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Axiomatic Theory of Sobolev Spaces

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Abstract We develop an axiomatic approach to the theory of Sobolev spaces on metric measure spaces and we show that this axiomatic construction covers the main known examples (Hajłasz Sobolev spaces, weighted Sobolev spaces, Upper-gradients, etc). We then introduce the notion of variational p−capacity and discuss its relation with the geometric properties of the metric space. The notions of p−parabolic and p−hyperbolic spaces are then discussed.

Introduction

Recent years have seen important developments in geometric analysis on metric measure spaces (MM-spaces). Motivating examples came from various subjects such as singular Riemannian manifolds, discrete groups and graphs, Carnot-Carathéodory geometries, hypoelliptic PDE's, ideal boundaries of Gromov-hyperbolic spaces, stochastic processes, fractal geometry etc. The recent books [22] and [19] are convenient references on the subject.

Suppose we are given a metric measure space (X, d, μ); how can we define in a natural way a first order Sobolev space $W^{1,p}(X)$?
Here is a simple construction. Let $\mathcal{F}$ be the class of Lipschitz functions with compact support $u : X \to \mathbb{R}$, and define for any $u \in \mathcal{F}$ and any point $x \in X$ the *infinitesimal stretching constant*

$$L_u(x) := \lim_{r\to 0} \sup_{d(y,x)\le r} \frac{|u(y) - u(x)|}{r}.$$

We can check that the formula

$$\|u\|_{1,p} := \|u\|_{L^p(X)} + \|L_u\|_{L^p(X)}$$

defines a norm on $\mathcal{F}$.
We then define $W^{1,p}(X)$ to be the completion of $\mathcal{F}$ for this norm. If X is the Euclidean space $\mathbb{R}^n$, then this construction gives the usual Sobolev space $W^{1,p}(\mathbb{R}^n)$.

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It is also a natural construction of the Sobolev space $W^{1,p}(X)$ for a large class of metric measure spaces.

However there are cases where this definition gives rise to a quite degenerate Sobolev space; here is an example based on an idea of S. Semmes : Let $X = \mathbb{R}^n$ with the Lebesgue measure μ and the metric $d_\alpha(x, y) := |y - x|^\alpha$ where $0 < \alpha < 1$ is a fixed number. In this case, it is not difficult to see that if the function $u : \mathbb{R}^n \to \mathbb{R}$ is differentiable (in the classic sense) at a point x, then $L_u(x) = 0$. In particular if a function $u : \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere, then we have $\|L_u\|_{L^p(X)} = 0$ hence $\|u\|_{1,p} = \|u\|_{L^p(X)}$; which is a rather degenerate behaviour.

The Sobolev space is called *non degenerate* if $\|L_u\|_{L^p(X)} > 0$ for any non constant function u. We just observed that this condition is not always satisfied; it must therefore be assumed axiomatically in order to develop a general theory.

An alternative notion of Sobolev spaces on metric spaces has been developed in [4], [24] and [39] starting from the notion of *upper gradient* (see section 2.6 for a description of this Sobolev space). This approach is well adapted to the case of length spaces (these are metric spaces such that the distance is defined in terms of the length of curves) or more generally to quasi-convex spaces.
A Poincaré inequality (see §sec.poinc) is often assumed or proved. It follows from this inequality that the Sobolev space is non degenerate.

Another approach is the Sobolev space on metric measure space defined by Piotr Hajłasz in [18]. The Hajłasz Sobolev space is in some sense a globally defined Sobolev space (unlike the constructions above), it is always non degenerate.

Other concepts of Sobolev spaces where motivated by the study of first order differential operators on homogeneous spaces (see for example the discussion in [11]) and by graph theory.

Our goal in this paper, is to develop an axiomatic version of the theory of Sobolev spaces on metric measure spaces. This axiomatic description covers many examples such as the Hajłasz Sobolev spaces, the weighted Sobolev spaces, the Sobolev spaces based on Hörmander systems of vector fields and on more abstract upper gradients.

The basic idea of this axiomatic description is the following: Given a metric space X with a measure μ, we associate (by some unspecified mean) to each function $u : X \to \mathbb{R}$ a set $D[u]$ of functions called the *pseudo-gradients* of u; intuitively a pseudo-gradient $g \in D[u]$ is a function which exerts some control on the variation of u (for instance in the classical case of $\mathbb{R}^n$: $D[u] = \{g \in L^1_{loc}(\mathbb{R}^n) : g \geq |\nabla u| \ a.e.\}$). A function $u \in L^p(X)$ belongs then to $W^{1,p}(X)$ if it admits a pseudo-gradient $g \in D[u] \cap L^p(X)$. Depending on the type of control required, the construction yields different versions of Sobolev spaces in metric spaces.

Instead of specifying how the pseudo-gradients are actually defined, we require them to satisfy some axioms. Our axioms can be divided in two independent groups: The first group (axioms A1–A4) is a formal description on the set $D[u]$ of pseudo-gradients and the second group (axioms A5 and A6) gives a meaning to the p−integrability of the pseudo-gradients.

A correspondence $u \to D[u]$ satisfying the axioms is called a *D-structure* on the metric measure space X. We look at such a structure as an ersatz for a theory of differentiation of the functions on the space (hence the name). A metric measure space (X, d, μ) equipped with a D-structure is called a MMD-space.

The *p-Dirichlet energy* $\mathcal{E}_p(U)$ of a function u is the greatest lower bound of the p'th power of the L_p-norms of all the pseudo-gradients of u and the p-Dirichlet space $\mathcal{L}^{1,p}(X)$ is the space of locally integrable functions with finite p-energy. The Sobolev space is then the space $W^{1,p}(X) := \mathcal{L}^{1,p}(X) \cap L^p(X)$. We can prove from the axioms that $W^{1,p}(X)$ is a Banach space; however, due to the fact that the definition of pseudo-gradient is not based on a linear operation, we can't generally prove that it is a reflexive Banach space for $1 < p < \infty$. Using this theory, we obtain a classification of metric spaces into p-parabolic/p-hyperbolic types similar to the case of Riemannian manifolds.

We now briefly describe the content of the paper:

In section 1, we give the axiomatic construction of Sobolev spaces on metric measure spaces and the basic properties of these spaces. The setting is the following: we fix a metric measure space (X, d, μ) and we choose a Boolean ring $\mathcal{K}$ of bounded subsets of X which plays the role of relatively compact subsets in the classical situation (the precise conditions that $\mathcal{K}$ must satisfy are specified in the next section). The space $L^p_{loc}(X)$ is defined to be the space of all measurable functions u such that $u|_A \in L^p(A)$ for all sets $A \in \mathcal{K}$. We then define the notion of D-structure by a set of axioms and we list some basic properties of the axiomatic Sobolev spaces.

In section 2, we show that familiar examples of Sobolev spaces on metric spaces such as the classical Sobolev spaces on Riemannian manifolds, weighted Sobolev spaces, Sobolev spaces on graphs, Hajłasz Sobolev spaces and Sobolev spaces based on upper gradients are examples of axiomatic Sobolev spaces.

In section 3, we develop the basics of non linear potential theory on metric spaces. We denote by $\mathcal{L}^{1,p}_0(X)$ the closure of the set of continuous functions $u \in \mathcal{L}^{1,p}(X)$ with support in a $\mathcal{K}$-set and we define the *variational p-capacity* of a set $F \in \mathcal{K}$ by

$$\mathrm{Cap_p}(F) := \inf\{\, \mathcal{E}_p(u) \,|\, u \in \mathcal{A}_p(F)\, \},$$

where $\mathcal{A}_p(F) := \{u \in \mathcal{L}^{1,p}_0(X) \,|\, u \geq 0 \text{ and } u \geq 1 \text{ on a neighbourhood of } F\}$. A metric space X is said to be *p-hyperbolic* if it contains a set $Q \subset \mathcal{K}$ of positive p-capacity and *p-parabolic* otherwise. One of our results (Theorem 3.1) says that the space X is p−parabolic if and only if $1 \in \mathcal{L}^{1,p}_0(X)$.

In the last section, we quote without proof a few recent results from the theory of Sobolev spaces on metric spaces.

Let us conclude this introduction by mentioning a few important, very active and related topics which are not discussed in this paper. First there is the theory of Sobolev mappings between two metric spaces which is a natural extension of the present work. Some papers on this subject are [32], [38] and [44]. Then there are papers dealing with a generalized notion of (co)tangent bundle on metric spaces such as [4], [37], [45] and [46]. Finally, there is the theory of Dirichlet forms and analysis in Wiener spaces such as exposed in [2] and [35].

Finally, it is our pleasure to thank Piotr Hajłasz and Khaled Gafaïti for their friendly and useful comments.

Content :

1 Axiomatic theory of Sobolev Spaces

1.1 The basic setting

An MM-space is a metric space (X, d) equipped with a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for any ball $B \subset X$ of positive radius (recall, that an outer measure μ is *Borel regular* if every Borel set is μ-measurable and for every set $E \subset X$, there is a Borel subset $A \subset E$ such that $\mu(A) = \mu(E)$, see [9, page 6]).

Our first aim is to introduce a notion of local Lebesgue space $L^p_{loc}(X)$. For this purpose we need the following concept :

Definition 1.1 (a) A *local Borel ring* in the MM-space (X, d, μ) is a Boolean ring[1] $\mathcal{K}$ of bounded Borel subsets of X satisfying the following three conditions:

K1) $\bigcup_{A \in \mathcal{K}} A = X$;

K2) if $A \in \mathcal{K}$ and $B \subset A$ is a Borel subset, then $B \in \mathcal{K}$;

K3) for every $A \in \mathcal{K}$ there exists a finite sequence of open balls $B_1, B_2, ...B_m \in \mathcal{K}$ such that $A \subset \cup_{i=1}^m B_i$ and $\mu(B_i \cap B_{i+1}) > 0$ for $1 \leq i < m$.

(b) A subset $A \subset X$ is called a *$\mathcal{K}$-set* if $A \in \mathcal{K}$.

Basic examples of such rings are the ring of all bounded Borel subsets of X and the ring of all relatively compact subsets if X is locally compact and connected.

In the rest of this subsection, we discuss some of the properties of such a structure $(X, d, \mathcal{K}, \mu)$. The reader may prefer to go directly to the next subsection and come back to this one only when it is needed.

Lemma 1.1 *The properties (K1)-(K3) have the following consequences:*

i) X can be covered by open $\mathcal{K}$-sets.

ii) X has the following "connectivity" property: For any pair of points $x, y \in X$, there exists a finite collection $\{B_1, B_2, ...B_n\} \subset \mathcal{K}$ of balls such that $x \in B_1$, $y \in B_n$ and $\mu(B_i \cap B_{i+1}) > 0$ for all i.

iii) $\mathcal{K}$ contains all compact subsets of X.

[1] A collection of subsets $\mathcal{K}$ of X is a *Boolean ring*, if $A_1, A_2 \in \mathcal{K} \Rightarrow (A_1 \cup A_2)$ and $(A_1 \setminus A_2) \in \mathcal{K}$. Boolean rings are also closed under finite intersections and symmetric differences.

Proof (i) Follows directly from conditions (K1) and (K3) and (ii) follows from (K3) since $A := \{x, y\} \in \mathcal{K}$. To prove (iii), let $C \subset X$ be a compact subset; by (i) it can be covered by open $\mathcal{K}$-sets (in finite number, by compactness): $C \subset U = \cup_{i=1}^{m} U_i$, where $U_i \in \mathcal{K}$; since $\mathcal{K}$ is a ring we have $U \in \mathcal{K}$, hence $C \in \mathcal{K}$ by (K2).

□

This lemma has the following consequences :

1) A local Borel ring $\mathcal{K}$ is always contained between the ring of relatively compact Borel sets and the ring of all bounded Borel sets.
In particular if X is a proper metric space (i.e. every closed bounded set is compact), then both of these rings coincide and $\mathcal{K}$ is always the ring of relatively compact Borel sets.

2) If $X \subset \mathbb{R}^n$ is an open subset and $\mathcal{K}$ is the ring of relatively compact Borel sets, then X must be connected.

In the sequel, X will always be an MM-space with metric d, measure μ and a local Borel ring $\mathcal{K}$.

Definition 1.2 We say that the space X is a $\sigma\mathcal{K}$, or that it is a *$\mathcal{K}$-countable space*, if X is a countable union of $\mathcal{K}$-sets.

Examples (a) If $\mathcal{K}$ is the ring of all bounded Borel subsets of X, then X is always $\mathcal{K}$-countable.

(b) If X is locally compact and separable, and $\mathcal{K}$ is the ring of all relatively compact Borel subsets, then X is $\mathcal{K}$-countable.

Definition 1.3 (a) For $1 \leq p < \infty$, the space $L^p_{loc}(X) = L^p_{loc}(X, \mathcal{K}, \mu)$ is the space of measurable functions on X which are p-integrable on every $\mathcal{K}$-set.
(b) $L^\infty_{loc}(X)$ is the space of measurable functions on X which are essentially bounded on every $\mathcal{K}$-set.

The family of semi-norms $\left\{ \|u\|_{L^p(K)} : K \in \mathcal{K} \right\}$ defines a locally convex topology on $L^p_{loc}(X)$; and we have :

Lemma 1.2 *If X is a $\mathcal{K}$-countable space, then $L^p_{loc}(X)$ is a Frechet space.*

The proof is obvious.

□

Observe also that, trivially, if $X \in \mathcal{K}$, then $L^p_{loc}(X) = L^p(X)$ is in fact a Banach space.

Notations The notation $A \subset\subset \Omega$ (or $A \Subset \Omega$) means that there exists a closed $\mathcal{K}$-set K such that $A \subset K \subset \Omega$ (in particular $A \Subset X$ if and only if A is contained in a closed $\mathcal{K}$-set).

If $\Omega \subset X$ is open, we denote by $\mathcal{K}|_\Omega$ the set of all Borel sets A such that $A \Subset \Omega$. It is a Boolean ring which we call the *trace* of $\mathcal{K}$ on Ω. This ring satisfies conditions (K1) and (K2) above. If condition (K3) also holds, then we say that Ω is $\mathcal{K}$*-connected.*

We denote by $C(X)$ the space of all continuous functions $u : X \to \mathbb{R}$ and by $C_c(X) \subset C(X)$ the subspace of continuous functions whose support is contained in a $\mathcal{K}$-set. If $\Omega \subset X$ is an open subset, then $C_c(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\operatorname{supp}(u) \Subset \Omega$.

It is clear that for any function $u \in C_0(\Omega)$, there exists an extension $\tilde{u} \in C_0(X)$ which vanishes on $X \setminus \Omega$ and such that $\tilde{u} = u$ on Ω.

The space of bounded continuous functions on an open set $\Omega \subset X$ is denoted by $C_b(\Omega) = C(\Omega) \cap L^\infty(\Omega)$. It is a Banach space for the sup norm.

We conclude this section with a few more technical definitions:

Definition 1.4 A subset F of an MM-space X is *strongly bounded* if there exists a pair of open sets $\Omega_1 \subset \Omega_2 \subset X$ such that $\Omega_2 \in \mathcal{K}$, $\mu(X \backslash \Omega_2) > 0$, $\operatorname{dist}(\Omega_1, X \backslash \Omega_2) > 0$ and $F \subset \Omega_1$.

