Ann. Inst. Fourier, Grenoble
52, 6 (2002), 1753–1784

LIOUVILLE TYPE THEOREMS FOR MAPPINGS WITH BOUNDED (CO)-DISTORTION

by M. TROYANOV & S. VODOP'YANOV *

---

1. Introduction.

A mapping $f : M \to N$ between oriented n-dimensional Riemannian manifolds is said to have *bounded s-distortion* (or *s-dilatation*) $(1 \leqslant s < \infty)$ if $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ and

$$|df_x|^s \leqslant K J_f(x) \tag{1}$$

a.e. $x \in M$.

The Sobolev class of mappings $W^{1,1}_{\mathrm{loc}}(M, N)$ is defined in Section 3 below; these mappings have a formal differential $df_x : T_xM \to T_{f(x)}N$ almost everywhere; in the above inequality, $|df_x|$ denotes its operator norm and $J_f(x) = \det df_x$ its Jacobian.

Mappings with bounded s-distortion are generalizations of quasi-regular mappings; they have been studied (under various names and viewpoints) since about 30 years, see [6], [8], [24], [25], [28], [30], [41], [44] among other works. In the special case of homeomorphisms with bounded s-distortion with $s > n-1$, a metric characterization has been given in [8].

These mappings originated as suitable class of mappings in the change-of-variable formula for functions in the Sobolev spaces $\mathcal{L}^{1,s}$ (see Section 4). As it turns out, this class of mappings feels quite well the

---

* This work is partially supported by the INTAS, grant N° 10170.
Keywords: Mapping with bounded distortion – Capacity – Parabolicity.
Math. classification: 30C65 – 31B15 – 26B10.

asymptotic geometry of Riemannian manifolds. In [6], J. Ferrand was able to prove that a Riemannian manifold is characterized up to bilipschitz equivalence by its Royden algebra; the proof heavily uses the theory of mappings with bounded s-distortion. In [28], P. Pansu gave conditions on the geometry of manifolds implying that mappings with bounded s-distortion are quasi-isometries. The work of Ferrand and Pansu has been extended to the case of metric measure spaces in the recent thesis of K. Gafaïti.

Mapping with bounded s-distortion are a subclass of the so-called *mappings with finite distortion* which are defined by the condition that $J_f \in L^1_{\mathrm{loc}}$ and $|df_x|^n \leqslant \Phi(x)J_f(x)$ where $\Phi(x)$ is finite a.e. (see the argument in the proof of Corollary 7.1). Mappings with finite distortion play an important role in non-linear elasticity (see e.g. [27]) and they are now being intensively studied. See e.g. the papers [16], [18], [41] and the rich references therein.

Another important generalization of mappings with bounded s-distortion is given by the class of mappings such that $|\Lambda_k f_x|^q \leqslant KJ_f(x)$ where $\Lambda_k f$ is the k-th. exterior power of df, i.e. the effect of df_x at the level of k-forms. These mappings appear in L^p cohomology; see e.g. the recent paper [29] of P. Pansu, where flows of such mappings are used in the computation of L^p-cohomology of manifolds with negative curvature and solvable Lie groups.

In the present paper, we will consider the case $k = n - 1$; let us thus define a *mappings with bounded q-codistortion* $(1 < q < \infty)$ to be a mapping $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ for which there exists a constant K' such that

$$|\Lambda_{n-1} f_x|^q \leqslant K' J_f(x) \quad \text{a.e.} \tag{2}$$

We now state a number of questions, concerning mappings with bounded s-distortion, we are interested in

1) What are the obstructions to the existence of a non constant mapping with bounded s-distortion $f : M \to N$?

2) Describe the set of all $s \geqslant 1$ for which there exists a homeomorphism (or a diffeomorphism) $f : M \to N$ with bounded s-distortion.

3) Suppose that $f : M \to N$ is a non constant mapping with bounded s-distortion: How big may the omitted set $N \setminus f(M)$ be? (In particular, when can it be said that f is onto?)

4) Assuming that $f : M \to N$ is a continuous mapping with bounded

s-distortion. What can be said about the topological and or the geometrical properties of f?

Similar questions may be raised about mappings with bounded q-codistortion.

We will give some answers to all of these questions. The main techniques we use are based on potential theory: Recall that a *condenser* in M is a pair (D, A) where $A \subset M$ is a connected open subset and $D \subset A$ is compact. The *p-capacity* of the condenser (D, A) is defined as

$$\mathrm{Cap}_p(D, A) = \inf \left\{ \int_A |\nabla u|^p : u \in C_0^0(A) \cap W^{1,p}(A) \text{ and } u \geqslant 1 \text{ on } D \right\}.$$

DEFINITION. — *The manifold M is* p-parabolic *if* $\mathrm{Cap}_p(D, M) = 0$ *for all compact subsets $D \subset M$ and* p-hyperbolic *otherwise.*

We have included in Section 7.2 below a brief discussion of this notion.

An answer to the first question above is the following Liouville type theorem:

THEOREM. — *Let M and N be oriented n-dimensional Riemannian manifolds and let $f \in W^{1,n}_{\mathrm{loc}}(M, N)$ be a mapping with bounded s-distortion with $s > (n-1)$. Assume that M is p-parabolic, where $\frac{1}{p} + \frac{n-1}{s} = 1$. Then either f is constant a.e. or N is also p-parabolic.*

This result is a consequence of Theorem A and Corollary 7.1 in the present paper; it is in fact proved for a wider class than $W^{1,n}_{\mathrm{loc}}(M, N)$.

In the special case of quasi-regular mappings (i.e. $s = p = n$), this result has been obtained around 1968 by Y. Reshetnyak and, independently, by O. Martio, S. Rickman and J. Väisälä (see [32] and [22]). At the end of the paper we shortly recall the original argument of Reshetnyack.

Some answers to the other questions mentioned above are given in Sections 2, 4 and 7.

The paper is organized as follow: In Section 2, we give some additional definitions, state the main results of the paper and give some corollaries. In Section 3 we recall some basic facts about Sobolev mappings, in Section 4 we discuss homeomorphisms with bounded s-distortion and in Section 5 we prove a capacity inequality. After these preparations, we prove the main theorems in Section 6. Finally, in Section 7, we give some complementary information on mappings with bounded s-distortion.

2. Definitions and statement of the results.

Throughout the paper M and N are oriented, connected n-dimensional Riemannian manifolds. We denote by $d\mu$ and $d\nu$ the volume elements of M and N respectively.

In order to state our results, we need some additional definitions:

Definitions. — (1) *The map f has* essentially finite multiplicity *if* $\mathrm{N}_f(M) < \infty$, *where*

$$\mathrm{N}_f(A) := \operatorname{ess\,sup}_y \operatorname{Card}(f^{-1}(y) \cap A)$$

for any measurable subset $A \subset M$.

(2) *A continuous map is* open and discrete *if the image of any open set $U \subset M$ is an open set $f(U) \subset N$ and the inverse image $f^{-1}(y)$ of any point $y \in N$ is a discrete subset of M. The* branch set *of such a mapping is the set $B_f \subset M$ of points $x \in M$ such that f is not a local homeomorphism in a neighborhood of x.*

The next two definitions are regularity assumptions. They are always satisfied if one assumes e.g. that f is locally Lipschitz, or that $f \in W^{1,s}_{\mathrm{loc}}(M,N)$ for $s > n$, or that f is locally quasi-regular.

(3) *A measurable map $f : M \to N$ satisfies* Lusin's property *if the image of any set $E \subset M$ of measure zero is a set $f(E) \subset N$ of measure zero.*

An important and well-known result (see Proposition 3.2) states that for any map $f : M \to N$ belonging to $W^{1,1}_{\mathrm{loc}}(M,N)$ there exists a sequence of compact sets $A_i \subset M$ such that the restriction of f to each A_i is Lipschitz and the complementary set $E_f := M \setminus \cup_i A_i$ has measure zero. We call E_f the *exceptional set of f.*

(4) *The map $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ is* almost absolutely continuous *if it is continuous and for any bounded domain $\Omega \Subset M$ the following property holds: for any $\varepsilon > 0$ we can find $\delta = \delta(\Omega,\varepsilon) > 0$ such that for any finite or infinite sequence of pairwise disjoint balls $\{B(x_i,r_i)\}$ contained in Ω with center $x_i \in E_f$, we have*

$$\sum \operatorname{vol}(B_i) \leqslant \delta \quad \Longrightarrow \quad \sum \operatorname{diam}(fB_i)^n < \varepsilon.$$

Remark. — The notion of almost absolute continuity appeared in [41], [42]; it is a generalization of absolute continuity in the sense of Malý as defined in [19]. In particular any mapping in $W^{1,p}_{\mathrm{loc}}(M,N)$ with $p>n$ and any continuous mapping in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ with monotone coordinate functions is an example of almost absolutely continuous mapping, see [19].

In dimension 2, a mapping has bounded s-distortion if and only if it has bounded s-codistortion. In higher dimension, we have the following relation between distortion and codistortion:

LEMMA 2.1. — *Let $f : M \to N$ be a mapping with bounded s-distortion for some $s > n-1$, then f has bounded q-codistortion for $q = s/(n-1)$.*

Conversely, if $f : M \to N$ is a mapping with bounded q-codistortion for some $q < \frac{n-1}{n-2}$ such that $J_f > 0$ a.e., then f has bounded s-distortion for $s = \frac{q}{(n-1)-q(n-2)}$.

The exponents in this lemma are sharp.

Proof. — It is a trivial consequence of the inequalities

$$|\Lambda_{n-1} f_x| \leqslant |df_x|^{n-1} \qquad \text{and} \qquad |df_x|\, J_f(x)^{n-2} \leqslant |\Lambda_{n-1} f_x|^{n-1}.$$

□

We now state the main results of the present paper:

THEOREM A. — *Let $f \in W^{1,s}_{\mathrm{loc}}(M,N)$ be a continuous open and discrete mapping with bounded s-distortion, where $s > (n-1)$, satisfying Lusin's property. If M is p-parabolic with $p = \frac{s}{s-(n-1)}$, then N is also p-parabolic.*

Recall that a map $f \in W^{1,s}_{\mathrm{loc}}$ always satisfies Lusin's property if $s > n$. In Section 3 below we give other sufficient conditions. In Section 7.1 below, we will also give sufficient conditions for a continuous mapping with bounded s-distortion to be discrete and open.

