hp-OPTIMAL DISCONTINUOUS GALERKIN METHODS FOR LINEAR ELLIPTIC PROBLEMS

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Abstract. The aim of this paper is to overcome the well-known lack of p-optimality in hp-version discontinuous Galerkin (DG) discretizations for the numerical approximation of linear elliptic problems. For this purpose, we shall present and analyze a class of hp-DG methods that is closely related to other DG schemes, however, combines both p-optimal jump penalty as well as lifting stabilization. We will prove that the resulting error estimates are optimal with respect to both the local element sizes and polynomial degrees.

1. Introduction

In this paper, we will propose and analyze a class of hp-version discontinuous Galerkin (DG) methods for the numerical approximation of linear elliptic partial differential equations. The focus of this work is to prove that the methods under consideration are stable and converge optimally in both the local element sizes as well as the local polynomial degrees.

We will consider the model problem

$$-\Delta u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

with a unique solution $u \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is an open bounded Lipschitz domain, and $f \in L^2(\Omega)$. Here and in the sequel, we shall use the following standard notation: for a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$), we denote by $L^2(D)$ the space of all square-integrable functions on D. Furthermore, for an integer $k \in \mathbb{N}$, $H^k(D)$ signifies the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,D}$. The space $H_0^1(D)$ is defined as the subspace of $H^1(D)$ with zero trace on $\partial\Omega$.

The numerical approximation of second-order linear elliptic PDEs by DG methods was first studied in [1, 3, 9, 15, 23]. Later, additional DG formulations were proposed in the literature; see, e.g., [2] for an overview and a unified analysis. In recent years, some of the existent DG methods have been analyzed further within an hp-context; see, e.g., [13, 16, 20, 21, 25] (cf. also [11, 12] for a posteriori results and hp-adaptive DG schemes). Here, the possibility of dealing with discontinuous finite element functions of possibly varying local approximation orders (even on irregular meshes containing hanging nodes), results in a notable degree of flexibility and computational convenience. For example, for smooth problems with local singularities, the hp-spaces can be quite effectively adjusted to the behavior of the

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solution, and high-order algebraic (or even exponential) convergence rates can be attained; see, e.g., [24, 25].

The presence of discontinuous functions in DG finite element spaces is typically accounted for by introducing suitable numerical fluxes in the formulations of DG schemes. These are quantities which are defined on the boundaries of the elements in the finite element mesh and supply the weak continuity of the numerical solution. In the analysis of DG schemes, the numerical fluxes are usually handled by use of so-called inverse estimates. For example, such bounds make it possible to control the L^2-norm of the trace of a polynomial on the boundary of an element by the L^2-norm of the polynomial in the interior of the element. Unfortunately, such inverse bounds often suffer from a suboptimality with respect to the polynomial degree. Consequently, the application of inverse estimates in the analysis of the numerical fluxes appearing in DG schemes results in suboptimal effects. For instance, mild over-stabilization with respect to the (local) polynomial degrees becomes necessary in order to prove the stability of hp-DG schemes. This in turn leads to p-suboptimal error estimates. We remark that, for interior penalty DG methods, this suboptimality can be overcome, provided that the exact solution u of (1)–(2) belongs to an augmented Sobolev space; cf. [10].

The aim of this paper is to present a new class of DG methods, as well as an appropriate analysis which does not rely on inverse estimates. To this end, we will consider a DG formulation that is closely related to the well-known interior penalty (IP) DG methods (cf., e.g., [13, 25]) and the hp-LDG method [16], and features hp-optimal stabilization of the jump penalty terms. In order to avoid the necessity of applying inverse estimates, some of the numerical flux terms will be replaced by suitable lifting operators (see, e.g., [2, 4, 20]) in the analysis. Moreover, we note that, in addition to the penalty jump stabilization in the classical IP methods, the DG forms in this paper include a further term which penalizes the lifting operators. We will prove that the new schemes are coercive and continuous (with explicit constants that are independent of h and of p) and converge optimally with respect to h and p.

The paper is organized as follows: In Section 2, we will present the new class of hp-DG methods in this paper. Furthermore, the hp-optimal error analysis will follow in Section 3. In addition, Section 4 contains a number of numerical results illustrating the theoretical results in this work. Finally, we shall add some concluding remarks in Section 5.

2. hp-Discontinuous Galerkin FEM

In this section, we shall present a class of hp-version discontinuous Galerkin (DG) finite element methods for the discretization of (1)–(2). Furthermore, we will discuss the well-posedness of these schemes, and prove some standard stability properties with respect to a suitable DG energy norm.

2.1. Meshes, Spaces, and Element Edge Operators.

Let us first consider shape-regular meshes $\mathcal{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint parallelograms $\{K\}_{K\in\mathcal{T}_h}$, i.e. $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Each element $K \in \mathcal{T}_h$ can then be affinely mapped onto the reference square $\widehat{S} = (-1,1)^2$. We allow the meshes to be 1-irregular, i.e., elements may contain hanging nodes. By h_K, we denote the diameter of an element $K \in \mathcal{T}_h$. We assume that these quantities are of bounded variation, i.e., there is a constant $\rho_1 \geq 1$ such that

$$\rho_1^{-1} \leq {}^{h_{K_\sharp}}/_{h_{K_\flat}} \leq \rho_1, \tag{3}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge. We store the elemental diameters in a vector $\boldsymbol{h}$ given by $\boldsymbol{h} = \{h_K \,:\, K \in \mathcal{T}_h\}$. Similarly, to each each element $K \in \mathcal{T}_h$ we assign a polynomial degree $p_K \geq 1$ and define the degree vector $\boldsymbol{p} = \{p_K \,:\, K \in \mathcal{T}_h\}$. We suppose that $\boldsymbol{p}$ is also of bounded variation, i.e., there is a constant $\rho_2 \geq 1$ such that

$$\rho_2^{-1} \leq {}^{p_{K_\sharp}}/_{p_{K_\flat}} \leq \rho_2, \tag{4}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge.