Definition 1.5 An MM-space X is strongly $\mathcal{K}$-coverable if there exist two countable families of open $\mathcal{K}$-sets $\{U_i\}$ and $\{V_i\}$ such that $V_i \neq X$ for all i and

1) $X = \cup U_i$;
2) $U_i \subset V_i$ for all i;
3) $\operatorname{dist}(U_i, X \setminus V_i) > 0$ and
4) $\mu(V_i \setminus U_i) > 0$.

Observe that if $F \subset U_i$ for some i, then it is a strongly bounded set.

It is clear that every strongly $\mathcal{K}$-coverable metric space is also $\mathcal{K}$-countable. It is in general not difficult to check that a space is strongly $\mathcal{K}$-coverable. The next two lemmas give examples of such.

Lemma 1.3 *Suppose that* $X \in \mathcal{K}$ *and contains four closed* $\mathcal{K}$*-sets* A_1, A_1', A_2, A_2' *such that* $A_i \subset A_i'$, $\mu(A_i' \setminus A_i) > 0$, $\operatorname{dist}(A_i, X \setminus A_i') > 0$ *and* $A_1' \cap A_2' = \emptyset$. *Then* X *is strongly* $\mathcal{K}$*-coverable.*

Proof Let us set $U_i := X \setminus A'_i$ and $V_i := X \setminus A_i$. Then $\{U_1, U_2\}$ and $\{V_1, V_2\}$ are the required coverings. □

Lemma 1.4 *Let* X *be a separable metric space. Suppose that for each point* $z \in X$ *there exists* $r_z > 0$ *such that* $B(z, 2r_z) \in \mathcal{K}$ *and* $\mu(B(z,2r) \setminus B(z,r)) > 0$ *for any* $0 < r < r_z$, *then* X *is strongly* $\mathcal{K}$*-coverable.*

Proof Let $Q \subset X$ be a dense countable subset. For each $z \in X$, we choose a point $q = \phi(z) \in Q$ such that $d(z, \phi(z)) \le \frac{1}{5}\min\{1, r_z\}$ and $\phi(z) = z$ if $z \in Q$. We then define a function $s : Q \to \mathbb{R}_+$ by

$$s(q) = \sup_{\phi(z)=q} \min\{1, r_z\}$$

and for each point $q \in Q$, we choose $z = \psi(q) \in X$ such that

1) $\phi(z) = q$ and

2) $\frac{1}{2}s(q) \le r_z \le s(q)$.

Observe that the map $\psi : Q \to X$ is a left inverse of $\phi : X \to Q$ (i.e. $\phi \circ \psi = id|_Q$); we denote by σ the map $\sigma := \psi \circ \phi : X \to X$.
Observe also that for any point $x \in X$ we have $q = \phi(x) \Longrightarrow d(x, q) \le \frac{1}{5}\min\{1, r_x\} \le \frac{1}{5}s(q)$ and $d(q, \psi(q)) \le \frac{1}{5}\min\{1, r_{\psi(q)}\} \le \frac{1}{5}s(q)$.
Since $s(q) \le 2r_{\psi(q)}$ and $\psi(q) = \psi(\phi(x)) = \sigma(x)$, we obtain the estimate

$$d(x, \sigma(x)) \le d(x, q) + d(q, \psi(q)) \le \frac{2}{5}s(q) \le \frac{4}{5}r_{\sigma(x)}\,.$$

We have thus shown that for any point $x \in X$ we have $x \in B(\sigma(x), r_{\sigma(x)})$, i.e. $\{B(z, r_z)\}_{z \in \sigma(X)}$ is a covering of X.
This is clearly a countable covering since $\sigma(X) = \sigma(Q)$ is countable; thus the families of open $\mathcal{K}$-sets $\{U_z := B(z, r_z)\}_{z \in \sigma(X)}$ and $\{V_z := B(z, 2r_z)\}_{z \in \sigma(X)}$ satisfy all the conditions of Definition 1.5. □

The following are by now classic notions (see [22]):

Definition 1.6 **a)** The measure μ is called *Ahlfors-David regular* of dimension s if there exists a constant c such that for any ball $B(x, r) \subset X$ we have

$$\frac{1}{c}r^s \le \mu(B(x, r)) \le cr^s.$$

b) The measure μ is *locally* s*-regular* if for every point $x \in X$ there exists two constants c_x, R_x such that for $0 \le r \le R_x$ we have

$$\frac{1}{c_x}r^s \le \mu(B(x, r)) \le c_x r^s.$$

c) The measure μ has the *doubling property* if there exists a constant k such that for all balls $B \subset X$ we have

$$\mu(2B) \leq 2^k \mu(B)$$

(where $2B$ means the ball with same center and double radius).

1.2 D-structure and Sobolev spaces

To define an axiomatic Sobolev space on $(X, d, \mathcal{K}, \mu)$, we fix a number $1 \leq p < \infty$ and we associate to each function $u \in L^p_{loc}(X)$ a family $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$. An element $g \in D[u]$ is called a *pseudo-gradient*[2] of u. The correspondence $u \to D[u]$ is supposed to satisfy some or all of the following axioms:

Axiom A1 (Non triviality) *If $u : X \to \mathbb{R}$ is non negative and k-Lipschitz, then the function*

$$g := k\,\mathrm{sgn}(u) = \begin{cases} k & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases}$$

belongs to $D[u]$.

It follows from this axiom that $0 \in D[c]$ for any constant function $c \geq 0$.

Axiom A2 (Upper linearity) *If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \geq |\alpha| g_1 + |\beta| g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.*

This axiom implies in particular that $D[u]$ is always convex and $D[\alpha u] = |\alpha| D[u]$.

Axiom A3 (Leibniz rule) *Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function $h(x) = (\sup|\varphi| g(x) + Lip(\varphi)|u(x)|)$ belongs to $D[\varphi u]$.*

Axiom A4 (Lattice property) *Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^p_{loc}(X)$. If $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, then $g := \max\{g_1, g_2\} \in D[u] \cap D[v]$.*

The previous axioms fixed general properties of the set $D[u]$ of pseudo-gradients. The last two axioms concern the behavior of the p-integrable pseudo-gradients of locally p-integrable functions; they really are properties of Sobolev spaces rather than properties of individual functions.

Axiom A5 (Completeness) *Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all i. Assume that $u_i \to u$ in L^p_{loc} topology and $(g_i - g) \to 0$*

[2] This notion should not be confused with the pseudo-gradients in the sense of Palais as defined e.g. in [27, page 299].

in L^p topology , then $g \in D[u]$.

This axiom implies in particular that two functions which agree almost everywhere have the same set of pseudo-gradients. We henceforth always identify two functions which agree a.e.

We define a notion of energy as follow :

Definition 1.7 The *p–Dirichlet energy* of a function $u \in L^p_{loc}(X)$ is defined to be

$$\mathcal{E}_p(u) = \inf \left\{ \int_X g^p d\mu : g \in D\,[u] \right\}$$

Our final axiom states that if the energy of a function is small, then this function is close to being constant.

Axiom A6 (Energy controls variation) *Let $\{u_i\} \subset \mathcal{L}^{1,p}(X)$ be a sequence of functions such that $\mathcal{E}_p(u_i) \to 0$. Then for any metric ball $B \in \mathcal{K}$ there exists a sequence of constants $a_i = a_i(B)$ such that $\|u_i - a_i\|_{L^p(B)} \to 0$.*

Definition 1.8 a) A *D-structure* on $(X, d, \mathcal{K}, \mu)$ for the exponent p is an operation which associates to a function $u \in L^p_{loc}(X)$ a set $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ and which satisfies the Axioms A1–A5 above (for the corresponding p).
b) The D-structure is *non degenerate* if it also satisfies axiom A6.
c) A measure metric space equipped with a D-structure is called an *MMD-space.*

We now define the notion of Dirichlet and Sobolev spaces associated to a D-structure.

Definition 1.9 i) The *p-Dirichlet space* is the space $\mathcal{L}^{1,p}(X)$ of functions $u \in L^p_{loc}(X)$ with finite p-energy.
ii) The *Sobolev space* is then defined as

$$W^{1,p}(X) = W^{1,p}(X, d, \mathcal{K}, \mu, D) := \mathcal{L}^{1,p}(X) \cap L^p(X)\,.$$

Theorem 1.5 *Given a D-structure on $(X, d, \mathcal{K}, \mu)$ (for the exponent p), the corresponding Sobolev space $W^{1,p}(X)$ is a Banach space with norm*

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u) \right)^{1/p}.$$

Proof By axiom A1, we know that $0 \in W^{1,p}(X)$. It is then clear from axiom A2 that $W^{1,p}(X)$ is a vector space. Let us prove that $\|\cdot\|_{W^{1,p}(X)}$ is indeed a norm.

i) $\|u\|_{W^{1,p}(X)} = 0 \Rightarrow \|u\|_{L^p(X)} = 0 \Rightarrow u = 0$ a.e. is obvious.

ii) $\|\lambda u\|_{W^{1,p}(X)} = |\lambda| \, \|u\|_{W^{1,p}(X)}$ follows from axiom A2, since $g \in D[u] \Leftrightarrow |\lambda| g \in D[\lambda u]$ for all $\lambda \in \mathbb{R} \setminus \{0\}$; hence $\mathcal{E}_p(\lambda u) = |\lambda|^p \mathcal{E}_p(u)$.

iii) $\|u+v\|_{W^{1,p}(X)} \leq \|u\|_{W^{1,p}(X)} + \|v\|_{W^{1,p}(X)}$ also follows from axiom A2, since $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$ implies $(g_1 + g_2) \in D[u_1 + u_2]$.

We have to prove that $W^{1,p}(X)$ is complete for this norm.

Let $\{u_i\} \in W^{1,p}(X)$ be an arbitrary Cauchy sequence: we may (and do) assume (taking a subsequence if necessary) that $\|(u_i - u_{i+1})\|_{W^{1,p}(X)} \leq 2^{-i}$.
Let us set $v_j := (u_j - u_{j+1})$; by hypothesis we can find $h_j \in D[v_j] \cap L^p(X)$ such that $\|h_j\|_{L^p(X)} \leq 2^{-j}$.

We then choose an arbitrary element $g_1 \in D[u_1]$ and set $g_k := g_1 + \sum_{j=1}^{k-1} h_j$ for $k \geq 2$. It follows from the identity $u_k := u_1 - \sum_{j=1}^{k-1} v_j$ and axiom A2 that $g_k \in D[u_k]$ for all $k \in \mathbb{N}$.

Now $\{g_k\}$ and $\{u_k\}$ are both Cauchy sequences in $L^p(X)$; thus there exist limit functions $u = \lim u_k$ and $g = \lim g_k$ in the L^p sense. It follows then from axiom A5 that $g \in D[u]$; and therefore $u \in W^{1,p}(X)$.

It only remains to show that $u_k \to u$ in $W^{1,p}(X)$. In fact, since $\|(u_k - u)\|_{L^p(X)} \to 0$, we only need to prove the existence of a sequence $f_k \in D[(u_k - u)]$ such that $\| f_k\|_{L^p(X)} \to 0$.

Fix k, the sequence $v_{k,m} := (u_k - u_m)$ (where $m \geq k$) is a Cauchy sequence in $W^{1,p}(X)$ because $v_{k,m} - v_{k,n} = u_n - u_m$. Furthermore, we have the following estimate :

$$\|v_{k,m}\|_{W^{1,p}(X)} \leq \sum_{s=k}^{m-1} \|u_s - u_{s+1}\|_{W^{1,p}(X)} \leq \sum_{s=k}^{m-1} 2^{-s} \leq 2^{-(k-1)}.$$

By construction we have $\lim_{m\to\infty} v_{k,m} = (u_k - u)$ in the L^p sense and, by axiom A2, we have $g_{k,m} := \sum_{s=k}^{m-1} h_s \in D[v_{k,m}]$ since $v_{k,m} = \sum_{s=k}^{m-1} v_s$.

Now $\{g_{k,m}\}_{m=1,2,..}$ is a Cauchy sequence in $L^p(X)$ and

$$\|g_{k,m}\|_{L^p(X)} \leq \sum_{s=k}^{m-1} \|h_s\|_{L^p(X)} \leq \sum_{s=k}^{m-1} 2^{-s} \leq 2^{-(k-1)}.$$

By axiom A5 the function $f_m := \lim_{m\to\infty} g_{k,m}$ belongs to $D[(u_k - u)]$ and

$$\|f_k\|_{L^p(X)} = \lim_{m\to\infty} \|g_{k,m}\|_{L^p(X)} \leq 2^{-(k-1)}.$$

This proves the completeness of $W^{1,p}(X)$.

□

Observe that in this proof, we have only used Axioms A2 and A5.

From the proof we obtain the following

Corollary 1.6 *$\mathcal{L}^{1,p}(X)$ is a seminormed space with seminorm $\|u\|_{\mathcal{L}^{1,p}(X)} = (\mathcal{E}_p(u))^{1/p}$.*

□

If the D-structure is degenerate, then the p-energy may be trivial (i.e. $\mathcal{E}_p(u) = 0$ for any u). In that case the Sobolev space $W^{1,p}(X)$ reduces to the usual Lebesgue space $L^p(X)$. For this reason, it is necessary to understand the meaning of axiom A6.

We first observe that the sequence of constants $a_i(B)$ appearing in axiom A6 is in fact global, i.e. independent of the chosen ball $B \in \mathcal{K}$, as our next result shows:

Proposition 1.7 *Assume that axiom A6 holds. Let $\{u_i\} \subset \mathcal{L}^{1,p}(X)$ be a sequence of functions such that $\mathcal{E}_p(u_i) \to 0$. Then there exists a sequence of constants $\{c_i\} \subset \mathbb{R}$ such that for any $A \in \mathcal{K}$ we have $\|u_i - c_i\|_{L^p(A)} \to 0$.*

Proof The proof follows easily from Lemma 1.1, axiom A6, and from the following lemma.

□

Lemma 1.8 *Assume that D is a non degenerate D-structure. Let $\{u_i\} \subset \mathcal{L}^{1,p}(X)$ be a sequence of functions such that $\mathcal{E}_p(u_i) \to 0$ and let $B_1, B_2 \in \mathcal{K}$ be open balls such that $\mu(B_1 \cap B_2) > 0$.*
If $\| u_i - c_i\|_{L^p(B_1)} \to 0$ for some sequence $\{c_i\} \subset \mathbb{R}$ then $\|u_i - c_i\|_{L^p(B_1 \cup B_2)} \to 0$.