The next result is an analog of Theorem A. It holds without any topological restrictions but assumes that f has finite essential multiplicity:

THEOREM B. — *Let $f \in W^{1,s}_{\mathrm{loc}}(M,N)$ be a mapping of essentially finite multiplicity with bounded s-distortion where $s > (n-1)$. Assume either*

1) $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, *or*
2) f *is almost absolutely continuous and* $J_f \in L^1_{\mathrm{loc}}(M)$.

If M is p-parabolic with $p = \frac{s}{s-(n-1)}$, then either f is constant a.e. or N is also p-parabolic.

In Theorem B (under assumption 2) no continuity is assumed. The proofs of theorems A and B are based on quite different approaches; it would be interesting to have a unified method proving both results.

Remark 1. — These results are sharp. They say for instance that there is no mapping of finite essential multiplicity with bounded s-distortion from the Euclidean space to the hyperbolic space for $s > (n-1)$. This is optimal since the Riemannian exponential $\exp : T_{x_0}\mathbb{H}^n \to \mathbb{H}^n$ (where $\mathbb{H}^n$ is the hyperbolic space) is a diffeomorphism with bounded $(n-1)$-distortion. Other comments on the optimality of these results are given in [7].

Theorem B will be obtained as a consequence of the following result on mappings with bounded codistortion:

THEOREM C. — *Let $f : M \to N$ be a mapping of essentially finite multiplicity with bounded q-codistortion where $q > 1$. Suppose that $J_f > 0$ on some set of positive volume. Assume furthermore either*

1) *$f \in W^{1,n-1}_{\mathrm{loc}}(M,N)$ and $|\Lambda_{n-1}f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, or*

2) *f is almost absolutely continuous, $f \in W^{1,s}_{\mathrm{loc}}(M,N)$ for some $s > (n-1)$ and $J_f \in L^1_{\mathrm{loc}}(M)$.*

If M is p-parabolic with $p = q/(q-1)$, then N is also p-parabolic.

Remark 2. — The condition that $J_f > 0$ on some set of positive volume cannot be replaced by the weaker condition that f is not constant a.e. For instance, look at the hyperbolic three-space in the upper-half space model $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 \,|\, z > 0\}$ (with metric tensor $ds^2 = (dx^2 + dy^2 + dz^2)/z^2)$). Then the mapping $f : \mathbb{R}^3 \to \mathbb{H}^3$ given by $f(x,y,z) = (x,0,1)$ is of finite essential multiplicity and has bounded q-codistortion for all $q \geqslant 1$. Yet $\mathbb{H}^3$ is p-hyperbolic for all p and $\mathbb{R}^3$ is p-parabolic for all $p \geqslant 3$.

The next result goes in the other direction:

THEOREM D. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ be a continuous non constant proper mapping with bounded s-distortion of finite essential multiplicity. If M is s-hyperbolic, then so is N.*

Remark 3. — The hypothesis that f is proper is necessary. For instance if N is a compact manifold and $M \subset N$ is an open domain whose complement $N \setminus M$ has non empty interior, then N is s-parabolic for all s and M is s-hyperbolic for all $s \in [1, \infty]$. Yet the inclusion $f : M \hookrightarrow N$ has bounded s-distortion for all s.

We now give some applications of our results. We begin by a Picard type theorem for mappings with bounded s-distortion.

COROLLARY 2.1. — *Let $f : M \to N$ be a continuous mapping with bounded s-distortion, $s > (n-1)$ satisfying the hypothesis of Theorem A. Assume that the manifold M is p-parabolic where $p := \frac{s}{s-(n-1)}$. Then f is surjective if $p > n$, and the omitted set $N \setminus f(M)$ has Hausdorff dimension $\leqslant (n-p)$ if $p \leqslant n$.*

Proof. — Observe that f actually maps M onto $N' = f(M)$ (which is an open subset of N). By Theorem A, the manifold N' is thus p-parabolic and therefore the Hausdorff dimension of $N \setminus N'$ is $\leqslant n - p$. □

For a quasiregular mapping on Euclidean space $f : \mathbb{R}^n \to \mathbb{R}^n$, a stronger result is due to S. Rickman. He proved that f omits at most finitely many points (see theorem 2.1 in [34], chapter IV]).

COROLLARY 2.2. — *Let $f : M \to N$ be an injective C^1 mapping with bounded q-codistortion. Assume that $q < \frac{n}{n-1}$ and that M is p-parabolic with $p = \frac{q}{q-1}$. Then f is a diffeomorphism.*

For the proof of this corollary, will need a lemma. Recall that the *principal dilatation coefficients* (or *singular values*) at $x \in M$ of a mapping $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ are the square roots $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ of the eigenvalues of $df_x df_x^t$; they are defined almost everywhere. Observe the following useful inequalities:

$$|df_x| = \lambda_n, \qquad |J_f(x)| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n, \qquad |\Lambda_{n-1} f_x| = \lambda_2 \cdot \lambda_3 \cdots \lambda_n.$$

LEMMA 2.2. — *Let $f : M \to N$ be a mapping with bounded q-codistortion. If $q < \frac{n}{n-1}$ then either $J_f = 0$ a.e. or there exists a constant $\delta > 0$ such that all the principal dilatation coefficients are almost everywhere $\geqslant \delta$.*

Proof. — Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the dilatation coefficients of f at x. We have by hypothesis $|\Lambda_{n-1} f_x|^q \leqslant K\, J_f(x)$ a.e., i.e. $(\lambda_2 \cdot \lambda_3 \cdots \lambda_n)^q \leqslant$

$K\,(\lambda_1 \cdot \lambda_2 \cdots \lambda_n)$. This implies $\lambda_1^{(n-1)(q-1)} \leqslant (\lambda_2 \cdot \lambda_3 \cdots \lambda_n)^{q-1} \leqslant K\,\lambda_1$, from which one obtains $\lambda_1 \geqslant K^{1/(n+q-nq)}$, provided $q < \frac{n}{n-1}$. □

Proof of Corollary 2.2. — By the previous lemma, all principal dilatation coefficients are bounded below, in particular f is a local diffeomorphism. Assume now that f is not surjective. Then there exists a point $y_0 \in N \setminus f(M)$. Let $N' := N \setminus \{y_0\}$, this is a p-hyperbolic manifold (since $p > n$). By Theorem C, the manifold M must therefore be p-hyperbolic; but this contradicts the hypothesis and we thus conclude that f is surjective. □

If $M = N = \mathbb{R}^n$, we don't need to assume global injectivity in the previous corollary.

Corollary 2.3. — *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping with bounded q-codistortion where $q < \frac{n}{n-1}$ and such that $J_f \not\equiv 0$. Then f is a global diffeomorphism.*

Proof. — By Lemma 2.2 all the eigenvalues of $df_x^t df_t$ are uniformly bounded below. We thus conclude from a recent theorem of M. Chamberland and G. Meister that f is injective (see [1], th. 1.1).

Now set $p := \frac{q}{q-1}$, then $p > n$ and hence $\mathbb{R}^n$ is p-parabolic. We conclude the proof from the previous corollary. □

We also have similar results for mappings with bounded s-distortion.

Corollary 2.4. — *Let $f : M \to N$ be an injective C^1 mapping with bounded s-distortion.*

Assume that $(n-1) < s < n$ and that M is p-parabolic with $p = \frac{s}{s-(n-1)}$. Then f is a diffeomorphism.

The proof is similar to that of Corollary 2.2. □

Corollary 2.5. — *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non constant C^1 mapping with bounded s-distortion where $(n-1) < s < n$. Then f is a global diffeomorphism.*

Proof. — This is clear from Lemma 2.1 and the previous corollaries. □

This last result also holds for $s = n \geqslant 3$. Indeed, V.A. Zorich has proved that a quasi-regular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, which is a local homeomorphism is in fact a global homeomorphism provided $n \geqslant 3$, see [46].

3. Calculus of Sobolev mappings.

Since a mapping $f : M \to \mathbb{R}^m$ is given by its components which are n functions: $f = (f_1, f_2, \dots, f_m)$, it is natural to say that f belongs to the Sobolev space $W^{1,s}_{\text{loc}}(M, \mathbb{R}^m)$ if each component $f_i \in W^{1,s}_{\text{loc}}(M, \mathbb{R})$.

In the case of a continuous mapping $f : M \to N$ between Riemannian manifolds, we may define the condition $f \in W^{1,s}_{\text{loc}}(M, N)$ by the use of local coordinates charts; however, such a procedure is in general not possible for a discontinuous map and we have to proceed differently to define the class of Sobolev mappings between Riemannian manifolds.

We follow the approach of [33], [42].