Moreover, we shall define some suitable element edge operators that are required for the DG method. To this end, we denote by $\mathcal{E}_\mathcal{I}$ the set of all interior edges of the partition $\mathcal{T}_h$ of Ω, and by $\mathcal{E}_\mathcal{B}$ the set of all boundary edges of $\mathcal{T}_h$. In addition, let $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{B}$. The boundary ∂K of an element K and the sets $\partial K \setminus \partial\Omega$ and $\partial K \cap \partial\Omega$ will be identified in a natural way with the corresponding subsets of $\mathcal{E}$.

Let $K_\sharp$ and $K_\flat$ be two adjacent elements of $\mathcal{T}_h$, and $\boldsymbol{x}$ an arbitrary point on the interior edge $e \in \mathcal{E}_\mathcal{I}$ given by $e = \partial K_\sharp \cap \partial K_\flat$. Furthermore, let v and $\boldsymbol{q}$ be scalar- and vector-valued functions, respectively, that are sufficiently smooth inside each element $K_{\sharp/\flat}$. By $(v_{\sharp/\flat}, \boldsymbol{q}_{\sharp/\flat})$, we denote the traces of $(v, \boldsymbol{q})$ on e taken from within the interior of $K_{\sharp/\flat}$, respectively. Then, the averages of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$\langle\!\langle v \rangle\!\rangle = \frac{1}{2}(v_\sharp + v_\flat), \qquad \langle\!\langle \boldsymbol{q} \rangle\!\rangle = \frac{1}{2}(\boldsymbol{q}_\sharp + \boldsymbol{q}_\flat),$$

respectively. Similarly, the jumps of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$[\![v]\!] = v_\sharp \, \boldsymbol{n}_{K_\sharp} + v_\flat \, \boldsymbol{n}_{K_\flat}, \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_\sharp \cdot \boldsymbol{n}_{K_\sharp} + \boldsymbol{q}_\flat \cdot \boldsymbol{n}_{K_\flat},$$

respectively, where we denote by $\boldsymbol{n}_{K_{\sharp/\flat}}$ the unit outward normal vector on $\partial K_{\sharp/\flat}$, respectively. On a boundary edge $e \in \mathcal{E}_\mathcal{B}$, we set $\langle\!\langle v \rangle\!\rangle = v$, $\langle\!\langle \boldsymbol{q} \rangle\!\rangle = \boldsymbol{q}$, and $[\![v]\!] = v\boldsymbol{n}$, $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ denoting the unit outward normal vector on the boundary $\partial\Omega$.

Given a finite element mesh $\mathcal{T}_h$ and an associated polynomial degree vector $\boldsymbol{p} = (p_K)_{K \in \mathcal{T}_h}$, with $p_K \geq 1$ for all $K \in \mathcal{T}_h$, consider the hp-discretization space

$$V(\mathcal{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_K \in Q_{p_K}(K), K \in \mathcal{T}_h\}, \tag{5}$$

for the DG method. Here, for $K \in \mathcal{T}_h$, $Q_{p_K}(K)$ is the space of all polynomials of degree at most p_K in each variable on K.

In addition, consider the space

$$H^{\boldsymbol{s}}(\Omega, \mathcal{T}_h) = \left\{ u \in L^2(\Omega) :\, u|_K \in H^{s_K}(K), K \in \mathcal{T}_h \right\}.$$

Here, $\boldsymbol{s} = (s_K)_{K \in \mathcal{T}_h}$, $s_K \geq 1$ for all $K \in \mathcal{T}_h$, is an integer vector. Let $\mathbf{1}$ be the vector containing only ones. Then, we shall introduce the lifting operator

$$\mathsf{L} :\, H^{\mathbf{1}}(\Omega, \mathcal{T}_h) \to V(\mathcal{T}_h, \boldsymbol{p})^2,$$

defined by

$$\int_\Omega \mathsf{L}(w) \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} = \int_\mathcal{E} [\![w]\!] \cdot \langle\!\langle \boldsymbol{\phi} \rangle\!\rangle \, \mathrm{d}s \qquad \forall \boldsymbol{\phi} \in V(\mathcal{T}_h, \boldsymbol{p})^2; \tag{6}$$

see, e.g., [2, 4, 20].

2.2. ***hp*-DG Discretization.** We will now develop the *hp*-DG methods to be considered in this paper. To this end, we multiply the equation (1) by a test function $v \in V(\mathcal{T}_h, \boldsymbol{p})$ and integrate by parts. Then, applying standard manipulations and using that $[\![\nabla u]\!] = 0$ on all internal edges (since $\Delta u \in L^2(\Omega)$), we obtain

$$\int_\Omega \nabla u \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla u \rangle\!\rangle [\![v]\!] \,\mathrm{d}s = \int_\Omega f v \,\mathrm{d}\boldsymbol{x}.$$

Here, ∇_h is the elementwise gradient. Furthermore, because $u \in H_0^1(\Omega)$, there holds

$$[\![u]\!] = \mathbf{0} \text{ on } \mathcal{E} \qquad \text{and} \qquad \mathbf{L}(u) = \mathbf{0} \text{ on } \Omega.$$