Proof By axiom A6, there are sequences $\{a_i\}, \{a_i'\} \subset \mathbb{R}$ such that $\|u_i - a_i\|_{L^p(B_1)} \to 0$ and $\|u_i - a_i'\|_{L^p(B_2)} \to 0$. It is clear that $\lim_{i\to\infty}(a_i - c_i) = 0$, on the other hand we have $\lim_{i\to\infty}(a_i - a_i') = 0$ because $\mu(B_1 \cap B_2) > 0$ and

$$\|a_i - a_i'\|_{L^p(B_1 \cap B_2)} \leq \|u_i - a_i\|_{L^p(B_1)} + \|u_i - a_i'\|_{L^p(B_2)} \to 0\,.$$

It follows that the three sequences $\{a_i\}, \{a_i'\}$ and $\{c_i\}$ are equivalent (i.e. $\lim(a_i - c_i) = \lim(a_i' - c_i) = 0$).
Therefore

$$\begin{aligned} \|u_i - c_i\|_{L^p(B_1 \cup B_2)} &\leq \|u_i - c_i\|_{L^p(B_1)} + \|u_i - c_i\|_{L^p(B_2)} \\ &\leq \|u_i - a_i\|_{L^p(B_1)} + \|u_i - a_i'\|_{L^p(B_2)} \\ &\quad + \|c_i - a_i\|_{L^p(B_1)} + \|c_i - a_i'\|_{L^p(B_2)} \\ &\to 0\,. \end{aligned}$$

We conclude this section by stating some elementary properties of the spaces $\mathcal{L}^{1,p}(X)$ and $W^{1,p}(X)$.

Proposition 1.9 *Suppose that axioms A1–A6 hold then*

1) *If* $u \in \mathcal{L}^{1,p}(X)$ *has no energy, i.e.* $\mathcal{E}_p(u) = 0$, *then* u *is a.e. constant*

2) $W^{1,p}(X)$ *is a lattice, i.e. if* $u, v \in W^{1,p}(X)$, *then* $\max\{u, v\} \in W^{1,p}(X)$ *and* $\min\{u, v\} \in W^{1,p}(X)$.

3) $W^{1,p}(X)$ *contains all Lipschitz functions with support in a* $\mathcal{K}$*-set.*

4) *If* $u \in W^{1,p}(X)$ *and* φ *is a Lipschitz function with support in a* $\mathcal{K}$*-set, then* $\varphi u \in W^{1,p}(X)$.

5) *Truncation does not increase energy, i.e.* $\mathcal{E}_p(\max(u, c)) \leq \mathcal{E}_p(u)$.

Proof (1) is not difficult to prove from Proposition 1.7. It is also an obvious consequence of Proposition 1.11 below. (2) follows from axiom A4 and (3) follows from A1. Finally, (4) follows immediately from the axioms A1 and A3 and (5) is a direct consequence of axioms A1 and A4.

□

1.3 Poincaré inequalities

The next result gives us a practical way of checking axiom A6.

Proposition 1.10 *Suppose that for each metric ball* $B \in \mathcal{K}$ *there exists a constant* $C = C_B$ *such that the following inequality*

$$\int_B |u - u_B|^p \leq C_B\, \mathcal{E}_p(u) \tag{1}$$

holds for any function $u \in L^p_{loc}(X)$ *where* $u_B := \frac{1}{\mu(B)} \int_B u\, d\mu$. *Then axiom A6 holds.*

An inequality such as (1) is classically called a *Poincaré type inequality* or a *pseudo Poincaré inequality*

The proof of this proposition is obvious. What is interesting is that the converse also holds; in fact we have the following stronger result.

Proposition 1.11 (Floating Poincaré inequality) *Assume that axiom A6 holds and let* $Q \subset A \subset X$ *be two measurable sets such that* $A \in \mathcal{K}$ *and* $\mu(Q) > 0$. *Then the inequality*

$$\|u - u_Q\|_{L^p(A)} \leq C_{A,Q} \, \|g\|_{L^p(X)}$$

holds for any $u \in \mathcal{L}^{1,p}(X)$ *and* $g \in D[u]$; *where the constant* $C_{A,Q}$ *depends on* p, A *and* Q *only, and* $u_Q := \frac{1}{\mu(Q)} \int_Q u \, d\mu$.

We call this inequality *the floating Poincaré inequality*, because the function u is averaged on the "floating vessel" $Q \subset A$.

Proof Suppose by contradiction that for some $Q \subset A \subset X$ with $\mu(Q) > 0$ and $A \in \mathcal{K}$ no such constant exists. It means that there exists sequences $\{u_i\} \subset L^p_{loc}(X)$ and $g_i \in D[u_i]$ such that

$$\lim_{i \to \infty} \left(\frac{\|u_i - u_{i,Q}\|_{L^p(A)}}{\|g_i\|_{L^p(X)}} \right) = \infty .$$

Using axiom A2 we can renormalize the sequence $\{u_i\}$ in such a way that $\|u_i - u_{i,Q}\|_{L^p(A)} = 1$ for all i and thus $\|g_i\|_{L^p(X)} \to 0$ as $i \to \infty$.

By Proposition 1.7 there exist a sequence of constants a_i such that $\|u_i - a_i\|_{L^p(A)} \to 0$. By Hölder's inequality we have

$$\|u_i - a_i\|_{L^p(Q)} \geq \mu(Q)^{-\frac{1}{q}} \, \|u_i - a_i\|_{L^1(Q)} \geq \mu(Q)^{\frac{1}{p}} \mid u_{i,Q} - a_i \mid$$

where $1/p + 1/q = 1$. Therefore $(u_{i,Q} - a_i) \to 0$ and we thus have

$$1 = \|u_i - u_{i,Q}\|_{L^p(A)} \leq \|u_i - a_i\|_{L^p(A)} + \|a_i - u_{i,Q}\|_{L^p(A)} \to 0 .$$

This contradiction implies the desired result. □

We will sometimes use the following corollary.

Corollary 1.12 *Assume that axiom A6 holds and let* $Q, A \in \mathcal{K}$ *be two* $\mathcal{K}$*-sets such that* $\mu(Q) > 0$. *Then there exists a constant* $C_{A,Q} = C(A, Q, p)$ *such that the inequality*

$$\|u\|_{L^p(A)} \leq C_{A,Q} \, \|g\|_{L^p(X)}$$

holds for any $u \in \mathcal{L}^{1,p}(X)$ *such that* $u \equiv 0$ *on* Q *and all* $g \in D[u]$.

(Observe that we do not assume here that $Q \subset A$.)

Proof Apply Proposition 1.11 to the set $A_1 := A \cup Q$. □

The next result is a variant of the Corollary 1.12 where the constant in the inequality depends on A and p only:

Proposition 1.13 *Assume that axiom A6 holds and let* $A \subset X$ *be a measurable* $\mathcal{K}$*-sets such that* $\mu(A) > 0$ *and* $\mu(X \setminus A) > 0$. *Then there exists a constant* C_A *depending on* p *and* A *only for which the inequality*

$$\|u\|_{L^p(A)} \leq C_A \|g\|_{L^p(X)}$$

holds for any $u \in \mathcal{L}^{1,p}(X)$ *such that* $\operatorname{supp}(u) \subset A$ *and* $g \in D[u]$.

Proof We argue as in the proof of Proposition 1.11. If no such constant exists, then we can find two sequences $\{u_i\} \subset \mathcal{L}^{1,p}(X)$, $g_i \in D[u_i]$ such that $\operatorname{supp}(u_i) \subset A$, $\|u_i\|_{L^p(A)} = 1$ for all i and $\|g_i\|_{L^p(X)} \to 0$ for $i \to \infty$.
By Proposition 1.7 there exist a sequence of constants a_i such that $\|u_i - a_i\|_{L^p(B)} \to 0$ for all $B \in \mathcal{K}$. Choosing first a $\mathcal{K}$−set $B \subset X \setminus A$ of positive measure, we deduce that $\lim a_i = 0$; and choosing then $B = A$ yields the contradiction

$$1 = \|u_i\|_{L^p(A)} \leq \|u_i - a_i\|_{L^p(A)} + \|a_i\|_{L^p(A)} \to 0\,.$$

□

The various Poincaré inequalities we have discussed above are rather weak in the sense that no control of the constant involved is specified.
However the case where the constant depends linearly on the radius of the ball deserves special attention; following J. Heinonen and P. Koskela, we adopt the following

Definition 1.10 One says that a D-structure on an MM-space X *supports a* (q,p)*-Poincaré inequality* if there exists two constants $\sigma \geq 1$ and $C > 0$ such that

$$\left(\fint_{B(x,r)} |u - u_B|^q d\mu\right)^{1/q} \leq Cr\left(\fint_{B(x,\sigma r)} g^p d\mu\right)^{1/p} \tag{2}$$

for any ball $B(x,r) \in \mathcal{K}$, any $u \in L^p_{loc}(X)$ and any $g \in D[u]$.

The explicit dependence on the radius of the ball expresses a scaling property of the Poincaré inequality; therefore this inequality is sometimes also called a *scaled Poincaré inequality* (inégalité de Poincaré "à l'échelle des boules" in the terminology of [1]).

The inequality (2) is sometimes called a *weak* Poincaré inequality when $\sigma > 1$ and a *strong* one if $\sigma = 1$.

Remark If the MMD-space X supports a (q,p)-Poincaré inequality, then it also supports a (q,p')-Poincaré inequality for all $p' \geq p$ (this follows directly from Jensen's –or Hölder's– inequality).
Likewise, if X supports a (q,p)-Poincaré inequality, then it also supports a (q',p)-Poincaré inequality for all $q' \leq q$.

If the measure has the doubling property (Definition 1.6), then the Poincaré inequality has the following non trivial self-improving property (see Theorem 5.1 in [19] or [39, Theorem 5.2] in the particular case of Ahlfors regular spaces).

Theorem 1.14 *Suppose that the MMD space X supports a $(1,p)-$Poincaré inequality. If μ has the doubling property, then there exists $\bar{q} > p$ such that X supports a $(q,p)-$Poincaré inequality for all $1 \leq q < \bar{q}$. In particular it supports a $(p,p)-$Poincaré inequality.*

□

Corollary 1.15 *Suppose that the MMD space X is doubling and supports a $(1,p)-$Poincaré inequality, then it is non degenerate.*

Proof This follows from Proposition 1.10 and Theorem 1.14. □

Corollary 1.16 *If X is doubling and supports a $(1,1)-$Poincaré inequality, then it also supports a $(q,p)-$Poincaré inequality for any $1 \leq q \leq p < \infty$.*

Proof This is clear from the previous remark and Theorem 1.14. □

A quite complete investigation of the meaning of $(q,p)-$Poincaré inequalities can be found in [19].

1.4 Locality

The gradient of a smooth function in $\mathbb{R}^n$ depends only on the local behaviour of this function. This is still the case for a (classical) Sobolev function; for instance if a function $v \in W^{1,p}(\mathbb{R}^n)$ is constant on some set $A \subset \mathbb{R}^n$, then its weak gradient vanishes on that set.

This property is not always true in the context of axiomatic Sobolev spaces and there seems to be several natural ways to define a notion of locality for Sobolev spaces. We propose below three notions of local D-structures.

Definition 1.11 **a)** We say that a D-structure is *local* if, in addition to the axioms A1–A5, the following property holds: *If u is constant a.e. on a subset $A \in \mathcal{K}$, then $\mathcal{E}_p(u|A) = 0$ where*

$$\mathcal{E}_p(u|A) := \inf\left\{ \int_A g^p d\mu \,\middle|\, g \in D[u] \right\}$$

is the local p-Dirichlet energy of u.

b) The D-structure is *strictly local* if for any $g \in D[v]$, we have $(g\chi_{\{v>0\}}) \in D[v^+]$.

(Here $v^+ = \max\{v, 0\}$ and $\chi_{\{v>0\}}$ is the characteristic function of the set $\{v > 0\}$.)

Lemma 1.17 *If D is strictly local, then it is local.*

Proof Suppose that $u = c = const.$ on a subset $A \subset X$ and set $v := (u - c)$; so that $u = v^+ - v^- + c$. We have $g_1 := (g\chi_{\{v>0\}}) \in D[v^+]$, $g_2 := (g\chi_{\{v<0\}}) \in D[v^-]$ by hypothesis and $0 \in D[c]$ by axiom A1. We thus have from axiom A2, $h := (g\chi_{\{v\neq 0\}}) = g_1 + g_2 + 0 \in D[u]$, hence

$$\mathcal{E}_p(u|A) \leq \int_A h^p d\mu = 0$$

since $h = 0$ on A.

□

The difference between the two notions of locality can be illustrated by the next two lemmas.

Lemma 1.18 *A D-structure on the MM-space X is local if and only if for any subset $A \in \mathcal{K}$ and any pair of functions $u, v \in L^p_{loc}(X)$ such that $u = v$ on A we have*

$$2^{1-p}\mathcal{E}_p(v|A) \leq \mathcal{E}_p(u|A) \leq 2^{p-1}\mathcal{E}_p(v|A).$$

Proof $\boxed{\Longleftarrow}$ is clear because constant functions have zero energy.
$\boxed{\Longrightarrow}$ It is enough to prove the second inequality. Since the function $w := (u - v)$ vanishes on A, we have $\mathcal{E}_p(w|A) = 0$. We can thus find for any $\varepsilon > 0$ a pseudo-gradient $g_0 \in D[w]$ such that $\int_A g_0^p d\mu \leq \varepsilon$.
Let $g \in D[v]$ be an arbitrary pseudo-gradient of v, since $u = v + w$, we have $h := (g + g_0) \in D[u]$ by Axiom A2. Thus

$$\begin{aligned}\mathcal{E}_p(u|A) &\leq \int_A h^p d\mu = \int_A (g + g_0)^p d\mu \\ &\leq 2^{p-1}\left(\int_A g^p d\mu + \int_A g_0^p d\mu\right) \\ &\leq 2^{p-1}\left(\int_A g^p d\mu + \varepsilon\right)\end{aligned}$$

and therefore $\mathcal{E}_p(u|A) \leq 2^{p-1}\mathcal{E}_p(v|A)$.

□

Lemma 1.19 *Let X be a strictly local MMD-space. If $u, v \in L^p_{loc}(X)$ is a pair of functions such that $u = v$ on $A \in \mathcal{K}$, then*

$$\mathcal{E}_p(v|A) = \mathcal{E}_p(u|A).$$

Proof Let $w := (u - v)$ and choose an arbitrary pseudo-gradient $g_0 \in D[w]$. Now set $g_1 := g_0\chi_{\{w>0\}}$ and $g_2 := g_0\chi_{\{w<0\}}$; observe that $g_1 = g_2 = 0$ on A. Because the D-structure is strictly local, we have $g_1 \in D[w^+]$ and $g_2 \in D[w^-]$. Fix $\varepsilon > 0$ and choose $g \in D[v]$ such that $\int_A g^p d\mu \le \mathcal{E}_p(v|A) + \varepsilon$.
Since $u = v + w^+ + w^-$, we have $h := (g + g_1 + g_2) \in D[u]$ by Axiom A2. Thus

$$\begin{aligned}\mathcal{E}_p(u|A) &\le \int_A h^p d\mu = \int_A (g + g_1 + g_2)^p d\mu \\ &= \int_A g^p d\mu \le \mathcal{E}_p(v|A) + \varepsilon .\end{aligned}$$

and therefore $\mathcal{E}_p(u|A) \le \mathcal{E}_p(v|A)$. The converse inequality follows by symmetry. □

It is sometimes useful in some applications to have a notion of locality which is intermediate between the notions (a) and (b).

Definition 1.12 A D-structure is *absolutely local* if it is local (i.e. condition (a) of the previous definition holds) and if for any $g \in D[v]$, we have $(g\chi_{\{v\ge 0\}}) \in D[v^+]$.