Definitions. — 1) *The mapping $f : M \to N$ belongs to $L^s_{\text{loc}}(M, N)$, $1 \leqslant s \leqslant \infty$, if and only if the function $[f]_y : M \to \mathbb{R}$, defined by $[f]_y(x) = d(f(x), y)$, is in $L^s_{\text{loc}}(M, \mathbb{R})$ for all point $y \in N$.*

2) *The map f belongs to $W^{1,s}_{\text{loc}}(M, N)$ if and only if $[f]_y \in W^{1,s}_{\text{loc}}(M, \mathbb{R})$ and there exists a function $g \in L^s_{\text{loc}}(M, \mathbb{R})$ such that $|\nabla [f]_y|(x) \leqslant g(x)$ a.e. in M for any point $y \in N$.*

3) *The map f belongs to $ACL^s_{\text{loc}}(M, N)$ if it satisfies the following three conditions:*

i) *the function $M \ni x \to [f]_z(x) = d(f(x), z)$ belongs to $L^s_{\text{loc}}(M)$ for every point $z \in N$;*

ii) *the mapping $f : M \to N$ is absolutely continuous on lines in the following sense: for any coordinate chart $\varphi : U \to \mathbb{R}^n$ on M, the function*

$$(x, \tau) \to g_i(x, \tau) := \text{ length}\,(f \circ \varphi^{-1}([x, x + \tau \mathbf{e}_i]))$$

is absolutely continuous in the parameter τ for all i and almost all $x \in \mathbf{e}_i^{\perp}$.

iii) *the derivative $\partial_i g_i : x \to \lim\limits_{\tau \to +0} \frac{g_i(x,\tau)}{\tau}$, which exists almost everywhere in U, belongs to $L^s_{\text{loc}}(U)$ for all i.*

Proposition 3.1. — *The following assertions are equivalent:*

1) $f \in W^{1,s}_{\mathrm{loc}}(M,N)$;

2) $f \in ACL^s_{\mathrm{loc}}(M,N)$;

3) $f \in L^s_{\mathrm{loc}}(M,N)$ *and there exists a function* $g \in L^s_{\mathrm{loc}}(M,\mathbb{R})$*such that for any Lipschitz function* $\psi : N \to \mathbb{R}$, *the function* $\varphi := \psi \circ f : M \to \mathbb{R}$ *belongs to* $W^{1,s}_{\mathrm{loc}}(M,\mathbb{R})$ *and* $|\nabla\varphi(x)| \leqslant \operatorname{Lip}(\psi)\, g(x)$ *a.e. in M.*

4) *for any isometric embedding* $i : N \to \mathbb{R}^k$ *all coordinate functions of the composition* $i \circ f$ *belong to* $W^{1,s}_{\mathrm{loc}}(M,\mathbb{R})$.

Proof. — The proof follows the order (1) ⇒(2)⇒(3)⇒(4)⇒(2). Observe that (3)⇒(1) is trivial since distance functions are 1-Lipschitz.

Then (1) ⇒(2) and (2)⇒(3) are proven in [42, Proposition 3] (notice that (1)⇒(3) is also proven in [33, Theorem 5.1] by other arguments).

The proof of (4)⇒(2) is given in [34, Proposition 1.2] for the special case $N = \mathbb{R}^n$. Its extension to the case of a submanifold $N \subset \mathbb{R}^k$ is based on the formula

$$g_i(x,\tau) = \int_0^\tau \Big| \frac{d}{dt}(f \circ \varphi^{-1}([x, x + t\mathbf{e}_i])) \Big|\, dt$$

which holds for all absolutely continuous curves in the $\mathbb{R}^k$. The general case now follows from the fact that any Riemannian manifold admits an isometric embedding in some Euclidean space.

(3) ⇒(4). We consider an isometric embedding $i : N \to \mathbb{R}^k$ and some coordinate function z_j in $\mathbb{R}^k$. The restriction $z_j|_N$ is a Lipschitz function on N, thus the composition $z_j \circ f$ belongs to $W^{1,s}_{\mathrm{loc}}(M,\mathbb{R})$. □

The next proposition says that a Sobolev mapping is Lipschitz on a big set.

Proposition 3.2. — *Let* $f \in W^{1,1}_{\mathrm{loc}}(M,N)$. *Then there exists a measurable decomposition* $M = E_f \cup \cup_{i=1}^{\infty} A_i$ *such that* $\mu(E_f) = 0$, A_i *is compact for all* i *and* $f|_{A_i}$ *is Lipschitz.*

Proof. — Using the previous proposition (assertion 4) we can reduce the proof to the well-known Whitney's approximation theorem for Sobolev function (see e.g. [4, p. 254]). □

As a consequence of this proposition, we have the following version of the change of variables formula for integrals (also known as the *area formula*), recall that χ_A denotes the characteristic function of a set $A \subset M$.

Proposition 3.3. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ be a Sobolev mapping between Riemannian manifolds of the same dimension. Then there exists a subset $E_f \subset M$ of measure zero such that for all measurable function $\psi : M \to \mathbb{R}_+$ we have*

$$\int_M \psi(x)\,|J_f(x)|\,d\mu(x) = \int_N \left(\sum_{f(x)=y} \psi(x)\,\chi_{M\setminus E_f}(x) \right) d\nu(y).$$

If f satisfies Lusin's property, then one may take $E = \emptyset$.

See e.g. [11] for a proof. □

For the area formula to be useful, we need to work with mappings having a locally integrable Jacobian. Observe in particular that if $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ has bounded s-distortion and $J_f \in L^1_{\mathrm{loc}}(M)$, then we have in fact $f \in W^{1,s}_{\mathrm{loc}}(M, N)$.

The next two lemmas give us sufficient conditions for the local integrability of the Jacobian.

Lemma 3.1. — *Let $f : M \to N$ be a mapping such that $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$. Then $J_f \in L^1_{\mathrm{loc}}(M)$.*

Proof. — This is a trivial consequence of the inequality $J_f \leqslant |\Lambda_{n-1} f|^{n/(n-1)}$. □

Lemma 3.2. — *If $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ is continuous and has essentially finite multiplicity or is open and discrete, then $J_f \in L^1_{\mathrm{loc}}(M)$.*

Proof. — This follows directly from the area formula. □

We now give sufficient conditions for Lusin's property:

Lemma 3.3. — *Let $f : M \to N$ be a mapping satisfying one of the following conditions:*

1) *$f \in W^{1,s}_{\mathrm{loc}}(M, N)$ with $s \geqslant (n-1)$, $J_f > 0$ a.e. and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$;*

2) *$f \in W^{1,1}_{\mathrm{loc}}(M, N)$ is almost absolutely continuous;*

3) *$f \in W^{1,n}_{\mathrm{loc}}(M,N)$ is continuous open and discrete.*

Then it also satisfies Lusin's property.

Under hypothesis (1) this is Theorem 5.3 in [26]; see also [43] for the case $s = n$. In case (2), this is Theorem 8 from [41]. In case (3), this is a result from [20]; see also [40] for a short proof. ◻

We refer to [23] and [19] for further results on Lusin's condition.

Proposition 3.4. — *If the map f is continuous, open and discrete and has bounded s-distortion for $s > (n-1)$, then it is differentiable almost everywhere.*

See Lemma 4.4 in chapter VI of Rickman's book [34] or Proposition 1 in [41] for a more general result. ◻

Finally we will also need the following result about the exterior differential of the pull-back of a $(n-1)$-form:

Lemma 3.4. — *Let $f : M \to N$ be a mapping satisfying one of the following conditions:*

1) *$f \in W^{1,n-1}_{\mathrm{loc}}(M,N)$ and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$;*

2) *f is almost absolutely continuous, $f \in W^{1,s}_{\mathrm{loc}}(M,N)$ for some $s > (n-1)$ and $J_f \in L^1_{\mathrm{loc}}(M)$.*

Let β be a smooth $(n-1)$-form. Then $\alpha := f^*\beta \in L^1_{\mathrm{loc}}(M, \Lambda^{n-1})$ and

$$d\alpha = f^*(d\beta).$$

This result is proved in [26, Th. 3.2] under the first hypothesis and in [41, Th. 8] in the case of the second hypothesis ◻

4. On homeomorphisms with bounded s-distortion.

In this section, we discuss the special case of homeomorphisms with bounded s-distortion.

Definition. — *The* s-Dirichlet space *of a Riemannian manifold M is the space $\mathcal{L}^{1,s}(M)$ of functions $u \in W^{1,s}_{\mathrm{loc}}(M,\mathbb{R})$ such that $\int_M |\nabla u|^s \, d\mu < \infty$. This space is equipped with the semi-norm*

$$\|u\|_{\mathcal{L}^{1,s}(M)} = \|\nabla u\|_{L^s(M)} .$$

If $f : M \to N$ is a homeomorphism and $v : N \to \mathbb{R}$ is any function, we denote by $f^*v = v \circ f$ its pull back on M. If $u : M \to \mathbb{R}$, we denote by $f_\sharp u = u \circ f^{-1} : N \to \mathbb{R}$ its pushforward.

S. Vodop'yanov has proved the following result [38], [39] (see its generalized version in [44, Theorems 1 and 9]):

THEOREM 4.1. — *Let $f : M \to N$ be a homeomorphism between n-dimensional Riemannian manifolds. Fix $s \in [1, \infty)$, then the following assertions are equivalent:*

1) $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$ *is a bounded operator;*

2) $f \in W^{1,s}_{\mathrm{loc}}(M, N)$ *and f has bounded s-distortion:* $|df(x)|^s \leqslant K J_f(x)$ *a.e.* $x \in M$.

Moreover, if $s \in (1, \infty)$, then condition (1) or (2) are equivalent to

3) f^{-1} *decreases the s-capacities of condensers up to a constant:*

$$\mathrm{Cap}_s(C, A) \leqslant \mathrm{const.}\, \mathrm{Cap}_s(f(C), f(A))$$

for any condensers (C, A) in M.

Finally, if $s > (n - 1)$ and Lusin's property holds, then any condition (1)–(3) is equivalent to

4) $f_\sharp : \mathcal{L}^{1,p}(M) \to \mathcal{L}^{1,p}(N)$ *is a bounded operator where* $p = \frac{s}{s-(n-1)}$, *and* $|df^{-1}(y)|^p \leqslant K^{p-1} J_{f^{-1}}(y)$ *a.e.* $y \in N$, *consequently f^{-1} has bounded p-distortion.*

Proof. — We only give a short proof of the second part of assertion (4). By Proposition 3.4, the map f is differentiable a.e. and by [44, Theorem 9], we know that $g := f^{-1} : N \to M$ is ACL (see also Lemma 5.6 below). Thus we have $dg_{f(x)} \circ df_x = \mathrm{Id}$ a.e. in M. Notice also that $J_g(y) \neq 0$ a.e. in N since f has Lusin's property by hypothesis, we thus have almost everywhere

$$|dg_{f(x)}| \leqslant \frac{|df_x|^{n-1}}{J_f(x)},$$

and therefore

$$|dg_{f(x)}|^p \leqslant \frac{(|df_x|^s)^{\frac{p(n-1)}{s}}}{J_f^p(x)} = \left(\frac{|df_x|^s}{J_f(x)}\right)^{p-1} J_f^{-1}(x) \leqslant K^{p-1} J_g(f(x)).$$

□

A useful consequence of this theorem is the following

Corollary 4.1. — *If $f : M \to N$ and $g : N \to W$ are homeomorphisms with bounded s-distortion, then $g \circ f : M \to W$ also has bounded s-distortion.*

Special cases of the previous result where also obtained in [9.2 and 12.3], [24], [25, Section 6.4.3] and [30].

