LIOUVILLE TYPE THEOREMS FOR MAPPINGS WITH BOUNDED (CO)-DISTORTION

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1. Introduction.

A mapping \( f : M \to N \) between oriented \( n \)-dimensional Riemannian manifolds is said to have bounded \( s \)-distortion (or \( s \)-dilatation) \( (1 \leq s < \infty) \) if \( f \in W^{1,1}_{\text{loc}}(M, N) \) and

\[
|df_x|^s \leq K J_f(x)
\]

a.e. \( x \in M \).

The Sobolev class of mappings \( W^{1,1}_{\text{loc}}(M, N) \) is defined in Section 3 below; these mappings have a formal differential \( df_x : T_x M \to T_{f(x)} N \) almost everywhere; in the above inequality, \( |df_x| \) denotes its operator norm and \( J_f(x) = \det df_x \) its Jacobian.

Mappings with bounded \( s \)-distortion are generalizations of quasi-regular mappings; they have been studied (under various names and viewpoints) since about 30 years, see [6], [8], [24], [25], [28], [30], [41], [44] among other works. In the special case of homeomorphisms with bounded \( s \)-distortion with \( s > n - 1 \), a metric characterization has been given in [8].

These mappings originated as suitable class of mappings in the change-of-variable formula for functions in the Sobolev spaces \( L^{1,s} \) (see Section 4). As it turns out, this class of mappings feels quite well the

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asymptotic geometry of Riemannian manifolds. In [6], J. Ferrand was able to prove that a Riemannian manifold is characterized up to bilipschitz equivalence by its Royden algebra; the proof heavily uses the theory of mappings with bounded $s$-distortion. In [28], P. Pansu gave conditions on the geometry of manifolds implying that mappings with bounded $s$-distortion are quasi-isometries. The work of Ferrand and Pansu has been extended to the case of metric measure spaces in the recent thesis of K. Gafaïti.

Mapping with bounded $s$-distortion are a subclass of the so-called mappings with finite distortion which are defined by the condition that $J_f \in L^1_{\text{loc}}$ and $|df_x|^n \leq \Phi(x)J_f(x)$ where $\Phi(x)$ is finite a.e. (see the argument in the proof of Corollary 7.1). Mappings with finite distortion play an important role in non-linear elasticity (see e.g. [27]) and they are now being intensively studied. See e.g. the papers [16], [18], [41] and the rich references therein.

Another important generalization of mappings with bounded $s$-distortion is given by the class of mappings such that $|\Lambda_k f_x|^q \leq K J_f(x)$ where $\Lambda_k f$ is the $k$-th. exterior power of $df$, i.e. the effect of $df_x$ at the level of $k$-forms. These mappings appear in $L^p$ cohomology; see e.g. the recent paper [29] of P. Pansu, where flows of such mappings are used in the computation of $L^p$-cohomology of manifolds with negative curvature and solvable Lie groups.

In the present paper, we will consider the case $k = n - 1$; let us thus define a mappings with bounded $q$-codistortion (1 $q$ $\infty$) to be a mapping $f \in W^{1,1}_{\text{loc}}(M, N)$ for which there exists a constant $K'$ such that

$$|\Lambda_{n-1} f_x|^q \leq K' J_f(x) \quad \text{a.e.}$$

We now state a number of questions, concerning mappings with bounded $s$-distortion, we are interested in

1) What are the obstructions to the existence of a non constant mapping with bounded $s$-distortion $f : M \to N$?

2) Describe the set of all $s \geq 1$ for which there exists a homeomorphism (or a diffeomorphism) $f : M \to N$ with bounded $s$-distortion.

3) Suppose that $f : M \to N$ is a non constant mapping with bounded $s$-distortion: How big may the omitted set $N \setminus f(M)$ be? (In particular, when can it be said that $f$ is onto?)

4) Assuming that $f : M \to N$ is a continuous mapping with bounded
s-distortion. What can be said about the topological and or the geometrical properties of $f$?

Similar questions may be raised about mappings with bounded $q$-codistortion.

We will give some answers to all of these questions. The main techniques we use are based on potential theory: Recall that a condenser in $M$ is a pair $(D, A)$ where $A \subset M$ is a connected open subset and $D \subset A$ is compact. The $p$-capacity of the condenser $(D, A)$ is defined as

$$\text{Cap}_p(D, A) = \inf \left\{ \int_A |\nabla u|^p : u \in C^0_0(A) \cap W^{1,p}(A) \text{ and } u \geq 1 \text{ on } D \right\}.$$ 

**Definition.** — The manifold $M$ is $p$-parabolic if $\text{Cap}_p(D, M) = 0$ for all compact subsets $D \subset M$ and $p$-hyperbolic otherwise.

We have included in Section 7.2 below a brief discussion of this notion.

An answer to the first question above is the following Liouville type theorem:

**Theorem.** — Let $M$ and $N$ be oriented $n$-dimensional Riemannian manifolds and let $f \in W^{1,n}_{\text{loc}}(M, N)$ be a mapping with bounded $s$-distortion with $s > (n - 1)$. Assume that $M$ is $p$-parabolic, where $\frac{1}{p} + \frac{n-1}{s} = 1$. Then either $f$ is constant a.e. or $N$ is also $p$-parabolic.

This result is a consequence of Theorem A and Corollary 7.1 in the present paper; it is in fact proved for a wider class than $W^{1,n}_{\text{loc}}(M, N)$.

In the special case of quasi-regular mappings (i.e. $s = p = n$), this result has been obtained around 1968 by Y. Reshetnyak and, independently, by O. Martio, S. Rickman and J. Väisälä (see [32] and [22]). At the end of the paper we shortly recall the original argument of Reshetnyack.

Some answers to the other questions mentioned above are given in Sections 2, 4 and 7.

The paper is organized as follow: In Section 2, we give some additional definitions, state the main results of the paper and give some corollaries. In Section 3 we recall some basic facts about Sobolev mappings, in Section 4 we discuss homeomorphisms with bounded $s$-distortion and in Section 5 we prove a capacity inequality. After these preparations, we prove the main theorems in Section 6. Finally, in Section 7, we give some complementary information on mappings with bounded $s$-distortion.
2. Definitions and statement of the results.

Throughout the paper $M$ and $N$ are oriented, connected $n$-dimensional Riemannian manifolds. We denote by $d\mu$ and $d\nu$ the volume elements of $M$ and $N$ respectively.

In order to state our results, we need some additional definitions:

**Definitions.**

1. The map $f$ has essentially finite multiplicity if $N_f(M) < \infty$, where
   
   $$N_f(A) := \text{ess sup}_y \text{Card}(f^{-1}(y) \cap A)$$
   
   for any measurable subset $A \subset M$.

2. A continuous map is open and discrete if the image of any open set $U \subset M$ is an open set $f(U) \subset N$ and the inverse image $f^{-1}(y)$ of any point $y \in N$ is a discrete subset of $M$. The branch set of such a mapping is the set $B_f \subset M$ of points $x \in M$ such that $f$ is not a local homeomorphism in a neighborhood of $x$.

The next two definitions are regularity assumptions. They are always satisfied if one assumes e.g. that $f$ is locally Lipschitz, or that $f \in W^{1,s}_{\text{loc}}(M,N)$ for $s > n$, or that $f$ is locally quasi-regular.

3. A measurable map $f : M \to N$ satisfies Lusin’s property if the image of any set $E \subset M$ of measure zero is a set $f(E) \subset N$ of measure zero.

An important and well-known result (see Proposition 3.2) states that for any map $f : M \to N$ belonging to $W^{1,1}_{\text{loc}}(M,N)$ there exists a sequence of compact sets $A_i \subset M$ such that the restriction of $f$ to each $A_i$ is Lipschitz and the complementary set $E_f := M \setminus \cup_i A_i$ has measure zero. We call $E_f$ the exceptional set of $f$.

4. The map $f \in W^{1,1}_{\text{loc}}(M,N)$ is almost absolutely continuous if it is continuous and for any bounded domain $\Omega \subset M$ the following property holds: for any $\varepsilon > 0$ we can find $\delta = \delta(\Omega, \varepsilon) > 0$ such that for any finite or infinite sequence of pairwise disjoint balls $\{B(x_i, r_i)\}$ contained in $\Omega$ with center $x_i \in E_f$, we have
   
   $$\sum \text{vol}(B_i) \leq \delta \quad \Rightarrow \quad \sum (\text{diam}(fB_i))^n < \varepsilon.$$ 

**Remark.** — The notion of almost absolute continuity appeared in [41], [42]; it is a generalization of absolute continuity in the sense of Malý.
as defined in [19]. In particular any mapping in $W^{1,p}_{\text{loc}}(M,N)$ with $p > n$ and any continuous mapping in $W^{1,n}_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ with monotone coordinate functions is an example of almost absolutely continuous mapping, see [19].

In dimension 2, a mapping has bounded $s$-distortion if and only if it has bounded $s$-codistortion. In higher dimension, we have the following relation between distortion and codistortion:

**Lemma 2.1.** — Let $f : M \to N$ be a mapping with bounded $s$-distortion for some $s > n - 1$, then $f$ has bounded $q$-codistortion for $q = s/(n - 1)$.