Hence, for any constants θ, γ, δ, there holds

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla u \rangle\!\rangle \cdot [\![v]\!] \,\mathrm{d}s - \theta \int_{\mathcal{E}} [\![u]\!] \cdot \langle\!\langle \nabla_h v \rangle\!\rangle \,\mathrm{d}s \\ &\qquad + \delta \int_\Omega \mathbf{L}(u) \cdot \mathbf{L}(v) \,\mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{E}} \sigma [\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{7}$$

for any $v \in V(\mathcal{T}_h, \boldsymbol{p})$, where

$$\sigma = \frac{\mathtt{p}}{\mathtt{h}} \tag{8}$$

is defined through the two functions $\mathtt{h} \in L^\infty(\mathcal{E})$ and $\mathtt{p} \in L^\infty(\mathcal{E})$ given by

$$\mathtt{h}(\boldsymbol{x}) = \begin{cases} \min(h_{K_\sharp}, h_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathcal{E}_{\mathcal{I}} \\ h_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{B}} \end{cases},$$

$$\mathtt{p}(\boldsymbol{x}) = \begin{cases} \max(p_{K_\sharp}, p_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathcal{E}_{\mathcal{I}} \\ p_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{B}} \end{cases}.$$

An *hp*-DG discretization for (1)–(2) is now obtained by restricting (7) to the *hp*-space $V(\mathcal{T}_h, \boldsymbol{p})$. More precisely, for $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$, let

$$\begin{aligned} a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w, v) = &\int_\Omega \nabla_h w \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla_h w \rangle\!\rangle \cdot [\![v]\!] \,\mathrm{d}s - \theta \int_{\mathcal{E}} [\![w]\!] \cdot \langle\!\langle \nabla_h v \rangle\!\rangle \,\mathrm{d}s \\ &+ \delta \int_\Omega \mathbf{L}(w) \cdot \mathbf{L}(v) \,\mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{E}} \sigma [\![w]\!] \cdot [\![v]\!] \,\mathrm{d}s, \end{aligned} \tag{9}$$

and

$$\ell_{\mathrm{DG}}(v) = \int_\Omega f v \,\mathrm{d}\boldsymbol{x}, \tag{10}$$

and define an approximate solution $u_{\mathrm{DG}} \in V(\mathcal{T}_h, \boldsymbol{p})$ of (1)–(2) by

$$a_{\mathrm{DG}}^{\gamma,\delta,\theta}(u_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \qquad \forall v \in V(\mathcal{T}_h, \boldsymbol{p}). \tag{11}$$

We note that, recalling the definition (6) of the lifting operator $\mathbf{L}$, there holds

$$\begin{aligned} a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w, v) = &\int_\Omega \nabla_h w \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} - \int_\Omega \nabla_h w \cdot \mathbf{L}(v) \,\mathrm{d}\boldsymbol{x} - \theta \int_\Omega \mathbf{L}(w) \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} \\ &+ \delta \int_\Omega \mathbf{L}(w) \cdot \mathbf{L}(v) \,\mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{E}} \sigma [\![w]\!] \cdot [\![v]\!] \,\mathrm{d}s \end{aligned} \tag{12}$$

for all $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$.

Remark 2.1. We note that the method proposed in this paper is closely related to other DG schemes in the literature; see, e.g., [2]. In the following, we shall display how different choices of the parameters γ, δ and θ appearing in the form $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$ from (12) result in previously known DG bilinear forms (considered in their *hp*-version primal form):

	stability parameters		DG method
$\theta = 1$	$\gamma = 0$	$\delta > 0$	Bassi-Rebay [4]
	$\gamma = \mathcal{O}(\mathtt{p})$	$\delta = 0$	SIPG [1, 9, 25]
	$\gamma = \mathcal{O}(\mathtt{p})$	$\delta = 1$	LDG (with $\beta = 0$ in [2]) [7, 16]
$\theta = -1$	$\gamma = 0$	$\delta = 0$	Baumann-Oden [5, 14, 19]
	$\gamma = \mathcal{O}(\mathtt{p})$	$\delta = 0$	NIPG [13, 19, 25]
$\theta = 0$	$\gamma = \mathcal{O}(\mathtt{p})$	$\delta = 0$	IIPG [8]

Furthermore, we remark that, although *hp*-version analyses for most of the above methods are available, the corresponding error estimates usually encounter a sub-optimality with respect to p. This is mainly due to the fact that the parameter γ in the jump stabilization terms of the DG forms is chosen of order p in order the ensure the stability (i.e. coercivity) of the methods. In this paper, we will prove that, choosing $\gamma \sim \delta \sim \mathcal{O}(1)$ (and $\gamma, \delta > 0$), the proposed DG method (11) is stable and converges optimally in h and p.

We conclude this section by introducing an energy norm for the DG method (11):

$$\|\!|w|\!\|_{\mathrm{DG},\gamma,\delta}^2 = \int_\Omega |\nabla_h w|^2 \,\mathrm{d}\boldsymbol{x} + \delta \int_\Omega |\mathbf{L}(w)|^2 \,\mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{E}} \sigma |[\![w]\!]|^2 \,\mathrm{d}s. \tag{13}$$

Henceforth, we shall always suppose that

$$\gamma > 0, \qquad \delta \geq 0, \tag{14}$$

i.e., $\|\!|\cdot|\!\|_{\mathrm{DG},\gamma,\delta}$ is indeed a norm.

2.3. **Stability.** The aim of this section is to discuss the stability and well-posedness of the DG method (11). Particularly, we will prove that the bilinear form $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$ from (12) is continuous and coercive on $V(\mathcal{T}_h, \boldsymbol{p}) \times V(\mathcal{T}_h, \boldsymbol{p})$. We note that the corresponding continuity and coercivity constants can be represented explicitly.

