The Journal of Geometric Analysis
Volume 16, Number 4, 2006

Sobolev Inequalities for Differential Forms and $L_{q,p}$-Cohomology

By *Vladimir Gol'dshtein and Marc Troyanov*

ABSTRACT. *We study the relation between Sobolev inequalities for differential forms on a Riemannian manifold (M, g) and the $L_{q,p}$-cohomology of that manifold.*

The $L_{q,p}$-cohomology of (M, g) is defined to be the quotient of the space of closed differential forms in $L^p(M)$ modulo the exact forms which are exterior differentials of forms in $L^q(M)$.

1. Introduction

Let us start by stating a Sobolev-type inequality for differential forms on a compact manifold.

Theorem 1.1. *Let (M, g) be a smooth n-dimensional compact Riemannian manifold, $1 \leq k \leq n$ and $p, q \in (1, \infty)$. Then there exists a constant C such that for any differential form θ of degree $k-1$ on M with coefficients in L^q, we have*

$$\inf_{\zeta \in Z^{k-1}} \|\theta - \zeta\|_{L^q(M)} \leq C \|d\theta\|_{L^p(M)} , \tag{1.1}$$

if and only if

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n} . \tag{1.2}$$

Here Z^{k-1} denotes the set of smooth closed $(k-1)$-forms on M.

The differential $d\theta$ in the inequality above is to be understood in the sense of currents.

Note that condition (1.2) is equivalent to

$$p \geq n \qquad \text{or} \qquad p < n \ \text{ and } \ q \leq p^* = \frac{np}{n-p} . \tag{1.3}$$

In the case of zero forms (i.e., $k = 1$), this theorem can be deduced from the corresponding result for functions with compact support in $\mathbb{R}^n$ by a simple argument using a partition of unity.

Math Subject Classifications. 58J10, 58A12, 46E35, 35J15.
Key Words and Phrases. Sobolev inequality, differential forms, $L_{q,p}$-cohomology.

ISSN 1050-6926

The case of differential forms of higher degree can be proved using more involved reasoning based on standard results from the Hodge-De Rham theory and L^p-elliptic estimates obtained in the 1950' by various authors. We give a sketch of such a proof in the Appendix of this article.

In the case of a non-compact manifold, the inequality (1.1) is still meaningful if the differential form θ belongs to L^q. Although, the condition (1.2) is still necessary in the non compact case, it is no longer sufficient and additional conditions must be imposed on the geometry of the manifold (M, g) for a Sobolev inequality to hold.

The main goal of this article is to investigate these conditions. Our Theorem 6.2 below gives a necessary and sufficient condition based on an invariant called the *$L_{q,p}$–cohomology of* (M, g) and which is defined as

$$H^k_{q,p}(M) = Z^k_p(M)/d\Omega^{k-1}_{q,p}(M)\ ,$$

where $Z^k_p(M)$ is the Banach space of closed k-forms θ in $L^p(M)$ and $\Omega^{k-1}_{q,p}(M)$ is the space of all $(k-1)$-forms ϕ in $L^q(M)$ such that $d\phi \in L^p$.

We will also prove a regularization theorem saying that any $L_{q,p}$-cohomology class can be represented by a smooth form, provided that (1.2) holds (see Theorem 12.7). This implies in particular that the $L_{q,p}$-cohomology of a compact manifold M coincides with the usual De Rham cohomology M and it gives us a new proof of Theorem 1.1 above. This new proof is perhaps simpler than the classical one sketched in the Appendix (at least it does not rely on the rather deep elliptic estimate).

The techniques of this article also provide a proof of the following result which is a complement to Theorem 1.1.

Theorem 1.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension n and $p, q \in (1, \infty)$. There exists a constant C such that for all closed differential forms ω of degree k with coefficients in $L^p(M)$, there exists a differential form θ of degree $k-1$ such that $d\theta = \omega$ and*

$$\|\theta\|_{L^q} \leq C\, \|\omega\|_{L^p}\ , \tag{1.4}$$

if and only if p, q satisfy the condition (1.2) *and* $H^k_{\mathrm{DeRham}}(M) = 0$.

Both Theorems 1.1 and 1.2 are proved at the end of Section 12. In the non compact case, we prove in Theorem 6.1 below that the inequality (1.4) holds if and only if $H^k_{q,p}(M, g) = 0$.

The Sobolev inequality is important because it is a key ingredient in solving partial differential equations. To illustrate this point, we show in Section 13 how Theorem 6.2 can be used to solve the non linear equation

$$\delta\big(\, \| \, d\theta \|^{p-2}\ d\theta\big) = \alpha \tag{1.5}$$

for differential forms. Here δ is the formal adjoint to the exterior differential d.

Although, it is certainly a nice observation that such Sobolev type inequalities for differential forms have interpretations in $L_{q,p}$-cohomology, this will not lead us very far unless we are able to compute some of this cohomology. Unfortunately, this is not an easy task and only few examples of $L_{q,p}$-cohomology groups are presently known. It is thus also one of our goals in this article to begin developing some of the basic facts from the theory. In particular, we present here some results in the direction of duality (see Section 8), a proof of the Poincaré Lemma for $L_{q,p}$-cohomology and a non vanishing result for the $L_{q,p}$-cohomology of the hyperbolic plane

$\mathbb{H}^2$. This non vanishing result says in particular that the Sobolev inequality (1.4) for one-forms never holds on $\mathbb{H}^2$ for any $p, q \in (1, \infty)$.

Let us shortly describe what is contained in the article. In Sections 2 and 3, we give the necessary definitions and we prove some elementary properties of $L_{q,p}$-cohomology. Then we present some basic facts of the theory of Banach complexes and we derive the cohomological interpretation of Sobolev inequalities for differential forms (Sections 4, 5, and 6). In Section 7, we prove some monotonicity properties for the $L_{q,p}$-cohomology of finite-dimensional manifolds and in Section 8 we introduce a notion of "almost duality" techniques (a standard Poincaré duality holds only when $p = q$). We apply these techniques to compute the $L_{q,p}$-cohomology of the line (Section 9) and the hyperbolic plane (Section 10) and to prove a version of the Poincaré Lemma (Section 11). In Section 12, we show that the $L_{q,p}$-cohomology of a manifold can be represented by smooth forms under the condition (1.2). Finally, we show in Section 13 how the $L_{q,p}$-cohomology can be relevant in the study of some non linear PDE, and in Section 14 we give a relation between the L_2-cohomology and the Laplacian on complete manifolds. The article ends with an Appendix describing an alternative proof of Theorems 1.1 based on L^p elliptic estimates.

Remark. The reader might prefer to call the inequality (1.1) a *Poincaré inequality* and use the term *Sobolev inequality* only for the inequality (1.4). In fact there are various uses of the terms Poincaré and Sobolev inequalities. According to [7], the Poincaré inequality is simply a special case of the Sobolev one (it is in fact the case $p = q$). In this article, we avoid the name Poincaré inequality.

2. Definitions

Let us recall the notion of weak exterior differential of a differential form on a Riemannian manifold (M, g).

We denote by $C_c^\infty(M, \Lambda^k)$ the vector space of smooth differential forms of degree k with compact support on M and by $L^1_{\text{loc}}(M, \Lambda^k)$ the space of differential k-forms whose coefficients (in any local coordinate system) are locally integrable.

Definition 2.1. One says that a form $\theta \in L^1_{\text{loc}}(M, \Lambda^k)$ is the *weak exterior differential* of a form $\phi \in L^1_{\text{loc}}(M, \Lambda^{k-1})$ and one writes $d\phi = \theta$ if for each $\omega \in C_c^\infty(M, \Lambda^{n-k})$, one has

$$\int_M \theta \wedge \omega = (-1)^k \int_M \phi \wedge d\omega \,.$$

Clearly, $d\phi$ is uniquely determined up to sets of Lebesgue measure zero, because $d\phi$ is the exterior differential (in the sense of currents) of the current ϕ. It is also clear that $d \circ d = 0$, and this fact allows us to define various cohomology groups.

Let $L^p(M, \Lambda^k)$ be the space of differential forms in $L^1_{\text{loc}}(M, \Lambda^k)$ such that

$$\|\theta\|_p := \left(\int_M |\theta|^p \, dx\right)^{\frac{1}{p}} < \infty \,.$$

We then set $Z_p^k(M) := L^p(M, \Lambda^k) \cap \ker d$ (= the set of weakly closed forms in $L^p(M, \Lambda^k)$) and

$$B_{q,p}^k(M) := d\left(L^q\big(M, \Lambda^{k-1}\big)\right) \cap L^p\big(M, \Lambda^k\big) \,.$$

Lemma 2.2. *$Z_p^k(M) \subset L^p(M, \Lambda^k)$ is a closed linear subspace. In particular, it is a Banach space.*

Proof. We need to show that an arbitrary element $\phi \in \overline{Z}_p^k(M)$ in the closure of $Z_p^k(M)$ is a weakly closed form. Choose a sequence $\phi_i \in Z_p^k(M)$ such that $\phi_i \to \phi$ in L^p-norm. Since ϕ_i are weakly closed forms, we have

$$\int_M \phi_i \wedge d\omega = 0 \,,$$

for any smooth differential forms ω of degree $n-k-1$ with compact support on M. Using Hölder's inequality, we obtain

$$\int_M \phi \wedge d\omega = \int_M (\phi - \phi_i) \wedge d\omega \leq \|\phi - \phi_i\|_{L^p(M)} \|d\omega\|_{L^{p'}(M)} \to 0 \,.$$

Here $1/p + 1/p' = 1$.

Thus, $\int_M \phi \wedge d\omega = 0$ for any $\omega = C_c^\infty(M, \Lambda^{n-k-1})$ and hence $\phi \in Z_p^k(M)$. □

Observe that $B_{q,p}^k(M) \subset Z_p^k(M)$ (because $d \circ d = 0$), we thus have

$$B_{q,p}^k(M) \subset \overline{B}_{q,p}^k(M) \subset Z_p^k(M) = \overline{Z}_p^k(M) \subset L^p\big(M, \Lambda^k\big) \,.$$

Definition 2.3. The *$L_{q,p}$-cohomology* of (M, g) (where $1 \leq p, q \leq \infty$) is defined to be the quotient

$$H_{q,p}^k(M) := Z_p^k(M)/B_{q,p}^k(M) \,,$$

and the *reduced $L_{q,p}$-cohomology* of (M, g) is

$$\overline{H}_{q,p}^k(M) := Z_p^k(M)/\overline{B}_{q,p}^k(M) \,,$$

(where $\overline{B}_{q,p}^k(M)$ is the closure of $B_{q,p}^k(M)$). We also define the *torsion* as

$$T_{q,p}^k(M) := \overline{B}_{q,p}^k(M) \,/\, B_{q,p}^k(M) \,.$$

We thus have the exact sequence

$$0 \to T_{q,p}^k(M) \to H_{q,p}^k(M) \to \overline{H}_{q,p}^k(M) \to 0 \,.$$

The reduced cohomology is naturally a Banach space. The unreduced cohomology is a Banach space if and only if the torsion vanishes.

By Lemma 4.4 below, we see that the torsion $T_{q,p}^k(M)$ can be either $\{0\}$ or infinite dimensional. Indeed, if $\dim T_{q,p}^k(M) < \infty$ then $B_{q,p}^k(M)$ is closed, hence $T_{q,p}^k(M) = \{0\}$. In particular, if $\dim T_{q,p}^k(M) \neq 0$ then $\dim H_{q,p}^k(M) = \infty$.

When $p = q$, we simply speak of L_p-cohomology and write $H_p^k(M)$ and $\overline{H}_p^k(M)$.

Example. The $L_{q,p}$-cohomology of the bounded interval $M = (0, 1)$ is easily computed: We clearly have $H_{q,p}^0((0,1)) = \mathbb{R}$ and $H_{q,p}^1((0,1)) = 0$ for any $1 \leq q, p \leq \infty$.

Indeed, if $\omega = a(x)\,dx$ belongs to $L^p((0,1)) \subset L^1((0,1))$, then $f(x) := \int_{-\infty}^x a(s)\,ds$ belongs to $L^q((0,1))$ for any $1 \leq q \leq \infty$.

The $L_{q,p}$-cohomology of the unbounded intervals and other examples will be computed below.

3. Some elementary properties of $L_{q,p}$-cohomology

3.1. Zero-dimensional cohomology

We have $H^0_{q,p}(M) = \overline{H}^0_{q,p}(M) = Z^0_p(M) = H^0_p(M)$ and these spaces have the following interpretation: dim $H^0_\infty(M)$ is the number of connected components of M and dim $H^0_p(M)$ is the number of connected components with finite volume of M if $1 \leq p < \infty$.

3.2. Conformal invariance

Let (M, g) be a Riemannian manifold of dimension n. Recall that a new metric g_1 is a conformal deformation of g if $g_1 := \rho^2 g$ where $\rho : M \to \mathbb{R}_+$ is a smooth function.

The pointwise norms of a k-form ω with respect to the metrics g_1 and g are related by the identity $|\omega|_{g_1} = \rho^{-k}|\omega|_g$. The volume elements are related by $d\text{vol}_{g_1} = \rho^n \, d\text{vol}_g$. In particular,

$$|\omega|^p_{g_1} \, d\text{vol}_{g_1} = \rho^{n-pk}|\omega|^p_g \, d\text{vol}_g$$

for any k-form; likewise, $|\theta|^q_{g_1} \, d\text{vol}_{g_1} = \rho^{n-q(k-1)}|\theta|^q_g \, d\text{vol}_g$ for any $k-1$-form θ. It follows that $H^k_{q,p}(M, g_1) = H^k_{q,p}(M, g)$ if $n - pk = n - q(k-1) = 0$.

We thus have the following.

Theorem 3.1. *If $q = \frac{n}{k-1}$ and $p = \frac{n}{k}$, then $H^k_{q,p}(M, g)$ and $\overline{H}^k_{q,p}(M, g)$ are conformal invariants.*

4. Banach complexes

The abstract theory of Banach complexes is based on a combination of techniques from homological algebra and functional analysis; this theory is the natural framework of $L_{q,p}$-cohomology and we shall take this point of view to show the connections between Sobolev inequalities and $L_{q,p}$-cohomology.

There is not much literature on Banach complexes, we therefore give below all necessary definitions. The reader may look in [11] for more information.

4.1. Cohomology of Banach complexes and abstract Sobolev inequalities

Definition 4.1. A *Banach complex* is a sequence $F^* = \{F^k, d_k\}_{k\in\mathbb{N}}$ where F^k is a Banach space, $d_k : F^k \to F^{k+1}$ is a bounded operator and $d_{k+1} \circ d_k = 0$.