Definition. — *The* Royden algebra *of M is the subspace $\mathcal{R}^s(M) \subset \mathcal{L}^{1,s}(M)$ of bounded continuous functions; it is a Banach algebra with norm*

$$\|u\|_{\mathcal{R}^s} = \|u\|_{L^\infty} + \|\nabla u\|_{L^s}\,.$$

We denote by $K_{\mathcal{R}}$ the norm of the operator $f^* : \mathcal{R}^s(N) \to \mathcal{R}^s(M)$ and by $K_{\mathcal{L}}$ the norm of the operator $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$.

Proposition 4.1. — *Suppose $1 < s < \infty$, then for any homeomorphism $f : M \to N$ we have $K_{\mathcal{R}} = \max\{1, K_{\mathcal{L}}\}$.*

We will need the following

Lemma 4.1. — *Let $v \in \mathcal{R}^s(N)$ be a non constant function, and fix $\varepsilon > 0$. If $1 < s < \infty$, then for any $t \in (\alpha, \beta)$, where $\alpha := \inf v$ and $\beta := \sup v$, there exists $r = r(t, \varepsilon) > 0$ such that $r < \min\{t - \alpha, \beta - t\}$ and*

$$\varepsilon^{-1}(t'' - t') \leqslant \|\max(\min(v, t''), t') - t'\|_{\mathcal{L}^{1,s}(N)} \tag{3}$$

for all $t', t'' \in (\alpha, \beta)$ such that $t - r < t' \leqslant t \leqslant t'' < t + r$.

Proof. — Suppose the lemma false, then the function

$$v_{t',t''} := \frac{\max(\min(v, t''), t') - t'}{t'' - t'}$$

satisfies $\|v_{t',t''}\|_{\mathcal{L}^{1,s}(N)} \leqslant \frac{1}{\varepsilon}$ for some $\varepsilon > 0$ and all $t', t'' \in (\alpha, \beta)$ such that $t - r < t' \leqslant t \leqslant t'' < t + r$. Consider a bounded domain $A \subset N$ such that $A_0 := \{x \in A : v(x) < t\}$ and $A_1 := \{x \in A : v(x) > t\}$ are non empty open subsets.

The family $\{v_{t',t''}\}$ is bounded in $W^{1,s}(A)$ and hence weakly compact. It follows that there is a sequence $v_n := v_{t'_n, t''_n}$ such that $t'_n \leqslant t \leqslant t''_n$ and $(t''_n - t'_n) \to 0$, which converges weakly to some function $w \in W^{1,s}(A)$. We can furthermore assume that the sequence $\lambda_n := \frac{t - t'_n}{t''_n - t'_n} \in [0, 1]$ converges to some number λ. Using Mazur's Lemma, we can produce convex combinations of the v_n converging strongly to w. Hence $w = 0$ a.e. on A_0,

$w = 1$ a.e. on A_1 and $w = \lambda$ a.e. on the level set $A_t := \{x \in A : v(x) = t\}$. But then $\nabla w = 0$ a.e. in A, hence w is constant a.e. in A which is impossible since A_0 and A_1 have positive measure. □

Proof of Proposition 4.1. — Observe that $K_{\mathcal{R}} \geqslant 1$ since constant functions belong to the Royden algebras. So we only need to prove the inequalities $K_{\mathcal{L}} \leqslant K_{\mathcal{R}} \leqslant \max\{1, K_{\mathcal{L}}\}$. Since f is a homeomorphism, f^* defines an isometry $f^* : L^\infty(N) \to L^\infty(M)$ and the inequality $K_{\mathcal{R}} \leqslant \max\{1, K_{\mathcal{L}}\}$ follows immediately.

To prove the inequality $K_{\mathcal{L}} \leqslant K_{\mathcal{R}}$ it suffices, by density of $\mathcal{R}^s(N)$ in $\mathcal{L}^{1,s}(N)$, to show that

$$\|f^* v\|_{\mathcal{L}^{1,s}(M)} \leqslant (1+\varepsilon) K_{\mathcal{R}} \, \|v\|_{\mathcal{L}^{1,s}(N)} \tag{4}$$

for any $\varepsilon > 0$ and any function $v \in \mathcal{R}^s(N)$.

Set $\alpha := \inf v$ and $\beta := \sup v$. By compactness of the interval $[\alpha, \beta]$, we can find a subdivision $\tau = \{\alpha = t_0 < t_1 < \ldots < t_l < t_{l+1} = \beta\}$, such that $(t_{i+1} - t_i) < r_i$ for $i = 1, \ldots, l-1$, where $r_i = r(t, \varepsilon)$ satisfies the property of the previous lemma for some $t \in (t_i, t_{i+1})$.

Set $v_\tau := \alpha + \sum_{i=1}^{l-1} v_i$, where $v_i := \max(\min(v, t_{i+1}), t_i) - t_i$. By the lemma we have $\|v_i\|_{L^\infty} \leqslant \varepsilon \, \|v_i\|_{\mathcal{L}^{1,s}(N)}$ for $i = 1, \ldots, l-1$, hence

$$\begin{aligned}
\|f^* v_\tau\|^s_{\mathcal{L}^{1,s}(M)} \leqslant \|f^* v_\tau\|^s_{\mathcal{R}^s(M)} &\leqslant \sum_{i=0}^{l-1} \|f^* v_i\|^s_{\mathcal{R}^s(M)} \\
&\leqslant \sum_{i=0}^{l-1} K^s_{\mathcal{R}} \left(\|v_i\|_{L^\infty(N)} + \|v_i\|_{\mathcal{L}^{1,s}(N)} \right)^s \\
&\leqslant K^s_{\mathcal{R}} (1+\varepsilon)^s \sum_{i=1}^{l-1} \|v_i\|^s_{\mathcal{L}^{1,s}(N)} \\
&\leqslant K^s_{\mathcal{R}} (1+\varepsilon)^s \, \|v_\tau\|^s_{\mathcal{L}^{1,s}(N)}
\end{aligned}$$

because $\|v_\tau\|^s_{\mathcal{L}^{1,s}(N)} = \sum_{i=1}^{l-1} \|v_i\|^s_{\mathcal{L}^{1,s}(N)}$. The inequality (4) now follows since $\|v - v_\tau\|_{\mathcal{L}^{1,s}(N)} \to 0$ and $\|f^* v - f^* v_\tau\|_{\mathcal{L}^{1,s}(N)} \to 0$ as $\max\{t_1 - t_0, t_{l+1} - t_l\} \to 0$. □

Remark. — Pierre Pansu has defined in [28, p. 475] the notion of *homeomorphism of bounded s-dilatation* as homeomorphism such that $K_{\mathcal{R}} \leqslant \infty$. It follows from the results of this section that the definition of homeomorphism of bounded s-dilatation used by Pansu, coincides with our notion of homeomorphism with bounded s-distortion if $1 < s < \infty$.

It also follows from Theorem 4.1 that if f is a homeomorphism satisfying Lusin's property with bounded s-dilatation in Pansu's sense, then f^{-1} is a homeomorphism with bounded p-dilatation where $1/s+(n-1)/p=1$. This gives a positive answer to question 10.3 in [28] in the case where Lusin's property holds.

5. Pushing functions forward.

The proof of Theorem A is based on a capacity estimate for the pushforward operator (Corollary 5.1) which is important in itself. It is the goal of this section to prove this capacity estimate.

Let $f : M \to N$ be a continuous mapping and $u : M \to \mathbb{R}$ a bounded function. We define the *pushforward* of u to be the function $v = f_\sharp u : N \to \mathbb{R}$ given by

$$v(y) := \begin{cases} \sup\{u(x) \,:\, f(x) = y\} & \text{if} \quad y \in f(M), \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5.1. — *If f is continuous discrete and open, and $u : M \to \mathbb{R}$ is continuous with compact support, then the function $v = f_\sharp u : N \to \mathbb{R}$ is also continuous and* $\operatorname{supp} v \subset f(\operatorname{supp} u)$.

This is Lemma 7.6 in [22]. □

If the mapping f has bounded s-distortion and $u \in C_0^1(M,\mathbb{R})$ then $v = f_\sharp u$ belongs to $W^{1,p}_{\mathrm{loc}}(N,\mathbb{R})$ where $p = \frac{s}{s-(n-1)}$ provided $s > (n-1)$. More precisely:

THEOREM 5.1. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ be a continuous open and discrete mapping with bounded s-distortion, $(n-1) < s < \infty$. Assume also that f satisfies Lusin's property if $n-1 < s < n$. Then the operator $f_\#$ possesses the following properties:*

1) $f_\sharp : C_0^1(M) \to W^{1,p}(N) \cap C_0^0(N)$,

2) $\int_N |df_\sharp(u)|^p \, d\nu \leqslant K^{p-1} \int_M |du|^p \, d\mu$, *for any $u \in C_0^1(M)$. where $p = \frac{s}{s-n+1}$ and K is the constant in (1).*

Remarks. — 1) If f is a continuous open mapping and $f \in W^{1,n}_{\mathrm{loc}}(M,N)$, then it always satisfies Lusin's property [20] (see also [40] for a short proof).

2) This theorem is known for $s = n$ (see [22]). It is also known for general values of s when f is a homeomorphism [44]. Our proof will be based on techniques borrowed from these two papers.