Proposition 2.2. *Suppose that* (14) *is satisfied.*

a) *If* $\delta > \frac{(1+\theta)^2}{4}$ *for* $\theta \neq -1$ *and* $\delta \geq 0$ *for* $\theta = -1$, *then the form* $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$ *is coercive in the norm* $\|\!|\cdot|\!\|_{\mathrm{DG},\gamma,\delta}$. *More precisely, we have*

$$a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w,w) \geq C_{\mathrm{coer}} \|\!|w|\!\|_{\mathrm{DG},\gamma,\delta}^2 \qquad \forall w \in V(\mathcal{T}_h, \boldsymbol{p}),$$

where

$$C_{\mathrm{coer}} = \begin{cases} 1 - \frac{|1+\theta|}{2\sqrt{\delta}} & \text{for } \theta \neq -1 \\ 1 & \text{for } \theta = -1 \end{cases}.$$

b) *If* $\delta > 0$, *the form* $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$ *is continuous, i.e.,*

$$\left| a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w,v) \right| \leq \left(1 + \frac{\max(1,|\theta|)}{\sqrt{\delta}} \right) \|\!|w|\!\|_{\mathrm{DG},\gamma,\delta} \|\!|v|\!\|_{\mathrm{DG},\gamma,\delta}$$

for any $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$.

Proof. Let $w,v \in V(\mathcal{T}_h,\boldsymbol{p})$.

Proof of a): We use the representation (12) of the DG form $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$. The case $\theta=-1$ follows directly from the definition of the norm $\|\!|\cdot|\!\|_{\mathrm{DG},\gamma,\delta}$. Hence, consider $\theta\neq -1$ and choose $\varepsilon>0$. Then, using Young's inequality,

$$|ab| \leq \frac{1}{2\varepsilon}a^2+\frac{\varepsilon}{2}b^2, \qquad a,b\in\mathbb{R}, \tag{15}$$

leads to

$$\begin{aligned}
a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w,w) &\geq \|\nabla_h w\|_{0,\Omega}^2-(1+\theta)\|\mathbf{L}(w)\|_{0,\Omega}\|\nabla_h w\|_{0,\Omega}\\
&\qquad +\delta\|\mathbf{L}(w)\|_{0,\Omega}^2+\gamma\left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2\\
&\geq \left(1-\frac{\varepsilon}{2}\right)\|\nabla_h w\|_{0,\Omega}^2+\left(1-\frac{(1+\theta)^2}{2\varepsilon\delta}\right)\delta\|\mathbf{L}(w)\|_{0,\Omega}^2+\gamma\left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2.
\end{aligned}$$

Then, with $\varepsilon=\frac{|1+\theta|}{\sqrt{\delta}}$, it holds that

$$1-\frac{\varepsilon}{2}=1-\frac{(1+\theta)^2}{2\varepsilon\delta}=1-\frac{|1+\theta|}{2\sqrt{\delta}},$$

and thus,

$$a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w,w)\geq\left(1-\frac{|1+\theta|}{2\sqrt{\delta}}\right)\|\!|w|\!\|_{\mathrm{DG},\gamma,\delta}^2.$$

We need the above constant to be positive, i.e., $\delta>\frac{(1+\theta)^2}{4}$.

Proof of b): Using a weighted Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
&\left|a_{\mathrm{DG}}^{\gamma,\delta,\theta}(w,v)\right|\\
&\quad\leq\int_\Omega|\nabla_h w|\,|\nabla_h v|\,\mathrm{d}\boldsymbol{x}+\int_\Omega|\nabla_h w||\mathbf{L}(v)|\,\mathrm{d}\boldsymbol{x}+|\theta|\int_\Omega|\mathbf{L}(w)|\,|\nabla_h v|\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\delta\int_\Omega|\mathbf{L}(w)|\,|\mathbf{L}(v)|\,\mathrm{d}\boldsymbol{x}+\gamma\int_{\mathcal{E}}\sigma|[\![w]\!]||[\![v]\!]|\,\mathrm{d}s\\
&\quad\leq\left(\left(1+\frac{1}{\sqrt{\delta}}\right)\|\nabla_h w\|_{0,\Omega}^2+\left(1+\frac{|\theta|}{\sqrt{\delta}}\right)\delta\|\mathbf{L}(w)\|_{0,\Omega}^2+\gamma\left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2\right)^{\frac12}\\
&\qquad\times\left(\left(1+\frac{|\theta|}{\sqrt{\delta}}\right)\|\nabla_h v\|_{0,\Omega}^2+\left(1+\frac{1}{\sqrt{\delta}}\right)\delta\|\mathbf{L}(v)\|_{0,\Omega}^2+\gamma\left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2\right)^{\frac12}\\
&\quad\leq\left(1+\frac{\max(1,|\theta|)}{\sqrt{\delta}}\right)\|\!|w|\!\|_{\mathrm{DG},\gamma,\delta}\|\!|v|\!\|_{\mathrm{DG},\gamma,\delta}.
\end{aligned}$$

This completes the proof. □

Moreover, we shall discuss the continuity of the linear form ℓ_{DG} from (10) with respect to the DG energy norm.

Proposition 2.3. *The linear form ℓ_{DG} is continuous, i.e., there exists a constant $C>0$ independent of $\boldsymbol{h}$ and of $\boldsymbol{p}$ such that*

$$|\ell_{\mathrm{DG}}(v)|\leq C\|f\|_{0,\Omega}\|\!|v|\!\|_{\mathrm{DG},\gamma,\delta} \tag{16}$$

for all $v\in V(\mathcal{T}_h,\boldsymbol{p})$.