Remark.

(1) It would be more correct to call such an object a Banach cocomplex (and to use the name complex for the case where $d_k : F^k \to F^{k-1}$), but for simplicity, we shall speak of complexes.

(2) To simplify notations, we usually note d for any of the operators d_k.

Definition 4.2. Given a Banach complex $\{F^k, d\}$ we introduce the following vector spaces:

- $Z^k := \ker(d : F^k \to F^{k+1})$, it is a closed subspace of F^k;
- $B^k :=\text{Im}\,(d : F^{k-1} \to F^k) \subset Z^k$;

- $H^k(F^*) := Z^k/B^k$ is the *cohomology* of the complex $F^* = \{F^k, d\}$;
- $\overline{H}^k(F^*) := Z^k/\overline{B}^k$ is *the reduced cohomology* of the complex F^*;
- $T^k(F^*) := \overline{B}^k/B^k = H^k/\overline{H}^k$ is the *torsion* of the complex F^*.

Let us make a few elementary observations:

(a) $\overline{H}^k$, Z^k and $\overline{B}^k$ are Banach spaces;

(b) The natural (quotient) topology on $T^k := \overline{B}^k/B^k$ is coarse (any closed set is either empty or T^k);

(c) We have the exact sequence

$$0 \to T^k \to H^k \to \overline{H}^k \to 0 \,.$$

There is a natural notion of subcomplex.

Definition 4.3. A *subcomplex* G^* of a Banach complex $\{F^*, d\}$ is a sequence of linear subspaces $G^k \subset F^k$ (not necessarily closed) such that $d(G^k) \subset G^{k+1}$. If all G^k are closed subspaces, we say that G^* is a *Banach-subcomplex* of F^*.

The cohomology of the subcomplex G^* is defined as

$$H^k\big(G^*\big) = \big(G^k \cap \ker d\big)/d\big(G^{k-1}\big) \,.$$

Observe that in general $H^k(G^*)$ is not a Banach space, but there is no way to define a reduced cohomology of G^*, unless $G^* \subset F^*$ is a Banach-subcomplex.

Lemma 4.4.

For any Banach complex $\{F^k, d\}$, the following conditions are equivalent

(i) $T^k = 0$*;*

(ii) $\dim T_k < \infty$*;*

(iii) $B^k \subset F^k$ *is closed.*

Proof.

Proof (i)⇒(ii) is obvious and (ii)⇒(iii) follows e.g., from [4, Th. 3.2 page 27]. The implication (iii)⇒(i) follows directly from the definition of the torsion. □

Proposition 4.5. *The following are equivalent:*

(i) $H^k = 0$*;*

(ii) *the operator* $d_{k-1} : F^{k-1}/Z^{k-1} \to Z^k$ *admits a bounded inverse* d_{k-1}^{-1}*;*

(iii) *there exists a constant* C_k *such that for any* $\theta \in Z^k$ *there is an element* $\eta \in F^{k-1}$ *with* $d\eta = \theta$ *and*

$$\|\eta\|_{F^{k-1}} \leq C_k \|\theta\|_{F^k} \,.$$

Proof.
(i)⇒(ii). Suppose $H^k = 0$. Then $d_{k-1} : F^{k-1}/Z^{k-1} \to Z^k$ is a bijective bounded linear operator and by the open mapping theorem, the inverse map

$$d_{k-1}^{-1} : Z^k \to F^{k-1}/Z^{k-1}$$

is also a bounded operator.
(ii)⇒(iii). Let γ be the norm of $d_{k-1}^{-1} : Z^k \to F^{k-1}/Z^{k-1}$, then for any $\theta \in Z^k$ we can find $\xi \in F^{k-1}$ such that $d_{k-1}\xi = \theta$. Furthermore,

$$\|[\xi]\|_{F^{k-1}/Z^{k-1}} = \inf_{\zeta \in Z^{k-1}} \|\xi - \zeta\|_{F^{k-1}} \le \gamma \, \|\theta\|_{F^k} \ .$$

In particular, there exists $\zeta \in Z^{k-1}$ such that $\|\xi - \zeta\|_{F^{k-1}} \le 2\gamma \, \|\theta\|_{F^k}$. Let us set $\eta := (\xi - \zeta)$, then $d_{k-1}\eta = \theta$ and $\|\eta\|_{F^{k-1}} \le C_k \, \|\theta\|_{F^k}$ with $C_k = 2\gamma = 2\left\|d_{k-1}^{-1}\right\|_{Z^k \to F^{k-1}/Z^{k-1}}$.

The implication (iii)⇒(i) is clear. □

Proposition 4.6. *The following conditions are equivalent:*

(i) $T^k = 0$;
(ii) *The operator* $d_{k-1} : F^{k-1}/Z^{k-1} \to B^k$ *admits a bounded inverse* d_{k-1}^{-1}.

And any one of these conditions imply

(iii) *There exists a constant* $C_k^{'}$ *such that for any* $\xi \in F^{k-1}$ *there is an element* $\zeta \in Z^{k-1}$ *such that*

$$\|\xi - \zeta\|_{F^{k-1}} \le C_k^{'} \| \, d\xi\|_{F^k} \ . \tag{4.1}$$

Proof. The conditions (i) and (ii) are equivalent, because the existence of a bounded inverse operator is equivalent to the closedness of B^{k-1} by the open mapping theorem.

Let us assume that $T^k = 0$ and prove (iii). By hypothesis, B^k is a Banach space and $d_{k-1} : F^{k-1}/Z^{k-1} \to B^k$ is a bijective bounded linear operator. Thus, by the open mapping theorem, the inverse $d_{k-1}^{-1} : B^k \to F^{k-1}/Z^{k-1}$ is also a bounded operator.

Let γ be the norm of $d_{k-1}^{-1} : B^k \to F^{k-1}/Z^{k-1}$, then for any $\xi \in F^{k-1}$ we have

$$\|[\xi]\|_{F^{k-1}/Z^{k-1}} = \inf_{\zeta \in Z^{k-1}} \|\xi - \zeta\|_{F^{k-1}} \le \gamma \, \|d_{k-1}\xi\|_{F^k}$$

in particular, there exists $\zeta \in Z^{k-1}$ such that $\|\xi - \zeta\|_{F^{k-1}} \le 2\gamma \, \|d_{k-1}\xi\|_{F^k}$. □

Proposition 4.7. *If* F^{k-1} *is a reflexive Banach space, then the three conditions of the previous proposition are equivalent.*

Proof. We only need to show that (iii)⇒(i) i.e., $B^k = \overline{B}^k \subset F^k$ provided (4.1) holds and F^{k-1} is a reflexive. Let $\theta \in \overline{B}^k$, then there exists a sequence $\xi_i \in F^{k-1}$ such that $d_{k-1}\xi_i \to \theta$ in F^k. By hypothesis there exists a sequence $\zeta_i \in Z^{k-1}$ such that $\|\xi_i - \zeta_i\|_{F^{k-1}} \le C_k' \, \|d\xi_i\|_{F^k}$. In particular, the sequence $\{\eta_i := (\xi_i - \zeta_i)\}$ is bounded, we may thus find a subsequence (still denoted $\{\eta_i\}$) which converges weakly to an element $\eta \in F^{k-1}$.

Using the Mazur Lemma (see, e.g., Chapter V Section 1, Theorem 2, p. 120 in [19]), we may construct a sequence $\{\widetilde{\eta}_i = \sum_{j=i}^{N(i)} a_i \eta_j\}$ of convex combinations of η_i such that $\widetilde{\eta}_i$ converges strongly to η. We then have

$$d_{k-1}\eta = \lim_{i\to\infty} d_{k-1}\widetilde{\eta}_i = \lim_{i\to\infty} \sum_{j=i}^{N(i)} a_i d_{k-1}\eta_i = \lim_{i\to\infty} \sum_{j=i}^{N(i)} a_i d_{k-1}\xi_j = \theta$$

hence $\theta \in \operatorname{Im}\ (d) = B^k$. We proved that B^k is closed, i.e., $T^k = 0$. □

4.2. Morphisms and homotopies of Banach complexes

This part will be useful to regularize $L_{q,p}$-cohomology, see Section 12.

Definition.

(1) A *morphism* R^* between two Banach complexes $F^* = \{F^k, d\}$ and $E^* = \{E^k, d\}$ is a family of bounded operators $R^k : F^k \to E^k$ such that

$$d_k \circ R^k = R^{k+1} \circ d_k \ .$$

(2) A *homotopy* between two morphisms R^* and $S^* : F^* \to E^*$ is a family of bounded operators $A^k : F^k \to E^{k-1}$ such that

$$S^k - R^k = d_{k-1} \circ A^k + A^{k+1} \circ d_k \ .$$

(3) A *weak homotopy* between two morphisms R^* and $S^* : F^* \to E^*$ is a sequence of families of bounded operators $A^k_j : F^k \to E^{k-1}$ such that for any element $x \in F^k$ we have

$$\lim_{j\to\infty} \left\| \big(d_{k-1} \circ A^k_j + A^{k+1}_j \circ d_k\big)x - \big(S^k - R^k\big)x \right\| = 0 \ .$$

Observe that, if $R^* = \{R^k : F^k \to E^k\}$ is a morphism, then its image is a subcomplex of E^* and it is a Banach-subcomplex if and only if all R^k are closed operators. The kernel of R^* is always a Banach-subcomplex of F^*.

Proposition 4.8. *Let $R^* : F^* \to F^*$ be an endomorphism of a Banach complex $\{F^*, d\}$ such $R^*(F^*) \subset G^*$ where G^* is a subcomplex.*

If there exists a homotopy $\{A^k : F^k \to F^{k-1}\}$ between R^* and the identity operator $I : F^* \to F^*$, then

$$H^k\big(F^*\big) = H^k\big(G^*\big) \ .$$

Proof. Given $\xi \in Z^k(F^*)$, we observe that $R^k\xi \in Z^k(G^*)$ because $dR\xi = R\,d\xi = 0$. If $\xi = d\eta \in B^k(F^*)$, then $R^k\xi = R^k d\eta = dR^k\eta \in B^k(G^*)$.

This proves that $[R\xi]$ is a well-defined cohomology class in $H^k(G^*)$ for any cohomology class $[\xi] \in H^k(F^*)$.

But since

$$\xi - R\xi = dA\xi + A\,d\xi = dA\xi$$

for any $\xi \in Z^k(F^*)$, we see that in fact $[R\xi] = [\xi] \in H^k(F^*)$ and the proposition is proved. □

The following result is a generalization of the previous proposition.

Proposition 4.9.

(1) *Any morphism $R^* : F^* \to E^*$ between two Banach complexes induces a sequence of linear homomorphisms $H^k R^* : H^k(F^*) \to H^k(E^*)$ from the cohomology of F^* to the cohomology of E^*.*

(2) *The morphism $R^* : F^* \to E^*$ induces a sequence of bounded operators $\overline{H}^k R^* : \overline{H}^k(F^*) \to \overline{H}^k(E^*)$ from the reduced cohomology of F^* to the reduced cohomology of E^*.*

(3) *If there exists a homotopy between two morphisms R^* and $S^* : F^* \to E^*$, then the corresponding homomorphisms on the cohomology groups coincide:*

$$H^k R^* = H^k S^* : H^k\big(F^*\big) \to H^k\big(E^*\big) .$$

(4) *If there exists a weak homotopy between two morphisms R^* and $S^* : F^* \to E^*$, then the corresponding morphisms on the reduced cohomology groups coincide:*

$$\overline{H}^k R^* = \overline{H}^k S^* : \overline{H}^k\big(F^*\big) \to \overline{H}^k\big(E^*\big) .$$

Proof.

(1) Because $dR^* = R^*d$, the image $R^*([\omega])$ of any cohomology class $[\omega]$ of the complex F^* is a well-defined cohomology class of the complex E^*.

(2) Using the continuity of R^* and $dR^* = R^*d$, we see that closure of the image $R^*([\omega])$ of a reduced cohomology class of F^* is a well-defined reduced cohomology class of E^*. By the boundedness of R^k, the operators $\overline{H}^k R^* : \overline{H}^k(F^*) \to \overline{H}^k(E^*)$ is also bounded.

(3) The condition $S^k - R^k = d \circ A^k + A^{k+1} \circ d$ implies that for any $\xi \in Z^k(F^*)$ we have $\big(S^k\xi - R^k\xi\big) = d(A^k\xi) \in B^k(E^*)$.

(4) The condition $\lim_{j\to\infty} \left\|(d \circ A_j^k + A_j^{k+1} \circ d)x - (S^k - R^k)x\right\| = 0$ for any $x \in F^k$ implies that for any $\xi \in Z^k(F^*)$ we have

$$\lim_{j\to\infty} \left\| S^k\xi - R^k\xi - d\big(A_j^k\xi\big)\right\| = 0 . \qquad \square$$

A special case of the previous proposition is given in the following definitions.

Definition 4.10.

(a) A Banach complex $F^* = \{F^k, d\}$ is *acyclic* if there exists a family of bounded operators $A^k : F^k \to F^{k-1}$ such that

$$\mathrm{Id} = d \circ A^k + A^{k+1} \circ d .$$

(b) The Banach complex F^* is *weakly acyclic* if for any k there exists a sequence of bounded operators $A_j^k : F^k \to F^{k-1}$ such that for any element $x \in F^k$ we have

$$\lim_{j\to\infty} \left\|\big(d \circ A_j^k + A_j^{k+1} \circ d\big)x - x\right\| = 0 .$$

In other words, F^* is (weakly) acyclic if and only if there exists a (weak) homotopy from the identity $\mathrm{Id} : F^* \to F^*$ to the trivial morphism $0 : F^* \to F^*$ It is thus clear that an acyclic complex has trivial cohomology and a weakly acyclic complex has trivial reduced cohomology.

5. $L_{q,p}$-cohomology and Banach complexes

In this section, we explain how the $L_{q,p}$-cohomology of a Riemannian manifold (M,g) can be formally seen as the cohomology of some complex of Banach spaces. Let us start by introducing the notation

$$\Omega^k_{q,p}(M) := \left\{ \omega \in L^q\big(M,\Lambda^k\big) \,\middle|\, d\omega \in L^p \right\} .$$

This is a Banach space for the graph norm

$$\|\omega\|_{\Omega_{q,p}} := \|\omega\|_{L^q} + \|d\omega\|_{L^p} \ . \tag{5.1}$$

By standard arguments of functional analysis (see e.g., [2]), it can be proved that $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < p, q < \infty$. We will also prove in Section 12 that smooth forms are dense in $\Omega^k_{q,p}(M)$ for any $1 \leq p, q < \infty$.