If f is continuous and open, then the image $(f(C), f(A))$ of a condenser (C, A) in M is again a condenser in N.

COROLLARY 5.1. — *For any condenser (C, A) in M we have*

$$\mathrm{Cap}_p(f(C), f(A))) \leqslant K^{p-1}\,\mathrm{Cap}_p(C, A).$$

□

Proof. — Choose a non negative function $u \in C_0^1(M)$ such that $u = 1$ on C, $\mathrm{supp}(u) \subset A$ and $\int_A |du|^p \leqslant \mathrm{Cap}_p(C, A) + \varepsilon$ where $\varepsilon > 0$ is arbitrary.

Let us set $v = f_\sharp u : N \to \mathbb{R}$. Then, by Theorem 5.1 we have $v \in W^{1,p}(A) \cap C_0^0(A)$. Since $v \geqslant 1$ on C, we have

$$\mathrm{Cap}_p(fC, fA) \leqslant \int_{fA} |dv|^p \leqslant K^{p-1} \int_A |du|^p \leqslant K^{p-1}(\mathrm{Cap}_p(C, A) + \varepsilon).$$

□

We begin the proof of Theorem 5.1 by some lemmas on capacities of condensers:

LEMMA 5.2. — *The inequality*

$$\mathrm{Cap}_s(C, A) \leqslant \frac{|A|}{\mathrm{dist}(C, \partial A)^s}$$

holds for the capacity of any bounded condenser $(C, A) \subset \mathbb{R}^n$.

Proof. — Take $u(x) := \min\left\{\frac{\mathrm{dist}(\partial A, x)}{\mathrm{dist}(\partial A, C)}, 1\right\}$ as a test function. □

LEMMA 5.3. — *Let $(C, A) \subset \mathbb{R}^n$ be a condenser such that C is connected. If $(n-1) < s < \infty$, then*

$$\mathrm{Cap}_s^{n-1}(C, A) \geqslant b(n, s)\,(\mathrm{diam}\, C)^s\,|A|^{(n-1-s)}$$

where the constant $b(n, s)$ depends on n and s only.

Proof. — See Lemma 5 of [44]. □

Recall that a domain $\Omega \subset M$ is said to be a *normal domain* for f if $\overline{\Omega}$ is compact and $\partial(f(\Omega)) = f(\partial\Omega)$. For any normal domain $\Omega \subset M$ we have $N_f(\Omega) < \infty$. A condenser (C, A) is a *normal condenser* if A is a normal domain of f.

LEMMA 5.4. — *If $\Omega \subset M$ is a normal domain then* $\operatorname{Cap}_s(C, A) \leqslant KN_f(\Omega) \operatorname{Cap}_s f(C, A)$ *for any condenser (C, A) in Ω.*

This is a direct consequence of Lemma 6.2 below. See also [44, Th. 4]. □

The next lemma sums up the basic topological properties of a discrete and open mapping $f : M \to N$. If $x \in M$ and $r > 0$, then we denote by $U(x, f, r)$ the connected component of $f^{-1}(B(f(x), r))$ containing x.

LEMMA 5.5. — *Let $f : M \to N$ be a continuous discrete and open mapping. Then $\lim_{r\to 0} \operatorname{diam} U(x, f, r) = 0$ for every $x \in M$. If $\overline{U}(x, f, r)$ is compact then $U(x, f, r)$ is a normal domain and $f(U(x, f, r)) = B(f(x), r)$. Furthermore, for every point $x \in N$ there is a positive number σ_x such that the following conditions are satisfied for $0 < r \leqslant \sigma_x$:*

i) *$U(x, f, r)$ is a normal neighborhood of x,*

ii) *$U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}(B(f(x), r))$,*

iii) *$\partial U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}(S(f(x), r))$ if $r < \sigma_x$,*

iv) *$M \setminus U(x, f, r)$ is connected if M is connected,*

v) *$M \setminus \overline{U}(x, f, r)$ is connected if M is connected,*

vi) *if $0 < r < s \leqslant \sigma_x$, then $\overline{U}(x, f, r) \subset U(x, f, s)$, and $U(x, f, s) \setminus \overline{U}(x, f, r)$ is a ring.*

See [22], [34] or [12] for a proof. □

LEMMA 5.6. — *Let $f : M \to N$ be as in Theorem 5.1 and $u \in C_0^1(M)$. Then the function $v = f_\sharp u$ is ACL.*

Recall that a function $v : N \to \mathbb{R}$ is *absolutely continuous on lines* (ACL) if for any local parametrization $\varphi : Q \to N$ (where $Q = \{y \in \mathbb{R}^n : a_i \leqslant y_i \leqslant b_i\} \subset \mathbb{R}^n$ is some n-interval) and for almost all $z \in P_k(Q)$ (= the projection of Q on the hyperplane $y_k = 0$), the one-variable function $t \to v(\varphi(z + t\mathbf{e}_k))$ is absolutely continuous.

Proof. — Let us fix some notations. Fix a local parametrization $\varphi : Q \to N$ (where $Q = \{t \in \mathbb{R}^n : a_i \leqslant t_i \leqslant b_i\} \subset \mathbb{R}^n$ is some closed n-interval). Choose Q small enough so that for any ball $B(y,r) \subset \varphi(Q)$ the domains $U_i := U(x_i, f, r)$ are disjoint normal neighborhoods of x_i for $1 \leqslant i \leqslant q$ where $\{x_1, \dots, x_q\} = f^{-1}(y) \cap \operatorname{supp} u$.

The function $v \circ \varphi^{-1}$ will be simply denoted by $v : Q \to \mathbb{R}$. We need to show that for any $l = 1, \dots, n$ and for almost all $z \in P_l(Q)$, the function v is absolutely continuous on the line segment $\beta_z : [a_l, b_l] \to Q$ defined by $\beta_z(t) = z + t\mathbf{e}_l$.

To this aim, we define a set function φ on $P_l(Q)$ by

$$\Phi(A) := \left|U \cap f^{-1}\left(\varphi(A \times [a_l, b_l])\right)\right|$$

where $U = \cup_{i=1}^{q} U_i$ and $A \subset P_l(Q)$ is any Borel set. Then Φ is a completely additive set function in $P_l(Q)$ and from Lebesgue's differentiation theorem, we know that $\Phi'(z) < \infty$ for almost all $z \in P_l(Q)$.

It is known (see [22, Lemma 2.7]) that for every point $x_0 \in U \cap f^{-1}(z + a_l\mathbf{e}_l)$ there exists a path $\alpha : [a_l, b_l] \to U$ such that $\alpha(a_l) = x_0$ and $f \circ \alpha = \varphi \circ \beta_z$. We call such a path a *lift* of $\beta_z(t) = z + t\mathbf{e}_l$ with base point x_0; clearly the number of lifts does not exceed $N_f(U)$.

Claim. — *Let $\alpha : [a_l, b_l] \to U$ be any lift of β_z. If $\Phi'(z) < \infty$, then α is absolutely continuous.*

Since the ACL-property is local it suffices to show that α is ACL in a neighborhood of every point. We may thus restrict our considerations to the case of mappings $f : U \to Q$ where U is a bounded domain in $\mathbb{R}^n$.

To prove the claim, we fix some arbitrary pairwise disjoint closed segments $\Delta_1, \dots, \Delta_k \subset (a_l, b_l)$ of lengths $b_1, \dots, b_k$. Choose $r > 0$ small enough so that the sets

$$R_i := \{y \in \mathbb{R}^n \,|\, \operatorname{dist}(y, \Delta_i) < r\}$$

are pairwise disjoint. Let $T_i := \bigcup_{z \in \Delta_i} U(\alpha(z), f, r)$, then $(\alpha(\Delta_i), T_i)$ and (Δ_i, R_i) are condensers and $(\Delta_i, R_i) = (f(\alpha(\Delta_i)), f(T_i))$; indeed, we have $f(\alpha(\Delta_i)) = \Delta_i$ and

$$f(T_i) = f\Big(\bigcup_{z \in \Delta_i} U(\alpha(z), f, r)\Big) = \bigcup_{z \in \Delta_i} B(z, f, r) = R_i.$$

From Lemmas 5.2 and 5.3, we have

$$\operatorname{Cap}_s(\Delta_i, R_i) \leqslant \frac{|R_i|}{r^s} \leqslant c_1 b_i r^{n-1-s}$$

and

$$\mathrm{Cap}_s(\alpha(\Delta_i), T_i) \geqslant c_2 \frac{(\operatorname{diam}\alpha(\Delta_i))^{s/(n-1)}}{|T_i|^{(1-n+s)/(n-1)}}.$$

These inequalities, together with Lemma 5.4, imply

$$\operatorname{diam}\alpha(\Delta_i) \leqslant c_3\, b_i^{\frac{n-1}{s}} \left(\frac{|U \cap T_i|}{r^{n-1}}\right)^{\frac{1-n+s}{s}}$$

where the constant c_3 depends on previous constants, K and $N_f(\operatorname{supp} u)$.

Set $E(z,r) = \{y \in Q : \operatorname{dist}(y, \beta_z([a_i, b_i])) < r\}$, then $\bigcup_{i=1}^k T_i \subset f^{-1}(E(z,r))$. Summing the previous inequality over $i = 1, \dots, k$ and applying Hölder's inequality we obtain

$$\sum_{i=1}^k \operatorname{diam}\alpha(\Delta_i) \leqslant c_4 \left(\frac{|U \cap f^{-1}(E(z,r))|}{r^{n-1}}\right)^{\frac{1-n+s}{s}} \left(\sum_{i=1}^k b_i\right)^{\frac{n-1}{s}}.$$

Letting $r \to 0$, we find that

$$\sum_{i=1}^k \operatorname{diam}\alpha(\Delta_i) \leqslant c_5\, \varphi'(z) \left(\sum_{i=1}^k b_i\right)^{\frac{n-1}{s}},$$

hence α is absolutely continuous if $\varphi'(z) < \infty$.