Some Sobolev spaces are local and some are not. For example, classical Sobolev spaces on Euclidean domains or Riemannian manifolds are local, while trace spaces of Sobolev spaces on bad domains are not local Sobolev spaces. The version of Sobolev spaces on metric spaces introduced by P.Hajłasz (see section 2.3) is another example of global Sobolev space.

A similar notion of locality appears in [19, page 9] under the name *truncation property*. A different, albeit related, notion of locality also appears in the theory of Dirichlet forms, see [2, page 28].

1.5 Topology on the Dirichlet space

Recall that the Dirichlet space $\mathcal{L}^{1,p}(X)$ is a semi-normed space with semi-norm

$$\|u\|_{\mathcal{L}^{1,p}(X)} = (\mathcal{E}_p(u))^{1/p} = \inf\left\{ \|g\|_{L^p(X)} \,\middle|\, g \in D[u] \right\}.$$

The space $\mathcal{L}^{1,p}(X)$ is also equipped with a locally convex topology defined as follow: one says that a sequence $\{u_i\}$ converges to some function $u \in \mathcal{L}^{1,p}(X)$ if $\mathcal{E}_p(u - u_i) \to 0$ and $\|u - u_i\|_{L^p(A)} \to 0$ for all $A \in \mathcal{K}$.

Proposition 1.20 *The quotient space* $\mathcal{L}^{1,p}(X)/\mathbb{R}$ *is a Banach space for the norm* $\|\cdot\| = \mathcal{E}_p(\cdot)^{1/p}$.

Proof By Proposition 1.9 any function $u \in \mathcal{L}^{1,p}(X)$ such that $\mathcal{E}_p(u) = 0$ is a constant, thus $\|\cdot\| = \mathcal{E}_p(\cdot)^{1/p}$ is a norm in the space $\mathcal{L}^{1,p}(X)/\mathbb{R}$.
To show that $\mathcal{L}^{1,p}(X)/\mathbb{R}$ is complete under this norm we consider a Cauchy sequence $\{u_i\}$ in $\mathcal{L}^{1,p}(X)/\mathbb{R}$. By Proposition 1.7, there exists a sequence $\{c_i\} \subset \mathbb{R}$ such that $v_i := (u_i - c_i)$ converges in $L^p_{loc}(X)$.
Arguing as in the proof of Theorem 1.5, we see that the function $v := \lim_{i\to\infty} v_i \in \mathcal{L}^{1,p}(X)$. Thus $\{u_i\}$ converges to v in $\mathcal{L}^{1,p}(X)/\mathbb{R}$.

□

It is also convenient to introduce a norm on $\mathcal{L}^{1,p}(X)$: to define this norm, we fix a set $Q \in \mathcal{K}$ such that $\mu(Q) > 0$ and we set

$$\|u\|^p_{\mathcal{L}^{1,p}(X,Q)} := \left(\int_Q |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p} \tag{3}$$

Theorem 1.21 *This norm turns $\mathcal{L}^{1,p}(X)$ into a Banach space. Furthermore the locally convex topology on $\mathcal{L}^{1,p}(X)$ defined above and the topology defined by this norm coincide; in particular the Banach space structure is independent of the choice of $Q \in \mathcal{K}$.*

Proof The proof of the first assertion is the same as the proof of Theorem 1.5. We prove the second assertion. Let Q' be another $\mathcal{K}$-set of positive measure and choose a $\mathcal{K}$-set $A \supset Q \cup Q'$. Since $|u_Q| \leq \frac{1}{\mu(Q)^{1/p}} \|u\|_{L^p(Q)}$, we have

$$\|u_Q\|_{L^p(Q')} = \mu(Q')^{1/p} |u_Q| \leq \left(\frac{\mu(Q')}{\mu(Q)}\right)^{1/p} \|u\|_{L^p(Q)} ;$$

By the floating Poincaré inequality (Proposition 1.11) we have $\int_A |u - u_Q|^p \, d\mu \leq C\mathcal{E}_p(u)$, thus

$$\begin{aligned} \|u\|_{L^p(Q')} &= \|u - u_Q + u_Q\|_{L^p(Q')} \leq \|u - u_Q\|_{L^p(A)} + \|u_Q\|_{L^p(Q')} \\ &\leq (C\mathcal{E}_p(u))^{1/p} + \left(\frac{\mu(Q')}{\mu(Q)}\right)^{1/p} \|u\|_{L^p(Q)} \\ &= \text{const.} \|u\|_{\mathcal{L}^{1,p}(X,Q)} \,. \end{aligned}$$

The proposition follows

□

1.6 Minimal pseudo-gradient

Proposition 1.22 *Assume that $1 < p < \infty$. Then for any function $u \in \mathcal{L}^{1,p}(X)$, there exists a unique function $g_u \in D[u]$ such that $\int_X g_u^p d\mu = \mathcal{E}_p(u)$.*

Definition 1.13 The function g_u is called the *minimal pseudo-gradient* and is denoted by $\underline{D}_p u$ (or simply $\underline{D}u$ if explicit reference to p is not needed).

We have thus defined a map $\underline{D} : W^{1,p}(X) \to L^p(X)$; this map is generally non linear.

The proposition is an immediate consequence of the next two lemmas:

Lemma 1.23 *For any function u, the set $D[u] \cap L^p(X)$ is convex and closed in $L^p(X)$.*

Proof Convexity follows from axiom A2 and closedness follows from axiom A5. □

Lemma 1.24 *In any nonempty closed convex subset $A \subset E$ of a uniformly convex Banach space E, there exists a unique element $x^* \in A$ with minimal norm: $\|x^*\| = \inf_{x\in A}\|x\|$.*

Recall that a Banach space E is said to be *uniformly convex* if for any pair of sequences $\{x_n\}, \{y_n\} \subset E$ satisfying

$$\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\| = \lim_{n\to\infty}\frac{1}{2}\|x_n + y_n\| = 1\,,$$

we have $\lim\limits_{n\to\infty}\|x_n - y_n\| = 0$.

This definition is due to Clarkson. A basic example of uniformly convex Banach space is $L^p(X, d\mu)$ (see e.g. [7], [25] or [26]).

Lemma 1.24 can be found in [26, Satz 16.4]. We repeat the proof for the convenience of the reader :

Proof If $0 \in A$, then there is nothing to prove, we thus assume that $\alpha := \inf_{x\in A}\|x\| > 0$.

Existence: Set $A_1 = \frac{1}{\alpha}A$ and choose a minimizing sequence $\{x_n\} \subset A_1$ such that $\|x_n\| \to 1$ for $n \to \infty$. Because A_1 is a convex set, we have $\frac{1}{2}(x_n + x_m) \in A_1$, hence $\frac{1}{2}\|x_n + x_m\| \geq 1$. On the other hand $\frac{1}{2}\|x_n + x_m\| \leq \frac{1}{2}(\|x_n\| + \|x_m\|) \to 1$ for $n, m \to \infty$, hence $\frac{1}{2}\|x_n + x_m\| \to 1$ for $n, m \to \infty$. Thus, by definition of uniform convexity $\|x_n - x_m\| \to 0$, i.e. $\{x_n\}$ is a Cauchy sequence. Since E is complete, there exists a limit $x^* = \lim\limits_{n\to\infty} x_n$. Since A_1 is closed, we have $x^* \in A_1$ and the existence of a minimal element is proved.

Uniqueness: Suppose x^*, y^* are minimal elements of A. Then $\|x^*\| = \|y^*\| = \alpha := \inf_{x\in A}\|x\|$. By convexity of the norm we have $\left\|\frac{1}{2}(x^* + y^*)\right\| \leq \alpha$. By convexity of A, we have $\frac{1}{2}(x^* + y^*) \in A$, thus (by definition of α) $\frac{1}{2}\|(x^* + y^*)\| \geq \alpha$. Therefore $\|x^* + y^*\| = 2\alpha = \|x^*\| + \|y^*\|$. By uniform convexity, this equality implies $\|x^* - y^*\| = 0$, hence $x^* = y^*$. □

Remark In the case where the pseudo-gradient is local (definition 1.11), then the minimal pseudo-gradient $\underline{D}u$ of a function $u \in \mathcal{L}^{1,p}(X)$ vanishes a.e. on any $\mathcal{K}$-set where u is constant.

1.7 Defining a D-structure by completion

The classical Sobolev space on $\mathbb{R}^n$ can either be defined through the notion of weak gradient or by completion of smooth functions for the Sobolev norm. A similar completion is sometimes useful in the context of axiomatic Sobolev spaces on metric spaces.

Let us first choose a class $\mathcal{F}$ of functions $u : X \to \mathbb{R}$ such that

F1) $\mathcal{F}$ is a vector space and a lattice;

F2) $\mathcal{F}$ contains all Lipschitz functions;

F3) if $u \in \mathcal{F}$ and φ is a bounded Lipschitz function, then $\varphi u \in \mathcal{F}$.

In particular, $\mathcal{F}$ is a module over the algebra of bounded Lipschitz functions. As an example, $\mathcal{F}$ may be the class of all locally Lipschitz functions.

We then assume that a family $D[u]$ of pseudo-gradients has been defined for all functions $u \in \mathcal{F}$ in such a way that axioms A1–A4 hold for the correspondence $\mathcal{F} \ni u \to D[u]$.

We finally define $\widetilde{D}[u]$ for all functions $u \in L^p_{loc}(X)$ by the following completion procedure:

Definition 1.14 Let $u \in L^p_{loc}(X)$ and $g : X \to \mathbb{R}_+ \cup \{\infty\}$ be a measurable function. Then $g \in \widetilde{D}[u]$ if and only if either $g \equiv \infty$ or there exist two sequences $\{u_i\}$ and $\{g_i\}$ of measurable functions such that $u_i \in \mathcal{F}$, $g_i \in D[u_i]$, $u_i \to u$ in L^p_{loc} topology and $(g - g_i) \to 0$ in L^p topology.

(If no such sequence exists, then $\widetilde{D}[u]$ contains only the function $g \equiv \infty$.)

An element of $\widetilde{D}[u]$ is called a *generalized pseudo-gradient.* Observe that $\widetilde{D}[u]$ depends on the choice of $\mathcal{F}$ and of p.

Proposition 1.25 *The correspondence* $u \to \widetilde{D}[u]$ *satisfies axioms A1–A5 for* $u \in L^p_{loc}(X)$.

Proof Axiom A1 is obvious because Lipschitz functions belong to $\mathcal{F}$.
To prove axiom A2, we consider two functions $u, v \in L^1_{loc}(X)$, pseudo-gradients $g \in \widetilde{D}[u]$, $h \in \widetilde{D}[v]$ and a function $f \geq |\alpha|g + |\beta|h$. By definition, we can find sequences $\{u_i\}, \{v_i\} \subset \mathcal{F}$ and $\{g_i\}, \{h_i\}$ such that $g_i \in D[u_i]$, $h_i \in D[v_i]$, $u_i \to u$, $v_i \to v$ in $L^p_{loc}(X)$ and $(g - g_i), (h - h_i) \to 0$ in $L^p(X)$.
Let $f_i := |\alpha|g_i + |\beta|h_i + (f - |\alpha|g - |\beta|h)$; since axiom A2 holds on $\mathcal{F}$ and $f_i \geq |\alpha|g_i + |\beta|h_i$, we have $f_i \in D[\alpha u_i + \beta v_i]$.

Because $(f - f_i) = (|\alpha|(g - g_i) + |\beta|(h - h_i)) \to 0$, we conclude that $f \in \widetilde{D}[\alpha u + \beta v]$. To verify axioms A3 and A4 we only need to observe that these properties are stable under L^p_{loc} convergence.

To prove axiom A5, we consider two sequences of functions $\{u_i\}$ and $\{g_i\}$ such that $g_i \in \widetilde{D}[u_i]$ for all i and $u_i \to u$ in L^p_{loc} and $(g - g_i) \to 0$ in L^p. If, for some subsequence, we have $g_{i_j} = \infty$ a.e. for all i, then $g = \infty$ almost everywhere and thus $g \in \widetilde{D}[u]$. Otherwise there exists for each i two sequences $\{u_{ij}\} \subset \mathcal{F}$, and $\{g_{ij}\}$ of measurable functions such that $g_{ij} \in D[u_{ij}]$, $u_{ij} \to u_i$ in L^p_{loc} and $(g - g_{ij}) \to 0$ in L^p as $j \to \infty$. Using a diagonal type process, we can find a sequence $\{v_k\} \subset \mathcal{F}$ and a sequence $\{f_k\}$ such that $f_k \in D[v_k]$ for all k and $v_k \to u$ in L^p_{loc} topology and $(g - f_k) \to 0$ in L^p. Therefore $g \in \widetilde{D}[u]$ by definition. □

Proposition 1.26 *If axiom A6 holds for the correspondence $u \to D[u]$ where $u \in \mathcal{F}$, then it also holds for $u \to \widetilde{D}[u]$ for any $u \in L^p_{loc}(X)$.*

Proof Suppose that $u_i \in L^p_{loc}(X)$, $g_i \in \widetilde{D}[u_i] \cap L^p(X)$ such that $\|g_i\|_{L^p(X)} = (\mathcal{E}_p(u_i))^{1/p} \to 0$. Fix a ball $B \in \mathcal{K}$. Using the definition of generalized pseudo-gradient and a diagonal argument as in the previous proof, we can find sequences $v_i \in \mathcal{F}$ and $f_i \in D[u_i]$, such that $\|g_i - f_i\|_{L^p(X)} \to 0$ and $\|u_i - v_i\|_{L^p(B)} \to 0$ in $L^p(B)$. Hence $\|f_i\|_{L^p(X)} \to 0$. Since axiom A6 holds for sequences $\{v_i\} \in \mathcal{F}$ there exists a sequence a_i such that $\|v_i - a_i\|_{L^p(B)} \to 0$. We thus conclude that $\|u_i - a_i\|_{L^p(B)} \leq \|u_i - v_i\|_{L^p(B)} + \|v_i - a_i\|_{L^p(B)} \to 0$. □

Since generalized pseudo-gradients behave like pseudo-gradients we usually drop the tilde and write simply $D[u]$ instead of $\widetilde{D}[u]$.

1.8 Relaxed topology, m-topology and density

Definition 1.15 Fix $1 \leq p \leq \infty$. A sequence $\{u_j\} \subset W^{1,p}(X)$ is said to *converge to the function* $u \in W^{1,p}(X)$ *in the relaxed topology* of $W^{1,p}(X)$ if $u_j \to u$ in $L^p(X)$ and $\mathcal{E}_p(u_j) \to \mathcal{E}_p(u)$.