To define a Banach complex, we choose an arbitrary finite sequence of numbers

$$\pi = \{p_0, p_1, \cdots, p_n\} \subset [1,\infty] ,$$

and define

$$\Omega^k_\pi(M) := \Omega^k_{p_k, p_{k+1}}(M) .$$

Observe that $\Omega^n_\pi(M) = L^{p_n}(M,\Lambda^n)$ and $\Omega^1_{p,p}(M)$ coincides with the Sobolev space $W^{1,p}(M)$.

Since the exterior differential is a bounded operator $d : \Omega^{k-1}_\pi \to \Omega^k_\pi$, we have constructed a Banach complex.

$$0 \to \Omega^0_\pi \xrightarrow{d} \cdots \xrightarrow{d} \ \Omega^{k-1}_\pi \xrightarrow{d} \ \Omega^k_\pi \xrightarrow{d} \cdots \xrightarrow{d} \ \Omega^n_\pi \to 0 \, .$$

Definition 5.1. *The (reduced) L_π-cohomology of M is the (reduced) cohomology of the Banach complex $\{\Omega^k_\pi(M), d_k\}$.*

The L_π-cohomology space $H^k_\pi(M)$ depends only on p_k and p_{k-1} and we have in fact

$$H^k_\pi(M) = H^k_{p_{k-1},p_k}(M) \quad \text{and} \quad \overline{H}^k_\pi(M) = \overline{H}^k_{p_{k-1},p_k}(M) \, .$$

Two cases are of special interest:

(1) The L_p-cohomology, which corresponds to the constant sequence $\pi = \{p, p, \ldots, p\}$.
(2) The *conformal cohomology,* which corresponds to the sequence $p_0 = \infty$, and $p_k = \frac{n}{k}$ for $k = 1, \ldots, n$. The cohomology associated to this sequence is a conformal invariant of the manifold by Theorem 3.1.
Let us remark here that $\left(\frac{1}{p_k} - \frac{1}{p_{k-1}}\right) = \frac{1}{n}$.

6. $L_{q,p}$-cohomology and Sobolev inequality

We are now in position to give the interpretation of $L_{q,p}$-cohomology in terms of a Sobolev type inequality for differential forms on a Riemannian manifold (M,g).

Theorem 6.1. *$H^k_{q,p}(M,g) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed p-integrable differential form ω of degree k there exists a differential form θ of degree $k-1$ such that $d\theta = \omega$ and*

$$\|\theta\|_{L^q} \le C\,\|\omega\|_{L^p}\ .$$

This result is a direct consequence of Proposition 4.5.

Theorem 6.2.

(A) *If $T^k_{q,p}(M) = 0$, then there exists a constant C' such that for any differential form $\theta \in \Omega^{k-1}_{q,p}(M)$ of degree $k-1$ there exists a closed form $\zeta \in Z^{k-1}_q(M)$ such that*

$$\|\theta - \zeta\|_{L^q} \le C'\,\|d\theta\|_{L^p}\ . \tag{6.1}$$

(B) *Conversely, if $1 < q < \infty$, and if there exists a constant C' such that for any form $\theta \in \Omega^{k-1}_{q,p}(M)$ of degree $k-1$ there exists $\zeta \in Z^{k-1}_q(M)$ such that* (6.1) *holds, then $T^k_{q,p}(M) = 0$.*

This statement follows immediately from Propositions 4.6 and 4.7.

7. Manifolds with finite volume and monotonicity

The $L_{q,p}$-cohomology of a manifold with finite volume has some monotonicity properties. In the next statement, the symbol $H_2 \twoheadrightarrow H_1$ (where H_1, H_2 are vector spaces) means that H_1 is a quotient of H_2.

Proposition 7.1. *If (M,g) has finite volume, $1 \le p \le \infty$ and $1 \le q_1 \le q_2 \le \infty$, then $\overline{H}^k_{q_2,p}(M) \twoheadrightarrow \overline{H}^k_{q_1,p}(M)$ and $H^k_{q_2,p}(M) \twoheadrightarrow H^k_{q_1,p}(M)$.*

Proof. Since $1 \le q_1 \le q_2$ and M has finite volume, we have $L^{q_1}(M,\Lambda^k) \supset L^{q_2}(M,\Lambda^k)$, hence $\Omega^{k-1}_{q_1,p} \supset \Omega^{k-1}_{q_2,p}$ and thus

$$\begin{aligned}
\overline{B}^k_{q_1,p}(M) &= \overline{d\left(\Omega^{k-1}_{q_1,p}\right)} \cap L^p\big(M,\Lambda^k\big) \\
&\supset \overline{d\left(\Omega^{k-1}_{q_2,p}\right)} \cap L^p\big(M,\Lambda^k\big) \\
&= \overline{B}^k_{q_2,p}(M)\ .
\end{aligned}$$

Since $B_2 \subset B_1 \subset Z$ implies $Z/B_1 \twoheadrightarrow Z/B_2$, we have

$$\overline{H}^k_{q_2,p}(M) = Z^k_p/\overline{B}^k_{q_2,p}(M) \twoheadrightarrow Z^k_p/\overline{B}^k_{q_1,p}(M) = \overline{H}^k_{q_1,p}(M)\ .$$

The proof for unreduced cohomology is the same. □

We also have some kind of monotonicity with respect to p.

Proposition 7.2. *If (M,g) has finite volume $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$, then*

$$H^k_{q_2,p_2}(M) = 0 \ \Rightarrow\ H^k_{q_1,p_1}(M) = 0\ .$$

Proof. Since M has finite volume, $q_1 \le q_2$ and $p_2 \le p_1$, we have[1] for any q_2-integrable form θ and any p_1-integrable form ω

$$\|\theta\|_{L^{q_1}} \lesssim \|\theta\|_{L^{q_2}} \qquad \text{and} \qquad \|\omega\|_{L^{p_2}} \lesssim \|\omega\|_{L^{p_1}} \ .$$

Since $H^k_{q_2,p_2}(M) = 0$, we know from Theorem 6.1 that for any closed p_2-integrable form ω of degree k there exists a differential form θ of degree $k-1$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^{q_2}} \lesssim \|\omega\|_{L^{p_2}} \ .$$

Combining this inequality with two previous inequalities we get

$$\|\theta\|_{L^{q_1}} \lesssim \|\omega\|_{L^{p_1}}$$

and the result immediately follows from the same Theorem 6.1. □

For the torsion, we need to avoid the values $q = 1$ and $q = \infty$.

Proposition 7.3. *If (M, g) has finite volume $1 \le p_2 \le p_1 \le \infty$ and $1 < q_1 \le q_2 < \infty$, then*

$$T^k_{q_2,p_2}(M) = 0 \ \Rightarrow \ T^k_{q_1,p_1}(M) = 0 \, .$$

Proof. Again, since $q_1 \le q_2$ we have $\zeta \in Z^{k-1}_{q_2}(M) \Rightarrow \zeta \in Z^{k-1}_{q_1}(M)$ and

$$\|\theta - \zeta\|_{L^{q_1}} \lesssim \|\theta - \zeta\|_{L^{q_2}} \qquad \text{and} \qquad \|d\theta\|_{L^{p_2}} \lesssim \| d\theta\|_{L^{p_1}} \ .$$

We may thus argue as in the previous proof using Theorem 6.2. □

8. Almost duality

It has been proved in [10] that for complete manifolds the dual space of $\overline{H}^k_p(M)$ coincides with $\overline{H}^{n-k}_{p'}(M)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ (there is also a duality result for non complete manifolds). The duality is based on the pairing $\int_M \alpha \wedge \beta$ where $\alpha \in \Omega^k_p(M)$ and $\beta \in \Omega^k_{p'}(M)$.

For $L_{q,p}$-cohomology we have no convenient description of dual spaces, but the notion of *almost duality* which we now introduce is sufficient for many calculations.

We start with a rather elementary result about the non vanishing of $L_{q,p}$-cohomology.

Lemma 8.1. *Let (M, g) be an arbitrary Riemannian manifold of dimension n. Let $\alpha \in Z^k_p(M)$. If there exists $\gamma \in C^\infty_c(M, \Lambda^{n-k})$ such that $d\gamma = 0$ and $\int_M \alpha \wedge \gamma \ne 0$, then $[\alpha] \ne 0$ in $\overline{H}^k_{q,p}(M)$ for any $1 \le q \le \infty$.*

Proof. Suppose that $\alpha \in \overline{B}^k_{q,p}(M)$. Then $\alpha = \lim_{j\to\infty} d\beta_j$ (where the limit is in L^p-topology) for some $\beta_j \in L^q(M, \Lambda^{k-1})$ with $d\beta_j \in L^p(M, \Lambda^k)$. We then have for any closed form with compact support $\gamma \in C^\infty_c(M, \Lambda^{n-k})$

$$\int_M \gamma \wedge \alpha = \lim_{j\to\infty} \int_M \gamma \wedge d\beta_j = \lim_{j\to\infty} (-1)^{n-k+1} \int_M d\gamma \wedge \beta_j = 0$$

[1] The symbol $\lesssim$ means that the inequality holds up to some constant.

in contradiction to the assumption. □

There are several generalizations of this result.

Proposition 8.2. *Let (M, g) be an arbitrary Riemannian manifold of dimension n. Let $\alpha \in Z_p^k(M)$. Then*

(A) *If there exists a sequence $\{\gamma_i\} \subset C_c^\infty(M, \Lambda^{n-k})$ such that*

(i) $\lim_{i\to\infty} \inf \int_M \alpha \wedge \gamma_i > 0$;

(ii) $\lim_{i\to\infty} \|d\gamma_i\|_{q'} = 0$ *where* $q' = \frac{q}{q-1}$.

Then $[\alpha] \neq 0$ *in* $H_{q,p}^k(M)$.

(B) *If there exists a sequence $\{\gamma_i\} \subset C_c^\infty(M, \Lambda^{n-k})$ satisfying the conditions (i) and (ii) above and*

(iii) $\|\gamma_i\|_{p'}$ *is a bounded sequence for* $p' = \frac{p}{p-1}$.

Then $[\alpha] \neq 0$ *in* $\overline{H}_{q,p}^k(M)$.

Proof.

(A) Suppose that $\alpha = d\beta$ for some $\beta \in L^q(M, \Lambda^{k-1})$, then by Hölder inequality we have for any $\gamma \in C_c^\infty(M, \Lambda^{n-k})$

$$\left|\int_M \alpha \wedge \gamma\right| = \left|\int_M d\beta \wedge \gamma\right| = \left|\int_M \beta \wedge d\gamma\right| \leq \|\beta\|_q \cdot \|d\gamma\|_{q'} \ .$$

It follows that for any sequence $\{\gamma_i\} \subset C_c^\infty(M, \Lambda^{n-k})$ such that $\lim_{i\to\infty} \|d\gamma\|_{q'} = 0$, we have $\lim_{i\to\infty} \left|\int_M \alpha \wedge \gamma\right| \leq \lim_{i\to\infty} \|\beta\|_q \cdot \|d\gamma_i\|_{L^{q'}(M)} = 0$.

(B) Suppose that $\alpha \in \overline{B}_{q,p}^k(M)$. Then $\alpha = \lim_{j\to\infty} d\beta_j$ for $\beta_j \in L^q(M, \Lambda^{k-1})$ with $d\beta_j \in L^p(M, \Lambda^k)$. We have for any i, j

$$\int_M \gamma_i \wedge \alpha = \int_M \gamma_i \wedge d\beta_j + \int_M \gamma_i \wedge (\alpha - d\beta_j) \ .$$

For each $j \in \mathbb{N}$, we can find $i = i(j)$ large enough so that $\|d\gamma_{i(j)}\|_{q'} \|\beta_j\|_q \leq 1/j$, we thus have

$$\left|\int_M \gamma_{i(j)} \wedge d\beta_j\right| \quad \leq \quad \left|\int_M d\gamma_{i(j)} \wedge \beta_j\right| \quad \leq \|d\gamma_{i(j)}\|_{q'} \|\beta_j\|_q \leq \frac{1}{j} \ .$$

On the other hand,

$$\lim_{j\to\infty} \left|\int_M \gamma_{i(j)} \wedge (\alpha - d\beta_j)\right| \quad \leq \lim_{j\to\infty} \|\gamma_{i(j)}\|_{p'} \|(\alpha - d\beta_j)\|_p = 0$$

since $\|\gamma_{i(j)}\|_{p'}$ is a bounded sequence and $\|(\alpha - d\beta_j)\|_p \to 0$. It follows that $\int_M \gamma_{i(j)} \wedge \alpha \to 0$ in contradiction to the hypothesis. □

8.1. The case of complete manifolds

If M is a complete manifold, we don't need to assume that the form γ from the previous discussion has compact support.

Proposition 8.3. *Assume that M is complete. Let $\alpha \in Z^k_p(M)$, and assume that there exists a smooth closed $(n-k)$-form γ such that $\gamma \in Z^{n-k}_{q'}(M)$, for $q' = \frac{q}{q-1}$, $\gamma \wedge \alpha \in L^1(M)$ and*

$$\int_M \gamma \wedge \alpha \neq 0 ,$$

then $\alpha \notin B^k_{q,p}(M)$. In particular, $H^k_{q,p}(M) \neq \emptyset$.

This proposition has also version for reduced $L_{q,p}$-cohomology.

Proposition 8.4. *Assume that M is complete. Let $\alpha \in Z^k_p(M)$, and assume that there exists a smooth closed $(n-k)$-form $\gamma \in Z^{n-k}_{p'}(M) \cap Z^{n-k}_{q'}(M)$, where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$, such that*

$$\int_M \gamma \wedge \alpha \neq 0 ,$$

then $\alpha \notin \overline{B}^k_{q,p}(M)$ where $q' = \frac{q}{q-1}$. In particular, $\overline{H}^k_{q,p}(M) \neq \emptyset$.

The proofs are based on the following integration by part lemma.

Lemma 8.5. *Assume that M is complete. Let $\beta \in L^q(M, \Lambda^{k-1})$ be such that $d\beta \in L^p(M, \Lambda^k)$, and $\gamma \in L^{p'}(M, \Lambda^{n-k})$ be such that $d\gamma \in L^{q'}(M, \Lambda^{n-k+1})$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*

If γ is smooth and $\gamma \wedge d\beta \in L^1(M)$, then

$$\int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta . \tag{8.1}$$

In particular, if $\gamma \in L^{n-k}_{p'}(M) \cap L^{n-k+1}_{q'}(M)$, then the above conclusion holds.