We now conclude the proof of the lemma as follows: Let $\alpha_1, \alpha_2, \dots \alpha_d$ be all the lifts of the segment β_z. If $\Phi'(z) < \infty$, then $u \circ \alpha_i$ is absolutely continuous since u is C^1 and α_i is absolutely continuous. We conclude that $v \circ \beta_z$ is absolutely continuous since

$$v \circ \beta_z = \max_i u \circ \alpha_i.$$

□

Lemma 5.7. — *Let $f : M \to N$ be as in Theorem 5.1, then $J_f = 0$ almost everywhere on the branch set and the image of the branch set has measure zero.*

Proof. — Because f has bounded s-distortion and $s > (n-1)$, $f \in W^{1,s}_{\mathrm{loc}}$, it then follows from 3.4 that f is differentiable almost everywhere.

Suppose that f is differentiable at x and $J_f(x) > 0$, then the index $j(x,f) = 1$ (because the map is continuous open and discrete and the topological degree is stable under homotopy, see e.g. pp. 15-21 in [34]).

If $j(x,f) = 1$, then $x \notin B_f$ (see [34, Proposition 4.10]); it follows that $J_f = 0$ a.e. on B_f.

Because f is assumed to satisfy Lusin's property, we can use the area formula (Proposition 3.3) to conclude that $f(B_f)$ has measure zero:

$$\nu(f(B_f)) \leqslant \int_N \left(\sum_{f(x)=y} \chi_{B_f(x)} \right) d\nu(y) = \int_M \chi_{B_f(x)} J_f(x) d\mu(x) = 0.$$

□

Proof of Theorem 5.1. — To conclude the proof of the theorem it only remains to check the integrability of dv. To do this we first observe that Vitali's covering Theorem implies

$$\operatorname{supp} v \setminus f(B_f \cap \operatorname{supp} u) \subset \bigcup_{i=1}^{\infty} B(y_i, r_i) \cup A,$$

where B_f is the branch set of f, $A \subset N$ is a set with $\nu(A) = 0$ and $B(y_i, r_i)$, $i \in \mathbb{N}$, are mutually disjoint balls small enough so that the components of $f^{-1}(B(y_i, r_i))$ which meet the support of u form a finite disjoint collection $D_{i_1}, D_{i_2}, \dots, D_{i_k}$ of open subsets of M such the restrictions of f define homeomorphisms $f_j : D_{i_j} \to B(y_i, r_i)$, $j = 1, \dots k$.

By Theorem 4.1, the inverse of f_i, i.e. the map $g_j := f_j^{-1} : B(y_i, r_i) \to D_{i_j}$ is ACL, furthermore, we have $|dg_j|^p \leqslant K^{p-1} J_{g_j}$ a.e. Hence we obtain

$$|dv(z)|^p \leqslant \max_{1 \leqslant j \leqslant k} |du(g_j(z))|^p |dg_j(z)|^p \leqslant K^{p-1} \sum_{j=1}^{k} |du(g_j(z))|^p J_{g_i}(z)$$

for almost every $z \in B(y_i, r_i)$. This implies

$$\int_{B(y_i,r_i)} |dv(z)|^p \, d\nu \leqslant K^{p-1} \sum_{j=1}^{k} \int_{B(y_i,r_i)} |du(g_j(z))|^p J(z, g_j) \, d\nu \leqslant K^{p-1} \cdot \int_{f^{-1}(B(y_i,r_i))} |du|^p \, d\mu.$$

From Lemma 5.7, we know that $\nu(f(B_f)) = 0$ and $J_f = 0$ a.e. on B_f; we thus have from the area formula

$$\begin{aligned} \int_N |dv(z)|^p \, d\nu = \sum_{i=1}^{\infty} \int_{B(y_i,r_i)} |dv(z)|^p \, d\nu &\leqslant K^{p-1} \sum_{i=1}^{\infty} \int_{f^{-1}(B(y_i,r_i))} |du|^p \, d\mu \\ &\leqslant K^{p-1} \int_M |du|^p \, d\mu. \end{aligned}$$

□

6. Proofs of the main theorems.

6.1. Proof of Theorem A.

Let us recall the statement :

THEOREM A. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ be a continuous open and discrete mapping with bounded s-distortion where $s > (n-1)$. Assume also that f satisfies Lusin's property. If M is p-parabolic with $p = \frac{s}{s-(n-1)}$, then N is also p-parabolic.*

Proof. — Let $D \subset M$ be a compact subset with non empty interior. Because f is a continuous and open map, $f(D) \subset N$ is also a compact set with non empty interior. By Corollary 5.1 we have

$$\operatorname{Cap}_p(f(D),N) \leqslant \operatorname{Cap}_p(f(D),fM) \leqslant K^{p-1}\operatorname{Cap}_p(D,M),$$

hence if M is p-parabolic then so is N. □

6.2. Proofs of Theorems C and B.

The proofs will use the following criterion for hyperbolicity which is due to V. Gol'dshtein and M. Troyanov (see [9]).

THEOREM 6.1. — *Let M be an oriented connected Riemannian manifold M. Then the following are equivalent $\left(\frac{1}{p}+\frac{1}{q}=1\right)$:*

1) *M is p-hyperbolic;*

2) *there exists a smooth form $\alpha \in L^q(M,\Lambda^{n-1})$ such that $d\alpha \geqslant 0$ and $\int_M d\alpha \neq 0$;*

3) *there exists a form $\alpha \in L^q(M,\Lambda^{n-1})$ such that $d\alpha \geqslant 0$ and $\int_M d\alpha \neq 0$;*

4) *$H^n_{\mathrm{comp},q}(M) = 0$.*

□

The cohomology space $H^n_{\mathrm{comp},q}(M)$ is the space of all closed differential forms of degree n with compact support modulo the differential of $(n-1)$-forms in L^q.

We will also need the following lemma :

LEMMA 6.1. — *Let $f : M \to N$ be a mapping of class $W^{1,1}_{\mathrm{loc}}$ with essentially finite multiplicity and bounded q-codistortion: $|\Lambda_{n-1}f|^q \leqslant K\, J_f$. Then*

$$\Lambda_{n-1}f : L^q(N, \Lambda^{n-1}) \to L^q(M, \Lambda^{n-1})$$

is a bounded operator with norm $\leqslant (K \cdot N_f(M))^{1/q}$.

(Recall that $N_f(A) = \operatorname{ess\,sup}_y \operatorname{Card}(A \cap f^{-1}(y))$ for any set $A \subset M$.)

Proof. — Let $\beta \in L^q(N, \Lambda^{n-1})$, then

$$\begin{aligned}\int_M |\Lambda_{n-1}f(\beta)|^q d\mu &\leqslant K \int_M |\beta_{f(x)}|^q J_f(x)\, d\mu \\ &= K \int_N \left(\sum_{f(x)=y} |\beta_{f(x)}|^q\, \chi_{M\setminus E_f}(x) \right) d\nu \\ &\leqslant (K \cdot N_f(M)) \int_N |\beta|^q d\nu.\end{aligned}$$

□

We now prove Theorem C; we restate it in the following form:

THEOREM C. — *Let $f : M \to N$ be a mapping of essentially finite multiplicity with bounded q-codistortion where $q > 1$ and such that $J_f > 0$ on some set of positive measure. Assume furthermore either*

1) *$f \in W^{1,n-1}_{\mathrm{loc}}(M, N)$ and $|\Lambda_{n-1}f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, or*

2) *f is almost absolutely continuous, $f \in W^{1,s}_{\mathrm{loc}}(M, N)$ for some $s > (n-1)$ and $J_f \in L^1_{\mathrm{loc}}(M)$.*

If N is p-hyperbolic with $p = \frac{q}{q-1}$, then M is also p-hyperbolic

Proof. — Let us choose a bounded Borel set $U \subset M$ such that U has positive measure, $f(U)$ is bounded and $J_f > 0$ on U. Observe that, by the area formula, $\nu(f(U)) > 0$.

Choose a non negative smooth function $h : N \to \mathbb{R}$ with compact support and such that $h > 0$ in a neighborhood of $f(U)$. Since N is p-hyperbolic, $H^n_{\mathrm{comp},q}(N) = 0$, hence there exists an $(n-1)$-form $\beta \in L^q(N, \Lambda^{n-1})$ such that $d\beta = h \cdot \omega_N$ (ω_M and ω_N are the volume forms of M and N respectively).

By Lemma 6.1, we have $\alpha := f^*\beta \in L^q(M, \Lambda^{n-1})$. We then have from Lemma 3.4

$$d\alpha = f^*(d\beta) = (h \circ f) \cdot f^*\omega_N = (h \circ f) \cdot J_f\, \omega_M.$$

Thus $d\alpha \geqslant 0$ and $\int_M d\alpha \geqslant \int_U (h \circ f) \cdot J_f\, d\mu > 0$ and we conclude by Theorem 6.1 that M is p -hyperbolic. □

Finally, we deduce Theorem B from Theorem C.

Theorem B. — *Let $f \in W^{1,s}_{\mathrm{loc}}(M, N)$ be a mapping of essentially finite multiplicity with bounded s-distortion where $s > (n-1)$. Assume either*

1) $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, *or*

2) f *is almost absolutely continuous and* $J_f \in L^1_{\mathrm{loc}}(M)$.

If M is p-parabolic and N is p-hyperbolic with $p = s/(s-(n-1))$, then f is constant a.e.