Conversely, if $f : M \to N$ is a mapping with bounded $q$-codistortion for some $q < \frac{n - 1}{2}$ such that $J_f > 0$ a.e., then $f$ has bounded $s$-distortion for $s = \frac{q}{(n-1)-q(n-2)}$.

The exponents in this lemma are sharp.

**Proof.** — It is a trivial consequence of the inequalities

$$|\Lambda_{n-1}f_x| \leq |df_x|^{n-1} \quad \text{and} \quad |df_x| J_f(x)^{n-2} \leq |\Lambda_{n-1}f_x|^{n-1}.$$  

We now state the main results of the present paper:

**Theorem A.** — Let $f \in W^{1,s}_{\text{loc}}(M,N)$ be a continuous open and discrete mapping with bounded $s$-distortion, where $s > (n - 1)$, satisfying Lusin’s property. If $M$ is $p$-parabolic with $p = \frac{s}{s-(n-1)}$, then $N$ is also $p$-parabolic.

Recall that a map $f \in W^{1,s}_{\text{loc}}$ always satisfies Lusin’s property if $s > n$. In Section 3 below we give other sufficient conditions. In Section 7.1 below, we will also give sufficient conditions for a continuous mapping with bounded $s$-distortion to be discrete and open.

The next result is an analog of Theorem A. It holds without any topological restrictions but assumes that $f$ has finite essential multiplicity:

**Theorem B.** — Let $f \in W^{1,s}_{\text{loc}}(M,N)$ be a mapping of essentially finite multiplicity with bounded $s$-distortion where $s > (n - 1)$. Assume either

1) $|\Lambda_{n-1}f| \in L^{n/(n-1)}_{\text{loc}}(M)$, or

2) $f$ is almost absolutely continuous and $J_f \in L^1_{\text{loc}}(M)$. 

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If $M$ is $p$-parabolic with $p = \frac{s}{s-(n-1)}$, then either $f$ is constant a.e. or $N$ is also $p$-parabolic.

In Theorem B (under assumption 2) no continuity is assumed. The proofs of theorems A and B are based on quite different approaches; it would be interesting to have a unified method proving both results.

Remark 1. — These results are sharp. They say for instance that there is no mapping of finite essential multiplicity with bounded $s$-distortion from the Euclidean space to the hyperbolic space for $s > (n-1)$. This is optimal since the Riemannian exponential $\exp : T_0 \mathbb{H}^n \to \mathbb{H}^n$ (where $\mathbb{H}^n$ is the hyperbolic space) is a diffeomorphism with bounded $(n-1)$-distortion. Other comments on the optimality of these results are given in [7].

Theorem B will be obtained as a consequence of the following result on mappings with bounded codistortion:

**Theorem C.** — Let $f : M \to N$ be a mapping of essentially finite multiplicity with bounded $q$-codistortion where $q > 1$. Suppose that $J_f > 0$ on some set of positive volume. Assume furthermore either

1) $f \in W^{1,n-1}_{loc}(M,N)$ and $|\Lambda_{n-1}f| \in L^{n/(n-1)}_{loc}(M)$, or

2) $f$ is almost absolutely continuous, $f \in W^{1,s}_{loc}(M,N)$ for some $s > (n-1)$ and $J_f \in L^1_{loc}(M)$.

If $M$ is $p$-parabolic with $p = q/(q-1)$, then $N$ is also $p$-parabolic.

Remark 2. — The condition that $J_f > 0$ on some set of positive volume cannot be replaced by the weaker condition that $f$ is not constant a.e. For instance, look at the hyperbolic three-space in the upper-half space model $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 | z > 0\}$ (with metric tensor $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$). Then the mapping $f : \mathbb{R}^3 \to \mathbb{H}^3$ given by $f(x,y,z) = (x,0,1)$ is of finite essential multiplicity and has bounded $q$-codistortion for all $q \geq 1$. Yet $\mathbb{H}^3$ is $p$-hyperbolic for all $p$ and $\mathbb{R}^3$ is $p$-parabolic for all $p \geq 3$.

The next result goes in the other direction:

**Theorem D.** — Let $f \in W^{1,1}_{loc}(M,N)$ be a continuous non constant proper mapping with bounded $s$-distortion of finite essential multiplicity. If $M$ is $s$-hyperbolic, then so is $N$.

Remark 3. — The hypothesis that $f$ is proper is necessary. For
instance if $N$ is a compact manifold and $M \subset N$ is an open domain whose complement $N \setminus M$ has non empty interior, then $N$ is $s$-parabolic for all $s$ and $M$ is $s$-hyperbolic for all $s \in [1, \infty]$. Yet the inclusion $f : M \hookrightarrow N$ has bounded $s$-distortion for all $s$.

We now give some applications of our results. We begin by a Picard type theorem for mappings with bounded $s$-distortion.

**Corollary 2.1.** — *Let $f : M \rightarrow N$ be a continuous mapping with bounded $s$-distortion, $s > (n-1)$ satisfying the hypothesis of Theorem A. Assume that the manifold $M$ is $p$-parabolic where $p := \frac{s}{s-(n-1)}$. Then $f$ is surjective if $p > n$, and the omitted set $N \setminus f(M)$ has Hausdorff dimension $\leq (n-p)$ if $p \leq n$.*

**Proof.** — Observe that $f$ actually maps $M$ onto $N' = f(M)$ (which is an open subset of $N$). By Theorem A, the manifold $N'$ is thus $p$-parabolic and therefore the Hausdorff dimension of $N \setminus N'$ is $\leq n - p$. ⊓⊔

For a quasiregular mapping on Euclidean space $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a stronger result is due to S. Rickman. He proved that $f$ omits at most finitely many points (see theorem 2.1 in [34], chapter IV]).

**Corollary 2.2.** — *Let $f : M \rightarrow N$ be an injective $C^1$ mapping with bounded $q$-codistortion. Assume that $q < \frac{n}{n-1}$ and that $M$ is $p$-parabolic with $p = \frac{q}{q-1}$, then $f$ is a diffeomorphism.*

For the proof of this corollary, will need a lemma. Recall that the principal dilatation coefficients (or singular values) at $x \in M$ of a mapping $f \in W^{1,1}_{\text{loc}}(M, N)$ are the square roots $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of the eigenvalues of $df_x df_x^T$; they are defined almost everywhere. Observe the following useful inequalities:

$$
|df_x| = \lambda_n, \quad |J_f(x)| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n, \quad |\Lambda_{n-1} f_x| = \lambda_2 \cdot \lambda_3 \cdots \lambda_n.
$$

**Lemma 2.2.** — *Let $f : M \rightarrow N$ be a mapping with bounded $q$-codistortion. If $q < \frac{n}{n-1}$ then either $J_f = 0$ a.e. or there exists a constant $\delta > 0$ such that all the principal dilatation coefficients are almost everywhere $\geq \delta$. 

**Proof.** — Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the dilatation coefficients of $f$ at $x$. We have by hypothesis $|\Lambda_{n-1} f_x|^q \leq K J_f(x)$ a.e., i.e. $(\lambda_2 \cdot \lambda_3 \cdots \lambda_n)^q \leq$
$K (\lambda_1 \cdot \lambda_2 \cdots \lambda_n)$. This implies $\lambda_1^{(n-1)(q-1)} \leq (\lambda_2 \cdot \lambda_3 \cdots \lambda_n)^{q-1} \leq K \lambda_1$, from which one obtains $\lambda_1 \geq K^{1/(n+q-nq)}$, provided $q < \frac{n}{n-1}$. □

Proof of Corollary 2.2. — By the previous lemma, all principal dilatation coefficients are bounded below, in particular $f$ is a local diffeomorphism. Assume now that $f$ is not surjective. Then there exists a point $y_0 \in N \setminus f(M)$. Let $N' := N \setminus \{y_0\}$, this is a $p$-hyperbolic manifold (since $p > n$). By Theorem C, the manifold $M$ must therefore be $p$-hyperbolic; but this contradicts the hypothesis and we thus conclude that $f$ is surjective. □

If $M = N = \mathbb{R}^n$, we don’t need to assume global injectivity in the previous corollary.

Corollary 2.3. — Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ mapping with bounded $q$-codistortion where $q < \frac{n}{n-1}$ and such that $J_f \neq 0$. Then $f$ is a global diffeomorphism.

Proof. — By Lemma 2.2 all the eigenvalues of $df_x^t df_t$ are uniformly bounded below. We thus conclude from a recent theorem of M. Chamberland and G. Meister that $f$ is injective (see [1], th. 1.1).

Now set $p := \frac{n}{q-1}$, then $p > n$ and hence $\mathbb{R}^n$ is $p$-parabolic. We conclude the proof from the previous corollary. □

We also have similar results for mappings with bounded $s$-distortion.

Corollary 2.4. — Let $f : M \to N$ be an injective $C^1$ mapping with bounded $s$-distortion.

Assume that $(n-1) < s < n$ and that $M$ is $p$-parabolic with $p = \frac{s}{s-(n-1)}$. Then $f$ is a diffeomorphism.

The proof is similar to that of Corollary 2.2. □

Corollary 2.5. — Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non constant $C^1$ mapping with bounded $s$-distortion where $(n-1) < s < n$. Then $f$ is a global diffeomorphism.