Proof. The integrability of $d\gamma \wedge \beta$ and $\gamma \wedge d\beta$ is a direct consequence of Hölder's inequality.

By Hölder's inequality, the forms $d\gamma \wedge \beta$ and $\gamma \wedge d\beta$ both belong to $L^1(M)$.

If γ is a smooth form with compact support, then the Equation (8.1) follows from the definition of the weak exterior differential (of β).

If the support of γ is not compact, we set $\gamma_i := \psi_i \gamma$ where $\{\psi_i\}$ is a sequence of smooth functions with compact support such that $\psi_i(x) \to 1$ uniformly on every compact subset, $0 \leq \psi_i(x) \leq 1$ and $|d\psi_i|_x \leq 1$ for all $x \in M$ (such a sequence exists on any complete manifold).

The formula (8.1) holds for each γ_i (since these forms have compact support).

Using $|d\psi_i|_x \leq 1$, we have the estimate

$$|\gamma_i \wedge d\beta + (-1)^{n-k} d\gamma_i \wedge \beta| \leq |d\gamma \wedge \beta| + |\gamma \wedge d\beta| + |\gamma \wedge \beta| \in L^1(M) .$$

By Lebesgue's dominated convergence theorem, we thus have

$$\int_M \left(\gamma \wedge d\beta + (-1)^{n-k} d\gamma \wedge \beta\right) = \lim_{i\to\infty} \int_M \left(\gamma_i \wedge d\beta + (-1)^{n-k} d\gamma_i \wedge \beta\right) = 0 . \qquad \square$$

Proof of Proposition 8.3. Suppose that $\alpha \in B^k_{q,p}(M)$. Then $\alpha = d\beta$ for some $\beta \in L^q(M, \Lambda^{k-1})$. By the previous lemma, we have

$$\int_M \gamma \wedge \alpha = \int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta = 0$$

(since γ is closed) in contradiction to the assumption. □

Proof of Proposition 8.4. Suppose that $\alpha \in \overline{B}^k_{q,p}(M)$. Then $\alpha = \lim_{j\to\infty} d\beta_j$ (where the limit is in L^p-topology) for some $\beta_j \in L^q(M, \Lambda^{k-1})$ with $d\beta_j \in L^p(M, \Lambda^k)$. Since $d\gamma = 0$, we have

$$\int_M \gamma \wedge \alpha = \lim_{j\to\infty} \int_M \gamma \wedge d\beta_j = \lim_{j\to\infty} (-1)^{n-k+1} \int_M d\gamma \wedge \beta_j = 0\,,$$

which contradicts our hypothesis. □

9. The $L_{q,p}$-cohomology of the line

In the following three sections, we compute the $L_{q,p}$-cohomology of the line, the hyperbolic plane and the ball. We will see in particular that the only case where $H^1_{q,p}(\mathbb{R})$ vanishes is when $q = \infty$, $p = 1$.

Proposition 9.1. $H^1_{\infty,1}(\mathbb{R}) = 0$.

Proof. If $\omega = a(x)\,dx$ belongs to $L^1(\mathbb{R})$, then $f(x) := \int_{-\infty}^x a(s)\,ds$ belongs to $L^\infty(\mathbb{R})$, hence $H^1_{1,\infty}(\mathbb{R}) = 0$. □

Proposition 9.2. *$T^1_{q,p}(\mathbb{R}) \neq 0$ for any $1 \leq p, q \leq \infty$ with the only exception of $q = \infty$, $p = 1$.*

Proof. Assume first that $q < \infty$. We know from Theorem 6.2 that if we had $T^1_{q,p}(\mathbb{R}) = 0$, then there would exist a Sobolev inequality for functions on the real line $\mathbb{R}$:

$$\inf_{z\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |f(x) - z|^q\,dx\right)^{1/q} \leq C \cdot \left(\int_{-\infty}^{\infty} \left|f'(x)\right|^p dx\right)^{1/p} \tag{9.1}$$

for some constant $C < \infty$.

To see that no such inequality is possible, consider a family of smooth functions with compact support $f_a : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $x \in [1, a]$ and $f_a(x) = 0$ if $x \notin [0, a+1]$. We may also assume that $\|f_a'\|_{L^\infty} \leq 2$. Assume now that the inequality (9.1) holds. Then the constant z must be zero and we have

$$\int_{-\infty}^{\infty} |f_a(x)|^q\,dx \geq a - 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \left|f_a'(x)\right|^p dx \leq 2^{1+p}\,,$$

hence

$$C \geq 2^{-1-\frac{1}{p}}(a-1)^{\frac{1}{q}}$$

for all $a > 0$ and we conclude that $C = \infty$.

Assume now that $q = \infty$ and $p > 1$. Again, if we had $T^1_{\infty,p}(\mathbb{R}) = 0$, there would exist $C < \infty$ such that for any $f \in L^p(\mathbb{R})$:

$$\inf_{z\in\mathbb{R}} \|f(x) - z\|_\infty \le C \cdot \left\|f'(x)\right\|_{L^p(\mathbb{R})} . \tag{9.2}$$

Let us consider the functions $g_k(x) := e^{-\pi k x^2}$ and $f(x) := \int_{-\infty}^x g(u)\,du$.

We have $0 \le f(x) < \sup f = \int_{-\infty}^{\infty} g(u)\,du = \frac{1}{\sqrt{k}}$, hence $\inf_{z\in\mathbb{R}} \|f(x) - z\|_\infty = \frac{1}{2\sqrt{k}}$. On the other hand, $\|f'(x)\|_{L^p(\mathbb{R})} = (kp)^{-1/2p}$, hence the constant in (9.2) satisfies

$$\frac{1}{2}k^{-1/2} \le C \cdot (kp)^{-1/2p}$$

for all $k > 0$, i.e., $C = \infty$ since $p > 1$.

Finally, we have $T^1_{\infty,1}(\mathbb{R}) = 0$ since $H^1_{\infty,1}(\mathbb{R}) = 0$. □

Let us turn to the reduced cohomology.

Proposition 9.3. *$\overline{H}^1_{q,p}(\mathbb{R}) \ne 0$ if and only if $p = 1$ and $1 \le q < \infty$.*

Proof. For $p = 1$, $q = \infty$, we have $\overline{H}^1_{\infty,1}(\mathbb{R}) = H^1_{\infty,1}(\mathbb{R}) = 0$.

Assume $1 \le q \le \infty$ and $1 < p \le \infty$ and let $\omega = a(x)\,dx \in L^p(\mathbb{R})$. For each $m \in \mathbb{N}$, we set $\omega_m := \chi_{[-m,m]}\omega = (\chi_{[-m,m]}(x)a(x))\,dx$. Let us choose a continuous function $\lambda_m(x)$ with compact support in $[0,\infty)$ such that $\int_{\mathbb{R}} \lambda_m(x)\,dx = \int_{-m}^m a(x)\,dx$ and $\|\lambda_m\|_{L^p(\mathbb{R})} < \frac{1}{m}$.

Let $b_m(x) := \int_{-\infty}^x \left(\chi_{[-m,m]}(t)a(t) - \lambda_m(t)\right)\,dt$, then $b_m \in L^q(\mathbb{R})$ (in fact b_m has compact support) and $\|db_m - \omega\|_{L^p(\mathbb{R})} \le \|a\|_{L^p(\mathbb{R}\setminus[-m,m])} + \|\lambda_m\|_{L^p(\mathbb{R})} \to 0$ as $m \to \infty$. This shows that $\overline{H}^1_{q,p}(\mathbb{R}) = 0$.

Assume now that $p = 1$ and $1 \le q < \infty$ and let $\omega = a(x)\,dx$ be a 1-form on $\mathbb{R}$ such that $\int_{\mathbb{R}} f\omega = 1$ and $a(x)$ is smooth with compact support (say $\mathrm{supp}(a) \subset [1,2]$). Let $f_j : \mathbb{R} \to \mathbb{R}$ be a sequence of smooth functions with compact support such that $f_j = 1$ on $[1,2]$, $\|f_j\|_{L^\infty} = 1$ and $\|f'_j\|_{L^{q'}} \le \frac{1}{j}$ where $q' = q/(q-1)$.

Using Proposition 8.2, we see that $[\omega] \ne 0 \in \overline{H}^1_{q,1}(\mathbb{R})$, because $\omega \in L^1(\mathbb{R})$ and the sequence $\{f_j\} \subset C^\infty_c(\mathbb{R})$ satisfies the three conditions of that proposition. □

Remark.
(1) In degree 0, the $L_{q,p}$-cohomology is controlled by the volume: $\overline{H}^0_{q,p}(\mathbb{R}) = H^0_{q,p}(\mathbb{R}) = 0$ if and only if $p < \infty$ and $\overline{H}^0_{q,\infty}(\mathbb{R}) = H^\infty_{q,\infty}(\mathbb{R}) = \mathbb{R}$.
(2) All the results of this section also hold for the half-line $\mathbb{R}_+$.

10. The cohomology of the hyperbolic plane

We treat in this section the case of the hyperbolic plane.

Recall that the hyperbolic plane is the Riemannian manifold $\mathbb{H}^2 = \{(u,v) \in \mathbb{R}^2 : v > 0\}$ with the metric $ds^2 = v^{-2}(du^2 + dv^2)$.

Theorem 10.1. *For any $q, p \in (1, \infty)$ we have*

$$\dim\left(\bar{H}^1_{q,p}(\mathbb{H}^2)\right) = \infty .$$

It will be convenient to introduce new coordinates (the so-called "horocyclic coordinates") $y := u$, $z := -\log(v)$, so that $\mathbb{H}^2 = \{(y, z) \in \mathbb{R}^2\}$ with $ds^2 = e^{2z}\, dy^2 + dz^2$.

Lemma 10.2. *There exist two smooth functions f and g on $\mathbb{H}^2$ such that:*

(1) *f and g are non negative;*
(2) *$f(y, z) = g(y, z) = 0$ if $z \le 0$ or $|y| \ge 1$;*
(3) *df and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \le \infty$;*
(4) *the support of $df \wedge dg$ is contained in $\{(y, z) : |y| \le 1\,,\ 0 \le z \le 1\}$;*
(5) *$df \wedge dg \ge 0$;*
(6) $\displaystyle\iint_{\mathbb{H}^2} df \wedge dg = 1$;
(7) $\dfrac{\partial f}{\partial y}$ *and* $\dfrac{\partial g}{\partial y} \in L^\infty(\mathbb{H}^2)$;
(8) $\dfrac{\partial f}{\partial z}$ *and* $\dfrac{\partial g}{\partial z}$ *have compact support.*

Remark. The forms df and dg cannot have compact support, otherwise, by Stokes theorem, we would have $\int_{\mathbb{H}^2} df \wedge dg = 0$.

Proof. Choose smooth functions h_1, h_2, and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:
(1) h_1, h_2 and k are ≥ 0;
(2) $h_i(y) = 0$ if $|y| \ge 1$;
(3) $h_1'(y)h_2(y) \ge 0$ and $h_1(y)h_2'(y) \le 0$ for all $y \in \mathbb{R}$;
(4) the function $(h_1'(y)h_2(y) - h_1(y)h_2'(y))$ has non empty support;
(5) $k'(z) \ge 0$ for all $z \in \mathbb{R}$;
(6) $k(z) = 1$ if $z \ge 1$ and $k(z) = 0$ if $z \le 0$.

We set $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$. Properties (1) and (2) of the lemma are then clear. We prove (3) (i.e., that $df \in L^r$ for any $1 < r \le \infty$).

Indeed,

$$df = h_1(y)k'(z)\, dz + k(z)h_1'(y)\, dy .$$

The first term $h_1(y)k'(z)\, dz$ has compact support, and the second term $k(z)h_1'(y)\, dy$ has its support in the infinite rectangle $Q = \{|y| \le 1\ z \ge 0\}$.

Choose $D < \infty$ such that $|k(z)h_1'(y)| \le D$ on Ω. We have

$$\left|k(z)h_1'(y)\, dy\right| \le D\, |dy| = D\, e^{-z} ,$$

thus, since the element of area of $\mathbb{H}^2$ is $dA = e^z\, dy\, dz$, we have

$$\int_{\mathbb{H}^2} \left|k(z)h_1'(y)\, dy\right|^r dA \le D^r \int_Q e^{-rz}\, e^z\, dy\, dz \le 2C D^r \int_0^\infty e^{(1-r)z}\, dz < \infty ,$$

from which one gets $df \in L^r$.

Now observe that

$$df \wedge dg = \Big(\big(k(z)k'(z)\big)\big(h_1'(y)h_2(y) - h_1(y)h_2'(y)\big)\ dy \wedge dz\,,$$

hence the properties (4) and (5) follow from the construction of h_1, h_2, and k.

Property (6) is only a normalization. It can be achieved by multiplying f (or g) by a suitable constant.

Properties (7) and (8) are easy to check. □

Proof of Theorem 10.1. Define the 1-forms $\alpha = df$ and $\gamma = dg$ on $\mathbb{H}^2$ (where f and g are as in Lemma 10.2). It is clear that $d\alpha = d\gamma = 0$. We also know that $\alpha \in L^p$ for any $1 < p < \infty$ and that γ is smooth and $\gamma \in L^{p'} \cap L^{q'}$ for all $1 < p', q' < \infty$.

Since $\int_{\mathbb{H}^2} \alpha \wedge \gamma \neq 0$, we see by Proposition 8.4 that $\alpha \notin \overline{B}^1_{q,p}(\mathbb{H}^2)$.

Now using the isometry group of $\mathbb{H}^2$, we produce an infinite family of linearly independent classes in $\overline{H}^1_{q,p}(\mathbb{H}^2)$. □

11. The cohomology of the ball

Since the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ has finite volume, we have for all $1 \leq p, q \leq \infty$ $H^0_{q,p}(\mathbb{B}^n) = \overline{H}^0_{q,p}(\mathbb{B}^n) = \mathbb{R}$.

In higher degree, the vanishing of the De Rham cohomology of $\mathbb{B}^n$ is traditionally called the Poincaré Lemma; it is proved by explicitly constructing a primitive to any closed form. To prove the vanishing of the $L_{q,p}$-cohomology of the ball, we need to control the L^q-norm of the primitive of a closed L^p-norm. For the case $p = q$, this was done by Gol'dshtein, Kuz'minov, and Shvedov in [8, Lemma 3.2] and for more general q by Iwaniec and Lutoborski in [12]. They proved the following.