Proof. — Let $q = p/(p-1)$. Then $s = q(n-1)$ and from Lemma 2.1 we know that if f has bounded s-distortion, then it has bounded q-codistortion. Hence by Theorem C, we have $J_f = 0$ a.e. and thus $|df| = 0$ a.e. since $|df|^s \leqslant K J_f$. As f is a Sobolev mapping, we conclude that f is constant a.e. □

6.3. Proof of Theorem D.

Lemma 6.2. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ be a mapping with bounded s-distortion and essential finite multiplicity. Then $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$ is a bounded operator with operator norm at most $(K\, N_f(M))^{1/s}$.*

Proof. — Let us first consider a function $v \in C^1(N) \cap \mathcal{L}^{1,s}(M)$. Then $u := f^*v \in W^{1,1}_{\mathrm{loc}}(M)$ and $du_x = df_x^t(dv_{f(x)})$. Hence we have almost everywhere $|du|^s \leqslant |dv|^s |df|^s \leqslant K |dv|^s\ J_f$. From the area formula we thus obtain

$$\begin{aligned}
\int_M |du_x|^s\, d\mu(x) &\leqslant K \int_M |dv_{f(x)}|^s J_f(x)\, d\mu(x) \\
&= K \int_N \left(\sum_{f(x)=y} |dv_{f(x)}|^s\, \chi_{M\setminus E_f}(x) \right) d\nu(y) \\
&\leqslant K\, N_f(M) \int_N |dv|^s\, d\nu(y).
\end{aligned}$$

Thus $u \in \mathcal{L}^{1,s}(M)$ and $\|u\|_{\mathcal{L}^{1,s}(M)} \leqslant (K\,N_f(M))^{1/p}\,\|v\|_{\mathcal{L}^{1,s}(N)}$.

Using the argument on page 673 of [44], we can extend this estimate from functions $v \in C^1(N) \cap \mathcal{L}^{1,s}(M)$ to all functions $v \in \mathcal{L}^{1,s}(N)$. This proves that the norm of the operator $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$ is bounded by $(K\,N_f(M))^{1/s}$. ◻

Recall the statement of Theorem D:

THEOREM D. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ be a continuous non constant proper mapping with bounded s-distortion of finite essential multiplicity. If N is s-parabolic then so is M.*

Proof. — Let $D \Subset M$ be a compact set; then $D' = f(D) \subset N$ is also compact and, by hypothesis, it has zero p-capacity. For each $\varepsilon > 0$, one can thus find a continuous function $v \in \mathcal{L}^{1,s}(N)$ with compact support and such that $v \equiv 1$ on D' and $\int_N |dv|^s \leqslant \varepsilon$.

Since f is a proper map, the function $u := f^*(v)$ also has compact support and, clearly, $u \equiv 1$ on D. Let A be the norm of the operator $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$; we know by Lemma 6.2 that A is finite. We then have $\int_M |du|^s \leqslant A^s \int_N |dv|^s \leqslant A^s\varepsilon$. Hence D has zero p-capacity and we conclude that M is p-parabolic. ◻

7. Complements.

7.1. A topological result.

A famous theorem of Yu. Reshetnyak states that a non constant quasi-regular mapping is open and discrete. We formulate below a generalization of this theorem established recently by S. Vodop'yanov's in [41], which provides topological properties for mappings with integrable distortion.

THEOREM 7.1. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M, N)$ be a continuous non constant mapping with nonnegative Jacobian $J_f(x) \geqslant 0$ and $K(x) = \frac{|df_x|^n}{J_f(x)} \in L^p_{\mathrm{loc}}(M)$ for some $n-1 < p \leqslant \infty$. Assume either*

1) *$|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, or*

2) *f is almost absolutely continuous and $J_f \in L^1_{\mathrm{loc}}(M)$.*

Then f is discrete and open.

Remarks. — 1) If the manifolds are two-dimensional, then the condition $n-1 < p \leqslant n$, can be relaxed to $1 \leqslant p \leqslant 2$.

(2) This result was also proven in [13] and [21] under the assumption $f \in W^{1,n}_{\mathrm{loc}}(M,N)$. It has been also recently proved in [18] under different analytical assumptions.

As a consequence of Theorem 7.1 we obtain topological properties for mappings with bounded s-distortion. The next assertion gives a positive answer to the question 10.8 of [28].

Corollary 7.1. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ be a continuous non constant mapping with bounded s-distortion where $n-1 < s \leqslant n$. Assume either*

1) $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, *or*

2) f *is almost absolutely continuous and* $J_f \in L^1_{\mathrm{loc}}(M)$.

Then f is discrete and open.

Remark. — This result does not hold if $s > n$. Consider for instance the map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$f(x) = \begin{cases} 0 & \text{if } \ |x| \leqslant 1, \\ \big||x|-1\big|^{\alpha} \frac{x}{|x|} & \text{if } \ 1 \leqslant |x| \leqslant 2, \\ \big||x|-1\big| \frac{x}{|x|} & \text{if } \ |x| \geqslant 2, \end{cases}$$

for some $\alpha > 1$. Then f is Lipschitz and has bounded s-distortion for $s = \frac{n\alpha - 1}{\alpha - 1} > n$. Clearly f is neither open nor discrete; however f has finite essential multiplicity.

Proof. — We suppose that $|df|^s \leqslant C J_f$ a.e. for some $n-1 < s < n$. Let us define the function

$$K_f(x) = \begin{cases} \frac{|df_x|^n}{J_f(x)} & \text{if } \ J_f(x) \neq 0, \\ 1 & \text{else.} \end{cases}$$

Set $p = \frac{s}{n-s}$, we have at almost all points where $J_f(x) \neq 0$,

$$|K_f|^p = \frac{|df|^{np}}{J_f^p} \leqslant C^{np/s} \frac{J_f^{np/s}}{J_f^p} = C^{np/s} J_f^{p(n/s-1)} \leqslant C^{np/s} J_f.$$

Thus $K_f \in L^p_{\mathrm{loc}}$. Since $n - s < 1$, we have $p > s > n-1$ and we can conclude the proof from Theorem 7.1. □

COROLLARY 7.2. — *Let $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ be a mapping with bounded q-codistortion where*

$$\frac{(n-1)^2}{1+(n-1)(n-2)} < q \leqslant \frac{n(n-1)}{1+n(n-2)}. \tag{5}$$

Assume that $J_f > 0$ a.e. and either

1) *$|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\mathrm{loc}}(M)$, or*

2) *f is almost absolutely continuous and $J_f \in L^1_{\mathrm{loc}}(M)$.*

Then f is discrete and open.

Proof. — By Lemma 2.1, f has bounded s-distortion for $s = \frac{q}{(n-1)-q(n-2)}$; observe that the inequalities (5) are equivalent to $n-1 < s \leqslant n$. Thus the corollary follows from Corollary 7.1. □

7.2. On p-parabolic manifolds.

A connected oriented Riemannian n-manifold M is called p-parabolic, $1 \leqslant p < \infty$, if $\mathrm{Cap}_p(C,M) = 0$ for all compact subsets $C \Subset M$ and p-hyperbolic otherwise. In this section, we list some facts concerning p-parabolicity. We refer to [37], [10], and [45] for further information on this notion.

a) If M contains one compact subset with nonempty interior having zero p-capacity then M is p-parabolic.

b) The Euclidean space $\mathbb{R}^n$ is p-hyperbolic for $p < n$ and p-parabolic for any $p \geqslant n$.

c) If M is p-hyperbolic, then any domain $\Omega \subset M$ is also p-hyperbolic.

d) If a closed subset $S \subset M$ with Hausdorff dimension $> (n-p)$ is removed from any manifold M and if $M \setminus S$ is connected, then $M \setminus S$ is p-hyperbolic.

e) In particular, if one removes a point x_0, then $M \setminus \{x_0\}$ is p-hyperbolic for all $p > n$ and if one removes a non separating closed subset with nonempty interior $D \subset M$, then $M \setminus D$ is p-hyperbolic for all $p \geqslant 1$.

f) If the manifold is complete and $\mathrm{Vol}(B(x_0,r)) \leqslant \mathrm{const.}\, r^d$ then M is p-parabolic for any $p \geqslant d$ (finer estimates relating the volume growth to parabolicity are in fact available).

g) If the isoperimetric inequality

$$\operatorname{Area}(\partial\Omega)^{d/(d-1)} \geqslant \text{const.} \operatorname{Vol}(\Omega)$$

holds for any big smooth domain $\Omega \subset M$, then M is p-hyperbolic for $p < d$.

h) Suppose that a Sobolev inequality

$$\|u\|_{L^q} \leqslant \text{const.} \|\nabla u\|_{L^p}$$

holds for some $1 \leqslant q \leqslant \infty$ and all functions $u \in C_0^1(M)$. Then M is p-hyperbolic.

Recall that the p-Laplacian is the operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. *A function is called p-superharmonic if* $\Delta_p u \leqslant 0$.

i) M is p-parabolic if and only if every positive p-superharmonic function on M is constant.

j) M is p-hyperbolic if and only if there exists a positive Green function for the p-Laplacian.

k) M is 2-hyperbolic if and only if the Brownian motion is transient.

l) If M has finite volume, then there exists a number $d \in [1, \infty]$ such that M is p-parabolic for $1 \leqslant p < d$ and p-hyperbolic for $p > d$.

m) For a non compact manifold with bounded geometry, we have the opposite behaviour: there exists a index d , called the *parabolic dimension* of M, such that M is p-hyperbolic for $1 \leqslant p < d$ and p-parabolic for $p > d$.

n) The parabolic dimension is a quasi-isometric invariant of manifolds with bounded geometry.

o) n-parabolicity is a quasi-conformal invariant for any manifolds.

Proof. — The proofs of (a)-(h) and (l)-(n) can be found in [37]. The proofs of (i) and (j) are in [14] (see also [17]). We refer to [10] for (k) and [45] for (o).

7.3. An improvement of a result by Pierre Pansu.

The following result gives an improvement of our Theorem B for Sobolev homeomorphisms with Lusin's property between manifolds with bounded geometry. It was proved by P. Pansu for diffeomorphisms, see [28, corollaire 2.1].

THEOREM. — *Let M and N be Riemannian manifolds with bounded geometry, and assume that N satisfies an isoperimetric inequality of order $d > n$:*

$$\mathrm{Area}(\partial\Omega)^{d/(d-1)} \geqslant \mathrm{const.}\,\mathrm{Vol}(\Omega)$$

for all smooth compact domain $\Omega \subset N$ of volume $\geqslant 1$ (in particular N is n-hyperbolic).