Proof. — This is clear from Lemma 2.1 and the previous corollaries. □

This last result also holds for $s = n \geq 3$. Indeed, V.A. Zorich has
proved that a quasi-regular mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \), which is a local homeomorphism is in fact a global homeomorphism provided \( n \geq 3 \), see [46].

3. Calculus of Sobolev mappings.

Since a mapping \( f : M \to \mathbb{R}^m \) is given by its components which are \( n \) functions: \( f = (f_1, f_2, \ldots, f_m) \), it is natural to say that \( f \) belongs to the Sobolev space \( W^{1,s}_{\text{loc}}(M, \mathbb{R}^m) \) if each component \( f_i \in W^{1,s}_{\text{loc}}(M, \mathbb{R}) \).

In the case of a continuous mapping \( f : M \to N \) between Riemannian manifolds, we may define the condition \( f \in W^{1,s}_{\text{loc}}(M, N) \) by the use of local coordinates charts; however, such a procedure is in general not possible for a discontinuous map and we have to proceed differently to define the class of Sobolev mappings between Riemannian manifolds.

We follow the approach of [33], [42].

**Definitions.**

1) The mapping \( f : M \to N \) belongs to \( L^s_{\text{loc}}(M, N) \), \( 1 \leq s \leq \infty \), if and only if the function \( [f]_y : M \to \mathbb{R} \), defined by \( [f]_y(x) = d(f(x), y) \), is in \( L^s_{\text{loc}}(M, \mathbb{R}) \) for all point \( y \in N \).

2) The map \( f \) belongs to \( W^{1,s}_{\text{loc}}(M, N) \) if and only if \( [f]_y \in W^{1,s}_{\text{loc}}(M, \mathbb{R}) \) and there exists a function \( g \in L^s_{\text{loc}}(M, \mathbb{R}) \) such that \( |\nabla [f]_y|(x) \leq g(x) \) a.e. in \( M \) for any point \( y \in N \).

3) The map \( f \) belongs to \( ACL^s_{\text{loc}}(M, N) \) if it satisfies the following three conditions:
   i) the function \( M \ni x \to [f]_z(x) = d(f(x), z) \) belongs to \( L^s_{\text{loc}}(M) \) for every point \( z \in N \);
   ii) the mapping \( f : M \to N \) is absolutely continuous on lines in the following sense: for any coordinate chart \( \varphi : U \to \mathbb{R}^n \) on \( M \), the function
      \[
      (x, \tau) \to g_i(x, \tau) := \text{length}(f \circ \varphi^{-1}([x, x + \tau e_i]))
      \]
      is absolutely continuous in the parameter \( \tau \) for all \( i \) and almost all \( x \in \mathbb{R}^n \).
   iii) the derivative \( \partial_i g_i : x \to \lim_{\tau \to +0} \frac{g_i(x, \tau)}{\tau} \), which exists almost everywhere in \( U \), belongs to \( L^s_{\text{loc}}(U) \) for all \( i \).
Proposition 3.1. — The following assertions are equivalent:

1) \( f \in W^{1,s}_{\text{loc}}(M,N) \);

2) \( f \in ACL^s_{\text{loc}}(M,N) \);

3) \( f \in L^s_{\text{loc}}(M,N) \) and there exists a function \( g \in L^s_{\text{loc}}(M,R) \) such that for any Lipschitz function \( \psi : N \to \mathbb{R} \), the function \( \varphi := \psi \circ f : M \to \mathbb{R} \) belongs to \( W^{1,s}_{\text{loc}}(M,R) \) and \( |\nabla \varphi(x)| \leq \text{Lip}(\psi) g(x) \) a.e. in \( M \).

4) for any isometric embedding \( i : N \to \mathbb{R}^k \) all coordinate functions of the composition \( i \circ f \) belong to \( W^{1,s}_{\text{loc}}(M,R) \).

Proof. — The proof follows the order (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (2). Observe that (3) \( \Rightarrow \) (1) is trivial since distance functions are 1-Lipschitz.

Then (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are proven in [42, Proposition 3] (notice that (1) \( \Rightarrow \) (3) is also proven in [33, Theorem 5.1] by other arguments).

The proof of (4) \( \Rightarrow \) (2) is given in [34, Proposition 1.2] for the special case \( N = \mathbb{R}^n \). Its extension to the case of a submanifold \( N \subset \mathbb{R}^k \) is based on the formula

\[
g_i(x, \tau) = \int_0^\tau \left| \frac{d}{dt}(f \circ \varphi^{-1}([x, x + t\mathbf{e}_i])) \right| dt
\]

which holds for all absolutely continuous curves in the \( \mathbb{R}^k \). The general case now follows from the fact that any Riemannian manifold admits an isometric embedding in some Euclidean space.

(3) \( \Rightarrow \) (4). We consider an isometric embedding \( i : N \to \mathbb{R}^k \) and some coordinate function \( z_j \) in \( \mathbb{R}^k \). The restriction \( z_j|_N \) is a Lipschitz function on \( N \), thus the composition \( z_j \circ f \) belongs to \( W^{1,s}_{\text{loc}}(M,R) \). \( \square \)

The next proposition says that a Sobolev mapping is Lipschitz on a big set.

Proposition 3.2. — Let \( f \in W^{1,1}_{\text{loc}}(M,N) \). Then there exists a measurable decomposition \( M = E_f \cup \bigcup_{i=1}^\infty A_i \) such that \( \mu(E_f) = 0 \), \( A_i \) is compact for all \( i \) and \( f|_{A_i} \) is Lipschitz.

Proof. — Using the previous proposition (assertion 4) we can reduce the proof to the well-known Whitney’s approximation theorem for Sobolev function (see e.g. [4, p. 254]). \( \square \)
As a consequence of this proposition, we have the following version of the change of variables formula for integrals (also known as the area formula), recall that $\chi_A$ denotes the characteristic function of a set $A \subset M$.

**Proposition 3.3.** — Let $f \in W^{1,1}_{\text{loc}}(M,N)$ be a Sobolev mapping between Riemannian manifolds of the same dimension. Then there exists a subset $E_f \subset M$ of measure zero such that for all measurable function $\psi : M \to \mathbb{R}_+$ we have
\[
\int_M \psi(x) |J_f(x)| \, d\mu(x) = \int_N \left( \sum_{f(x) = y} \psi(x) \chi_{M \setminus E_f}(x) \right) \, d\nu(y).
\]

If $f$ satisfies Lusin’s property, then one may take $E = \emptyset$.

See e.g. [11] for a proof.

For the area formula to be useful, we need to work with mappings having a locally integrable Jacobian. Observe in particular that if $f \in W^{1,1}_{\text{loc}}(M,N)$ has bounded $s$-distortion and $J_f \in L^1_{\text{loc}}(M)$, then we have in fact $f \in W^{1,s}_{\text{loc}}(M,N)$.

The next two lemmas give us sufficient conditions for the local integrability of the Jacobian.

**Lemma 3.1.** — Let $f : M \to N$ be a mapping such that $f \in W^{1,1}_{\text{loc}}(M,N)$ and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$. Then $J_f \in L^1_{\text{loc}}(M)$.

**Proof.** — This is a trivial consequence of the inequality $J_f \leq |\Lambda_{n-1} f|^{n/(n-1)}$.

**Lemma 3.2.** — If $f \in W^{1,1}_{\text{loc}}(M,N)$ is continuous and has essentially finite multiplicity or is open and discrete, then $J_f \in L^1_{\text{loc}}(M)$.

**Proof.** — This follows directly from the area formula.

We now give sufficient conditions for Lusin’s property:

**Lemma 3.3.** — Let $f : M \to N$ be a mapping satisfying one of the following conditions:

1) $f \in W^{1,s}_{\text{loc}}(M,N)$ with $s \geq (n-1)$, $J_f > 0$ a.e. and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$;

2) $f \in W^{1,1}_{\text{loc}}(M,N)$ is almost absolutely continuous;
3) $f \in W^{1,n}_{\text{loc}}(M,N)$ is continuous open and discrete.

Then it also satisfies Lusin’s property.

Under hypothesis (1) this is Theorem 5.3 in [26]; see also [43] for the case $s = n$. In case (2), this is Theorem 8 from [41]. In case (3), this is a result from [20]; see also [40] for a short proof.

We refer to [23] and [19] for further results on Lusin’s condition.

PROPOSITION 3.4. — If the map $f$ is continuous, open and discrete and has bounded $s$-distortion for $s > (n-1)$, then it is differentiable almost everywhere.

See Lemma 4.4 in chapter VI of Rickman’s book [34] or Proposition 1 in [41] for a more general result.

Finally we will also need the following result about the exterior differential of the pull-back of a $(n-1)$-form:

LEMMA 3.4. — Let $f : M \to N$ be a mapping satisfying one of the following conditions:

1) $f \in W^{1,n-1}_{\text{loc}}(M,N)$ and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$;

2) $f$ is almost absolutely continuous, $f \in W^{1,s}_{\text{loc}}(M,N)$ for some $s > (n-1)$ and $J_f \in L^1_{\text{loc}}(M)$.

Let $\beta$ be a smooth $(n-1)$-form. Then $\alpha := f^* \beta \in L^1_{\text{loc}}(M, \Lambda^{n-1})$ and $d\alpha = f^*(d\beta)$.