Theorem 11.1. *For any bounded convex domain $U \subset \mathbb{R}^n$ and any $k = 1, 2, \dots, n$, there exists an operator*

$$T = T_U : L^1_{\mathrm{loc}}\big(U, \Lambda^k\big) \to L^1_{\mathrm{loc}}\big(U, \Lambda^{k-1}\big)$$

with the following properties:

(a) $T(d\theta) + dT\theta = \theta$ *(in the sense of currents);*

(b) $|T\theta(x)| \leq C \displaystyle\int_U \frac{|\theta(y)|}{|y-x|^{n-1}}\, dy.$

Corollary 11.2. *The operator T maps $L^p(U, \Lambda^k)$ continuously to $L^q(U, \Lambda^{k-1})$ in the following cases:*

Either

(i) $1 \leq p, q \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$,

or

(ii) $1 < p, q \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$.

Remark. Note that condition (i) is equivalent to $p \geq n$ or $p < n$ and $q < \frac{np}{n-p}$ and condition (ii) is relevant to conformal cohomology $\frac{1}{p_k} - \frac{1}{p_{k-1}} = \frac{1}{n}$.

Proof. Assume first that $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ and recall the Young inequality for convolution (see [5, Proposition 8.9]), which says that if $1 \leq r, s, t \leq \infty$ satisfy $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{t}$, then $\|f * g\|_{L^t} \leq \|f\|_{L^r}\|g\|_{L^s}$. Applying this inequality to $f = |\theta|$ and $g = |x|^{1-n}$ with $r = p$, $t = q$ and $s = \frac{pq}{p+pq-q}$, and observing that

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n} \quad \Leftrightarrow \quad s(1-n) > -n \quad \Leftrightarrow \quad \|g\|_{L^s(U)} < \infty\,,$$

we conclude from previous proposition that $T : L^p(U, \Lambda^k) \to L^q(U, \Lambda^{k-1})$ is bounded with norm at most $\||x|^{1-n}\|_{L^s(U)}$.

If $p > 1$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, then the conclusion also holds by the Hardy-Litlewood-Sobolev inequality (see [16, p. 119]). □

Corollary 11.3. *The operator $T : \Omega^k_{p,r}(U) \to \Omega^{k-1}_{q,p}(U)$ is bounded and for any $\omega \in \Omega^k_{p,r}(U)$ we have $T\,d\omega + dT\omega = \omega$ provided either*
(i) $1 \leq p, q, r \leq \infty$ *such that* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ *and* $\frac{1}{r} - \frac{1}{p} < \frac{1}{n}$,

or

(ii) $1 < p, q, r \leq \infty$ *such that* $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ *and* $\frac{1}{r} - \frac{1}{p} \leq \frac{1}{n}$.

Proof. The proof is immediate from the previous theorem and corollary. □

The Corollary 11.2 implies the following Poincaré Lemma.

Proposition 11.4. *Suppose that p, q satisfy either*

(i) $1 \leq p, q \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$,

or

(ii) $1 < p, q \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$.

Then $H^k_{q,p}(\mathbb{B}^n) = 0$ for any $k = 1, \ldots, n$.

Proof. Let ω be an arbitrary element in $Z^k_p(\mathbb{B}^n)$. By Corollary 11.2, we have $T\omega \in L^q(\mathbb{B}^n, \Lambda^{k+1})$, since $\omega = dT\omega + T\,d\omega = d(T\omega)$ we conclude that $[\omega] = 0 \in H^k_{q,p}(\mathbb{B}^n)$ and thus $H^k_{q,p}(\mathbb{B}^n) = 0$. □

If $p, q > 1$, we have a necessary and sufficient condition.

Theorem 11.5. *If $1 < p, q \leq \infty$ and $k = 1, \ldots, n$, then $H^k_{q,p}(\mathbb{B}^n) = 0$ if and only if $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$.*

Proof. We know from the previous proposition that the condition is sufficient.

To prove that $H^k_{q,p}(\mathbb{B}^n) \neq 0$ if $p < n$ and $q > \frac{np}{n-p}$, we will use Proposition 8.2. Let us fix a number μ in the interval $k - \frac{n}{p} < \mu < k - 1 - \frac{n}{q}$ (which is possible since $\frac{1}{p} > \frac{1}{q} + \frac{1}{n}$); and choose two forms $\theta \in C^\infty(\mathbb{S}^{n-1}, \Lambda^{k-1})$ and $\varphi \in C^\infty(\mathbb{S}^{n-1}, \Lambda^{n-k-1})$ such that

$$\int_{\mathbb{S}^{n-1}} \varphi \wedge d\theta = 1 .$$

For any $0 < t < 1/4$, we choose a smooth function $h_t : \mathbb{R} \to \mathbb{R}$ such that $h(t,r) = 0$ if $r < t$ or $r > 1-t$ and $h(t,r) = \frac{1}{|\log 2t|}$ if $r < 1-2t$ or $r > 2t$.

Let us then consider the forms

$$\begin{aligned} \alpha &:= d\left(r^\mu \theta\right) \\ \gamma_t &:= h_t(r) r^{-(\mu+1)}\, dr \wedge \varphi . \end{aligned}$$

Step 1. The form α belongs to $L^p(\mathbb{B}^n, \Lambda^k)$.

We will use the same notation θ and φ for a pullback of corresponding forms from $\mathbb{S}^n$ to $\mathbb{B}^n \setminus \{0\}$ induced by the radial projection in polar coordinates.

We have

$$\alpha = r^\mu \left(d\theta + \mu \frac{1}{r}\, dr \wedge \theta \right) .$$

Because $|\theta| \lesssim r^{-(k-1)}$ and $|d\theta| \lesssim r^{-k}$ we have $|\alpha| \lesssim r^{\mu-k}$. Therefore

$$\int_{\mathbb{B}^n} |\alpha|^p\, dx \lesssim \int_0^1 \left(r^{\mu-k}\right)^p r^{n-1}\, dr < \infty$$

because $p(\mu - k) + n - 1 > p(k - \frac{n}{p} - k) + n - 1 > -1$.

Step 2. The quantity $\left|\int_{\mathbb{B}^n} \alpha \wedge \gamma_t\right|$ is bounded below.

We have $\alpha \wedge \gamma_t = h_t(r) r^{-1}\, dr \wedge \varphi \wedge d\theta$; since $\int_{\mathbb{S}^{n-1}} \varphi \wedge d\theta = 1$, we have by Fubini Theorem

$$\left| \int_{\mathbb{B}^n} \alpha \wedge \gamma_t \right| = \int_0^1 h_t(r) r^{-1}\, dr \geq \frac{1}{|\log 2t|} \int_{2t}^{1-2t} r^{-1}\, dr \to 1$$

as $t \to 0$. This implies that $\left|\int_{\mathbb{B}^n} \alpha \wedge \gamma_t\right|$ is bounded below for small values of t.

Step 3. We have $\|d\gamma_t\|_{L^{q'}(\mathbb{B}^n)} \to 0$ as $t \to 0$:

We have $d\gamma_t := h_t(r) r^{-(\mu+1)}\, dr \wedge \varphi$ with $0 \leq h_t \leq \frac{1}{|\log 2t|}$. Since $|dr \wedge \varphi| \lesssim r^{-n+k}$, we have

$$|d\gamma_t| \lesssim \frac{r^{-\mu-1+k-n}}{|\log 2t|}$$

and by Fubini Theorem

$$\begin{aligned} \int_{\mathbb{B}^n} |d\gamma_t|^{q'}\, dx &= \int_{\mathbb{B}^n} \left| h_t(r) r^{-(\mu+1)}\, dr \wedge \varphi \right|^{q'} dx \\ &\lesssim \left(\frac{1}{|\log 2t|} \right)^{q'} \int_0^1 \left(r^{-\mu-1+k-n} \right)^{q'} r^{n-1}\, dr . \end{aligned}$$

Because

$$q'(-\mu-1+k-n)+n = q'\Big(-\mu-1+k-n\Big(1-\frac{1}{q'}\Big)\Big) = q'\Big(-\mu-1+k-\frac{n}{q}\Big) > 0$$

we have

$$\int_0^1 \big(r^{-\mu-1+k-n}\big)^{q'} r^{n-1}\,dr < \infty\,.$$

Therefore

$$\lim_{t\to 0}\int_{\mathbb{B}^n} |d\gamma_t|^{q'}\,dx \lesssim \lim_{t\to 0}\left(\frac{1}{|\log 2t|}\right)^{q'}\int_0^1 \big(r^{-\mu-1+k-n}\big)^{q'} r^{n-1}\,dr = 0\,.$$

Since γ_t are smooth forms with compact support, Proposition 8.2 implies that $[\alpha] \neq 0$ in $H^k_{q,p}(\mathbb{B}^n)$. □

Corollary 11.6. *The conformal cohomology of the hyperbolic space $\mathbb{H}^n$ vanishes for any degree $k > 1$, i.e.,*

$$H^k_{\frac{n}{k-1},\frac{n}{k}}(\mathbb{H}^n) = 0\,.$$

Proof. Since the hyperbolic space $\mathbb{H}^n$ is conformally equivalent to the ball $\mathbb{B}^n \subset \mathbb{R}^n$, this result follows at once from the conformal invariance of conformal cohomology and the previous theorem. □

Remark 11.7. Because $H^1_{q,p}(\mathbb{H}^2) \neq 0$ for any q, p, the corollary does not hold for $k = 1$.

12. Regularization of forms and cohomology classes

In this section we investigate two different but related problems. The first one is a density result for smooth forms in $\Omega^*_{q,p}(M)$ and the second one is a result about representation of the cohomology $H^*_{q,p}(M)$ by smooth forms. We will use the De Rham regularization method [3] and its version for L_p-cohomology [9] in combination with the results of Section 11.

12.1. Regularization operators for differential forms

The standard way of smoothing a function in $\mathbb{R}^n$ is by convolution with a smooth mollifier. This procedure extends to differential forms and more generally to any tensor. In his book, De Rham proposes a clever way of localizing this construction and grafting it on manifolds.

Following De Rham, we associate to any vector $v \in \mathbb{R}^n$ the map $s_v : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$s_v(x) = \begin{cases} h^{-1}(h(x)+v) & \text{if} \quad \|x\| < 1\,, \\ x & \text{if} \quad \|x\| \geq 1\,, \end{cases}$$

where $h : \mathbb{B}^n \to \mathbb{R}^n$ is a radial diffeomorphism such that

$$h(x) = \begin{cases} x & \text{if} \quad \|x\| < 1/3\,, \\ \frac{1}{\|x\|}\exp\Big(\frac{1}{(1-\|x\|^2)}\Big)\cdot x & \text{if} \quad \|x\| \geq 2/3\,. \end{cases}$$

Lemma 12.1. *The map $v \to s_v$ defines an action of the group $\mathbb{R}^n$ on the space $\mathbb{R}^n$ satisfying the following properties:*

(a) *For every $v \in \mathbb{R}^n$, the map $s_v : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism;*
(b) *the mapping $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth;*
(c) *s_v is the identity outside of $\mathbb{B}^n$;*
(d) *for every $x \in \mathbb{B}^n$ the mapping $v \mapsto \alpha_x(v) := s_v(x)$ is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{B}^n$.*

Proof. For the first two assertions, see [3]. The assertions (c) and (d) are obvious. □

Let us fix an arbitrary bounded convex domain U such that $\overline{\mathbb{B}}^n \subset U \subset \mathbb{R}^n$. We now define the regularization operator $R_\epsilon : L^1_{\text{loc}}(U, \Lambda^k) \to L^1_{\text{loc}}(U, \Lambda^k)$ by

$$R_\varepsilon \omega := \int_{\mathbb{R}^n} s_v^*(\omega) \rho_\varepsilon(v)\, dv$$

where $\rho_\varepsilon(v) = \rho(v/\varepsilon)$ is a standard mollifier.

Proposition 12.2. *The regularization operator defined above satisfies the following properties:*

(1) *For any $\omega \in L^1_{\text{loc}}(U, \Lambda^k)$, the form $R_\epsilon \omega$ is smooth in $\mathbb{B}^n$ and $R_\epsilon \omega = \omega$ in $U \setminus \mathbb{B}^n$;*
(2) *for any $\omega \in \Omega^k_{q,p}(U)$, we have $d R_\varepsilon \omega = R_\varepsilon\, d\omega$.*
(3) *For any $1 \le p, q < \infty$ and any $\varepsilon > 0$, the operator*

$$R_\varepsilon : \Omega^k_{q,p}(U) \to \Omega^k_{q,p}(U)$$

is bounded and its norm satisfies $\lim_{\varepsilon \to 0} \|R_\varepsilon\|_{q,p} = 1$;
(4) *for any $1 \le p, q < \infty$ and any $\omega \in \Omega^k_{q,p}(U)$, we have*

$$\lim_{\varepsilon \to 0} \left\| R_\varepsilon^* \omega - \omega \right\|_p = 0\ .$$

Proof. The first two properties are proved in [3]. Property (3) follows from (2) and [9, Lemma 2] and (4) is a standard property of the regularization. □

12.2. Homotopy operator

Given a bounded convex domain $U \subset \mathbb{R}^n$ containing the closed unit ball, we introduce the homotopy

$$A_\epsilon := (I - R_\varepsilon) \circ T_U : \ L^1_{\text{loc}}\big(U, \Lambda^k\big) \to L^1_{\text{loc}}\big(U, \Lambda^{k-1}\big)\ ,$$

where T_U is the operator defined in Theorem 11.1.

Lemma 12.3. *The operator A_ε is a homotopy between the identity and the regularization operator R_ε, i.e., it satisfies*

$$(I - R_\varepsilon)\, \omega = d A_\varepsilon \omega + A_\varepsilon\, d\omega\ .$$

Proof. We know from Theorem 11.1 that $T\, d\omega + dT\omega = \omega$ for all $\omega \in L^1_{\text{loc}}(U, \Lambda^{k-1})$, hence we have

$$\begin{aligned} dA_\varepsilon\omega + A_\varepsilon\, d\omega &= d(I-R_\varepsilon)T\omega + (I-R_\varepsilon)T\, d\omega \\ &= dT\omega - dR_\varepsilon T\omega + T\, d\omega - R_\varepsilon T\, d\omega \\ &= (dT\omega + T\, d\omega) - R_\varepsilon(dT\omega + T\, d\omega) \\ &= (I-R_\varepsilon)(T\, d\omega + dT\omega) \\ &= (I-R_\varepsilon)\omega\, . \end{aligned}$$
□

Proposition 12.4. *Let $U \subset \mathbb{R}^n$ be a bounded convex domain containing the closed unit ball. Then $A_\varepsilon : \Omega^k_{p,r}(U) \to \Omega^{k-1}_{q,p}(U)$ is a bounded operator for any $k = 1, 2, \dots, n$ in the following two cases:*

(i) $1 \le p, q, r \le \infty$ *such that* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$,

(ii) $1 < p, q \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$ *and* $\frac{1}{r} - \frac{1}{p} \le \frac{1}{n}$.