Proposition 1.27 *Suppose that $W^{1,p}(X)$ is uniformly convex. Then any sequence $\{u_j\} \subset W^{1,p}(X)$ which converges in the relaxed topology contains a subsequence which converges in the usual topology (i.e. for the Sobolev norm).*

The proof will be based on the following:

Lemma 1.28 *Let $\{u_j\} \subset E$ be a sequence in a uniformly convex Banach space E. Assume that $\{u_j\}$ converges weakly to an element $u \in E$ and that $\lim_{j\to\infty} \|u_j\| = \|u\|$. Then $\{u_j\}$ converges strongly to u.*

Proof We have to show that $\lim_{j\to\infty}\|u_j - u\| = 0$. If $u = 0$ then there is nothing to prove; we may thus assume $u \neq 0$ and we normalize it to $\|u\| = 1$.
Set $x_j := u_j$ and $y_j := u$, we have $\lim_{j\to\infty}\|x_j\| = \|u\| = 1$ and $\lim_{j\to\infty}\|y_j\| = \|u\| = 1$, on the other hand $\frac{1}{2}(x_j + y_j)$ converges weakly to u, thus, from the lower semicontinuity of the norm in the weak topology in any Banach space, we have

$$1 = \|u\| \leq \liminf_{j\to\infty} \frac{1}{2}\|x_j + y_j\| \leq \limsup_{j\to\infty} \frac{1}{2}(\|x_j\| + \|y_j\|) = 1,$$

which implies $\lim_{j\to\infty}\|x_j + y_j\| = 2$.
By the uniform convexity of E, we conclude that $\lim_{j\to\infty}\|u_j - u\| = \lim_{j\to\infty}\|x_j - y_j\| = 0$.

□

Proof of Proposition 1.27 If $\{u_j\} \subset W^{1,p}(X)$ converges in the relaxed topology, then it is bounded in $W^{1,p}(X)$, hence it contains a subsequence which converges weakly. We conclude from Lemma 1.28.

□

Until the end of this section, we assume that the Sobolev space $W^{1,p}(X)$ is defined by the completion procedure described in definition 1.14 starting from some class of functions $\mathcal{F}$.

Lemma 1.29 *Assume that X is $\mathcal{K}$-countable and let $1 < p < \infty$. Then for any $u \in W^{1,p}(X)$, there exist sequences $\{u_k\} \subset \mathcal{F} \cap W^{1,p}_{loc}(X)$ and $\{g_k\} \subset L^p(X)$ such that $g_k \in D[u_k]$ and*

1) $u_k \to u$ in $L^p_{loc}(X)$ and

2) $g_k \to \underline{D}u$ in $L^p(X)$.

In particular $\lim_{k\to\infty} \mathcal{E}_p(u_i) = \mathcal{E}_p(u)$.

Recall that $\underline{D}u$ is the minimal pseudo-gradient of u.

Proof By definition, for any $k \in \mathbb{N}$, there exists sequences $\{w_{k,i}\} \subset \mathcal{F}$ and $\{g_{k,i}\} \subset L^p(X)$ such that $g_{k,i} \in D[w_{k,i}]$, $w_{k,i} \to u$ in $L^p_{loc}(X)$ and $(h_k - g_{k,i}) \to 0$ in $L^p(X)$ where $h_k \in \widetilde{D}[u]$ is a generalized pseudo-gradient for u and $\int_X h_k^p d\mu \leq \mathcal{E}_p(u) + \frac{1}{k}$.
Because X is assumed to be $\mathcal{K}$-countable, the space $L^p_{loc}(X)$ is a Frechet space; in particular it is metrizable. An example of metric on $L^p_{loc}(X)$ is given by

$$\rho(u,v) := \sum_{j=1}^{\infty} 2^{-j} \min\{1, \|u - v\|_{L^p(U_j)}\}$$

where $U_1 \subset U_2 \subset \cdots X$ is an exhaustion of X by $\mathcal{K}$-sets.
For each $k \in \mathbb{N}$, we can find $i(k)$ such that $\rho(w_{k,i(k)}, u) \leq 1/k$. Let us set $v_k := w_{k,i(k)}$ and $g_k := g_{k,i(k)}$, then $g_k \in D[v_k]$ and $v_k \to u$ in $L^p_{loc}(X)$. Furthermore $\limsup_{k\to\infty} \int_X g_k^p d\mu \leq \mathcal{E}_p(u)$.
In particular $\{g_k\}$ is a bounded sequence in the reflexive Banach space $L^p(X)$.

Passing to a subsequence if necessary, we may assume that $\{g_k\}$ converges weakly in $L^p(X)$. By Masur's lemma there exists a sequence of finite convex combinations $G_k := \sum_{\nu=k}^{m(k)} \alpha_{k,\nu} g_\nu$ $(\alpha_{k,\nu} \geq 0,\ \sum_{\nu=k}^{m(k)} \alpha_{k,\nu} = 1)$ which converges in $L^p(X)$ to a function $h \in L^p(X)$. Let us set $u_k := \sum_{\nu=k}^{m(k)} \alpha_{k,\nu} v_\nu \in \mathcal{F}$, then $u_k \to u$ in $L^p_{loc}(X)$ and $G_k \in D[u_k]$ by axiom A2. It follows by definition that $h \in \widetilde{D}[u]$ and thus

$$\mathcal{E}_p(u) \leq \int_X h^p d\mu \leq \limsup_{k\to\infty} \int_X G_k^p d\mu \leq \mathcal{E}_p(u)\,,$$

Hence $h := \lim G_k$ is the minimal pseudo-gradient of u. □

For $1 < p < \infty$, we introduce an intermediate topology between the relaxed topology and the Sobolev topology :

Definition 1.16 Fix $1 < p < \infty$. A sequence $\{u_j\} \subset W^{1,p}(X)$ is said to *converge to* u *in the m-topology* of $W^{1,p}(X)$ if $u_j \to u$ in $L^p(X)$ and $\underline{D}u_k \to \underline{D}u$ in $L^p(X)$.

This topology is metrizable; a compatible distance can be defined by

$$m(u,v) := \|u - v\|_{L^p(X)} + \|\underline{D}u - \underline{D}v\|_{L^p(X)}\ .$$

Observe that the m-topology is finer than the relaxed topology. The terminology comes from the fact that this topology is based on the notion of minimal pseudo-gradient.

It is clear that a sequence $\{u_j\} \subset W^{1,p}(X)$ converges to u in the m-topology if and only if $u_i \to u$ in $L^p(X)$ and, for each i, there exists $g_i \in D[u_i] \cap L^p(X)$, such that $g_i \to \underline{D}u$.

Let us denote by $\mathcal{F}_0 = \mathcal{F} \cap C_0(X)$ the set of those functions in $\mathcal{F}$ having their support in a $\mathcal{K}$-set.

Theorem 1.30 *Assume that $\mathcal{K}$ is the ring of all Borel bounded subsets of X and that the D-structure is absolutely local. If $1 < p < \infty$, then $\mathcal{F}_0 \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$ for the m-topology (in particular, it is dense for the relaxed topology).*

Proof Let $u \in W^{1,p}(X)$ be an arbitrary Sobolev function; since every ball in X is a $\mathcal{K}$-set by hypothesis, the metric space X is clearly $\mathcal{K}$-countable and we can thus apply the previous Lemma. We then know that there exist two sequences $\{u_i\}$ and $\{g_i\}$ of functions such that $u_i \in \mathcal{F} \cap W^{1,p}_{loc}(X)$, $g_i \in D[u_i] \cap L^p(X)$, $u_i \to u$ in $L^p_{loc}(X)$ and $g_i \to \underline{D}u$ in $L^p(X)$.
Fix $x_0 \in X$ and define

$$\varphi_k(x) := \min\left\{1, \frac{1}{k}\operatorname{dist}(x, (X \setminus B(x_0, 2k)))\right\},$$

this is a $\frac{1}{k}$-Lipschitz function such that $\varphi_k = 1$ in $B(x_0, k)$, and $\operatorname{supp}(\varphi_k) = \overline{B}(x_0, 2k)$.

Let us set $v_{k,i} := \varphi_k u_i$ and $h_{k,i} := \chi_{B(x_0,2k)}\left(g_k + \frac{1}{k}|u_i|\right)$.
Using axiom A3 and the fact that the D-structure is absolutely local together with $\operatorname{supp}(v_{k,i}) = \overline{B}(x_0, 2k) \in \mathcal{K}$; we conclude that $h_{k,i} \in D[v_{k,i}]$. It follows that $v_{k,i} \in W^{1,p}(X) \cap \mathcal{F}_0$.

Since $\lim_{i\to\infty} \|u_i - u\|_{L^p(B(x_0,2k))} = 0$ for all k, we can find $i_k \in \mathbb{N}$ such that $\|v_{k,i_k} - u\|_{L^p(B(x_0,2k))} < 2^{-k}$. Hence the function $w_k := v_{k,i_k}$ satisfies

$$\|w_k - u\|_{L^p(X)} < \frac{1}{2^k} + \|u\|_{L^p(X \setminus B(x_0,2k))}$$

Since $u \in L^p(X)$ we have $\lim_{k\to\infty} \|u\|_{L^p(X\setminus B(x_0,2k))} = 0$ and thus $\lim_{k\to\infty} \|w_k - u\|_{L^p(X)} = 0$. The function $f_k := h_{k,i_k} \in D[w_k]$ and it is clear that $\lim_{k\to\infty} f_k = \lim_{k\to\infty}(g_k\chi_{B(x_0,2k)}) = \underline{D}u$ in $L^p(X)$. This implies that $w_k \to u$ in m-topology. □

Corollary 1.31 *Assume that $\mathcal{K}$ is the ring of all Borel bounded subsets of X and that the Sobolev space is local. If $W^{1,p}(X)$ is uniformly convex, then $\mathcal{F}_0 \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$ for the usual topology.*

Proof Follows directly from the previous Theorem and Lemma 1.28. □

In the special case that $\mathcal{F}$ is the class of all locally Lipschitz functions, we have, still assuming that the Sobolev space $W^{1,p}(X)$ is defined by the completion procedure described in definition 1.14:

Corollary 1.32 *Let X be a proper metric space and assume that the Sobolev space is local. If $W^{1,p}(X)$ is uniformly convex, then the space of Lipschitz functions with compact support is dense in $W^{1,p}(X)$ (for the usual topology).*

Proof Since X is proper, $\mathcal{K}$ is the ring of all Borel bounded subsets of X. As $\mathcal{F}_0$ is the set of Lipschitz functions with compact support, we have $\mathcal{F}_0 \subset W^{1,p}(X)$ and the result follows from the previous corollary.

□

Remark The notion of relaxed energy has its origin in the theory of non-convex integrands in the calculus of variation (see e.g. section 5.2 in [8]).

1.9 Linear D-structures

Let X be a MM space, and let us choose a class $\mathcal{F}$ of functions $X \to \mathbb{R}$ satisfying the conditions (F1)–(F3) of section 1.7.

Definition. A *linear $D-$structure* X is given by the following data :

a) A Banach space $\mathbf{E}_x$ associated to each point $x \in X$ is given and

b) for any function $u \in \mathcal{F}$ and any point $x \in X$, an element $du(x) \in \mathbf{E}_x$ is given.

It is furthermore assumed that $x \to |du(x)|$ is a measurable function on X for all $u, \in \mathcal{F}$ and that

i) If $k-$Lipshitz, then $|du| \leq k$ a.e.

ii) d is linear.

iii) $d(uv) = udv + vdu$.

iv) If $u = \min\{u_1, u_2\}$, then $du = du_1$ a.e. on the set $\{u_1 \leq u_2\}$ and $du = du_2$ a.e. on $\{u_1 \geq u_2\}$.

Lemma 1.33 *If two functions $u, v \in \mathcal{F}$ coincide on a set A, then $du = dv$ a.e. on A.*

Proof It is a direct consequence of condition (iv).

□

Proposition 1.34 *The correspondence*

$$\mathcal{F} \ni u \to D[u] := \{g : X \to \mathbb{R} \mid g \textit{ is measurable and } g \geq |du| \textit{ a.e.}\}$$

satisfies the axioms A1–A4.

Proof Axiom A1 follows from the previous lemma and condition (i). Axiom A2 follows from the condition (ii) and A3 follows from (iii).
It is clear from (iv) that if $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, then $\max\{g_1, g_2\} \in D[\min\{u_1, u_2\}]$. Now using (ii), (iv) and the relation $\max\{u_1, u_2\} = u_1 + u_2 - \min\{u_1, u_2\}$, we conclude that $\max\{g_1, g_2\} \in D[\max\{u_1, u_2\}]$. □

The completion procedure of section 1.7 gives us an extension of the D-structure on all functions $u \in L^p_{loc}(X)$ and we obtain a corresponding Sobolev space $W^{1,p}(X)$.

Proposition 1.35 *This D-structure is a strictly local.*

Proof Just observe that the previous Lemma implies that $d(v^+) = 0$ a.e. on the set $\{v^+ = 0\} = X \setminus \{v > 0\}$. □

A slightly different notion of linear D-structure appears in [11] (see in particular Theorem 9 and 10), in [4] and in [45].

2 Examples of axiomatic Sobolev Spaces

2.1 Classical Sobolev Space

Let M be a Riemannian manifold and $\mathcal{K}$ be the class of relatively compact Borel subsets of M. We say that a measurable function $g : M \to \mathbb{R}$ is a *classic pseudo-gradient* of a function $u \in L^1_{loc}(M)$ if and only if either $g \equiv \infty$ or for any smooth vector field ξ with compact support we have

$$\left| \int_M u \operatorname{div}\xi \, d\mathrm{vol} \right| \leq \int_M g \, |\xi| \, d\mathrm{vol}. \tag{4}$$

We denote by $D[u]$ the set of all pseudo-gradients of u.

Lemma 2.1 *If u has a distributional gradient $\nabla u \in L^1_{loc}(M)$, then $g \in D[u]$ if and only if $g(x) \geq |\nabla u(x)|$ a.e.*

Proof The lemma is obvious for smooth functions because of the inequality

$$\left| \int_M \langle \nabla u \, , \, \xi \rangle \; d\,\mathrm{Vol} \right| = \left| \int_M u \, \operatorname{div} \xi \, d\,\mathrm{Vol} \right| \leq \int_M |\nabla u(x)| \, |\xi| \, d\,\mathrm{Vol}.$$

For general functions, it then follows from the density of smooth functions in the space $W^{1,1}_{loc}(M)$. □

Proposition 2.2 *This definition of pseudo-gradient satisfies all Axioms A1–A6.*

Proof Axioms A1–A4 are basic properties of weak gradients (see e.g. [23], [36] or [48]). Axiom A5 follows from the fact that inequality (4) is stable under L^p_{loc} convergence. Finally Axiom A6 is a consequence of Proposition 1.10 and the following classical lemma.

□

Lemma 2.3 *For any closed compact ball $B \subset M$, there exists a constant $C = C(B)$ such that the Poincaré inequality*

$$\int_B |u - u_B|^p dvol \leq C \int_B |\nabla u|^p dvol$$

holds for any $u \in W^{1,p}(B)$.

Proof A general (functional analysis) inequality of this type is proven in [48, Lemma 4.1.3]. In our case we need to set $X = W^{1,p}(B)$, $X_0 = L^p(B)$, $Y = \mathbb{R}$ (= constant functions) and the projection $L : X \to Y$ is given by averaging: $L(u) := u_B$. Observe that a ball in a Riemannian manifold has a Lipschitz boundary; hence we can apply, Rellich-Kondrachov's theorem which says that the embedding $X \subset X_0$ is compact, thus all hypothesis of [48, Lemma 4.1.3] are satisfied.