If $d\frac{n-1}{d-1} < s < n$, then every homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(M,N)$ with bounded s-distortion satisfying Lusin's property is a rough quasi-isometry.

Proof. — We know that if f has bounded s-distortion, $s > (n-1)$ and satisfies Lusin's property, then f^{-1} has bounded p-distortion where $p = \frac{s}{s-(n-1)}$ (see Theorem 4.1 and the Remark at the end of Section 4). The above theorem thus follows from [28, Théorème 1].

7.4. On Reshetnyak's proof for the case of quasi-regular mappings.

In order to illustrate the alternative approach based on methods of non-linear potential theory, we give a short proof of Liouville's theorem for quasi-regular mappings along Reshetnyak's ideas.

THEOREM. — *Let $f : M \to N$ be a non constant quasi-regular mapping between oriented n-dimensional Riemannian manifolds. Assume that M is n-parabolic, then so is N.*

Proof. — Assume that $f : M \to N$ is a non constant quasi-regular mapping, then it is known (see [31, Th. 6.4, chap. II]) that f is an open map; in particular $N' := f(M) \subset N$ is open. If N is n-hyperbolic, then so is N' and, by [14, Th. 5.2], we know that there exists a non constant positive n-superharmonic function $v : N' \to \mathbb{R}$. The function $u = f^*v = v \circ f : M \to \mathbb{R}$ is then $\mathcal{A}$-superharmonic where $\mathcal{A}$ is the pull back to M of the operator $TN' \to TN'$ given by $\eta \to |\eta|^{n-2}\eta$ (see [31, Th. 11.2, chap. II] or [12, Th. 14.42]). By [14, Th. 5.2] again, one concludes that M is also n-hyperbolic, contradicting the hypothesis. $\square$

Final remarks. — 1) The argument of Martio, Väisälä and Rickman are based on capacity estimates in the spirit of our proof of Theorem A (see [22]).

2) Another proof can be found in [3]. This paper gives other obstructions to the existence of quasi-regular mappings.

BIBLIOGRAPHY

[1] M. Chamberland and G. Meister, A Mountain Pass to the Jacobian Conjecture, Canad. Math. Bull., Vol. 41 (1998), 442–451.

[2] V. M. Chernikov and S. K. Vodop'yanov, Sobolev Spaces and Hypoelliptic Equations, II, Siberian Advances in Mathematics, 6 (1996), 64–96.

[3] T. Coulhon, I. Holopainen and L. Saloff-Coste, Harnack Inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems, Geom. Funct. Anal., 11 (2001), 1139–1191.

[4] L. C. Evans and R. F. Gariepy, Measure Theory and Fine properties of Functions, Studies in Advanced Mathematics, CRC press (1992).

[5] H. Federer, Geometric Measure Theory, Grundlehren der Mathematik, 153 (1969).

[6] J. Lelong-Ferrand, Étude d'une classe d'applications liées à des homomorphismes d'algèbres de fonctions et généralisant les quasi-conformes, Duke Math. J., 40 (1973), 163–186.

[7] K. Gafaïti, Algèbre de Royden et Homéomorphismes p-dilatation bornée entre espaces métriques mesurés, Thèse, EPFL Lausanne (2001).

[8] V. Gol'dshtein, L. Gurov and A. Romanov, Homeomorphisms that induce Monomorphisms of Sobolev Spaces, Israel J. of Math., 91 (1995), 31–60.

[9] V. Gol'dshtein and M. Troyanov, The Kelvin-Nevanlinna-Royden criterion for p-parabolicity, Math. Zeitschrift, 232 (1999), 607–619.

[10] A. Grigor'yan, Analytic and Geometric Background of Recurence and Non-Explosion of the Brownian Motion on Riemannian Manifolds, Bull. Amer. Math. Soc., 36 (1999), 135–242.

[11] P. Hajłasz, Change of Variables Formula under Minimal Assumptions, Colloquium Mathematicum, 64 (1993), 93–101.

[12] J. Heinonen, T. Kilpeläinen and O. Martio, Non Linear Potential Theory of Degenerate Elliptic Equations, Oxford Math. Monographs (1993).

[13] J. Heinonen and P. Koskela, Sobolev Mappings with Integrable Dilatation, Arch. Rational Mech.Anal., 125 (1993), 81–97.

[14] I. Holopainen, Non linear Potential Theory and Quasiregular Mappings on Riemannian Manifolds, Annales Academiae Scientiarum Fennicae Series A, 74 (1990).

[15] I. Holopainen, Rough isometries and p-harmonic functions with finite Dirichlet integral, Revista Mathemática Iberoamericana, 10 (1994), 143–175.

[16] T. Iwaniec, P. Koskela and J. Onninen, Mappings of finite distortion: Monotonicity and continuity, Preprint.

[17] M. Kanai, Rough isometries and the parabolicity of Riemannian manifolds, J. Math. Soc. Japan, 38, n° 2 (1986), 227–238.

[18] J. Kauhanen, P. Koskela and J. Malỳ, Mappings of finite distortion: discreteness and openess, University of Jyvv̈askylä (2000), preprint 226.

[19] J. Malỳ, Absolutely continuous functions of several variables, J. Math. Anal. Appl., 231 (1999), 492–508.

[20] J. Malỳ and O. Martio, Lusin's condition (N) and mappings of the class $W^{1,n}$, J. Reine Angew. Math., 458 (1995), 19–36.

[21] J. J. Manfredi and E. Villamor, An extension of Reshetnyak's theorem, Indiana Univ. Math. J., 47, n° 3 (1998), 1131–1145.

[22] O. Martio, S. Rickman and J. Väisälä, Definitions for Quasiregular Mappings, Ann. Acad. Sc. Fenn., 448 (1969), 5–40.

[23] O. Martio and W. P. Ziemer, Lusin's condition (N) and mappings of non negative Jacobian, Michigan. Math. J., 39 (1992), 495–508.

[24] V.G. Maz'ya, Sobolev Spaces, Springer Verlag, 1985.

[25] V. Maz'ya and T. Shaposhnikova, Theory of Multipliers in Spaces of Differentiable Functions, Pitman, 1985.

[26] S. Müller, T. Qi and B.S. Yan, On a new class of elastic deformations not allowing for cavitation, Ann. Inst. Henri Poincaré, 11 (1994), 217–243.

[27] S. Müller and S. Spector, An existence theory for nonlinear elasticity that allows for cavitation, Arch. Rational Mech. Anal., 131, n° 1 (1995), 1–66.

[28] P. Pansu, Difféomorphismes de p-dilatation bornées, Ann. Acad. Sc. Fenn., 223 (1997), 475–506.

[29] P. Pansu, Cohomologie L^p, espaces homogènes et pincement, preprint, Orsay 1999.

[30] M. Reiman, Über harmonishe Kapazität und quasikonforme Abbildungen in Raum., Comm. Math. Helv., 44 (1969), 284–307.

[31] Yu. G. Reshetnyak, Space Mappings with Bounded Distortion, Translation of Math. Monographs, 73 (1989).

[32] Yu. G. Reshetnyak, Mappings with bounded distortion as extremals of integrals of Dirichlet type, Siberian Math. Journal, 9 (1968), 652–666.

[33] Yu. G. Reshetnyak, Sobolev classes of functions with values in a metric space, Siberian Math. Journal, 38 (1997), 657–675.

[34] S. Rickman, Quasi-regular Mappings, Springer Ergebnisse der Mathematik, 26 (1993).

[35] S. Rickman, Topics in the Theory of Quasi-regular Mappings, Aspekte der Mathematik, E12 (Kulkarni and Pinkall eds) (1988).

[36] C. J. Titus and G. S. Young, The extension of interiority, with some applications, Trans. Amer. Math. Soc., 103 (1962), 329–340.

[37] M. Troyanov, Parabolicity of Manifolds, Siberian Adv. Math., 9, n° 4 (1999), 1–25.

[38] S. K. Vodop'yanov, Taylor Formula and Function Spaces (in Russian), Novosibirsk University (1988).

[39] S. K. Vodop'yanov, Geometric Properties of Function Spaces (in Russian), D. Sc. Thesis, Novosibirsk, Sobolev Institute of Mathematics (1992).

[40] S. K. VODOP'YANOV, Monotone Functions and Quaiconformal Mappings on Carnot Groups, Siberian Math. J., 37, n^{o} 6 (1996), 1269–1295.

[41] S. K. VODOP'YANOV, Topological and Geometric Properties of Mappings of Sobolev Spaces with Integrable Jacobian I, Siberian Math. J., 41, n^{o} 1 (2000), 23–48.

[42] S. K. VODOP'YANOV, Mappings with Bounded Distortion and with Finite Distortion on Carnot Groups, Siberian Math. J., 40, n^{o} 4 (1999), 764–804.

[43] S. VODOP'YANOV and V. GOL'DSHTEIN, Quasi-conformal mapping and spaces of functions with generalized first derivatives, Siberian Math. J., 17, n^{o} 4 (1977), 515–531.

[44] S. VODOP'YANOV and A. UKHLOV, Sobolev Spaces and (P, Q)-Quasiconformal Mappings of Carnot Groups, Siberian Math. J., 39, n^{o} 4 (1998), 665–682.

[45] V. A. ZORICH and V. M. KESEL'MAN, On the Conformal Type of a Riemannian Manifold, Func. Anal. and Appl., 30 (1996), 106–117.

[46] V. A. ZORICH, A theorem of M.A. Lavrent'ev on quasiconformal space maps, Mat.Sb., 74, 116 (1967), 417–433; English transl. in Math. USSR Sb., 3 (1967).

Manuscrit reçu le 7 novembre 2000,
révisé le 26 avril 2002,
accepté le 26 juin 2002.

Marc TROYANOV,
E.P.F.L.
Institut de Mathématiques
CH-1015 Lausanne (Switzerland).
marc.troyanov@epfl.ch

Sergei VODOP'YANOV,
Sobolev Institute of Mathematics
Novosibirsk 630090 (Russia).
vodopis@math.nsc.ru