Furthermore, we have $(I - R_\varepsilon)\omega = dA_\varepsilon\omega + A_\varepsilon\, d\omega$ for any $\omega \in \Omega^k_{p,r}(U)$ and $A_\varepsilon\omega = 0$ outside the unit ball.

Proof. The first assertion follows from Proposition 12.2 and Corollary 11.3 and the second one is the previous lemma. The last assertion follows from the fact that $R_\varepsilon = I$ outside of the unit ball. □

12.3. Globalization

This regularization operators R_ε and A_ε can be globalized as follow: Given a Riemannian manifold (M, g), we can find a countable atlas $\{\varphi_i : V_i \subset M \to U_i\}_{i\in\mathbb{N}}$ such that $U_i \subset \mathbb{R}^n$ is a bounded convex domain satisfying $\overline{\mathbb{B}}^n \subset U_i \subset \mathbb{R}^n$ for all i and that $\{B_i\}$ is a covering of M, where $B_i := \varphi_i^{-1}(B) \subset V_i$. We also assume that $\{V_i\}$ (and hence $\{B_i\}$) is a locally finite covering of M (we can in fact assume that any collection of $n+2$ different charts V_i has an empty intersection, where $n = \dim M$).

For any $m \in \mathbb{N}$, we define two operators

$$R^{(m)}_\varepsilon, A^{(m)}_\varepsilon : L^1_{\text{loc}}\big(M, \Lambda^m\big) \to L^1_{\text{loc}}\big(M, \Lambda^m\big)$$

as follows:

$$R^{(m)}_\varepsilon := R_{1,\varepsilon} \circ R_{2,\varepsilon} \circ \cdots \circ R_{m,\varepsilon}\ ,$$

and

$$A^{(m)}_\varepsilon := R_{1,\varepsilon} \circ R_{2,\varepsilon} \circ \cdots \circ R_{m-1,\varepsilon} \circ A_{m,\varepsilon}\ ,$$

where

$$R_{i,\varepsilon}(\theta) := \big(\varphi_i^{-1}\big)^* \circ R_\varepsilon \circ \varphi_i^*(\theta)\ ;$$

and

$$A_{i,\varepsilon}(\theta) := \big(\varphi_i^{-1}\big)^* \circ (R_{i,\varepsilon} - I)\, T_{U_i} \circ \varphi_i^*(\theta)\ .$$

Here T_{U_i} is the operator defined on the domain U_i in Theorem 11.1.

Observe that the operator $R_{i,\varepsilon}$ is a priori only defined on V_i, but it acts as the identity on $V_i \setminus \overline{B}_i$ and can thus be extended on the whole of M by declaring that $R_{i,\varepsilon} = \text{id}$ on $M \setminus \overline{B}_i$. Likewise, the operator $A_{i,\varepsilon}$ is a priori only defined on V_i, but it is zero on $V_i \setminus \overline{B}_i$ (because $R_\varepsilon = I$ outside of the unit ball). Hence, $A_{i,\varepsilon}$ can be extended on the whole of M by declaring $A_{i,\varepsilon} = 0$ on $M \setminus \overline{B}_i$.

We now define the global regularization operator and the global homotopy operator as follows:

$$R_\varepsilon^M := \lim_{m\to\infty} R_\varepsilon^{(m)}, \qquad A_\varepsilon^M := \sum_{m=1}^{\infty} A_\varepsilon^{(m)} . \tag{12.1}$$

By construction, the expressions $R_\varepsilon^M := \prod_i R_{i,\varepsilon}$ and $A_\varepsilon^M := \sum_l A_\varepsilon^{(k)}$ are really finite operations in any compact set and the operators R_ε^M, A_ε^M are thus well-defined on $L^1_{\mathrm{loc}}(M, \Lambda^k)$.

Theorem 12.5. *For every Riemannian manifold M there exists a family of regularization operators R_ε^M and homotopy operators A_ε^M such that:*

(1) *For any $\omega \in L^1_{\mathrm{loc}}(M, \Lambda^k)$, the form $R_\epsilon^M \omega$ is smooth in M;*

(2) *for any $\omega \in \Omega^k_{q,p}(M)$, we have $dR_\varepsilon^M \omega = R_\varepsilon^M \, d\omega$;*

(3) *for any $1 \le p, q < \infty$ and any $\varepsilon > 0$, the operator $R_\varepsilon^M : \Omega^k_{q,p}(M) \to \Omega^k_{q,p}(M)$ is bounded and its norm satisfies $\lim_{\varepsilon\to 0} \left\| R_\varepsilon^M \right\|_{q,p} = 1$;*

(4) *for any $1 \le p, q < \infty$ and any $\omega \in \Omega^k_{q,p}(M)$ we have*

$$\lim_{\varepsilon\to 0} \left\| R_\varepsilon^M \omega - \omega \right\|_p = 0 .$$

(5) *The operator $A_\varepsilon : \Omega^k_{pr}(M) \to \Omega^{k-1}_{q,p}(M)$ is bounded for any $k = 1, \dots, n$ in the following cases:*

   (i) $1 \le p, q, r \le \infty$ *such that* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ *and* $\frac{1}{r} - \frac{1}{p} < \frac{1}{n}$,

   (ii) $1 < p, q, r \le \infty$ *such that* $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$ *and* $\frac{1}{r} - \frac{1}{p} \le \frac{1}{n}$.

(6) *We have the homotopy formula*

$$\omega - R_\varepsilon^M \omega = dA_\varepsilon^M \omega + A_\varepsilon^M \, d\omega .$$

Proof. The first four assertions follow immediately from Proposition 12.2.

The fifth assertion follows from Proposition 12.2 and Corollary 11.3.

To prove the last assertion, observe that by Lemma 12.3, we have $\omega - R_{m,\varepsilon}\omega = dA_{m,\varepsilon}\omega + A_{m,\varepsilon}\, d\omega$. Multiplying this expression by $R_\varepsilon^{(m-1)}$, we obtain

$$R_\varepsilon^{(m-1)}\omega - R_\varepsilon^{(k)}\omega = dA_\varepsilon^{(k)}\omega + A_\varepsilon^{(m)}\, d\omega ,$$

summing this identities on $m = 1, 2, \dots,$ we obtain the assertion (6). □

Corollary 12.6. *For any $q, p \in [1, \infty)$, the space*

$$C^\infty\Omega^k_{q,p}(M) := C^\infty(M) \cap \Omega^k_{q,p}(M)$$

of smooth k-forms θ in L^p such that $d\theta \in L^q$ is dense in $\Omega^k_{q,p}(M)$.

Proof. This result follows immediately from the first three conditions in Theorem 12.5. □

12.4. L_π-cohomology and smooth forms

The previous theorem implies that under suitable assumptions on p, q, the L_π-cohomology of a Riemannian manifold can be represented by smooth forms.

To be more precise, for any sequence π, we denote by

$$C^\infty\Omega^k_\pi(M) := C^\infty(M) \cap \Omega^k_\pi(M)$$

the subcomplex of smooth forms in $\Omega^k_\pi(M)$ and by

$$C^\infty H^*_\pi(M) = H^*\big(C^\infty\Omega^k_\pi(M)\big)$$

its cohomology.

Theorem 12.7. *Let (M, g) be a n-dimensional Riemannian manifold and $\pi = \{p_0, p_1, \cdots, p_n\} \subset (1, \infty)$ a finite sequence of numbers such that $\frac{1}{p_k} - \frac{1}{p_{n-k}} \le \frac{1}{n}$ for $k = 1, 2, \ldots, n$. Then*

$$C^\infty H^*_\pi(M) = H^*_\pi(M) .$$

Proof. This result follows immediately from Proposition 4.8 and Theorem 12.5. □

It is perhaps useful to reformulate this theorem without the language of complexes.

Theorem 12.8. *Let (M, g) be a n-dimensional Riemannian manifold and suppose that $p, q \in (1, \infty)$ satisfy $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$. Then the cohomology $H^*_{q,p}(M)$ can be represented by smooth forms.*

More precisely, any closed form in $Z^k_p(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M) \cap Z^k_p(M)$ are cohomologous modulo $d\Omega^{k-1}_{q,p}(M)$, then they are cohomologous modulo $dC^\infty\Omega^{k-1}_{q,p}(M)$.

Corollary 12.9. *Let (M, g) be a n-dimensional Riemannian manifold and suppose that $p, q \in (1, \infty)$ satisfy $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$. Then any reduced cohomology class can be represented by a smooth form.*

Proof. This is clear from the previous theorem, since $\overline{H}^k_{q,p}(M)$ is a quotient of $H^k_{q,p}(M)$. □

12.5. The case of compact manifolds

From previous results, we now immediately have the following.

Theorem 12.10. *Let (M, g) be a compact n-dimensional Riemannian manifold and $\pi = \{p_0, p_1, \cdots, p_n\} \subset (1, \infty)$ a finite sequence of numbers such that $\frac{1}{p_k} - \frac{1}{p_{n-k}} \le \frac{1}{n}$ for $k = 1, 2, \ldots, n$. Then*

$$H^*_\pi(M) = H^*_{\mathrm{DeRham}}(M) .$$

In particular, $H^*_\pi(M)$ is finite-dimensional and thus $T^*_\pi(M) = 0$.

Proof. Recall that the De Rham cohomology $H^*_{\mathrm{DeRham}}(M)$ of M is the cohomology of the complex $(C^\infty(M, \Lambda^*), d)$. Any smooth form on a compact Riemannian manifold clearly belongs to L^p for any $p \in [0, \infty]$, hence $(C^\infty(M, \Lambda^*), d) = C^\infty\Omega^k_\pi(M)$ and by Theorem 12.7, we have

$$H^*_\pi(M) = C^\infty H^*_\pi(M) = H^*_{\mathrm{DeRham}}(M) .$$

It is well known that the De Rham cohomology of a compact manifold is finite-dimensional. Since $\dim T_\pi^*(M) \le \dim H_\pi^*(M) < \infty$, it follows from Lemma 4.4 that $T_\pi^*(M) = 0$. □

12.6. Proof of Theorems 1.1 and 1.2

Let us define the sequence $\pi = \{p_0, p_1, \cdots, p_n\}$ by $p_j = q$ if $j = 1, 2, \dots k-1$ and $p_j = p$ if $j = k, \dots, n$.

By hypothesis, we have $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$, hence the sequence π satisfies $\frac{1}{p_j} - \frac{1}{p_{j-1}} \le \frac{1}{n}$ for all j. Hence, we know by Theorem 12.10 that $H^k_{q,p}(M) = H^k_{\mathrm{DeRham}}(M)$ and $T^k_{q,p}(M) = 0$.

Thus, Theorem 1.1 follows from Theorem 6.2 and Theorem 1.2 follows from Theorem 6.1.

13. Relation with a nonlinear PDE

We show in this section that the vanishing of torsion gives sufficient condition to solving the nonlinear equation

$$\delta\big(\, \|d\theta\|^{p-2}\, d\theta\big) = \alpha \tag{13.1}$$

where δ is the operator defined for $\omega \in L^1_{\mathrm{loc}}(M, \Lambda^k)$ as

$$\delta\, \omega = (-1)^{nk+n+1} * d * \omega\, .$$

Recall that for any k-form ω, we have[2]

$$*\delta\omega = (-1)^k\, d * \omega\, . \tag{13.2}$$

This operator is the formal adjoint to the exterior differential d in the sense that

$$\int_M \langle \omega, d\varphi\rangle\, d\mathrm{vol} = \int_M \langle \delta\omega, \varphi\rangle\, d\mathrm{vol} \tag{13.3}$$

for any $\varphi \in C_c^\infty(M, \Lambda^{k-1})$.

Indeed, by definition of the Hodge $*$ operator, we have

$$\langle d\varphi, \omega\rangle\, d\mathrm{vol} = (d\varphi \wedge *\omega)$$

and from the definition of the weak exterior differential, it follows that

$$\int_M \langle d\varphi, \omega\rangle\, d\mathrm{vol} = \int_M d\varphi \wedge *\omega = (-1)^k \int_M \varphi \wedge\, d * \omega\, .$$

Thus, from (13.2):

$$\begin{aligned}\int_M \langle d\varphi, \omega\rangle\, d\mathrm{vol} &= (-1)^k \int_M \varphi \wedge\, d * \omega \\ &= \int_M \varphi \wedge *\delta\omega \\ &= \int_M \langle \varphi, \delta\omega\rangle\, d\mathrm{vol}\, .\end{aligned}$$

[2] Here is the proof: Since ω is a k form, $d*\omega$ is a form of degree $m = n-k+1$ and $**d*\omega = (-1)^{m(n-m)}\, d*\omega = (-1)^{nk+n+1+k}\, d * \omega$, therefore $(-1)^k\, d * \omega = (-1)^{nk+n+1} **d*\omega = *\delta\omega$.

Applying (13.3) to $\omega = |d\theta|^{p-2}\, d\theta$, we obtain the following.

Lemma 13.1. *$\theta \in L^1_{\mathrm{loc}}(M, \Lambda^k)$ is a solution to (1.5) if and only if*

$$\int_M \left\langle d\varphi, \|d\theta\|^{p-2}\ d\theta \right\rangle d\mathrm{vol} = \int_M \langle \varphi, \alpha \rangle\, d\mathrm{vol} \tag{13.4}$$

for any $\varphi \in C_c^\infty(M, \Lambda^k)$.

Equation (13.4) is just the weak form of (1.5).

Remark. In the scalar case, Equation (1.5) is just the p-Laplacian. The case of differential forms on the manifold $M = \mathbb{R}^n$ appears in Section 6.1 of [13] where it is investigated by the method of Hodge dual systems, see also [12, Section 8].

