It is also shown to be a Riemannian space with the length of a line-element written as $ds^2 = a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta dt^2$. The paper then gives explicit expressions of $Q_\gamma$ based on the expressions of the holonomic and nonholonomic constraints.

In the literature, there are both alternate derivations of Lagrange's equations and expositions of the dynamics of rigid bodies using the Riemannian manifold setting. For example, regarding the whole system as a simple point moving in an abstract space appeared in M. M. G. Ricci and T. Levi-Civita's work [Math. Ann. **54** (1900), no. 1-2, 125–201; MR1511109; JFM 31.0297.01]. J. L. Synge [Philos. Trans. Roy. Soc. London Ser. A **226** (1926), 31–106; JFM 52.0798.05] presented a similar exposition based on the opposite progression of thought as in the present paper, namely taking the Lagrange equations as an assumption (E. T. Whittaker's derivation [*A treatise on the analytical dynamics of particles and rigid bodies: With an introduction to the problem of three bodies*, Cambridge Univ. Press, New York, 1959; MR0103613 (21 #2381)]) and concluding that the dynamics can be expressed as geodesics in a suitable Riemannian space (i.e. the metric should involve the kinetic energy). H. Hertz [*The principles of mechanics*, Translation by D. E. Jones and J. T. Walley, Dover, New York, 1956; MR0077293 (17,1017f)] tried to bypass altogether the notion of force and energy so as to only rely on time, space, and mass. However, he admitted that the motion is along a stationary solution of the minimization problem for a metric. He admitted that to justify this would simply involve the acceptance of both the law of inertia and Gauss's principle of least constraint [H. Hertz, op. cit. (p. 28)] (the latter being a formulation of d'Alembert's principle leading to a minimum problem [C. Lánczos, *The variational principles of mechanics*, Fourth edition, Univ. Toronto Press, Toronto, Ont., 1970; MR0431821 (55 #4815)]). Synge and A. Schild presented an almost purely tensorial derivation of dynamics. However, once we look more closely at it, in [J. L. Synge and A. Schild, *Tensor Calculus*, Univ. Toronto Press, Toronto, Ont., 1949; MR0033165 (11,400f); reprint, Dover, New York, 1978; MR0521984 (80b:53001)], Lagrange's equations (5.521) (or (5.531)) are explicitly stated as a consequence of (2.431) and (2.438) which, in turn, follow from the equations of geodesics. The geodesic equations are a consequence of the calculus of variations presented earlier in the book, which is simply applied for solving the stationary problem involving the kinetic-energy Riemannian metric.

Finally, there is the original derivation of Lagrange himself, which rests on the principle of equilibrium stated as the principle of virtual speed (i.e. the principle of virtual displacement of Johann Bernouilli). A system is in equilibrium whenever the forces are in inverse correspondence with their potential speed (i.e. the speed that the system would gain should the equilibrium cease). D'Alembert rendered this static equilibrium principle to be extendable to the dynamic case, which then allowed Lagrange to find an appropriate analytical treatment unfolded in his famous treatise [J. L. Lagrange, *Méchanique analitique*, Desaint, Paris, 1788; reprint, ?d. Jacques Gabay, Paris, 1989]. Incidentally, d'Alembert presented the dynamics in his book [J. R. d'Alembert, *Traité de dynamique*, Reprint of the 1758 edition, Éd. Jacques Gabay, Sceaux, 1990; MR1451137 (98a:01011)] without resorting to the origin of the concept of force based on three principles: (1) the law of inertia, (2) the composition of movements, and (3) the equilibrium principle.

Hence, all derivations must start from unprovable statements that can only be justified by their agreement with experiments (here, Euler's balance equations are admitted). All these points of view complement and enlighten each other, forging a deeper understanding of mechanics. This paper contributes nicely to this mission. From the computational point of view, the pros and cons must be addressed based on the specific problem at hand.

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