□

Remark From Lemma 2.1, we conclude that the Sobolev space associated to these pseudo-gradients is the classical Sobolev space $W^{1,p}(M)$. It is a local Sobolev space in the sense of definition 1.11. If the manifold M is complete, it is a proper metric space and hence Lipschitz functions with compact support form a dense subset (Corollary 1.32).

2.2 Weighted Sobolev space

Let M be a Riemannian manifold and $w \in L^1_{loc}(M)$ be a weight (i.e. a non negative function). We then define the measure to be $d\mu = w\, d\text{vol}$. The ring $\mathcal{K}$ and the pseudo-gradients are defined as in the previous example.

Theorem 2.4 *Suppose that the weight w belongs to the Mückenhaupt class A_p, $1 < p < \infty$. Then all Axioms A1–A6 hold. Furthermore smooth functions are dense in the corresponding Sobolev space $W^{1,p}(M, w)$.*

Recall that w belongs to Mückenhaupt class A_p if there exists a constant $C_{w,p}$ such that for all balls $B \subset M$ we have

$$\left(\int_B w(x) d\text{vol}_x\right) \left(\int_B w(x)^{1/(p-1)} d\text{vol}_x\right)^{p-1} \leq C_{w,p}\,.$$

The proof of this Theorem can be found in the paper of T. Kilpeläinen, [29], see also [30].

□

The theory of Weighted Sobolev space has been also extended to the case of domains in Carnot groups (see [5] and [19, section 13.1]).

2.3 Hajłasz-Sobolev space

The following concept was introduced by P. Hajłasz in [18], see also [31]. In this example X is an arbitrary measure metric space and $\mathcal{K}$ is the ring of all bounded Borel subsets of X. A measurable function $g : X \to \mathbb{R}_+$ is said to be a *Hajłasz pseudo-gradient* of the function $u : X \to \mathbb{R}$, if

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y))$$

for all $x, y \in X \setminus F$ where $F \subset X$ is some set (called the exceptional set) with $\mu(F) = 0$.

We denote by $HD[u]$ the set of all *Hajłasz pseudo-gradients* of u.

Lemma 2.5 *Assume that* $1 \leq p < \infty$. *Let* $u \in L^1_{loc}(X)$ *and* $g \in HD[u]$, *then the floating Poincaré inequality*

$$\|u - u_Q\|_{L^p(A)} \leq 2\,\text{diam}(A) \left(\frac{\mu(A)}{\mu(Q)}\right) \|g\|_{L^p(A)}$$

holds for any bounded measurable subsets $Q \subset A \subset X$ *with* $\mu(Q) > 0$.

Proof Observe that

$$\begin{aligned} |u(x) - u_Q| &= \left|\frac{1}{\mu(Q)} \int_Q (u(x) - u(y))d\mu(y)\right| \leq \frac{1}{\mu(Q)} \int_Q |u(x) - u(y)|\, d\mu(y) \\ &\leq \frac{\mu(A)}{\mu(Q)} \frac{1}{\mu(A)} \int_A |u(x) - u(y)|\, d\mu(y) \\ &\leq \frac{\mu(A)}{\mu(Q)} \left(\frac{1}{\mu(A)} \int_A |u(x) - u(y)|^p\, d\mu(y)\right)^{1/p} \end{aligned}$$

by Jensen's inequality. Thus

$$|u(x) - u_Q|^p \leq \left(\frac{\mu(A)}{\mu(Q)}\right)^p \left(\frac{1}{\mu(A)} \int_A |u(x) - u(y)|^p\, d\mu(y)\right),$$

integrating this inequality gives us

$$\begin{aligned}
\int_A |u(x) - u_Q|^p \, d\mu(x) &\le \left(\frac{\mu(A)}{\mu(Q)}\right)^p \left(\frac{1}{\mu(A)} \int_A \int_A |u(x) - u(y)|^p \, d\mu(y) d\mu(x)\right) \\
&\le \operatorname{diam}(A)^p \left(\frac{\mu(A)}{\mu(Q)}\right)^p \left(\frac{1}{\mu(A)} \int_A \int_A (g(x) + g(y))^p \, d\mu(y) d\mu(x)\right) \\
&\le 2^{p-1} \operatorname{diam}(A)^p \left(\frac{\mu(A)}{\mu(Q)}\right)^p \left(\frac{1}{\mu(A)} \int_A \int_A (g(x)^p + g(y)^p) \, d\mu(y) d\mu(x)\right) \\
&\le 2^p \operatorname{diam}(A)^p \left(\frac{\mu(A)}{\mu(Q)}\right)^p \int_A g^p d\mu \ .
\end{aligned}$$

The proof is complete.

□

Remark The proof of this Poincaré inequality is only a simple generalization of the argument given in [18] and [31].

Proposition 2.6 *The correspondence $u \to HD[u]$ satisfies Axioms A1–A6.*

Proof For Axiom A1 we consider an arbitrary non negative K–Lipschitz function u. We have to check that for all $x, y \in X$, we have

$$|u(x) - u(y)| \le d(x,y)(K\operatorname{sgn}(u(x)) + K\operatorname{sgn}(u(y)))\,.$$

If $u(x) = u(y) = 0$ this inequality is trivial, otherwise either $\operatorname{sgn}(u(x)) = 1$, or $\operatorname{sgn}(u(y)) = 1$ and thus, from the definition of K. we have

$$|u(x) - u(y)| \le K\, d(x,y) \le d(x,y)(K\operatorname{sgn}(u(x)) + K\operatorname{sgn}(u(y))).$$

We leave the verification of Axiom A2 to the reader.

To prove Axiom A3, we let $F \subset X$ be the exceptional subset for (u, g) and set $g_1(x) = (\sup|\varphi| g(x) + Lip(\varphi)|u(x)|)$. We then have for all $x, y \notin F$

$$\begin{aligned}
|\varphi(x)u(x) - \varphi(y)u(y)| &= |\varphi(x)u(x) - \varphi(x)u(y) + \varphi(x)u(y) - \varphi(y)u(y)| \\
&\le \sup|\varphi|\,|u(x) - u(y)| + |u(y)|\,|\varphi(x) - \varphi(y)| \\
&\le (d(x,y) \sup|\varphi|(g(x) + g(y)) + \operatorname{Lip}(\varphi)|u(y)|) \\
&\le d(x,y)\ (g_1(x) + g_1(y))\,.
\end{aligned}$$

Axiom A4 is proved in [31, Lemma 2.4].

We now prove Axiom A5: Consider two sequences $\{u_i\}$ and $\{g_i\}$ converging a.e. to some functions u and g and such that $g_i \in HD[u_i]$ for all i.
We may assume (passing to a subsequence if necessary) that $u_i \to u$ and $(g_i - g) \to 0$ pointwise on $X \setminus G$ where $G \subset X$ is some set of measure zero.

Let $F_i \subset X$ be the exceptional set for g_i and set $F := G \cup (\cup_{i=1}^{\infty} F_i)$, then it is clear that F has measure zero and $|u(x) - u(y)| \leq (g(x) + g(y))d(x,y)$ for all $x, y \in X \setminus F$. We thus conclude that $g \in HD[u]$, and Axiom 5 (and in fact a more general statement since only a.e. convergence is needed) is thus proven.

Finally Axiom A6 is a consequence of Lemma 2.5 and Proposition 1.10. □

Remarks 1) The associated Sobolev space is called the Hajłasz-Sobolev space and denoted by $HW^{1,p}(X)$ (or $M^{1,p}(X)$ in the literature). It contains Lipschitz functions as a dense subset (see [18]).

2) HD is not a local D-structure. Indeed, consider a Lipschitz function u such that $u \equiv 0$ on a bounded open set $A \subset X$ and $u \equiv 1$ on a bounded open set $A' \subset X$. If $g \in HD[u]$, and $x \in A, y \in A'$, then we have $(g(x) + g(y)) \geq \frac{1}{\Delta}$ (where $\Delta := \sup\{d(x,y) : x \in A, y \in A'\}$). Integrating this inequality over $A \times A'$ yields.

$$\frac{1}{\Delta}\mu(A)\mu(A') \leq \mu(A) \int_{A'} g(y)d\mu(y) + \mu(A') \int_{A} g(x)d\mu(x)$$

which gives a positive lower bound for the local energy $\mathcal{E}_p(u|A) + \mathcal{E}_p(u|A')$, in contradiction to the definition of locality.

Let us finally mention that, if the measure is locally doubling, then there is a kind of converse to Lemma 2.5. Namely Hajłasz pseudo-gradients can be characterized by a Poincaré inequality. More precisely :

Theorem 2.7 *Assume that the measure μ is locally doubling and atomfree. If $u \in L^p_{loc}(X)$ and $g \in L^p(X)$; then Kg belongs to $HD[u]$ for some constant $K > 0$ if and only if there exists a constant C such that for any bounded measurable subset $A \subset X$ of positive measure we have*

$$⨍_A |u - u_A| d\mu \leq C \operatorname{diam}(A) \left(⨍_A g^p d\mu \right)^{1/p} . \tag{5}$$

The proof is given in [14]. □

2.4 Graphs (combinatorial Sobolev spaces)

Let $\Gamma = (V, E)$ be a locally finite connected graph. We define the combinatorial distance between two vertices to be the length of the shortest combinatorial path joining them.
The ring $\mathcal{K}$ is the class of all finite subsets of V and the measure μ is the counting measure given by $\mu(A) = |A| =$ cardinal of A. See e.g. [41] for more information on the geometry of graphs.

For any function $u : V \to \mathbb{R}$, we define $CD[u]$ to be the set of all functions $g : V \to \mathbb{R}$ such that

$$\text{If} \quad y \sim x \qquad \text{then} \qquad |u(y) - u(x)| \le (g(x) + g(y)) \tag{6}$$

where $y \sim x$ means that y is a neighbour of x (i.e. there is an edge joining x to y).

Axioms A1–A5 for the correspondence $u \to CD[u]$ are not difficult to prove using standard arguments, Axiom A6 is a direct consequence of the lemma below.

This construction gives us a combinatorial Sobolev spaces $CW^{1,p}(\Gamma)$. This is a local theory (in the sense of definition 1.11). Observe that all functions are trivially locally Lipschitz functions.

Observe also that it follows clearly from the definition that for any function u on V, we have $HD[u] \subset CD[u]$, hence $HW^{1,p}(\Gamma) \subset CW^{1,p}(\Gamma)$.

Lemma 2.8 *For all finite subsets $A \subset X$ and all non empty subsets $Q \subset A$ we have the floating Poincaré inequality :*

$$\sum_{x \in A} |u(x) - u_Q|^p \le C \left(\sum_{y \in X} g(y)^p \right) \tag{7}$$

for any $g \in CD[u]$, where $u_Q := \frac{1}{|Q|} \sum_{x \in Q} u(x)$ and the constant C depends on A only.

Proof Let us denote by A_1 the set of all vertices in X whose combinatorial distance to Q is $\le \operatorname{diam}(A)$.
For any $x \in A_1$ and $z \in Q$, we can find a combinatorial path $x = x_0, x_1, ..., x_n = z$ where $x_j \sim x_{j+1}$, the $x_j \in A_1$ are pairwise distinct points and $n \le \operatorname{diam}(A)$. We thus have

$$|u(x) - u(z)| \le \sum_{j=0}^{n-1} |u(x_{j+1}) - u(x_j)| \le \sum_{j=0}^{n-1} (g(x_j) + g(x_{j+1})) \le 2 \sum_{y \in A_1} g(y) .$$

Hence the following inequality holds for any $x \in A_1$

$$\begin{aligned} |u(x) - u_Q| &= \left| u(x) - \frac{1}{|Q|} \sum_{z \in Q} u(z) \right| \le \frac{1}{|Q|} \sum_{z \in Q} |u(x) - u(z)| \\ &\le 2 \sum_{y \in A_1} g(y) , \end{aligned}$$

and therefore

$$\sum_{x \in A} |u(x) - u_Q| \le 2\,|A| \sum_{y \in A_1} g(y).$$

Combining the previous inequalities with Hölder's inequality, we obtain

$$\begin{aligned}\sum_{x\in A}|u(x)-u_Q|^p &\leq \left(\sum_{x\in A}|u(x)-u_Q|\right)^p \leq \left(2|A|\sum_{y\in A_1} g(y)\right)^p \\ &\leq (2|A|)^p|A_1|^{(p-1)}\sum_{y\in A_1} g(y)^p\end{aligned}$$

□

From Proposition 1.10 we now have the

Corollary 2.9 *$CD[\cdot]$ satisfies also axiom A6.*

□

Remark. The condition (6) used in the definition of the combinatorial pseudo-gradients is often replaced by the following, simpler one:

$$\text{If} \quad y\sim x \quad \text{then} \quad |u(y)-u(x)|\leq g(x). \tag{8}$$

This would lead to an equivalent topology on the Sobolev space, however the axiom A1 would fail to be true.

2.5 Infinitesimal Stretch

The Hajłasz Sobolev space is in some sense a universal non local Sobolev space; it is universal because it is defined on any measure metric spaces (no additional structure on the space is beeing needed). We now give an example of universal local Sobolev space. In this example, X is a priori an arbitrary metric space and $\mathcal{K}$ is any ring of bounded sets satisfying the conditions (K1)–(K3) given in section 1.1.

Let us first introduce some notations. For a locally Lipschitz function $u: X \to \mathbb{R}$ and a ball $B(x,r)\subset X$ the *local stretching constant* is defined by

$$L_{u,r}(x) := \sup_{d(y,x)\leq r}\frac{|u(x)-u(y)|}{r}$$

and the "infinitesimal stretch" is the Borel measurable function

$$L_u(x) := \limsup_{r\to 0}\ L_{u,r}(x)\,.$$

(The infinitesimal stretch $L_u(x)$ is denoted Lip $u(x)$ in [4].)
For a locally Lipschitz function $u: X\to\mathbb{R}$, we define $SD[u]$ to be the set of all Borel measurable functions g such that

$$g(x)\geq L_u(x)$$

for almost all $x \in X$. We then define $SD[u]$ for any function $u \in L^p_{loc}$ using the completion procedure of section 1.7. In other words, for a function $u \in L^p_{loc}$, we have $g \in SD[u]$ if and only if there exists two sequences of functions $\{u_i\}$ and $\{g_i\}$ such that $u_i \to u$ in L^p_{loc} topology and $(g - g_i) \to 0$ in L^p topology such that u_i is locally Lipschitz and $g_i \in SD[u_i]$.

It is not difficult to check that Axioms A1–A5 hold for all locally Lipschitz functions. By the discussion in section 1.7 we know that Axioms A1–A5 hold for all functions $u \in L^p_{loc}(X)$.

The associated Sobolev space is denoted by $SW^{1,p}(X)$. It is a local Sobolev space.

Remark 1 Axiom A6 is a special property of the space (X, d, μ) which sometimes fail and must therefore be assumed or proved (usually it is in fact a Poincaré type inequality which is assumed or proved).

Example Recall the example in the introduction. Let $X = \mathbb{R}^n$ with the metric $d(x, y) = |x-y|^{1/2}$ and choose any measure on X. Let $u : X \to \mathbb{R}$ be any linear function; it is then easy to check that $0 \in SD[u]$ hence axiom A6 is not satisfied.