Proof. Eq. (1.8) in [6] (see also Theorem 6.2) implies that there exists a constant $C > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ such that the following discrete Poincaré-Friedrichs inequality is satisfied

$$\|w\|_{0,\Omega}^2 \leq C \left(\|\nabla_h w\|_{0,\Omega}^2 + \int_{\mathcal{E}_\mathcal{I}} \mathtt{h}^{-1} \left|[\![w]\!]\right|^2 \,\mathrm{d}s + \int_{\partial\Omega} |w|^2 \,\mathrm{d}s \right)$$

for all $w \in \{v \in L^2(\Omega) : v|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$. Therefore, we obtain

$$\|w\|_{0,\Omega} \leq C |\!|\!| w |\!|\!|_{\mathrm{DG},\gamma,\delta} \qquad \forall w \in V(\mathcal{T}_h, \boldsymbol{p}).$$

Hence, it follows that

$$|\ell_{\mathrm{DG}}(v)| \leq \|f\|_{0,\Omega} \|v\|_{0,\Omega} \leq C \|f\|_{0,\Omega} |\!|\!| v |\!|\!|_{\mathrm{DG},\gamma,\delta}$$

for all $v \in V(\mathcal{T}_h, \boldsymbol{p})$. □

The above results, Propositions 2.2–2.3, imply the well-posedness of the DG discretization (11).

Theorem 2.4. *The hp-DG method* (11) *has a unique solution* $u_{\mathrm{DG}} \in V(\mathcal{T}_h, \boldsymbol{p})$. *Furthermore, if the bound* (16) *in Proposition 2.3 holds, then we have*

$$|\!|\!| u_{\mathrm{DG}} |\!|\!|_{\mathrm{DG},\gamma,\delta} \leq C \|f\|_{0,\Omega},$$

where $C > 0$ *is a constant independent of* $\boldsymbol{h}$ *and of* $\boldsymbol{p}$.

3. *hp*-Error Analysis

The goal of this section is to show that the DG method (11) converges optimally with respect to both the local element sizes $\boldsymbol{h}$ and the polynomial degrees $\boldsymbol{p}$. To do so, we shall briefly collect some *hp*-approximation results that will play an important role in the subsequent error analysis. Furthermore, later on in this section, the main result of this paper will be given.

3.1. *hp*-Approximations.

The first of the following two lemmas shows that the elementwise L^2-projection on $V(\mathcal{T}_h, \boldsymbol{p})$ remains optimal on the edges of affine quadrilateral elements (in the corresponding L^2-norm). The second result is an optimal (with respect to the H^1-norm) conforming *hp*-interpolant in $V(\mathcal{T}_h, \boldsymbol{p})$.

Lemma 3.1. *Let* $K \in \mathcal{T}_h$ *and suppose that* $u \in H^s(K)$ *for some integer* $s \geq 1$. *Then, for* $1 \leq \widetilde{s} \leq \min(p+1, s)$, *and* $p \geq 0$, *we have that*

$$\|u - \pi_p u\|_{0,\partial K} \leq C \left(\frac{h_K}{p}\right)^{\widetilde{s}-\frac{1}{2}} |u|_{H^{\widetilde{s}}(K)}.$$

Here, $C > 0$ *is a constant independent of* h_K *and* p, *and* $\pi_p : L^2(K) \to Q_p(K)$ *is the* L^2*-projection of degree* p *on* K.

Proof. See [13, Lemma 3.9 and Remark 3.10]. □

Lemma 3.2. *Given* $u \in H^{\boldsymbol{s}}(\Omega, \mathcal{T}_h) \cap H^2(\Omega) \cap H_0^1(\Omega)$, *then there exists a continuous interpolant* $\mathsf{P}_{\boldsymbol{p}}(u) \in V(\mathcal{T}_h, \boldsymbol{p}) \cap H_0^1(\Omega)$ *of* u *such that*

$$\|u - \mathsf{P}_{\boldsymbol{p}}(u)\|_{H^1(\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p_K}\right)^{2\widetilde{s}_K - 2} |u|_{H^{\widetilde{s}_K}(K)}^2,$$

for $2 \leq \widetilde{s}_K \leq \min(p_K + 1, s_K)$, $K \in \mathcal{T}_h$. *Here,* $C > 0$ *is a constant independent of* $\boldsymbol{h}$ *and* $\boldsymbol{p}$.

Proof. See, e.g., [22, Theorem 4.72 and Remark 4.73] $\square$

3.2. *hp*-**Optimal Error Estimates.** Let us analyze the error

$$e_{\mathrm{DG}} = u - u_{\mathrm{DG}},$$

of the DG method in the energy norm $\|\!|\cdot|\!\|_{\mathrm{DG},\gamma,\delta}$. Here, u is the exact solution of (1)–(2), and u_{DG} is the DG approximation from (11). We will proceed in a similar way as in [13, 24], for example. More precisely, we split the DG error into two parts,

$$e_{\mathrm{DG}} = \eta + \xi,$$

where $\eta = u - \mathsf{P}_{\boldsymbol{p}}(u)$ and $\xi = \mathsf{P}_{\boldsymbol{p}}(u) - u_{\mathrm{DG}}$, and $\mathsf{P}_{\boldsymbol{p}}(u) \in V(\mathcal{T}_h, \boldsymbol{p}) \cap H_0^1(\Omega)$ is the conforming *hp*-interpolant of u from Lemma 3.2. We note that $\eta \in H_0^1(\Omega)$ and $\xi \in V(\mathcal{T}_h, \boldsymbol{p})$. Then, applying the triangle inequality, there holds

$$\|\!|e|\!\|_{\mathrm{DG},\gamma,\delta} \leq \|\!|\eta|\!\|_{\mathrm{DG},\gamma,\delta} + \|\!|\xi|\!\|_{\mathrm{DG},\gamma,\delta}. \tag{17}$$