This result is proved in [26, Th. 3.2] under the first hypothesis and in [41, Th. 8] in the case of the second hypothesis.

4. On homeomorphisms with bounded $s$-distortion.

In this section, we discuss the special case of homeomorphisms with bounded $s$-distortion.

DEFINITION. — The $s$-Dirichlet space of a Riemannian manifold $M$ is the space $\mathcal{L}^{1,s}(M)$ of functions $u \in W^{1,s}_{\text{loc}}(M, \mathbb{R})$ such that $\int_M |\nabla u|^s \, d\mu < \infty$. This space is equipped with the semi-norm

$$\|u\|_{\mathcal{L}^{1,s}(M)} = \|\nabla u\|_{L^s(M)}.$$
If \( f : M \to N \) is a homeomorphism and \( v : N \to \mathbb{R} \) is any function, we denote by \( f^* v = v \circ f \) its pull back on \( M \). If \( u : M \to \mathbb{R} \), we denote by \( f_* u = u \circ f^{-1} : N \to \mathbb{R} \) its pushforward.

S. Vodop’yanov has proved the following result [38], [39] (see its generalized version in [44, Theorems 1 and 9]):

**Theorem 4.1.** — Let \( f : M \to N \) be a homeomorphism between \( n \)-dimensional Riemannian manifolds. Fix \( s \in [1, \infty) \), then the following assertions are equivalent:

1) \( f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M) \) is a bounded operator;

2) \( f \in W^{1,s}_{loc}(M,N) \) and \( f \) has bounded \( s \)-distortion: 
\[
|df(x)|^s \leq K J_f(x) \quad \text{a.e.} \quad x \in M.
\]

Moreover, if \( s \in (1, \infty) \), then condition (1) or (2) are equivalent to

3) \( f^{-1} \) decreases the \( s \)-capacities of condensers up to a constant:
\[
\text{Cap}_s(C,A) \leq \text{const} \cdot \text{Cap}_s(f(C),f(A))
\]
for any condensers \((C,A)\) in \( M \).

Finally, if \( s > (n-1) \) and Lusin’s property holds, then any condition (1)–(3) is equivalent to

4) \( f^* : \mathcal{L}^{1,p}(M) \to \mathcal{L}^{1,p}(N) \) is a bounded operator where 
\[
p = \frac{s}{s-(n-1)},
\]
and \( |df^{-1}(y)|^p \leq K^{p-1} J_{f^{-1}}(y) \) a.e. \( y \in N \), consequently \( f^{-1} \) has bounded \( p \)-distortion.

**Proof.** — We only give a short proof of the second part of assertion (4). By Proposition 3.4, the map \( f \) is differentiable a.e. and by [44, Theorem 9], we know that \( g := f^{-1} : N \to M \) is ACL (see also Lemma 5.6 below). Thus we have \( dg_f(x) \circ df_x = \text{Id} \) a.e. in \( M \). Notice also that \( J_g(y) \neq 0 \) a.e. in \( N \) since \( f \) has Lusin’s property by hypothesis, we thus have almost everywhere
\[
|dg_f(x)| \leq \frac{|df_x|^n}{J_f(x)},
\]
and therefore
\[
|dg_f(x)|^p \leq \left( \frac{|df_x|^p}{J_f(x)} \right)^{p-1} J_f^{-1}(x) \leq K^{p-1} J_{g}(f(x)).
\]

\( \square \)

A useful consequence of this theorem is the following...
Corollary 4.1. — If \( f : M \to N \) and \( g : N \to W \) are homeomorphisms with bounded \( s \)-distortion, then \( g \circ f : M \to W \) also has bounded \( s \)-distortion.

Special cases of the previous result where also obtained in [9.2 and 12.3], [24], [25, Section 6.4.3] and [30].

Definition. — The Royden algebra of \( M \) is the subspace \( R^s(M) \subset L^{1,s}(M) \) of bounded continuous functions; it is a Banach algebra with norm

\[
\|u\|_{R^s} = \|u\|_{L^{\infty}} + \|\nabla u\|_{L^{1,s}}.
\]

We denote by \( K_R \) the norm of the operator \( f^* : R^s(N) \to R^s(M) \) and by \( K_L \) the norm of the operator \( f^* : L^{1,s}(N) \to L^{1,s}(M) \).

Proposition 4.1. — Suppose \( 1 < s < \infty \), then for any homeomorphism \( f : M \to N \) we have \( K_R = \max\{1, K_L\} \).

We will need the following

Lemma 4.1. — Let \( v \in R^s(N) \) be a non constant function, and fix \( \varepsilon > 0 \). If \( 1 < s < \infty \), then for any \( t \in (\alpha, \beta) \), where \( \alpha := \inf v \) and \( \beta := \sup v \), there exists \( r = r(t, \varepsilon) > 0 \) such that \( r < \min\{t - \alpha, \beta - t\} \) and

\[
\varepsilon^{-1}(t'' - t') \leq \|\max(\min(v, t''), t') - t'\|_{L^{1,s}(N)}
\]

for all \( t', t'' \in (\alpha, \beta) \) such that \( t - r < t' \leq t \leq t'' < t + r \).

Proof. — Suppose the lemma false, then the function

\[
v_{t', t''} := \frac{\max(\min(v, t''), t') - t'}{t'' - t'}
\]

satisfies \( \|v_{t', t''}\|_{L^{1,s}(N)} \leq \frac{1}{2} \) for some \( \varepsilon > 0 \) and all \( t', t'' \in (\alpha, \beta) \) such that \( t - r < t' \leq t \leq t'' < t + r \). Consider a bounded domain \( A \subset N \) such that \( A_0 := \{x \in A : v(x) < t\} \) and \( A_1 := \{x \in A : v(x) > t\} \) are non empty open subsets.

The family \( \{v_{t', t''}\} \) is bounded in \( W^{1,s}(A) \) and hence weakly compact. It follows that there is a sequence \( v_n := v_{t'_n, t''_n} \) such that \( t'_n \leq t \leq t''_n \) and \( (t''_n - t'_n) \to 0 \), which converges weakly to some function \( w \in W^{1,s}(A) \). We can furthermore assume that the sequence \( \lambda_n := \frac{t - t'_n}{t''_n - t'_n} \in [0, 1] \) converges to some number \( \lambda \). Using Mazur’s Lemma, we can produce convex combinations of the \( v_n \) converging strongly to \( w \). Hence \( w = 0 \) a.e. on \( A_0 \),
\textit{Proof of Proposition 4.1. —} Observe that $K_R \geq 1$ since constant functions belong to the Royden algebras. So we only need to prove the inequalities $K_L \leq K_R \leq \max\{1, K_L\}$. Since $f$ is a homeomorphism, $f^*$ defines an isometry $f^* : L^\infty(N) \to L^\infty(M)$ and the inequality $K_R \leq \max\{1, K_L\}$ follows immediately.

To prove the inequality $K_L \leq K_R$ it suffices, by density of $R^s(N)$ in $L^{1,s}(N)$, to show that
\begin{equation}
\|f^*v\|_{L^{1,s}(M)} \leq (1 + \varepsilon)K_R \|v\|_{L^{1,s}(N)}
\end{equation}
for any $\varepsilon > 0$ and any function $v \in R^s(N)$.

Set $\alpha := \inf v$ and $\beta := \sup v$. By compactness of the interval $[\alpha, \beta]$, we can find a subdivision $\tau = \{\alpha = t_0 < t_1 < \ldots < t_l = \beta\}$, such that $(t_{i+1} - t_i) < r_i$ for $i = 1, \ldots, l - 1$, where $r_i = r(t, \varepsilon)$ satisfies the property of the previous lemma for some $t \in (t_i, t_{i+1})$.

Set $v_\tau := \alpha + \sum_{i=1}^{l-1} v_i$, where $v_i := \max(\min(v, t_{i+1}), t_i) - t_i$. By the lemma we have $\|v_i\|_{L^\infty} \leq \varepsilon \|v_i\|_{L^{1,s}(N)}$ for $i = 1, \ldots, l - 1$, hence
\begin{align*}
\|f^*v_\tau\|_{L^{1,s}(M)}^s &\leq \|f^*v_\tau\|_{R^s(M)}^s \leq \sum_{i=0}^{l-1} \|f^*v_i\|_{R^s(M)}^s \\
&\leq \sum_{i=0}^{l-1} K_R^s \left(\|v_i\|_{L^\infty(N)} + \|v_i\|_{L^{1,s}(N)}\right)^s \\
&\leq K_R^s(1 + \varepsilon)^s \sum_{i=1}^{l-1} \|v_i\|_{L^{1,s}(N)}^s \\
&\leq K_R^s(1 + \varepsilon)^s \|v_\tau\|_{L^{1,s}(N)}^s
\end{align*}
because $\|v_\tau\|_{L^{1,s}(N)} = \sum_{i=1}^{l-1} \|v_i\|_{L^{1,s}(N)}^s$. The inequality (4) now follows since $\|v - v_\tau\|_{L^{1,s}(N)} \to 0$ and $\|f^*v - f^*v_\tau\|_{L^{1,s}(N)} \to 0$ as $\max\{t_1 - t_0, t_{i+1} - t_i\} \to 0$. \hfill \square

\textit{Remark. —} Pierre Pansu has defined in [28, p. 475] the notion of homeomorphism of bounded $s$-dilatation as homeomorphism such that $K_R \leq \infty$. It follows from the results of this section that the definition of homeomorphism of bounded $s$-dilatation used by Pansu, coincides with our notion of homeomorphism with bounded $s$-distortion if $1 < s < \infty$.
It also follows from Theorem 4.1 that if \( f \) is a homeomorphism satisfying Lusin’s property with bounded \( s \)-dilatation in Pansu’s sense, then \( f^{-1} \) is a homeomorphism with bounded \( p \)-dilatation where \( 1/s + (n-1)/p = 1 \). This gives a positive answer to question 10.3 in [28] in the case where Lusin’s property holds.