Remark 2 It is also possible to use an alternative definition; namely for a locally Lipschitz function $u : X \to \mathbb{R}$, we define $SD[u]$ to be the set of all Borel measurable functions g such that

$$g(x) \geq \lim_{r \to 0} \sup \operatorname{Lip}(u|_{B(x,r)})$$

for almost all $x \in X$, where $\operatorname{Lip}(u|_A)$ is the Lipschitz function of u on the set A.

2.6 Upper Gradients

This Sobolev space is studied in [4], [24] and [39]. In this section, we assume X to be a rectifiably connected metric space, i.e. any pair of points can be joined by a rectifiable curve. We fix is a ring $\mathcal{K}$ of subsets of X satisfying the conditions (K1)–(K3).

Definition Let $u : X \to \mathbb{R}$ be a locally Lipschitz function. A Borel measurable function $g : X \to \mathbb{R}$ is an *upper gradient* (also called *very weak gradient*) for u if for all Lipschitz paths $\gamma : [0, 1] \to X$ we have

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_0^1 g(\gamma(t))dt\,.$$

We denote by $UD[u]$ the set of all upper gradients for a locally Lipschitz function u; and we extend this definition by the procedure described in section 1.7. In other words, for a function $u \in L^p_{loc}(X)$, we have $g \in UD[u]$ if and only if there exists two sequences of functions $u_i \to u$ in L^p_{loc} topology and $g_i \to g$ in L^p topology such that u_i is locally Lipschitz and g_i is an upper gradient for u_i. (In the terminology of [4], we can say that g is a *generalized upper gradient* for u).

Proposition 2.10 *The correspondence $u \to UD[u]$ satisfies Axioms A1–A5.*

Proof Axioms A1–A4 can be checked by routine argumentation. Axiom A5 is a consequence of Proposition 1.25.

□

The associated Sobolev space is denoted by $UW^{1,p}(X)$. It is a local Sobolev space (see [40, Lemma C.19] or [19, Lemma 10.4]).

Remark In general Axiom A6 is not satisfied. Here is an example taken from [17]: Let $X = \mathbb{B}^n \subset \mathbb{R}^n$ be the unit ball in euclidean space with lebesgue measure λ and with the metric:

$$d(x_1, x_2) := |r_1 - r_2| + \min\{r_1, r_2\} \, \|\sigma_1 - \sigma_2\|^\alpha$$

where $0 < \alpha < 1$, and (r, σ) are polar coordinates on $\mathbb{R}^n$. This distance gives rise to the usual topology on $\mathbb{B}^n$; the MM-space X also enjoys the following properties:

1. $0 < \lambda(B(x, \rho)) < \infty$ for any ball of positive radius ρ;
2. X is compact;
3. X is rectifiably connected and the only rectifiable curves are contained in a union of radii from 0.

Let $w : S^{n-1} \to \mathbb{R}$ be an arbitrary non constant function on the sphere which is Lipschitz for the metric $\|\sigma_1 - \sigma_2\|^\alpha$ and set $u_k(x) = u_k(r, \sigma) := \phi_k(r) w(\sigma)$ where $\phi_k(r) = \min\{kr, 1\}$; the function u_k is then Lipschitz on (X, d).

We now define $g_k : X \to \mathbb{R}$ by

$$g_k(r, \sigma) := \phi'(r) = \begin{cases} k & \text{if } \ 0 \le r \le 1/k \\ 0 & \text{if } \ r > 1/k. \end{cases}$$

Because any rectifiable curve is contained in a union of radii, $g_k \in UD[u_k]$. We thus have

$$U\mathcal{E}_p(u_k) \le \int_X g_k^p d\lambda = \omega_{n-1} k^{p-n} \ .$$

Hence $\lim_{k\to\infty} U\mathcal{E}_p(u_k) = 0$ if $p < n$. But $\lim_{k\to\infty} u_k(r, \sigma) = u(r, \sigma) = w(\sigma)$ is not constant, it follows that Axiom A6 fails.

By Proposition 1.10, this problem is avoided if the space X supports a Poincaré inequality.

There are many spaces on which upper gradients are known to support a Poincaré inequality (see the discussion in §10.2 in [19]). Let us mention in particular the following recent result of Laakso [34] showing that there are examples in any (fractal) dimension :

Theorem 2.11 *For any real number $s > 0$ there exists an unbounded proper geodesic metric space X_s with an Ahlfors regular measure μ in dimension s and on which upper gradients support a weak $(1, 1)$-Poincaré inequality (see Definition 1.10).*

□

Remark *On the space X_s described above, UD is a non degenerate D-structure.*

Indeed, by Jensen's inequality, if UD supports a $(1,1)$-Poincaré inequality, then it also supports $(1,p)$-Poincaré inequality for all $p \geq 1$. On the other hand, since μ is Ahlfors regular, it is doubling. It thus follows from Corollary 1.15 that UD is non degenerate.

□

Thus Laakso's construction provides us with an example of non degenerate Sobolev space $UW^{1,p}(X_s)$ on a metric space of Hausdorff dimension s for any $s > 0$ and any $p > 1$.

Remark In [4], J. Cheeger constructs a Sobolev space based on upper gradients in a slightly different way. Namely let $\mathcal{F}$ be the set of all measurable functions u admitting an upper-gradient (i.e. such that there exists a Borel measurable function g such that $|u(\gamma(1))-u(\gamma(0))| \leq \int_0^1 g(\gamma(t))dt$ for all Lipschitz path $\gamma : [0,1] \to X$). We then denote by $\breve{U}D[u]$ the set of all upper gradients for a function $u \in \mathcal{F}$; and we extend this definition by the same approximation procedure as above. If X supports a weak p-Poicaré inequality, then we can define a corresponding Sobolev space $\breve{U}W^{1,p}(X)$.

We have the following

Proposition 2.12 *Suppose that locally Lipschitz functions are dense in both $UW^{1,p}(X)$ and $\breve{U}W^{1,p}(X)$. Then $\breve{U}W^{1,p}(X) = UW^{1,p}(X)$.*

Proof If $u : X \to \mathbb{R}$ is locally Lipschitz, then $UD[u] = \breve{U}D[u]$ and hence $\|u\|_{UW^{1,p}(X)} = \|u\|_{\breve{U}W^{1,p}(X)}$. It follows that the closure of locally Lipschitz functions in the spaces $UW^{1,p}(X)$ and $\breve{U}W^{1,p}(X)$ coincides.

□

In fact we always have $UW^{1,p}(X) \subset \breve{U}W^{1,p}(X)$. We don't know if there are cases where $UW^{1,p}(X) \neq \breve{U}W^{1,p}(X)$.

Cheeger has also proved that if the measure μ satisfies the doubling condition and if UD supports a $(1,p)$-Poincaré inequality, then the Sobolev norm in $UW^{1,p}(X)$ is equivalent to a uniformly convex norm, see [4, Proposition 4.48].

Let us finally mention that in [39], Nageswari Shanmugalingam develop another construction of a Sobolev space based on upper-gradients. Her approach is to consider the class of p–integrable functions $u : X \to \mathbb{R}$ which admits a function $g \in L^p(X)$ which is an upper-gradient of u for p-modulus almost all curves; two such functions are then identified if the norm of their difference vanishes. The resulting space is a Banach space denoted by $N^{1,p}(X)$ and is called the Newtonian space. For $1 < p < \infty$, $N^{1,p}(X)$ coincides with $\breve{U}W^{1,p}(X)$ (see [39, Theorem 4.10]).

2.7 Comparing different $D-$structures

Let AD and BD be two D-structures on a fix MM-space $(X, d, \mathcal{K}, \mu)$. Let us denote by $A\mathcal{E}_p(u)$ and $B\mathcal{E}_p(u)$ the corresponding energies and by $AW^{1,p}(X)$ and $BW^{1,p}(X)$ the corresponding Sobolev spaces.
We will write $AD \preceq BD$ if $AD[u] \subset BD[u]$ for all functions $u \in L^p_{loc}(X)$; we then have the following simple observation :

Proposition 2.13 *Assume $AD \preceq BD$, then*

i) $A\mathcal{E}_p(u) \geq B\mathcal{E}_p(u)$;

ii) $AW^{1,p}(X) \subset BW^{1,p}(X)$ *(closed subspace);*

iii) if AD is local, then so is BD;

iv) if BD is non degenerate, then so is AD.

The proof is not difficult and left to the reader.

□

Proposition 2.14 *Suppose that $1 < p < \infty$, then $AD \preceq BD$ if and only if for any function $u \in L^p_{loc}(X)$ we have $\underline{BD}_p u \leq \underline{AD}_p u$ a.e. (where $\underline{AD}_p u$ and $\underline{BD}_p u$ are the corresponding minimal pseudo-gradients).*

The proof is obvious.

□

Proposition 2.15 *Consider a (finite or infinite) collection $\mathcal{D} = \{D_t\}_{t \in T}$ of D-structures on X, then $D = \cap_{t \in T} D_t$ is again a D-structure.*
If one of the D_t is a non degenerate D-structure, then D is also non degenerate.

Proof It is just a routine to check that $D = \cap_{t \in T} D_t$ satisfy the axioms A1–A5. The last assertion follows from last Proposition 2.13 (iv).

□

Let us end this section with some examples :

Proposition 2.16 *a) If X is a graph, then $HD \preceq CD$;*

b) for any MM spaces we have $SD \preceq UD$;

c) if D is any D-structure on X, then $D \preceq \lambda D$ for any $0 < \lambda \leq 1$, where
$$\lambda D[u] := \{\lambda g \,:\, g \in D[u]\}\,.;$$

d) for any MM spaces we have $4HD \preceq UD$ (i.e. if $g \in HD[u]$ then $4g \in UD[u]$).

Proof (a) is clear from the definitions.
To prove (b), we observe that it is not difficult to check that $SD[v] \subset UD[v]$ if $v : X \to \mathbb{R}$ is a locally Lipschitz function (see e.g. [4, Proposition 1.11]). The inclusion $SD[u] \subset UD[u]$ follows then for all functions $u \in L^p_{loc}(X)$ by construction.
To prove (c), observe that by Axiom A2 we have $\lambda D[u] = \{sg \,:\, g \in D[u] \text{ and } s \geq \lambda\}$, hence $D[u] \subset \lambda D[u]$ for all u.
The proof of (d) is given in [39, Lemma 4.7]. □

Corollary 2.17 *$SW^{1,p}(X)$ and $HW^{1,p}(X)$ are closed subspace of $UW^{1,p}(X)$.*

□

A recent theorem of J. Cheeger says that if the measure μ satisfies the doubling condition and if UD supports a $(1,p)$-Poincaré inequality, then $SD = UD$, see section 4.4.

3 Capacities and Hyperbolicity

In this part we introduce a concept of variational p-capacity, and we study its relation with the geometry of X. The corresponding theory for Riemannian manifolds can be found in [16], [42] and [49].

3.1 Definition of the variational capacity

Let $\Omega \subset X$ be an open subset. Recall that $C_0(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\operatorname{supp}(u) \Subset \Omega$, i.e. $\operatorname{supp}(u)$ is a closed $\mathcal{K}$-subset of Ω.

Definition 3.1 a) We define $\mathcal{L}^{1,p}_0(\Omega)$ to be the closure of $C_0(\Omega) \cap \mathcal{L}^{1,p}(X)$ in $\mathcal{L}^{1,p}(X)$ for the norm defined in section 1.5 (recall that this norm is given by $\|u\|_{\mathcal{L}^{1,p}(\Omega,Q)} := \left(\int_Q |u|^p d\mu + \mathcal{E}_p(u|\Omega)\right)^{1/p}$ where $Q \Subset \Omega$ is a fixed $\mathcal{K}$-subset of positive measure).

b) The *variational p-capacity* of a pair $F \subset \Omega \subset X$ (where Ω is open and F is arbitrary) is defined as

$$\operatorname{Cap_p}(F,\Omega) := \inf\{\mathcal{E}_p(u) \,|\, u \in \mathcal{A}_p(F,\Omega)\} ,$$

where the set of admissible functions is defined by

$$\mathcal{A}_p(F,\Omega) := \{u \in \mathcal{L}^{1,p}_0(\Omega) |\; u \geq 1 \text{ on a neighbourhood of } F \text{ and } u \geq 0 \text{ a.e.}\}$$

If $\mathcal{A}_p(F,\Omega) = \emptyset$, then we set $\operatorname{Cap_p}(F,\Omega) = \infty$. If $\Omega = X$, we simply write $\operatorname{Cap_p}(F,X) = \operatorname{Cap_p}(F)$.

Remarks 1. The space $\mathcal{L}_0^{1,p}(\Omega)$ may depend on the ambient space $X \supset \Omega$, however we will avoid any heavier notation such as $\mathcal{L}_0^{1,p}(\Omega, X)$.
2. By definition capacity is decreasing with respect to the domain Ω : if $\Omega_1 \subset \Omega_2$, then $\mathrm{Cap_p}(F, \Omega_1) \geq \mathrm{Cap_p}(F, \Omega_2)$.

Theorem 3.1 *Let $Q \subset X$ be a $\mathcal{K}$-set such that $\mu(Q) > 0$. Then the following conditions are equivalent :*

1) there exists a constant C such that for any $u \in \mathcal{L}_0^{1,p}(X)$ we have

$$\|u\|_{L^p(Q)} \leq C \ (\mathcal{E}_p(u))^{1/p} \ ;$$

2) $\mathrm{Cap_p}(Q) > 0$;

3) $1 \notin \mathcal{L}_0^{1,p}(X)$;

4) $\mathcal{L}_0^{1,p}(X)$ is a Banach space for the norm $\|u\| := (\mathcal{E}_p(u))^{1/p}$.

A similar result in the case of Riemannian manifolds was obtained in [43]. See also [47] for the case of graphs.

Proof Observe that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. The proof of (3) $\Rightarrow$ (4) follows from the fact that $\mathcal{L}^{1,p}(X)/\mathbb{R}$ is a Banach space for the norm $\|u\| = \mathcal{E}_p(u)^{1/p}$ (Proposition 1.20) and that the canonical mapping $\mathcal{L}_0^{1,p}(X) \to \mathcal{L}^{1,p}(X)/\mathbb{R}$ is injective if and only if $1 \notin \mathcal{L}_0^{1,p}(X)$. Finally, the proof of (4) $\Rightarrow$ (1) is a consequence of the open mapping theorem applied to the identity map $Id : (\mathcal{L}_0^{1,p}(X), \|\cdot\|_{\mathcal{L}^{1,p}(X,Q)}) \to (\mathcal{L}_0^{1,p}(X), \mathcal{E}_p(\cdot)^{1/p})$.

□

Definition 3.2 The MMD space X is said to be $p-$*hyperbolic* if one of the above conditions holds and $p-$*parabolic* otherwise.

For instance if $X \in \mathcal{K}$, then X is $p-$parabolic for all p.