We first analyze the term $\|\!|\xi|\!\|_{\mathrm{DG},\gamma,\delta}$ and aim at bounding it in terms of η; this will make it possible to estimate the DG error by the interpolation error η only. Due to the consistent construction of the DG method (11), Galerkin orthogonality holds true. This and the coercivity of $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$, Proposition 2.2, imply

$$C\|\!|\xi|\!\|^2_{\mathrm{DG},\gamma,\delta} \leq a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\xi,\xi) = a_{\mathrm{DG}}^{\gamma,\delta,\theta}(e_{\mathrm{DG}} - \eta, \xi) = -a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\eta,\xi) \leq \left| a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\eta,\xi) \right|. \tag{18}$$

Because $\eta \in H_0^1(\Omega)$ we have that

$$[\![\eta]\!] = \mathbf{L}(\eta) = 0 \qquad \text{on } \mathcal{E}. \tag{19}$$

Hence, using the definition (9) of the bilinear form $a_{\mathrm{DG}}^{\gamma,\delta,\theta}$, results in

$$\left| a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\eta,\xi) \right| = \left| \int_\Omega \nabla\eta \cdot \nabla_h \xi \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla\eta \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s \right|$$

Applying the definition (6) of the lifting operator $\mathbf{L}$, there holds

$$\begin{aligned}
-\int_{\mathcal{E}} \langle\!\langle \nabla\eta \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s &= -\int_{\mathcal{E}} \langle\!\langle \nabla\eta - \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s - \int_{\mathcal{E}} \langle\!\langle \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s \\
&= -\int_{\mathcal{E}} \langle\!\langle \nabla\eta - \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s - \int_\Omega \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \cdot \mathbf{L}(\xi) \,\mathrm{d}\boldsymbol{x} \\
&= -\int_{\mathcal{E}} \langle\!\langle \nabla\eta - \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s - \int_\Omega \nabla\eta \cdot \mathbf{L}(\xi) \,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where $\boldsymbol{\Pi}_{\boldsymbol{p}}$ is the elementwise L^2-projection on $V(\mathcal{T}_h, \boldsymbol{p})^2$. Therefore, it follows that

$$\begin{aligned}
\left| a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\eta,\xi) \right| &= \left| \int_\Omega \nabla\eta \cdot \nabla_h \xi \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla\eta - \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s - \int_\Omega \nabla\eta \cdot \mathbf{L}(\xi) \,\mathrm{d}\boldsymbol{x} \right| \\
&\leq \|\nabla\eta\|_{0,\Omega} \|\nabla_h \xi\|_{0,\Omega} + \left\| \mathtt{p}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} \langle\!\langle \nabla\eta - \boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta) \rangle\!\rangle \right\|_{0,\mathcal{E}} \left\| \mathtt{p}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [\![\xi]\!] \right\|_{0,\mathcal{E}} \\
&\quad + \|\nabla\eta\|_{0,\Omega} \|\mathbf{L}(\xi)\|_{0,\Omega}.
\end{aligned}$$

Thus, if

$$\gamma, \delta > 0, \tag{20}$$

we conclude that

$$\begin{aligned}
&\left|a_{\mathrm{DG}}^{\gamma,\delta,\theta}(\eta,\xi)\right| \\
&\qquad \le C\left((1+\delta^{-1})\,\|\nabla\eta\|_{0,\Omega}^2+\gamma^{-1}\left\|\mathtt{h}^{\frac12}\mathtt{p}^{-\frac12}\langle\!\langle\nabla\eta-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta)\rangle\!\rangle\right\|_{0,\mathcal{E}}^2\right)^{\frac12}|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}.
\end{aligned}$$

Therefore, due to (18) and (3), (4), we have

$$\begin{aligned}
&|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}^2 \\
&\qquad \le C\left((1+\delta^{-1})\,\|\nabla\eta\|_{0,\Omega}^2+\gamma^{-1}\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K}\|\nabla\eta-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta)\|_{0,\partial K}^2\right)^{\frac12}|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}.
\end{aligned}$$

Dividing both sides of the above inequality by $|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}$, leads to

$$|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}\le C\left((1+\delta^{-1})\,\|\nabla\eta\|_{0,\Omega}^2+\gamma^{-1}\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K}\|\nabla\eta-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta)\|_{0,\partial K}^2\right)^{\frac12}.$$

Furthermore, since $\nabla(\mathsf{P}_{\boldsymbol{p}}(u))\in V(\mathcal{T}_h,\boldsymbol{p})^2$ and because the L^2-projection preserves polynomials, it follows that

$$\nabla\eta-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla\eta)=\nabla u-\nabla(\mathsf{P}_{\boldsymbol{p}}(u))-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla u)+\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla(\mathsf{P}_{\boldsymbol{p}}(u)))=\nabla u-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla u).$$

Hence,

$$|\!|\!|\xi|\!|\!|_{\mathrm{DG},\gamma,\delta}\le C\left((1+\delta^{-1})\,\|\nabla\eta\|_{0,\Omega}^2+\gamma^{-1}\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K}\|\nabla u-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2\right)^{\frac12},$$

and inserting this into (17), implies

$$\begin{aligned}
&|\!|\!|e|\!|\!|_{\mathrm{DG},\gamma,\delta} \\
&\qquad \le |\!|\!|\eta|\!|\!|_{\mathrm{DG},\gamma,\delta}+C\left((1+\delta^{-1})\,\|\nabla\eta\|_{0,\Omega}^2+\gamma^{-1}\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K}\|\nabla u-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2\right)^{\frac12}.
\end{aligned}$$

Then, using (19), we obtain

$$|\!|\!|e|\!|\!|_{\mathrm{DG},\gamma,\delta}\le C\max(1,\gamma^{-1},\delta^{-1})\left(\|\nabla\eta\|_{0,\Omega}^2+\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K}\|\nabla u-\boldsymbol{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2\right)^{\frac12}.$$

Finally, using the approximation properties of the interpolants $\mathsf{P}_{\boldsymbol{p}}$ and $\boldsymbol{\Pi}_{\boldsymbol{p}}$ (cf. Lemmas 3.1 and 3.2), and recalling (20) and Proposition 2.2, leads to the main result of this paper.