5. Pushing functions forward.

The proof of Theorem A is based on a capacity estimate for the pushforward operator (Corollary 5.1) which is important in itself. It is the goal of this section to prove this capacity estimate.

Let \( f : M \to N \) be a continuous mapping and \( u : M \to \mathbb{R} \) a bounded function. We define the \emph{pushforward} of \( u \) to be the function \( v = f_\#u : N \to \mathbb{R} \) given by

\[
v(y) := \begin{cases} 
  \sup\{u(x) : f(x) = y\} & \text{if } y \in f(M), \\
  0 & \text{otherwise}.
\end{cases}
\]

Lemma 5.1. — If \( f \) is continuous discrete and open, and \( u : M \to \mathbb{R} \) is continuous with compact support, then the function \( v = f_\#u : N \to \mathbb{R} \) is also continuous and \( \text{supp } v \subset f(\text{supp } u) \).

This is Lemma 7.6 in [22]. \( \square \)

If the mapping \( f \) has bounded \( s \)-distortion and \( u \in C^1_0(M, \mathbb{R}) \) then \( v = f_\#u \) belongs to \( W^{1,p}_{\text{loc}}(N, \mathbb{R}) \) where \( p = \frac{s}{s-(n-1)} \) provided \( s > (n-1) \). More precisely:

Theorem 5.1. — Let \( f \in W^{1,1}_{\text{loc}}(M, N) \) be a continuous open and discrete mapping with bounded \( s \)-distortion, \((n-1) < s < \infty \). Assume also that \( f \) satisfies Lusin’s property if \( n-1 < s < n \). Then the operator \( f_\# \) possesses the following properties:

1) \( f_\# : C^1_0(M) \to W^{1,p}(N) \cap C^0_0(N) \),

2) \( \int_N |df_\#(u)|^p \, d\nu \leq K^{p-1} \int_M |du|^p \, d\mu \), for any \( u \in C^1_0(M) \), where \( p = \frac{s}{s-(n-1)} \) and \( K \) is the constant in (1).

Remarks. — 1) If \( f \) is a continuous open mapping and \( f \in W^{1,n}_{\text{loc}}(M, N) \), then it always satisfies Lusin’s property [20] (see also [40] for a short proof).
2) This theorem is known for \( s = n \) (see [22]). It is also known for general values of \( s \) when \( f \) is a homeomorphism [44]. Our proof will be based on techniques borrowed from these two papers.

If \( f \) is continuous and open, then the image \((f(C), f(A))\) of a condenser \((C, A)\) in \( M \) is again a condenser in \( N \).

**Corollary 5.1.** — For any condenser \((C, A)\) in \( M \) we have

\[
\text{Cap}_p(f(C), f(A)) \leq K^{p-1} \text{Cap}_p(C, A).
\]

\[\square\]

**Proof.** — Choose a non negative function \( u \in C^1_0(M) \) such that \( u = 1 \) on \( C \), \( \text{supp}(u) \subset A \) and \( \int_A |du|^p \leq \text{Cap}_p(C, A) + \varepsilon \) where \( \varepsilon > 0 \) is arbitrary.

Let us set \( v = f_\# u : N \to \mathbb{R} \). Then, by Theorem 5.1 we have \( v \in W^{1,p}(A) \cap C^0(A) \). Since \( v \geq 1 \) on \( C \), we have

\[
\text{Cap}_p(fC, fA) \leq \int_{fA} |dv|^p \leq K^{p-1} \int_A |du|^p \leq K^{p-1}(\text{Cap}_p(C, A) + \varepsilon).
\]

\[\square\]

We begin the proof of Theorem 5.1 by some lemmas on capacities of condensers:

**Lemma 5.2.** — The inequality

\[
\text{Cap}_s(C, A) \leq \frac{|A|}{\text{dist}(C, \partial A)^s}
\]

holds for the capacity of any bounded condenser \((C, A) \subset \mathbb{R}^n\).

**Proof.** — Take \( u(x) := \min \left\{ \frac{\text{dist}(\partial A, x)}{\text{dist}(\partial A, C)}, 1 \right\} \) as a test function. \[\square\]

**Lemma 5.3.** — Let \((C, A) \subset \mathbb{R}^n\) be a condenser such that \( C \) is connected. If \((n - 1) < s < \infty\), then

\[
\text{Cap}_s^{n-1}(C, A) \geq b(n, s) (\text{diam} C)^s |A|^{(n-1-s)}
\]

where the constant \( b(n, s) \) depends on \( n \) and \( s \) only.

**Proof.** — See Lemma 5 of [44]. \[\square\]
Recall that a domain \( \Omega \subset M \) is said to be a normal domain for \( f \) if \( \overline{\Omega} \) is compact and \( \partial(f(\Omega)) = f(\partial \Omega) \). For any normal domain \( \Omega \subset M \) we have \( N_f(\Omega) < \infty \). A condenser \((C,A)\) is a normal condenser if \( A \) is a normal domain of \( f \).

**Lemma 5.4.** — If \( \Omega \subset M \) is a normal domain then \( \text{Cap}_a(C,A) \leq Kn_f(\Omega) \text{Cap}_s f(C,A) \) for any condenser \((C,A)\) in \( \Omega \).

This is a direct consequence of Lemma 6.2 below. See also [44, Th. 4].

The next lemma sums up the basic topological properties of a discrete and open mapping \( f : M \to N \). If \( x \in M \) and \( r > 0 \), then we denote by \( U(x,f,r) \) the connected component of \( f^{-1}(B(f(x),r)) \) containing \( x \).

**Lemma 5.5.** — Let \( f : M \to N \) be a continuous discrete and open mapping. Then \( \lim_{r \to 0} \text{diam}(U(x,f,r)) = 0 \) for every \( x \in M \). If \( U(x,f,r) \) is compact then \( U(x,f,r) \) is a normal domain and \( f(U(x,f,r)) = B(f(x),r) \). Furthermore, for every point \( x \in N \) there is a positive number \( \sigma_x \) such that the following conditions are satisfied for \( 0 < r \leq \sigma_x \):

i) \( U(x,f,r) \) is a normal neighborhood of \( x \),

ii) \( U(x,f,r) = U(x,f,\sigma_x) \cap f^{-1}(B(f(x),r)) \),

iii) \( \partial U(x,f,r) = U(x,f,\sigma_x) \cap f^{-1}(S(f(x),r)) \) if \( r < \sigma_x \),

iv) \( M \setminus U(x,f,r) \) is connected if \( M \) is connected,

v) \( M \setminus \overline{U(x,f,r)} \) is connected if \( M \) is connected,

vi) if \( 0 < r < s \leq \sigma_x \), then \( \overline{U(x,f,r)} \subset U(x,f,s) \), and \( U(x,f,s) \setminus \overline{U(x,f,r)} \) is a ring.

See [22], [34] or [12] for a proof.

**Lemma 5.6.** — Let \( f : M \to N \) be as in Theorem 5.1 and \( u \in C^0_0(M) \). Then the function \( v = f_2u \) is ACL.

Recall that a function \( v : N \to \mathbb{R} \) is absolutely continuous on lines (ACL) if for any local parametrization \( \varphi : Q \to N \) (where \( Q = \{y \in \mathbb{R}^n : a_i \leq y_i \leq b_i\} \subset \mathbb{R}^n \) is some \( n \)-interval) and for almost all \( z \in P_k(Q) \) (= the projection of \( Q \) on the hyperplane \( y_k = 0 \)), the one-variable function \( t \to v(\varphi(z + t\mathbf{e}_k)) \) is absolutely continuous.
Proof. — Let us fix some notations. Fix a local parametrization \( \varphi : Q \to N \) (where \( Q = \{ t \in \mathbb{R}^n : a_i \leq t_i \leq b_i \} \subset \mathbb{R}^n \) is some closed \( n \)-interval). Choose \( Q \) small enough so that for any ball \( B(y, r) \subset \varphi(Q) \) the domains \( U_i := U(x_i, f, r) \) are disjoint normal neighborhoods of \( x_i \) for \( 1 \leq i \leq q \) where \( \{ x_1, \ldots, x_q \} = f^{-1}(y) \cap \text{supp } u. \)

The function \( v \circ \varphi^{-1} \) will be simply denoted by \( v : Q \to \mathbb{R} \). We need to show that for any \( l = 1, \ldots, n \) and for almost all \( z \in P_l(Q) \), the function \( v \) is absolutely continuous on the line segment \( \beta_z : [a_l, b_l] \to Q \) defined by \( \beta_z(t) = z + te_l. \)

To this aim, we define a set function \( \varphi \) on \( P_l(Q) \) by

\[
\Phi(A) := |U \cap f^{-1}(\varphi(A \times [a_l, b_l]))|
\]

where \( U = \bigcup_{i=1}^q U_i \) and \( A \subset P_l(Q) \) is any Borel set. Then \( \Phi \) is a completely additive set function in \( P_l(Q) \) and from Lebesgue’s differentiation theorem, we know that \( \Phi'(z) < \infty \) for almost all \( z \in P_l(Q) \).