Remark. By Theorem 1.21, the space $\mathcal{L}_0^{1,p}(X)$ does not depend on the choice of the $\mathcal{K}$-set $Q \subset X$. It thus follows :
a) That condition (3) (or (4)) does not depend on the choice of Q. In particular the notion of $p-$hyperbolicity of a MMD space is well defined.
b) X is p-parabolic $\iff$ $\mathrm{Cap_p}(A) = 0$ for any $A \in \mathcal{K}$ $\iff$ there exists at least one K-set $Q \in \mathcal{K}$ of positive measure such that $\mathrm{Cap_p}(Q) = 0$.

For more information on the parabolic/hyperbolic dichotomy in the case of Riemannian manifolds, see [16], [49] and [42].

Proposition 3.2 *The variational p-capacity* $\mathrm{Cap_p}(\ )$ *satisfies the following properties:*

i) $\mathrm{Cap_p}(\ )$ *is an outer measure;*

ii) for any subset $F \subset X$ *we have* $\mathrm{Cap_p}(F) = \inf\{\mathrm{Cap_p}(U) : U \supset F \text{ open }\}$*;*

iii) If $X \supset K_1 \supset K_2 \supset K_3 \ldots$ *is a decreasing sequence of compact sets, then*

$$\lim_{i\to\infty} \mathrm{Cap_p}(K_i) = \mathrm{Cap_p}\left(\cap_{i=1}^{\infty} K_i\right) .$$

Proof If X is p-parabolic, then the $p-$capacity is trivial, we thus assume X to be $p-$hyperbolic.

i) Clearly $\mathrm{Cap_p}(\emptyset) = 0$ and $\mathrm{Cap_p}()$ is monotone : $A \subset B \Rightarrow \mathrm{Cap_p}(A) \leq \mathrm{Cap_p}(B)$. To prove countable subadditivity suppose that $\{F_n \subset X\}$ is a sequence of subsets of X such that $\sum_{n=1}^{\infty} \mathrm{Cap_p}(F_n) < \infty$. Let $F := \cup_{n=1}^{\infty} F_n$ and fix some $\varepsilon > 0$. By definition of the variational p-capacity, for each n we can find a function $u_n \in \mathcal{A}(F_n, X)$ and $g_n \in D[u_n]$ such that

$$\|g_n\|_{L^p(X)}^p < \mathrm{Cap_p}(F_n) + \frac{\varepsilon}{2^n} .$$

By axiom A4 the function $v_n := \max(u_1, ..., u_n)$ is admissible for $\cup_{s=1}^{n} F_s$; observe that by Fatou's lemma, the sequence $\{v_n\}$ converges to $v_0 = \sup_{i\in\mathbb{N}}(u_i)$ in $L^p_{loc}(X)$. If $m > n$, then $\max(u_1, ..., u_m) \leq \max(u_1, ..., u_n) + \max(u_{n+1}, ..., u_m)$. Using this inequality and Axiom A4, we have for any $m > n$:

$$\begin{aligned}
\| v_m - v_n \|_{\mathcal{L}^{1,p}(X)}^p &= \| \max(u_1, ..., u_m) - \max(u_1, ..., u_n) \|_{\mathcal{L}^{1,p}(X)}^p \\
&\leq \| \max(u_{n+1}, ..., u_m) \|_{\mathcal{L}^{1,p}(X)}^p \\
&\leq \| \max(g_{n+1}, ..., g_m) \|_{L^p(X)}^p \\
&\leq \sum_{i=n+1}^{m} \| g_i \|_{L^p(X)}^p \leq \sum_{i=n+1}^{m} \left(\mathrm{Cap_p}(F_i) + \frac{\varepsilon}{2^i}\right) \\
&\leq \varepsilon + \sum_{i=n+1}^{\infty} \mathrm{Cap_p}(F_i) .
\end{aligned}$$

Because the series $\sum_1^{\infty} \mathrm{Cap_p}(F_i)$ converges, the sequence $\{v_n\}$ is a Cauchy sequence in the Banach space $\mathcal{L}^{1,p}(X)$. Therefore $v_n \to v_0$ and $v_0 \in \mathcal{L}^{1,p}(X)$. Since we clearly have $v_0 \geq 1$ on a neighbourhood of $F := \cup_{n=1}^{\infty} F_n$, we have thus established that $v_0 \in \mathcal{A}_p(F, X)$, therefore

$$\begin{aligned}
\mathrm{Cap_p}(F) &\leq \|v_0\|_{\mathcal{L}^{1,p}(X)}^p = \lim_{n\to\infty} \|v_n\|_{\mathcal{L}^{1,p}(X)}^p = \lim_{n\to\infty} \| \max(u_1, ..., u_n) \|_{\mathcal{L}^{1,p}(X)}^p \\
&\leq \sum_{i=1}^{\infty} \| g_i \|_{L^p(X)}^p \leq \varepsilon + \sum_{i=1}^{\infty} \mathrm{Cap_p}(F_i) .
\end{aligned}$$

We have proved that the variational p-capacity is an outer measure.

ii) This assertion is clear from the definition of p-capacity.

iii) First we observe that the monotonicity of p-capacity implies $\lim_{i\to\infty} \mathrm{Cap_p}(K_i) \geq \mathrm{Cap_p}(K)$, where $K := \cap_{i=1}^{\infty} K_i$. To prove the converse inequality, choose an arbitrary open set $U \subset X$ containing K. By compactness of the K_i's, the open set U contains K_i for all sufficiently large i. Therefore $\lim_{i\to\infty} \mathrm{Cap_p}(K_i) \leq \mathrm{Cap_p}(U)$. By (ii), we now obtain the inequality $\lim_{i\to\infty} \mathrm{Cap_p}(K_i) \leq \mathrm{Cap_p}(K)$. □

3.2 Growth of balls and parabolicity

Theorem 3.3 *Let $\Omega \subset X$ be an open set and suppose that $B(x_0, r) \Subset B(x_0, R) \Subset \Omega$. Then*

$$\mathrm{Cap_p}(B(x_0, r), \Omega)) \leq \frac{\mu(B(x_0, R))}{(R-r)^p}.$$

Proof Let us set $\lambda(x) := \mathrm{dist}(x, x_0)$ and define the function $u = u_t : X \to \mathbb{R}$ (where $t \geq 1$) by

$$u(x) = \begin{cases} t & \text{if } x \in B(x_0, r) \\ \dfrac{t(R-\lambda(x))}{R-r} & \text{if } x \in B(x_0, R) \setminus B(x_0, r) \\ 0 & \text{if } x \notin B(x_0, R)\,. \end{cases}$$

It is clearly a Lipschitz function with $\mathrm{Lip}(u) \leq \frac{t}{R-r}$. By axiom A1 we thus have $u \in \mathcal{L}^{1,p}(X)$. Observe also that u is continuous and $\mathrm{supp}\, u \subset B(x_0, R) \Subset \Omega$; in particular $u \in \mathcal{L}_0^{1,p}(\Omega)$. A pseudo-gradient $g \in D[u]$ is given by $g(x) = \frac{t}{R-r}$ for $x \in B(x_0, R)$ and $g(x) = 0$ for all other x. If $t > 1$, then the function u is an admissible function for $\mathrm{Cap_p}(B(x_0, r), \Omega)$. Therefore

$$\mathrm{Cap_p}(B(x_0, r), \Omega)) \leq \frac{t^p}{(R-r)^p} \int_{B(x_0,R)} d\mu = t^p \frac{\mu(B(x_0, R))}{(R-r)^p} \to \frac{\mu(B(x_0, R))}{(R-r)^p}$$

as $t \to 1$. □

We immediately deduce the following sufficient condition for p-parabolicity.

Corollary 3.4 *Suppose that the metric space X is complete and unbounded and that $\mathcal{K}$ is the Boolean ring of all bounded Borel subsets of X. If there exists a point $x_0 \in X$ such that*

$$\liminf_{R\to\infty} R^{-p}\mu(B(x_0, R)) = 0\,,$$

then X is p-parabolic.

□

We also have the following consequence on the capacity of points :

Corollary 3.5 *Suppose that* $\lim_{R\to 0} R^{-p}\mu(B(x_0,R)) = 0$. *Then* $\mathrm{Cap_p}(\{x_0\},\Omega) = 0$ *for every open set* Ω *containing* x_0.

Proof For $R > 0$ small enough, we have $B(x_0,R/2) \Subset B(x_0,R) \Subset \Omega$. The previous Theorem implies then

$$\mathrm{Cap_p}(B(x_0,R/2),\Omega) \leq \frac{\mu(B(x_0,R))}{\left(\frac{1}{2}R\right)^p}.$$

Letting $R \to 0$ gives us the result.

□

4 A survey of some recent results

In this section we describe without proof some other recent results from the theory of MMD spaces.

4.1 A global Sobolev inequality

The following Sobolev inequality has been proved by K. Gafaïti in his thesis [12] using techniques of the paper [1].

Theorem 4.1 *Let* X *be a complete MMD space such that*

i) D *is absolutely local;*

ii) μ *has the doubling property;*

iii) there exists constants $s > 1$ *and* $c > 0$ *such that for all* $x \in X$ *we have*

$$\mu(B(x,r)) \geq cr^s.$$

iv) the $p-$*Poincaré type inequality*

$$\int_B |u(x) - u_B|^p d\mu(x) \leq Cr^p \int_{2B} g^p d\mu$$

holds for any locally Lipschitz function $u : X \to \mathbb{R}$, *any ball* $B \subset X$ *and any* $g \in D[u]$.

Then the following global Sobolev inequality

$$\left(\int_X |u|^{p^*} d\mu\right)^{1/p^*} \leq C\ (\mathcal{E}_p(u))^{1/p}$$

holds for any $u \in W^{1,p}(X)$ *where* $p^* = \frac{sp}{s-p}$.

□

4.2 Some results on p-capacity

We first mention that $p-$capacity satisfies the Choquet property :

Proposition 4.2 *Suppose that* $F \subset X$ *is a strongly bounded Borel set which is contained in a countable union of compact sets, then*

$$\mathrm{Cap_p}(F) = \sup\{\mathrm{Cap_p}(K)|\ K \subset F \ \ \textit{a compact subset}\}.$$

The proof is given in [15].

□

Recall that a set $F \subset X$ is strongly bounded if there exists a pair of open sets $\Omega_1 \subset \Omega_2 \subset X$ such that $\Omega_2 \in \mathcal{K}$, $\mu(X\setminus\Omega_2) > 0$, $\mathrm{dist}(\Omega_1, X\setminus\Omega_2) > 0$ and $F \subset \Omega_1$.

We now state a result about the existence and uniqueness of extremal functions for $p-$capacities. We first need a definition :

Definition A subset F is said to be *p–fat* if it is a Borel subset and there exists a probability measures τ on X which is absolutely continuous with respect to p-capacities (i.e. such that $\tau(S) = 0$ for all subsets $S \subset X$ of local $p-$capacity zero) and whose support is contained in F.

Theorem 4.3 *Let* (X, d) *be a* σ*-compact measure metric space and* $F \subset X$ *be a* $p-$*fat subset* $(1 < p < \infty)$. *Then there exists a unique function* $u^* \in \mathcal{L}_0^{1,p}(X)$ *such that* $u^* = 1$ *p-quasi-everywhere on* F *and* $\mathcal{E}_p(u^*) = \mathrm{Cap_p}(F)$. *Furthermore* $0 \leq u^*(x) \leq 1$ *for all* $x \in X$.

The proof is also given in [15].

□

4.3 Bilipschitz characterization of metric spaces

On a MMD space X we define $\mathcal{A}_0^p(X) := C_0(X) \cap \mathcal{L}^{1,p}(X)$ where $C_0(X)$ is the set of continuous functions converging to zero at infinity.

Lemma 4.4 *Suppose that the MMD space satisfies the condition*

$$(\mathcal{E}_p(uv))^{1/p} \leq \|v\|_{L^\infty} (\mathcal{E}_p(u))^{1/p} + \|u\|_{L^\infty} (\mathcal{E}_p(v))^{1/p} \tag{9}$$

for all $u, v \in \mathcal{A}_0^p(X)$.
Then $\mathcal{A}_0^p(X)$ *is a Banach algebra for the norm* $\|u\|_{\mathcal{A}_0^p(X)} := \|u\|_{L^\infty} + (\mathcal{E}_p(u))^{1/p}$.

This algebra is called the *Royden algebra* of the MMD space X.

All examples of D-structures we have previously given do satisfy the hypothesis of this Lemma.

We now consider two MMD spaces X and Y satisfying the following four conditions:

1) X and Y are proper and quasi-convex, i.e. there exists $Q \geq 1$ such that any pair of points a, b in X (or in Y) can be joined by a curve of length at most $Q \cdot d(a, b)$
2) X and Y are *uniformly locally* s*-regular*, i.e. for every point $x \in X$ there exists two constants $c, \eta > 0$ such that for any ball B of radius $r \leq \eta$ in X or Y we have

$$\frac{1}{c} r^s \leq \mu(B) \leq c r^s.$$

3.) The $(1,p)$-Poincaré inequality holds on any ball of radius $r < \eta$, i.e. there exists two constants $\sigma \geq 1$ and $C > 0$ such that

$$\left(\fint_B |u - u_B|^q d\mu \right)^{1/q} \leq Cr \left(\fint_{\sigma B} g^p d\mu \right)^{1/p} \tag{10}$$

for any ball B of radius $r < \eta$ in X or Y, for any continuous function $u : X \to \mathbb{R}$ and any $g \in D[u]$.
4.) X and Y satisfy the condition (9) above.

Theorem 4.5 *Let X and Y be as above. Suppose that $\mathcal{A}_0^p(X)$ and $\mathcal{A}_0^p(Y)$ are isomorphic Banach algebras for some $p > s$. Then X and Y are bilipschitz equivalent.*

This result has been obtained by Gafaïti in his thesis [12]. Note that in the special case of Riemannian manifolds, this is a Theorem of J. Ferrand (see [10]).

4.4 A theorem of J. Cheeger

The Sobolev space $UW^{1,p}(X)$ has a very rich structure on doubling metric spaces supporting a Poincaré inequality :

Theorem 4.6 *If the MM space X satisfies the doubling condition and if UD supports a $(1,p)$-Poincaré inequality, then*

i) $UD = SD$, *in particular* $UW^{1,p}(X) = SW^{1,p}(X)$;

ii) UD *can be defined from a linear D−structure, i.e. there exists a linear D-structure* $\{\mathbf{E}_x, d\}$ *such that for any locally Lipschitz function* v *on* X *we have a.e.*

$$|dv(x)| = L_v(x) = \underline{UD}v(x)$$

(where $\underline{UD}v$ *is the minimal upper-gradient of* v*).*

iii) If $1 < p < \infty$, *then the Sobolev norm on* $UW^{1,p}(X)$ *is equivalent to a uniformly convex norm (in particular* $UW^{1,p}(X)$ *is reflexive).*

This is a deep result of J. Cheeger proved in Theorems 4.38, 4.48 and 6.1 of [4], see also [45]. N. Weaver has developed an alternative construction in [46]. □

From this result and Corollary 1.32, we now have:

Corollary 4.7 *Let X be a proper MM space X satisfying the doubling condition. If UD supports a $(1,p)$-Poincaré inequality for some $p > 1$, then the space of Lipschitz functions with compact support is dense in $UW^{1,p}(X)$ for the usual topology.*

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