Theorem 3.3. *Suppose that $\gamma>0$ and $\delta>\frac{(1+\theta)^2}{4}$. Furthermore, let the exact solution u of* (1)–(2) *belong to $H^{\boldsymbol{s}}(\Omega,\mathcal{T}_h)\cap H^2(\Omega)\cap H_0^1(\Omega)$, with $s_K\ge 2$ for all $K\in\mathcal{T}_h$. Then, there exists a constant $C>0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ such that there holds the a priori hp-error estimate*

$$|\!|\!|u-u_{\mathrm{DG}}|\!|\!|_{\mathrm{DG},\gamma,\delta}^2\le C\sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p_K}\right)^{2\widetilde{s}_K-2}|u|_{H^{\widetilde{s}_K}(K)}^2, \tag{21}$$

for $2\le\widetilde{s}_K\le\min(p_K+1,s_K)$, $K\in\mathcal{T}_h$, where u_{DG} is the hp-DG solution from (11).

Remark 3.4. We note that the error estimate in Theorem 3.3 is optimal with respect to both the local element sizes and polynomial degrees. The main ingredients for the proof of this result are 1) the fact that the elementwise L^2-projection remains optimal on the edges of affine quadrilateral elements, and 2) that both stabilization parameters γ and δ are strictly positive. Hence, in order to obtain *hp*-optimal convergence of the DG method in this paper, we propose the use of jump penalty stabilization with $\gamma = \mathcal{O}(1) > 0$ (notice that IP and LDG methods feature similar jump penalty terms, however there, γ is typically chosen of order $\mathcal{O}(\mathtt{p})$, i.e., the stabilization becomes stronger as p increases), and, furthermore, a term that stabilizes the lifting operators (with $\delta = \mathcal{O}(1) > \frac{1}{4}(1+\theta)^2$).

Remark 3.5. The techniques used in the proof of Theorem 3.3 apply basically also in three (or even higher) space dimensions. In particular, the *hp*-interpolation results from Section 3.1 are based on tensor-product arguments and can be generalized to hexahedral elements in 3d, for example. The construction of a conforming *hp*-interpolant (that is optimal with respect to the H^1-norm) on meshes containing hanging nodes (respectively edges or faces) is, however, remarkably more technical in higher space dimensions.

4. Numerical Experiments

We shall present two test problems for the proposed DG methods in this paper. In both examples, we consider (1)–(2) on the open unit square $\Omega = (0,1)^2$. A sequence of structured meshes containing square elements with uniform meshsize h and uniform polynomial degree p for the numerical approximation will be used. The implementation uses the software library *life*, a unified C++ implementation of finite and spectral element methods in 1–3d; see [17, 18]. We shall mainly focus on the symmetric version of the proposed method, i.e., $\theta = 1$. Further computations for $\theta = 0$ and $\theta = -1$ show similar convergence behavior (under the conditions of Theorem 3.3).

Example 1: Smooth solution. Consider the exact solution

$$u(x,y) = \sin(\pi x)\sin(\pi y) \in C^\infty(\overline{\Omega}) \cap H_0^1(\Omega)$$

of (1)–(2). Then, the force term is given by $f(x,y) = 2\pi^2\sin(\pi x)\sin(\pi y)$, and the error bound from Theorem 3.3 implies that

$$|\!|\!| u - u_{\mathrm{DG}} |\!|\!|_{\mathrm{DG},\gamma,\delta} \leq C\left(\frac{h}{p}\right)^p |u|_{H^{p+1}(\Omega)}.$$

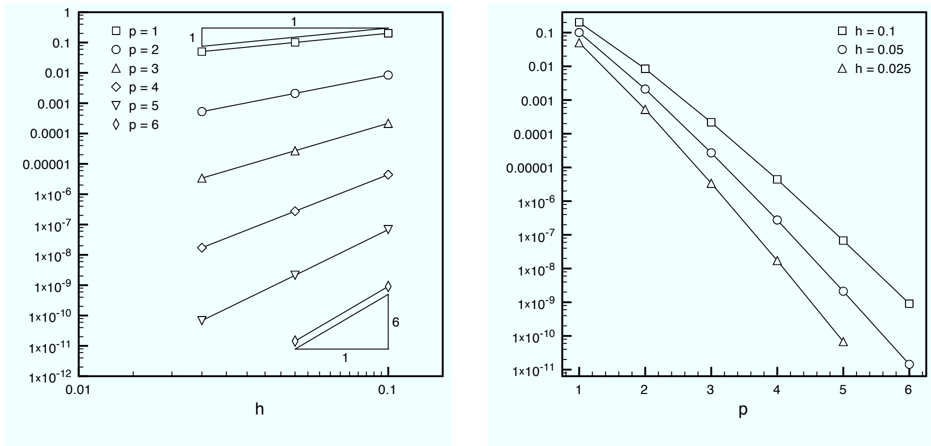
In Figure 1, the errors measured in the energy norm $|\!|\!| \cdot |\!|\!|_{\mathrm{DG},\gamma,\delta}$ for both h- and p-refinement have been plotted. We clearly observe algebraic convergence of order p with respect to h in Figure 1 (a), and exponential convergence with respect to p in Figure 1 (b) (lin-log plot).