Theorem 13.2. *Assume $T^k_{q,p}(M) = 0$, $(1 < q, p < \infty)$ and $\alpha \in L^{q'}(M, \Lambda^k)$ where $q' = q/(q-1)$.*
(A) *If $\int_M \langle \alpha, \varphi \rangle\, d\mathrm{vol} = 0$ for any $\varphi \in Z^k_q(M)$, then* (13.4) *has a solution $\theta \in \Omega^k_{q,p}(M)$.*
(B) *Conversely, if* (13.4) *is solvable in $\Omega^k_{q,p}(M)$, then $\int_M \langle \alpha, \varphi \rangle\, d\mathrm{vol} = 0$ for any $\varphi \in C_c^\infty(M, \Lambda^k)$ such that $d\varphi = 0$.*

Proof. Assertion (B) follows from the previous lemma, because for any $\varphi \in C_c^\infty(M, \Lambda^k) \cap \ker d$, we have

$$\int_M \langle \alpha, \varphi \rangle\, d\mathrm{vol} = \int_M \left\langle \|d\theta\|^{p-2}\ d\theta, d\varphi \right\rangle d\mathrm{vol} = 0\ .$$

Let us prove assertion (A). The variational functional corresponding to (13.4) reads

$$I(\theta) = \frac{1}{p} \int_M \|d\theta\|^p\ d\mathrm{vol} - \int_M \langle \alpha, \theta \rangle\, d\mathrm{vol}\ .$$

We first show that the functional $I(\theta) : \Omega^k_{q,p}(M) \to \mathbb{R}$ is bounded from below.

For any $\theta \in \Omega^k_{q,p}(M)$ there exists a unique element $z_q(\theta) \in Z^k_q(M)$ such that $\|\theta - z_q(\theta)\|_q \le \inf_{z \in Z^k_q(M)} \|\theta - z\|_q$; this follows from the uniform convexity of $\Omega^k_{q,p}(M)$. Since $T^k_{q,p}(M) = 0$, the Proposition 1.2 implies that

$$\left\|\theta - z_q(\theta)\right\|_q \le C\, \|d\theta\|_p \tag{13.5}$$

for some positive constant C. Using this inequality and Hölder's inequality, we obtain

$$I(\theta) \ge \frac{1}{p}\, \|d\theta\|_p^p - \|\alpha\|_{q'} \left\|\theta - z_q(\theta)\right\|_q \ge \frac{1}{p}\, \|d\theta\|_p^p - C\, \|\alpha\|_{q'}\, \|d\theta\|_p\ .$$

Since the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{p}|x|^p - ax$ is bounded below for $x \ge 0$, the previous inequality implies that

$$\inf_{\theta \in \Omega^k_{q,p}(M)} I(\theta) > -\infty\ .$$

We now prove the existence of a minimizer of I on $\Omega^k_{q,p}(M)$: Let $\{\theta_i\} \subset \Omega^k_{q,p}(M)$ be a sequence such that $I(\theta_i) \to \inf I(\theta)$. Because the function $f(x) = \frac{1}{p}|x|^p - ax$ is proper, the inequality

$$I(\theta_i) \geq \frac{1}{p} \|d\theta_i\|_p^p - C \|\alpha\|_{q'} \|d\theta_i\|_p$$

implies that $\{\|d\theta_i\|_p\} \subset \mathbb{R}$ is bounded and, by (13.5), $\{\|\theta_i - z_q(\theta_i)\|_q\}$ is also bounded. Hence, the sequence $\{\widetilde{\theta}_i := \theta_i - z_q(\theta_i)\}$ is bounded in $\Omega^k_{q,p}(M)$.

Since $\Omega^k_{q,p}(M)$ is reflexive there exists a subsequence (still noted $\{\widetilde{\theta}_i\}$) which converges weakly to some $\theta_0 \in \Omega^k_{q,p}(M)$. By the weak continuity of the functional $\int_M \langle \alpha, \theta \rangle \, d\mathrm{vol}$ in $\Omega^k_{q,p}(M)$ we have

$$\lim_{i\to\infty} \int_M \left\langle \alpha, \widetilde{\theta}_i \right\rangle d\mathrm{vol} = \int_M \langle \alpha, \theta_0 \rangle \, d\mathrm{vol} \, . \tag{13.6}$$

The lower semicontinuity of the norm under the weak convergence implies that

$$\|d\theta_0\|_p \leq \liminf_{i\to\infty} \left\| d\widetilde{\theta}_i \right\|_p \, .$$

Combining the last inequality with (13.6) we obtain

$$I(\theta_0) \leq \liminf_{i\to\infty} I(\theta_i)$$

and by the choice of θ_i we finally have $I(\theta_0) = \inf I(\theta)$.

It is now clear that θ_0 is a solution of (13.4), hence a weak solution of (1.5). $\square$

Definition. The Riemannian manifold (M, g) is *s-parabolic* if for any $\varepsilon > 0$, there exists a smooth function f_ε with compact support, such that $f_\varepsilon = 1$ on the ball $B(x_0, 1/\varepsilon)$ and $\|df_\varepsilon\|_{L^s(M)} \leq \varepsilon$. where $x_0 \in M$ is a fixed base point.

Some basic facts about this notion can be found in [17].

Corollary 13.3. *Assume as above that $T^k_{q,p}(M) = 0$ and $\alpha \in L^{q'}(M, \Lambda^k)$ where $q' = q/(q-1)$, $(1 < q, p < \infty)$.*

Assume furthermore that M is s-parabolic for $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$.

Then Equation (13.4) is solvable in $\Omega^k_{q,p}(M)$, if and only if $\int_M \langle \alpha, \varphi \rangle \, d\mathrm{vol} = 0$ for any $\varphi \in Z^k_q(M)$.

Proof. The condition is sufficient by the previous theorem. Now let $\varphi \in Z^k_q(M)$ be arbitrary and let R^M_ε be the smoothing operator and f_ε be as in the previous definition. Then

$$\varphi_\varepsilon := f_\varepsilon R^M_\varepsilon(\varphi) \in C^\infty_c\big(M, \Lambda^k\big) \, .$$

Let us observe that

$$\big\| |d\theta|^{p-2} \, d\theta \big\|_{L^{p'}(M)} = \|d\theta\|^{p/p'}_{L^p(M)}$$

where $p' = p/(p-1)$. Since $\frac{1}{s} = 1 - \frac{1}{p'} + \frac{1}{q}$, we have by Hölder's inequality:

$$\begin{aligned}
\int_M \langle \alpha, \varphi_\varepsilon \rangle \, d\mathrm{vol} &= \int_M \left\langle \|d\theta\|^{p-2} \, d\theta, d\varphi_\varepsilon \right\rangle d\mathrm{vol} \\
&= \int_M \left\langle \|d\theta\|^{p-2} \, d\theta, df_\varepsilon \wedge R_\varepsilon^M(\varphi) \right\rangle d\mathrm{vol} \\
&\leq \left\| |d\theta|^{p-2} \, d\theta \right\|_{L^{p'}(M)} \| df_\varepsilon \|_{L^s(M)} \left\| R_\varepsilon^M(\varphi) \right\|_{L^q(M)} \\
&\leq \left(\|d\theta\|_{L^p(M)}^{p'/p} \left\| R_\varepsilon^M(\varphi) \right\|_{L^q(M)} \right) \| df_\varepsilon \|_{L^s(M)} .
\end{aligned}$$

As $\varepsilon \to 0$, we have $\|df_\varepsilon\|_{L^s(M)} \to 0$ while $\left(\|d\theta\|_{L^p(M)}^{p'/p} \|R_\varepsilon^M(\varphi)\|_{L^q(M)}\right)$ remains bounded. On the other hand,

$$\lim_{\varepsilon \to 0} \int_M \langle \alpha, \varphi_\varepsilon \rangle \, d\mathrm{vol} = \int_M \langle \alpha, \varphi \rangle \, d\mathrm{vol}$$

and the result follows. □

14. Torsion in L_2-cohomology and the Hodge-Kodaira decomposition

In this section, we study some connection between the torsion in L_2-cohomology and the Laplacian Δ acting on differential forms on the complete Riemannian manifold (M, g).

Recall that $\Delta = d\delta + \delta d$ where δ is the formal adjoint operator to the exterior differential d. We look at Δ as an unbounded operator acting on the Hilbert space $L^2(M, \Lambda^k)$. In particular, all function spaces appearing in this section are subspaces of $L^2(M, \Lambda^k)$. We denote by $\mathcal{H}_2^k(M) = L^2(M, \Lambda^k) \cap \ker \Delta$ the space of L^2 harmonic forms.

We begin with the following result, which can be proved by standard arguments from functional analysis.

Theorem 14.1. *For any complete Riemannian manifold (M, g), the following conditions are equivalent:*

(a) $\mathrm{Im}\,\Delta$ *is a closed subspace in $L^2(M, \Lambda^k)$;*
(b) $\mathrm{Im}\,\Delta = \left(\mathcal{H}_2^k(M)\right)^\perp$*;*
(c) *there exists a bounded linear operator $G : L^2(M, \Lambda^k) \to L^2(M, \Lambda^k)$ such that for any $\alpha \in L^2(M, \Lambda^k)$ we have*

$$\Delta \circ G\alpha = G \circ \Delta\alpha = \alpha - H\alpha$$

where $H : L^2(M, \Lambda^k) \to \mathcal{H}_2^k(M)$ is the orthogonal projection onto the space of L^2 harmonic forms.

Remark. G is called the *Green operator.* It is not difficult to check that $d \circ G = G \circ d$ and $\delta \circ G = G \circ \delta$.

For the convenience of the reader, we briefly explain the proof of this theorem.

Proof.
(a)⇔(b): Because Δ is self-adjoint, we know by standard functional analysis (see, e.g., [2], p. 28) that $\overline{\mathrm{Im}\,\Delta} = \left(\mathcal{H}_2^k(M)\right)^\perp$,

(b)⇒(c): This follows from the Banach Open Mapping Theorem. More precisely, let us denote by

$$E := \left\{\omega \in L^2\big(M, \Lambda^k\big) \;\middle|\; \omega \perp \mathcal{H}_2^k(M) \text{ and } \Delta\omega \in L^2\big(M, \Lambda^k\big)\right\}$$

the domain of the Laplacian. This is a Hilbert space for the graph norm $\|\omega\|_E := \|\omega\|_{L^2} + \|\Delta\omega\|_{L^2}$ and the map $\Delta : E \to \operatorname{Im}\Delta = \big(\mathcal{H}_2^k(M)\big)^\perp$ is a continuous bijective operator.

From the Banach Open Mapping Theorem, we know that the map

$$G := \Delta^{-1} \circ (1 - H) : L^2\big(M, \Lambda^k\big) \to L^2\big(M, \Lambda^k\big)$$

given by the composition

$$L^2\big(M, \Lambda^k\big) \xrightarrow{1-H} \Big(\mathcal{H}_2^k(M)\Big)^\perp \xrightarrow{\Delta^{-1}} E \subset L^2\big(M, \Lambda^k\big)$$

is continuous. It is clear that G satisfies the required properties.

(c)⇒(b): Condition (c) obviously implies that $\operatorname{Im}\Delta \supset \big(\mathcal{H}_2^k(M)\big)^\perp$. The other inclusion $\operatorname{Im}\Delta \subset \big(\mathcal{H}_2^k(M)\big)^\perp$ always holds since Δ is self-adjoint. □

In the case of complete Riemannian manifolds, we have the following.

Theorem 14.2. *For any complete Riemannian manifold (M, g), we have*

$$\mathcal{H}_2^k(M) = \ker d \cap \ker \delta \cap L^2\big(M, \Lambda^k\big)\,,$$

and the orthogonal decomposition

$$L^2\big(M, \Lambda^k\big) = \overline{\operatorname{Im} d} \oplus \overline{\operatorname{Im} \delta} \oplus \mathcal{H}_2^k(M)\,.$$

The first part is due to Andreotti and Vesentini, the second part is the well known Hodge-Kodaira decomposition. A proof is given in [3, Theorem 24 and 26].

Using both previous theorems, we can now prove the following result.

Theorem 14.3. *For any complete Riemannian manifold (M, g), the following conditions are equivalent:*

(i) $\operatorname{Im}\Delta = \big(\mathcal{H}_2^k(M)\big)^\perp$;

(ii) *we have the orthogonal decomposition*

$$L^2\big(M, \Lambda^k\big) = \operatorname{Im} d \oplus \operatorname{Im} \delta \oplus \mathcal{H}_2^k(M)\,;$$

(iii) $\operatorname{Im} d$ *and* $\operatorname{Im}\delta$ *are closed in* $L^2(M, \Lambda^k)$;

(iv) $T_2^k(M) = 0$ *and* $T_2^{n-k}(M) = 0$.

We will also need the following lemma.

Lemma 14.4. *If $T_2^k(M) = 0$, then*

$$\operatorname{Im}(\delta d) = \operatorname{Im}(\delta)$$

as subsets of $L^2(M, \Lambda^k)$.

Proof. It is clear that $\operatorname{Im}(\delta d) \subset \operatorname{Im}(\delta)$. To prove the other inclusion, consider an arbitrary element $\alpha \in \operatorname{Im}\delta$. Because $\operatorname{Im}\delta \perp \ker d = Z_2^k(M)$, we know by Theorem 13.2 that we can find a form $\theta \in L^2(M, \Lambda^k)$ such that $\delta\, d\theta = \alpha$. In particular, $\alpha \in \operatorname{Im}\delta d$. □

Remark. Using the formula $\delta = \pm * d *$, we see that this lemma also says that $\operatorname{Im}(d\delta) = \operatorname{Im}(d)$, provided $T_{2,2}^{n-k}(M) = 0$.

Proof of Theorem 14.3.

(i)⇒(ii): Condition (i) is equivalent to (c) of Theorem 14.1. Hence, assuming (i), we know that any $\alpha \in L^2(M, \Lambda^k)$ can be written as

$$\alpha - H\alpha = \Delta \circ G\,\alpha = d(\delta G\alpha) + \delta(dG\alpha)$$

and the decomposition (ii) follows.

(ii)⇒(iii): Is clear from Theorem 14.2.

(iii)⇔(iv): Follows from the definition of torsion and the formula $\delta = \pm * d *$.