It is known (see [22, Lemma 2.7]) that for every point \( x_0 \in U \cap f^{-1}(z + a_l e_l) \) there exists a path \( \alpha : [a_l, b_l] \to U \) such that \( \alpha(a_l) = x_0 \) and \( f \circ \alpha = \varphi \circ \beta_z \). We call such a path a lift of \( \beta_z(t) = z + te_l \) with base point \( x_0 \); clearly the number of lifts does not exceed \( \mathcal{N}_f(U) \).

Claim. — Let \( \alpha : [a_l, b_l] \to U \) be any lift of \( \beta_z \). If \( \Phi'(z) < \infty \), then \( \alpha \) is absolutely continuous.

Since the ACL-property is local it suffices to show that \( \alpha \) is ACL in a neighborhood of every point. We may thus restrict our considerations to the case of mappings \( f : U \to Q \) where \( U \) is a bounded domain in \( \mathbb{R}^n \).

To prove the claim, we fix some arbitrary pairwise disjoint closed segments \( \Delta_1, \ldots, \Delta_k \subset (a_l, b_l) \) of lengths \( b_1, \ldots, b_k \). Choose \( r > 0 \) small enough so that the sets

\[
R_i := \{ y \in \mathbb{R}^n | \text{dist}(y, \Delta_i) < r \}
\]

are pairwise disjoint. Let \( T_i := \bigcup_{z \in \Delta_i} U(\alpha(z), f, r) \), then \( (\alpha(\Delta_i), T_i) \) and \( (\Delta_i, R_i) \) are condensors and \( (\Delta_i, R_i) = (f(\alpha(\Delta_i)), f(T_i)) \); indeed, we have

\[
f(T_i) = f \left( \bigcup_{z \in \Delta_i} U(\alpha(z), f, r) \right) = \bigcup_{z \in \Delta_i} B(z, f, r) = R_i.
\]

From Lemmas 5.2 and 5.3, we have

\[
\text{Cap}_s(\Delta_i, R_i) \leq \frac{|R_i|}{r^s} \leq c_1 b_i r^{n-1-s}
\]

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and
\[ \text{Cap}_e(\alpha(\Delta_i), T_i) \geq c_2 \frac{(\text{diam} \alpha(\Delta_i))^{s/(n-1)}}{|T_i|^{(1-n+s)/(n-1)}}. \]
These inequalities, together with Lemma 5.4, imply
\[ \text{diam} \alpha(\Delta_i) \leq c_3 b_i^{\frac{n-1}{s}} \left( \frac{|U \cap T_i|}{r^{n-1}} \right)^{\frac{1-n+s}{s}} \left( \sum_{i=1}^{k} b_i \right)^{\frac{n-1}{s}}. \]
where the constant $c_3$ depends on previous constants, $K$ and $N_f(\text{supp} \ u)$.

Set $E(z,r) = \{ y \in Q : \text{dist}(y, \beta_z([a_i,b_i])) < r \}$, then $\bigcup_{i=1}^{k} T_i \subset f^{-1}(E(z,r))$. Summing the previous inequality over $i = 1, \ldots, k$ and applying Hölder’s inequality we obtain
\[ \sum_{i=1}^{k} \text{diam} \alpha(\Delta_i) \leq c_4 \left( \frac{|U \cap f^{-1}(E(z,r))|}{r^{n-1}} \right)^{\frac{1-n+s}{s}} \left( \sum_{i=1}^{k} b_i \right)^{\frac{n-1}{s}}. \]
Letting $r \to 0$, we find that
\[ \sum_{i=1}^{k} \text{diam} \alpha(\Delta_i) \leq c_5 \varphi'(z) \left( \sum_{i=1}^{k} b_i \right)^{\frac{n-1}{s}}, \]
hence $\alpha$ is absolutely continuous if $\varphi'(z) < \infty$.

We now conclude the proof of the lemma as follows: Let $\alpha_1,\alpha_2,\ldots,\alpha_d$ be all the lifts of the segment $\beta_z$. If $\Phi'(z) < \infty$, then $u \circ \alpha_i$ is absolutely continuous since $u$ is $C^1$ and $\alpha_i$ is absolutely continuous. We conclude that $v \circ \beta_z$ is absolutely continuous since
\[ v \circ \beta_z = \max_i u \circ \alpha_i. \]
\[ \square \]

**Lemma 5.7.** — Let $f : M \to N$ be as in Theorem 5.1, then $J_f = 0$ almost everywhere on the branch set and the image of the branch set has measure zero.

**Proof.** — Because $f$ has bounded $s$-distortion and $s > (n-1)$, $f \in W_{\text{loc}}^{1,s}$, it then follows from 3.4 that $f$ is differentiable almost everywhere.

Suppose that $f$ is differentiable at $x$ and $J_f(x) > 0$, then the index $j(x,f) = 1$ (because the map is continuous open and discrete and the topological degree is stable under homotopy, see e.g. pp. 15-21 in [34]).

If $j(x,f) = 1$, then $x \notin B_f$ (see [34, Proposition 4.10]); it follows that $J_f = 0$ a.e. on $B_f$. 

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Because \( f \) is assumed to satisfy Lusin’s property, we can use the area formula (Proposition 3.3) to conclude that \( f(B_f) \) has measure zero:

\[
\nu(f(B_f)) \leq \int_N \left( \sum_{f(x)=y} \chi_{B_f(x)} \right) dv(y) = \int_M \chi_{B_f(x)} J_f(x) d\mu(x) = 0.
\]

\( \square \)

**Proof of Theorem 5.1.** — To conclude the proof of the theorem it only remains to check the integrability of \( dv \). To do this we first observe that Vitali’s covering Theorem implies

\[
\text{supp } v \setminus f(B_f \cap \text{supp } u) \subset \bigcup_{i=1}^{\infty} B(y_i, r_i) \cup A,
\]

where \( B_f \) is the branch set of \( f \), \( A \subset N \) is a set with \( \nu(A) = 0 \) and \( B(y_i, r_i) \), \( i \in \mathbb{N} \), are mutually disjoint balls small enough so that the components of \( f^{-1}(B(y_i, r_i)) \) which meet the support of \( u \) form a finite disjoint collection \( D_{i_1}, D_{i_2}, \ldots, D_{i_k} \) of open subsets of \( M \) such the restrictions of \( f \) define homeomorphisms \( f_j : D_{i_j} \to B(y_i, r_i) \), \( j = 1, \ldots, k \).

By Theorem 4.1, the inverse of \( f_i \), i.e. the map \( g_j := f_i^{-1} : B(y_i, r_i) \to D_{i_j} \) is ACL, furthermore, we have

\[
|dg_j|^p \leq K_{g_j} \quad \text{a.e.}
\]

for almost every \( z \in B(y_i, r_i) \). This implies

\[
\int_{B(y_i, r_i)} |dv(z)|^p \, dv \leq K_{g_j}^{-1} \sum_{j=1}^k \int_{B(y_i, r_i)} |du(g_j(z))|^p J(z, g_j) \, d\nu \leq K_{g_j}^{-1} \int_{f^{-1}(B(y_i, r_i))} |du|^p \, d\mu.
\]

From Lemma 5.7, we know that \( \nu(f(B_f)) = 0 \) and \( J_f = 0 \) a.e. on \( B_f \); we thus have from the area formula

\[
\int_N |dv(z)|^p \, dv = \sum_{i=1}^{\infty} \int_{B(y_i, r_i)} |dv(z)|^p \, dv \leq K_{g_j}^{-1} \sum_{i=1}^{\infty} \int_{f^{-1}(B(y_i, r_i))} |du|^p \, d\mu
\]

\[
\leq K_{g_j}^{-1} \int_M |du|^p \, d\mu.
\]

\( \square \)
6. Proofs of the main theorems.

6.1. Proof of Theorem A.

Let us recall the statement:

**Theorem A.** — Let \( f \in W^{1,1}_{\text{loc}}(M,N) \) be a continuous open and discrete mapping with bounded \( s \)-distortion where \( s > (n - 1) \). Assume also that \( f \) satisfies Lusin’s property. If \( M \) is \( p \)-parabolic with \( p = \frac{s}{s-(n-1)} \), then \( N \) is also \( p \)-parabolic.

**Proof.** — Let \( D \subset M \) be a compact subset with non empty interior. Because \( f \) is a continuous and open map, \( f(D) \subset N \) is also a compact set with non empty interior. By Corollary 5.1 we have

\[
\text{Cap}_p(f(D), N) \leq \text{Cap}_p(f(D), fM) \leq K^{n-1} \text{Cap}_p(D, M),
\]

hence if \( M \) is \( p \)-parabolic then so is \( N \).