Example 2: Solution with local singularity. We now choose the exact solution of (1)–(2) to be

$$u(x,y) = x^\alpha \sin(\pi x)\sin(\pi y), \tag{22}$$

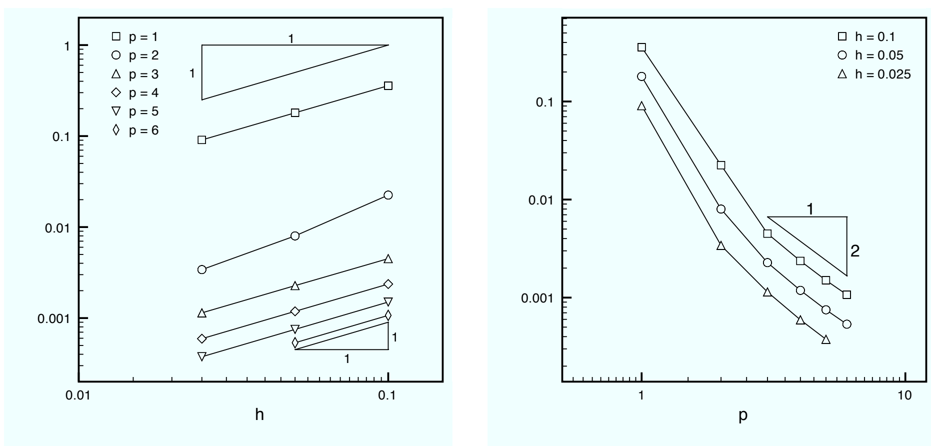
with a corresponding force term f. Note that, for $\alpha < 1$, the Hessian $\mathsf{D}^2 u$ of u has a singularity at the origin $(0,0)$. More precisely,

$$u \in H^{\alpha+\frac{3}{2}-\varepsilon}(\Omega) \cap H_0^1(\Omega)$$



(a) h-refinement (b) p-refinement

FIGURE 1. Example 1: Error in the DG energy norm for (a) h- and (b) p-refinement; $\theta = 1$, $\delta = \gamma = 10$.



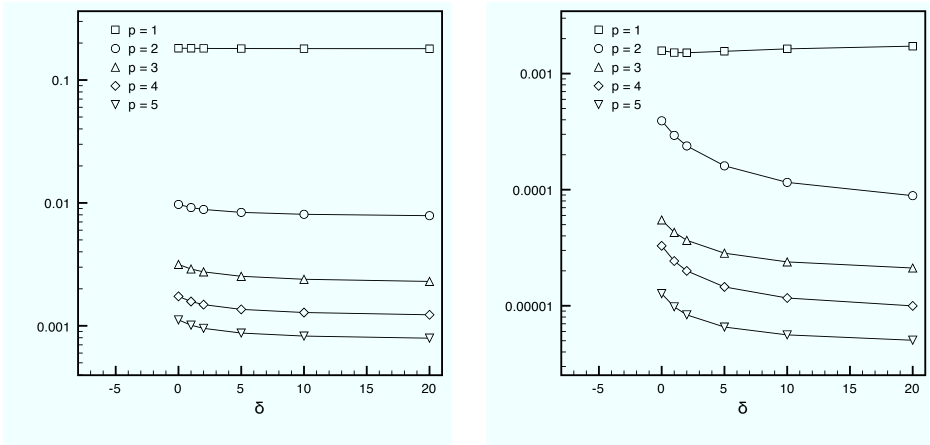
(a) h-refinement (b) p-refinement

FIGURE 2. Example 2: Error in the DG energy norm for (a) h- and (b) p-refinement; $\theta = 1$, $\delta = \gamma = 10$.

for all arbitrarily small $\varepsilon > 0$.

Let us consider the case $\alpha = 0.5$. Then, we expect that the error in the energy norm, i.e., $|||u - u_{\mathrm{DG}}|||_{\mathrm{DG},\gamma,\delta}$, decays at a rate of $\left({}^{h}/{}_{p}\right)^{1-\varepsilon}$. In Figure 2, the error for h- and p-refinement, with $\delta = \gamma = 10$, is presented. We see that, essentially, algebraic convergence of order 1 with respect to h is obtained. Furthermore, we notice that we observe superconvergence of order 2 with respect to p.

The nonsymmetric method $\theta = -1$. We note that the DG method (11) with $\theta = -1$ remains coercive robustly in $\mathtt{p}$ even in the case $\delta = 0$; cf. Proposition 2.2.



(a) Energy norm (b) L^2-norm

FIGURE 3. Example 2: Error of the DG method in (a) the DG energy norm and (b) the L^2-norm, for $\theta = -1$, $h = 0.05$, $\gamma = 10$, and different values of δ and p.

Theorem 3.3, however, suggests $\delta > 0$ for optimal convergence (with a constant of order δ^{-1} in the error estimate (21) for small δ). Indeed, in Figure 3, we observe a mild loss of accuracy in the DG approximations (for $\theta = -1$) of the singular solution u from (22) with $\delta \to 0$.

5. Concluding Remarks

In this paper we have presented a new class of *hp*-version discontinuous Galerkin methods for the numerical solution of linear elliptic partial differential equations. The schemes are stable and optimally convergent with respect to both the local element sizes and polynomial degrees (provided that the involved parameters are chosen appropriately). Our analysis indicates that the use of combined (optimal-order) jump penalization and lifting stabilization might be essential for *hp*-optimal convergence. Future work includes the a posteriori error analysis of the proposed DG methods and the application to nonlinear elliptic PDEs.

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