(iv)⇒(i): We know from the previous lemma and the orthogonality of $\operatorname{Im} d$ and $\operatorname{Im}\delta$ that

$$\operatorname{Im}\Delta = \operatorname{Im}(d\delta + \delta d) = \operatorname{Im}(d\delta) + \operatorname{Im}(\delta d) = \operatorname{Im}(d) + \operatorname{Im}(\delta)\ ,$$

provided $T_2^k(M) = T_2^{n-k}(M) = 0$. In particular, $\operatorname{Im}\Delta$ is closed, since $\operatorname{Im} d$ and $\operatorname{Im}\delta$ are closed, and we conclude by Theorem 14.1 that $\operatorname{Im}\Delta = \left(\mathcal{H}_2^k(M)\right)^{\perp}$. □

Corollary 14.5. *If (M, g) is complete, then the equation $\Delta\omega = \alpha \in L^2(M, \Lambda^k)$ is solvable in $L^2(M, \Lambda^k)$ for any $\alpha \perp \mathcal{H}_2^k(M)$, if and only if*

$$T_2^k(M) = 0 \quad \text{and} \quad T_2^{n-k}(M) = 0\ .$$

The proof is immediate.

In conclusion, we formulate the following version of Hodge Theorem and Poincaré duality for L^2-cohomology.

Corollary 14.6. *If (M, g) is a complete Riemannian manifold such that $T_2^k(M) = T_2^{n-k}(M) = 0$, then*

$$\overline{H}_2^k(M) = H_2^k(M) \cong \mathcal{H}_2^k(M) \cong \mathcal{H}_2^{n-k}(M) \cong H_2^{n-k}(M) = \overline{H}_2^{n-k}(M)\ .$$

Proof. The equality $\overline{H}_2^k(M) = H_2^k(M)$ is equivalent to $T_2^k(M) = 0$.

From Theorem 14.3, we know that if the torsion vanishes, then

$$\ker d = (\operatorname{Im}\delta)^{\perp} = \operatorname{Im} d \oplus \mathcal{H}_2^k(M)\ ,$$

i.e., $H_2^k(M) \cong \mathcal{H}_2^k(M)$ by definition of cohomology.

The isomorphism $\mathcal{H}_2^k(M) \cong \mathcal{H}_2^{n-k}(M)$ is given by the Hodge $*$ operator and the proof now ends as it begins. □

Appendix

A. A "classic" proof of Theorem 1.1 in the compact case

In this Appendix, we shortly give another proof of Theorem 1.1 for compact manifolds which is based on the Hodge-De Rham theory and the regularity theory for elliptic systems, together with some techniques from functional analysis. All these tools were available 40 years ago, however, we did not find a written proof in the literature.

We start with the fact that the space of harmonic currents on a compact Riemannian manifold (M, g) is finite-dimensional and that we can construct two linear operators acting on currents on M

$$G, H : \mathcal{D}'(M) \to \mathcal{D}'(M) ,$$

and such that

(i) $\ker \Delta = \operatorname{Im} H = \ker(I - H)$;
(ii) $\ker \Delta \cap \operatorname{Im}(I - H) = \{0\}$;
(iii) $\Delta \circ G = (I - H)$;
(iv) $\Delta \circ (I - H) = \Delta$;
(v) $d \circ G = G \circ d$.

This result is Theorem 23 in [3], the operator H is the projection onto the space of harmonic forms and G is the Green operator.

Using elliptic regularity, we can prove the following theorem.

Theorem A.1. *The Green operator defines a bounded linear operator*

$$G : W^{m,p}\big(M, \Lambda^k\big) \to W^{m+2,p}\big(M, \Lambda^k\big)$$

for any $m \in \mathbb{N}$. Here $W^{m,p}(M, \Lambda^k)$ is the Sobolev space of differential forms of degree k on M with coefficients in $W^{m,p}$.

Assuming this result for the time being, let us conclude the proof of Theorem 1.1. We first state the following corollary.

Corollary A.2. *For any compact Riemannian manifold (M, g), there exists a constant C_1 such that*

$$\|\theta - \zeta\|_{W^{1,p}(M)} \le C_1 \|d\theta\|_{L^p(M)} , \tag{A.1}$$

where $\zeta := H\theta + d\delta G\theta$.

Proof. From previous theorem, we see that $\delta \circ G : L^p(M, \Lambda^k) \to W^{1,p}(M, \Lambda^{k+1})$ is a bounded operator.

Since $\Delta G = (d\delta + \delta d)G = (I - H)$, we have $\theta - \zeta = \delta d G\,\theta = \delta G\, d\theta$ and thus

$$\|\theta - \zeta\|_{W^{1,p}(M)} = \|\delta G\, d\theta\|_{W^{1,p}(M)} \le C_1 \|d\theta\|_{L^p(M)} ,$$

where C_1 is the operator norm $C_1 := \|\delta G\|_{L^p \to W^{1,p}}$. □

Proof of Theorem 1.1

The classical Sobolev embedding theorem on compact manifolds, states in particular that there is a constant C_2 such that

$$\|\omega\|_{L^q(M)} \le C_2 \|\omega\|_{W^{1,p}(M)} , \tag{A.2}$$

provided that conditions (1.2), are satisfied.

Combining (A.1) and (A.2) and observing that, by the Sobolev embedding theorem and (1.2), we have $\zeta = H\theta + d\delta G\theta \in Z_q^k(M)$, we obtain (1.1) with $C = C_1 C_2$.

Proof of Theorem A.1

The proof is in several steps.

Step 1. The elliptic estimate for the Laplacian acting on forms on a compact manifold says that there exists a constant A_m such that for any form $\theta \in W^{m+2,p}(M, \Lambda^k)$ we have

$$\|\theta\|_{W^{m+2,p}(M)} \leq A_m \left(\|\Delta\theta\|_{W^{m,p}(M)} + \|\theta\|_{W^{m,p}(M)}\right) . \tag{A.3}$$

This result is deep. The case $p = 2$ is proved in proved in [18, Section 6.29], the scalar case for any $p \in (0, \infty)$ can be found in [7, Section 9.5] and the general case in [1, Chapter IV].

Step 2. A first consequence of this estimates is the hypoellipticity of the Laplacian, i.e., the fact if $\Delta\theta$ is a smooth form, then θ itself is smooth (the proof follows from a bootstrap argument based on (A.3) and the fact that $\cap_{m\geq 1} W^{m,p}(M) = C^{\infty}(M)$.) It follows in particular that the Green operator G maps smooth forms to smooth forms.

Step 3. Using (A.3), we show that for any sequence $\{\theta_i\} \subset W^{m+2,p}$, we have

$$\|\Delta\theta_i\|_{W^{m,p}(M)} \quad \text{bounded} \;\Rightarrow\; \|(I - H)\theta_i\|_{W^{m,p}(M)} \quad \text{bounded} . \tag{A.4}$$

Indeed, otherwise there exists a sequence such $\|\Delta\theta_i\|_{W^{m,p}(M)}$ is bounded and $\|(I - H)\theta_i\|_{W^{m,p}(M)} \to \infty$. Let us set

$$\varphi_i := \frac{(I - H)\theta_i}{\|(I - H)\theta_i\|_{W^{m,p}(M)}} \in W^{m+2,p}(M) ,$$

we then have $\|\varphi_i\|_{W^{m,p}(M)} = 1$ and

$$\lim_{i\to\infty} \|\Delta\varphi_i\|_{W^{m,p}(M)} = \frac{\|\Delta\theta_i\|_{W^{m,p}(M)}}{\|(I - H)\theta_i\|_{W^{m,p}(M)}} = 0 .$$

The elliptic estimate (A.3) gives us

$$\|\varphi_i\|_{W^{m+2,p}(M)} \leq A_m \left(\|\Delta\varphi_i\|_{W^{m,p}(M)} + \|\varphi_i\|_{W^{m,p}(M)}\right)$$

and thus $\{\varphi_i\}$ is bounded in $W^{m+2,p}(M)$.

Because $W^{m+2,p}(M)$ is reflexive, there exists a subsequence which converges weakly in $W^{m+2,p}(M)$. We still denote this subsequence by $\{\varphi_i\}$. Let $\varphi \in W^{m+2,p}(M)$ be the weak limit of this subsequence, we then have by the lower semi-continuity of the norm

$$\|\Delta\varphi\|_{W^{m,p}(M)} \leq \liminf_{i\to\infty} \|\Delta\varphi_i\|_{W^{m,p}(M)} = 0 ,$$

hence $\varphi \in \ker \Delta$. Since we also have $\varphi \in \mathrm{Im}\,(I - H)$ we must have $\varphi = 0$.

By the compactness of the embedding $W^{m+2,p}(M) \subset W^{m,p}(M)$, we may assume that this subsequence converges strongly in $W^{m,p}(M)$. In particular, we have

$$1 = \lim_{i\to\infty} \|\varphi_i\|_{W^{m,p}(M)} = \|\lim_{i\to\infty} \varphi_i\|_{W^{m,p}(M)} = 0 ,$$

This contradiction proves (A.4).

Step 4. We now show that:

$$\Delta\left(W^{m+2,p}(M)\right) \quad \text{is closed in} \qquad W^{m,p}(M)\ .$$

Indeed, for any $\omega \in W^{m,p}(M)$ in the closure of $\Delta\left(W^{m+2,p}\right)$, there exists a sequence $\{\theta_i\} \subset W^{m+2,p}$, such that $\Delta\theta_i \to \omega$. By Step 3, $\{(I-H)\theta_i\}$ is bounded in $W^{m,p}$, and by (A.3), this sequence is also bounded in $W^{m+2,p}$ (recall that $\Delta(I-H)\theta_i = \Delta\theta_i$).

By the compactness of the embedding $W^{m+2,p}(M) \subset W^{m,p}(M)$, there exists a subsequence such that $\{(I-H)\theta_i\}$ converges strongly in $W^{m,p}$, and by (A.3) again, $\{(I-H)\theta_i\}$ converges in $W^{m+2,p}$.

Let us denote by $\psi = \lim\limits_{i\to\infty}(1-H)\theta_i$, we then have $\omega = \Delta\psi \in \Delta(W^{m+2,p}(M))$.

Step 5. Let us denote by $\mathcal{E}^{m,p} = \ker H \cap W^{m,p}(M,\Lambda^k) = \mathrm{Im}\,(I-H)\cap W^{m,p}(M,\Lambda^k)$. Then $\Delta : \mathcal{E}^{m+2,p} \to \mathcal{E}^{m,p}$ is continuous, injective and has closed image by previous step. Furthermore, $\mathrm{Im}\,\Delta \subset \mathcal{E}^{m,p}$ is dense because any smooth form in $\mathcal{E}^{m,p}$ is the image under Δ of a smooth form in $\mathcal{E}^{m+2,p}$. To sum up, we have proved that

$$\Delta : \mathcal{E}^{m+2,p} \to \mathcal{E}^{m,p}$$

is a continuous linear bijection.

Step 6. By the Banach open mapping theorem, we finally see that

$$G = \Delta^{-1}\circ(1-H) : W^{m,p}\big(M,\Lambda^k\big) \to \mathcal{E}^{m+2,p} \subset W^{m+2,p}\big(M,\Lambda^k\big)$$

is a bounded operator.

Acknowledgments

Part of this research has been done in the autumn of 2001, when both authors stayed at IHES in Bures-Sur-Yvette. We are happy to thank the Institute for its warm hospitality. We also thank Pierre Pansu for his interest in our work and for the kindness and patience with which he explained us his viewpoint on the subject.

References

[1] Agmon, S., Douglis, A., and Nirenberg, L. *Comm. Pure Appl. Math.* **17**, 35–92, (1964).

[2] Brezis, H. *Analyse Fonctionnelle, Théorie et Applications,* Dunod, Paris, (1999).

[3] de Rham, G. *Grundlehren der Mathematischen Wissenschaften,* **266**, Springer-Verlag, Berlin, (1984).

[4] Edmunds, D. E. and Evans, W. D. *Spectral theory and differential operators,* Oxford Science Publications, (1987).

[5] Folland, G. *Real Analysis. Modern Techniques and their Applications,* 2nd ed., John Wiley & Sons, Inc., New York, (1999).

[6] Gaffney, M. A special Stoke's Theorem for complete Riemannian manifolds, *Ann. of Math.* **60**, 140–145, (1954).

[7] Gilbarg, D. and Trudinger, N. *Elliptic Partial Differential Equations of Second Order,* 2nd ed., Grundlehren der Mathematischen Wissenschaften, **224**, Springer-Verlag, Berlin, (1983).

[8] Gol'dshtein, V. M., Kuz'minov, V. I., and Shvedov, I. A. Differential forms on Lipschitz manifolds, *Siberian Math. J.* **23**(2), (1984).

[9] Gol'dshtein, V. M., Kuz'minov, V. I., and Shvedov, I. A. A property of De Rham regularization operators, *Siberian Math. J.* **25**(2), (1984).

[10] Gol'dshtein, V. M., Kuz'minov, V. I., and Shvedov, I. A. Dual spaces of spaces of differential forms, *Siberian Math. J.* **54**(1), (1986).

[11] Gromov, M. Asymptotic invariants of infinite groups, in *Geometric Group Theory,* Vol. 2, London Math. Soc. Lecture Notes, **182**, Cambridge University Press, (1992).

[12] Iwaniec, T. and Lutoborski, A. Integral estimates for null Lagrangians, *Arch. Rat. Mech. Anal.* **125**, 25–79, (1993).

[13] D'Onofrio, L. and Iwaniec, T. Notes on p-harmonic analysis, *Cont. Math.* to appear.

[14] Pansu, P. Cohomologie L^p des variétés à courbure négative, cas du degré 1, *Rend. Sem. Mat. Univ. Politec. Torino,* Fascicolo Speciale P.D.E and Geometry, 95–119, (1989).

[15] Pansu, P. Cohomologie L^p, espaces homogènes et pincement, preprint, Orsay, (1999).

[16] Stein, E. M. *Princeton Mathematical Series,* **30**, Princeton University Press, Princeton, NJ, (1970).

[17] Troyanov, M. Parabolicity of manifolds, *Siberian Adv. Math.* **9**, 125–150, (1999).

[18] Warner, F. Foundations of differentiable manifolds and Lie groups, *Graduate Texts in Mathematics,* **94**, Springer-Verlag, New York-Berlin, (1983).

[19] Yosida, K. *Functional Analysis,* reprint of the 6th ed., (1980), Classics in Mathematics, Springer-Verlag, Berlin, (1995).

[20] Zucker, S. L^p-cohomology: Banach spaces and homological methods on Riemannian manifolds, *Proc. Symp. Pure Math.* **54**(2), 637–655, (1993).

Received July 13, 2005

Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, Beer Sheva, Israel
e-mail: vladimir@bgumail.bgu.ac.il

Institut de Géométrie, algèbre et topologie (IGAT) Bâtiment BCH,
École Polytechnique Féderale de Lausanne, 1015 Lausanne—Switzerland
e-mail: marc.troyanov@epfl.ch