6.2. Proofs of Theorems C and B.

The proofs will use the following criterion for hyperbolicity which is due to V. Gol’dshtein and M. Troyanov (see [9]).

**Theorem 6.1.** — Let \( M \) be an oriented connected Riemannian manifold \( M \). Then the following are equivalent \( \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \):

1) \( M \) is \( p \)-hyperbolic;

2) there exists a smooth form \( \alpha \in L^q(M, \Lambda^{n-1}) \) such that \( d\alpha \geq 0 \) and \( \int_M d\alpha \neq 0 \);

3) there exists a form \( \alpha \in L^q(M, \Lambda^{n-1}) \) such that \( d\alpha \geq 0 \) and \( \int_M d\alpha \neq 0 \);

4) \( H^{n}_{\text{comp}, q}(M) = 0 \).

The cohomology space \( H^{n}_{\text{comp}, q}(M) \) is the space of all closed differential forms of degree \( n \) with compact support modulo the differential of \( (n-1) \)-forms in \( L^q \).
We will also need the following lemma:

**Lemma 6.1.** — Let $f : M \to N$ be a mapping of class $W_{\text{loc}}^{1,1}$ with essentially finite multiplicity and bounded $q$-codistortion: $|\Lambda_{n-1} f|^q \leq K J_f$. Then

$$\Lambda_{n-1} f : L^q(N, \Lambda^{n-1}) \to L^q(M, \Lambda^{n-1})$$

is a bounded operator with norm $\leq (K \cdot N_f(M))^{1/q}$.

(Recall that $N_f(A) = \text{ess sup}_y \text{Card}(A \cap f^{-1}(y))$ for any set $A \subset M$.)

**Proof.** — Let $\beta \in L^q(N, \Lambda^{n-1})$, then

$$\int_M |\Lambda_{n-1} f(\beta)|^q d\mu \leq K \int_M |\beta_f(x)|^q J_f(x) d\mu$$

$$= K \int_N \left( \sum_{f(x) = y} |\beta_f(x)|^q \chi_{M \setminus E_f}(x) \right) d\nu$$

$$\leq (K \cdot N_f(M)) \int_N |\beta|^q d\nu.$$

We now prove Theorem C; we restate it in the following form:

**Theorem C.** — Let $f : M \to N$ be a mapping of essentially finite multiplicity with bounded $q$-codistortion where $q > 1$ and such that $J_f > 0$ on some set of positive measure. Assume furthermore either

1) $f \in W_{\text{loc}}^{1,n-1}(M, N)$ and $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$, or

2) $f$ is almost absolutely continuous, $f \in W_{\text{loc}}^{1,s}(M, N)$ for some $s > (n-1)$ and $J_f \in L^1_{\text{loc}}(M)$.

If $N$ is $p$-hyperbolic with $p = \frac{q}{q-1}$, then $M$ is also $p$-hyperbolic.

**Proof.** — Let us choose a bounded Borel set $U \subset M$ such that $U$ has positive measure, $f(U)$ is bounded and $J_f > 0$ on $U$. Observe that, by the area formula, $\nu(f(U)) > 0$.

Choose a non negative smooth function $h : N \to \mathbb{R}$ with compact support and such that $h > 0$ in a neighborhood of $f(U)$. Since $N$ is $p$-hyperbolic, $H^n_{\text{comp}, q}(N) = 0$, hence there exists an $(n-1)$-form $\beta \in L^q(N, \Lambda^{n-1})$ such that $d\beta = h \cdot \omega_N$ ($\omega_M$ and $\omega_N$ are the volume forms of $M$ and $N$ respectively).
By Lemma 6.1, we have $\alpha := f^* \beta \in L^q(M, \Lambda^{n-1})$. We then have from Lemma 3.4
\[
d\alpha = f^*(d\beta) = (h \circ f) \cdot f^* \omega_N = (h \circ f) \cdot J_f \omega_M.
\]
Thus $d\alpha \geq 0$ and $\int_M d\alpha \geq \int_U (h \circ f) \cdot J_f \ d\mu > 0$ and we conclude by Theorem 6.1 that $M$ is $p$-hyperbolic. \qed

Finally, we deduce Theorem B from Theorem C.

**Theorem B.** — Let $f \in W^{1,s}_{\text{loc}}(M, N)$ be a mapping of essentially finite multiplicity with bounded $s$-distortion where $s > (n - 1)$. Assume either
1) $|\Lambda_{n-1}f| \in L^{n/(n-1)}_{\text{loc}}(M)$, or
2) $f$ is almost absolutely continuous and $J_f \in L^1_{\text{loc}}(M)$.

If $M$ is $p$-parabolic and $N$ is $p$-hyperbolic with $p = s/(s - (n - 1))$, then $f$ is constant a.e.

**Proof.** — Let $q = p/(p - 1)$. Then $s = q(n - 1)$ and from Lemma 2.1 we know that if $f$ has bounded $s$-distortion, then it has bounded $q$-codistortion. Hence by Theorem C, we have $J_f = 0$ a.e. and thus $|df| = 0$ a.e. since $|df|^s \leq K J_f$. As $f$ is a Sobolev mapping, we conclude that $f$ is constant a.e. \qed

6.3. Proof of Theorem D.

**Lemma 6.2.** — Let $f \in W^{1,1}_{\text{loc}}(M, N)$ be a mapping with bounded $s$-distortion and essential finite multiplicity. Then $f^* : L^{1,s}(N) \rightarrow L^{1,s}(M)$ is a bounded operator with operator norm at most $(KN_f(M))^{1/s}$.

**Proof.** — Let us first consider a function $v \in C^1(N) \cap L^{1,s}(M)$. Then $u := f^* v \in W^{1,1}_{\text{loc}}(M)$ and $du_x = df^*_x (dv_f(x))$. Hence we have almost everywhere $|du|^s \leq |dv|^s |df|^s \leq K |dv|^s J_f$. From the area formula we thus obtain
\[
\int_M |du_x|^s \ d\mu(x) \leq K \int_M |dv_f(x)|^s J_f(x) \ d\mu(x)
\]
\[
= K \int_N \left( \sum_{f(x) = y} |dv_f(x)|^s \chi_{M \setminus E_f}(x) \right) \ d\nu(y)
\]
\[
\leq K N_f(M) \int_N |dv|^s \ d\nu(y).
\]
Thus $u \in \mathcal{L}^{1,s}(M)$ and $\|u\|_{\mathcal{L}^{1,s}(M)} \leq (K N_f(M))^{1/p} \|v\|_{\mathcal{L}^{1,s}(N)}$.

Using the argument on page 673 of [44], we can extend this estimate from functions $v \in C^1(N) \cap \mathcal{L}^{1,s}(M)$ to all functions $v \in \mathcal{L}^{1,s}(N)$. This proves that the norm of the operator $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$ is bounded by $(K N_f(M))^{1/s}$.

Recall the statement of Theorem D:

**Theorem D.** — Let $f \in W^{1,1}_{\text{loc}}(M,N)$ be a continuous non constant proper mapping with bounded $s$-distortion of finite essential multiplicity. If $N$ is $s$-parabolic then so is $M$.

**Proof.** — Let $D \subseteq M$ be a compact set; then $D' = f(D) \subset N$ is also compact and, by hypothesis, it has zero $p$-capacity. For each $\varepsilon > 0$, one can thus find a continuous function $v \in \mathcal{L}^{1,s}(N)$ with compact support and such that $v \equiv 1$ on $D'$ and $\int_N |dv|^s \leq \varepsilon$.

Since $f$ is a proper map, the function $u := f^*(v)$ also has compact support and, clearly, $u \equiv 1$ on $D$. Let $A$ be the norm of the operator $f^* : \mathcal{L}^{1,s}(N) \to \mathcal{L}^{1,s}(M)$; we know by Lemma 6.2 that $A$ is finite. We then have $\int_M |du|^s \leq A^s \int_N |dv|^s \leq A^s \varepsilon$. Hence $D$ has zero $p$-capacity and we conclude that $M$ is $p$-parabolic.

**7. Complements.**

**7.1. A topological result.**

A famous theorem of Yu. Reshetnyak states that a non constant quasi-regular mapping is open and discrete. We formulate below a generalization of this theorem established recently by S. Vodop’yanov’s in [41], which provides topological properties for mappings with integrable distortion.

**Theorem 7.1.** — Let $f \in W^{1,1}_{\text{loc}}(M,N)$ be a continuous non constant mapping with nonnegative Jacobian $J_f(x) \geq 0$ and $K(x) = \frac{|df_x|^n}{J_f(x)} \in L^p_{\text{loc}}(M)$ for some $n-1 < p \leq \infty$. Assume either

1) $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$, or

2) $f$ is almost absolutely continuous and $J_f \in L^1_{\text{loc}}(M)$.

Then $f$ is discrete and open.
Remarks. — 1) If the manifolds are two-dimensional, then the condition $n - 1 < p < n$, can be relaxed to $1 < p < 2$.

(2) This result was also proven in [13] and [21] under the assumption $f \in W^{1,n}_{\text{loc}}(M,N)$. It has been also recently proved in [18] under different analytical assumptions.

As a consequence of Theorem 7.1 we obtain topological properties for mappings with bounded $s$-distortion. The next assertion gives a positive answer to the question 10.8 of [28].

Corollary 7.1. — Let $f \in W^{1,1}_{\text{loc}}(M,N)$ be a continuous non constant mapping with bounded $s$-distortion where $n - 1 < s \leq n$. Assume either

1) $|\Lambda_{n-1} f| \in L^{n/(n-1)}_{\text{loc}}(M)$, or

2) $f$ is almost absolutely continuous and $J_f \in L^1_{\text{loc}}(M)$.

Then $f$ is discrete and open.

Remark. — This result does not hold if $s > n$. Consider for instance the map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ |x| - 1 & \text{if } 1 \leq |x| \leq 2, \\ |x|^\alpha & \text{if } |x| \geq 2, \end{cases}$$

for some $\alpha > 1$. Then $f$ is Lipschitz and has bounded $s$-distortion for $s = \frac{na}{\alpha-1} > n$. Clearly $f$ is neither open nor discrete; however $f$ has finite essential multiplicity.

Proof. — We suppose that $|df|^s \leq CJ_f$ a.e. for some $n - 1 < s < n$. Let us define the function

$$K_f(x) = \begin{cases} |df|^s \frac{J_f}{J_f(x)} & \text{if } J_f(x) \neq 0, \\ 1 & \text{else}. \end{cases}$$

Set $p = \frac{s}{n-s}$, we have at almost all points where $J_f(x) \neq 0$,

$$|K_f|^p = \frac{|df|^np}{J_f} \leq C^{np/s} \frac{J_f^{np/s}}{J_f} = C^{np/s} J_f^{p(n/s-1)} \leq C^{np/s} J_f.$$

Thus $K_f \in L^p_{\text{loc}}$. Since $n - s < 1$, we have $p > s > n - 1$ and we can conclude the proof from Theorem 7.1. □
Corollary 7.2. — Let \( f \in W_{1,1}^{1,1}(M,N) \) be a mapping with bounded \( q \)-codistortion where

\[
\frac{(n-1)^2}{1+(n-1)(n-2)} < q \leq \frac{n(n-1)}{1+n(n-2)}.
\]

Assume that \( J_f > 0 \) a.e. and either

1) \( |A_{n-1}f| \in L_{loc}^{n/(n-1)}(M) \), or
2) \( f \) is almost absolutely continuous and \( J_f \in L_{loc}^1(M) \).

Then \( f \) is discrete and open.

Proof. — By Lemma 2.1, \( f \) has bounded \( s \)-distortion for \( s = \frac{q}{(n-1)-(q(n-2))} \); observe that the inequalities (5) are equivalent to \( n-1 < s \leq n \). Thus the corollary follows from Corollary 7.1.

\( \Box \)

7.2. On \( p \)-parabolic manifolds.

A connected oriented Riemannian \( n \)-manifold \( M \) is called \( p \)-parabolic, \( 1 \leq p < \infty \), if \( \text{Cap}_p(C,M) = 0 \) for all compact subsets \( C \subseteq M \) and \( p \)-hyperbolic otherwise. In this section, we list some facts concerning \( p \)-parabolicity. We refer to [37], [10], and [45] for further information on this notion.

a) If \( M \) contains one compact subset with nonempty interior having zero \( p \)-capacity then \( M \) is \( p \)-parabolic.

b) The Euclidean space \( \mathbb{R}^n \) is \( p \)-hyperbolic for \( p < n \) and \( p \)-parabolic for any \( p \geq n \).

c) If \( M \) is \( p \)-hyperbolic, then any domain \( \Omega \subseteq M \) is also \( p \)-hyperbolic.

d) If a closed subset \( S \subseteq M \) with Hausdorff dimension \( > (n-p) \) is removed from any manifold \( M \) and if \( M \setminus S \) is connected, then \( M \setminus S \) is \( p \)-hyperbolic.

e) In particular, if one removes a point \( x_0 \), then \( M \setminus \{x_0\} \) is \( p \)-hyperbolic for all \( p > n \) and if one removes a non separating closed subset with nonempty interior \( D \subseteq M \), then \( M \setminus D \) is \( p \)-hyperbolic for all \( p \geq 1 \).

f) If the manifold is complete and \( \text{Vol}(B(x_0,r)) \leq \text{const.} r^d \) then \( M \) is \( p \)-parabolic for any \( p \geq d \) (finer estimates relating the volume growth to parabolicity are in fact available).
g) If the isoperimetric inequality
\[ \text{Area}(\partial \Omega)^{d/(d-1)} \geq \text{const. Vol(\Omega)} \]
holds for any big smooth domain \( \Omega \subset M \), then \( M \) is \( p \)-hyperbolic for \( p < d \).

h) Suppose that a Sobolev inequality
\[ \|u\|_{L^q} \leq \text{const.} \|\nabla u\|_{L^p} \]
holds for some \( 1 \leq q \leq \infty \) and all functions \( u \in C^1_0(M) \). Then \( M \) is \( p \)-hyperbolic.

Recall that the \( p \)-Laplacian is the operator \( \Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) \).

A function is called \( p \)-superharmonic if \( \Delta_p u \leq 0 \).

i) \( M \) is \( p \)-parabolic if and only if every positive \( p \)-superharmonic function on \( M \) is constant.

j) \( M \) is \( p \)-hyperbolic if and only if there exists a positive Green function for the \( p \)-Laplacian.

k) \( M \) is 2-hyperbolic if and only if the Brownian motion is transient.

l) If \( M \) has finite volume, then there exists a number \( d \in [1, \infty] \) such that \( M \) is \( p \)-parabolic for \( 1 \leq p < d \) and \( p \)-hyperbolic for \( p > d \).

m) For a non compact manifold with bounded geometry, we have the opposite behaviour: there exists an index \( d \), called the \emph{parabolic dimension} of \( M \), such that \( M \) is \( p \)-hyperbolic for \( 1 \leq p < d \) and \( p \)-parabolic for \( p > d \).

n) The parabolic dimension is a quasi-isometric invariant of manifolds with bounded geometry.

o) \( n \)-parabolicity is a quasi-conformal invariant for any manifolds.

\textit{Proof.} — The proofs of (a)-(h) and (l)-(n) can be found in [37]. The proofs of (i) and (j) are in [14] (see also [17]). We refer to [10] for (k) and [45] for (o).

7.3. An improvement of a result by Pierre Pansu.

The following result gives an improvement of our Theorem B for Sobolev homeomorphisms with Lusin’s property between manifolds with...
bounded geometry. It was proved by P. Pansu for diffeomorphisms, see [28, corollaire 2.1].

**Theorem.** — Let $M$ and $N$ be Riemannian manifolds with bounded geometry, and assume that $N$ satisfies an isoperimetric inequality of order $d > n$:

$$\text{Area}(\partial \Omega)^{d/(d-1)} \geq \text{const} \cdot \text{Vol}(\Omega)$$

for all smooth compact domain $\Omega \subset N$ of volume $\geq 1$ (in particular $N$ is $n$-hyperbolic).

If $\frac{n-1}{n-1} < s < n$, then every homeomorphism $f \in W^{1,1}_{\text{loc}}(M, N)$ with bounded $s$-distortion satisfying Lusin’s property is a rough quasi-isometry.

**Proof.** — We know that if $f$ has bounded $s$-distortion, $s > (n-1)$ and satisfies Lusin’s property, then $f^{-1}$ has bounded $p$-distortion where $p = \frac{s}{s-(n-1)}$ (see Theorem 4.1 and the Remark at the end of Section 4). The above theorem thus follows from [28, Théorème 1].

### 7.4. On Reshetnyak’s proof for the case of quasi-regular mappings.

In order to illustrate the alternative approach based on methods of non-linear potential theory, we give a short proof of Liouville’s theorem for quasi-regular mappings along Reshetnyak’s ideas.

**Theorem.** — Let $f : M \to N$ be a non constant quasi-regular mapping between oriented $n$-dimensional Riemannian manifolds. Assume that $M$ is $n$-parabolic, then so is $N$.

**Proof.** — Assume that $f : M \to N$ is a non constant quasi-regular mapping, then it is known (see [31, Th. 6.4, chap. II]) that $f$ is an open map; in particular $N' := f(M) \subset N$ is open. If $N$ is $n$-hyperbolic, then so is $N'$ and, by [14, Th. 5.2], we know that there exists a non constant positive $n$-superharmonic function $v : N' \to \mathbb{R}$. The function $u = f^* v = v \circ f : M \to \mathbb{R}$ is then $\mathcal{A}$-superharmonic where $\mathcal{A}$ is the pull back to $M$ of the operator $T N' \to T N'$ given by $\eta \to |\eta|^{n-2} \eta$ (see [31, Th. 11.2, chap. II] or [12, Th. 14.42]). By [14, Th. 5.2] again, one concludes that $M$ is also $n$-hyperbolic, contradicting the hypothesis. $\square$
Final remarks. — 1) The argument of Martio, Väisälä and Rickman are based on capacity estimates in the spirit of our proof of Theorem A (see [22]).

2) Another proof can be found in [3]. This paper gives other obstructions to the existence of quasi-regular mappings.

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