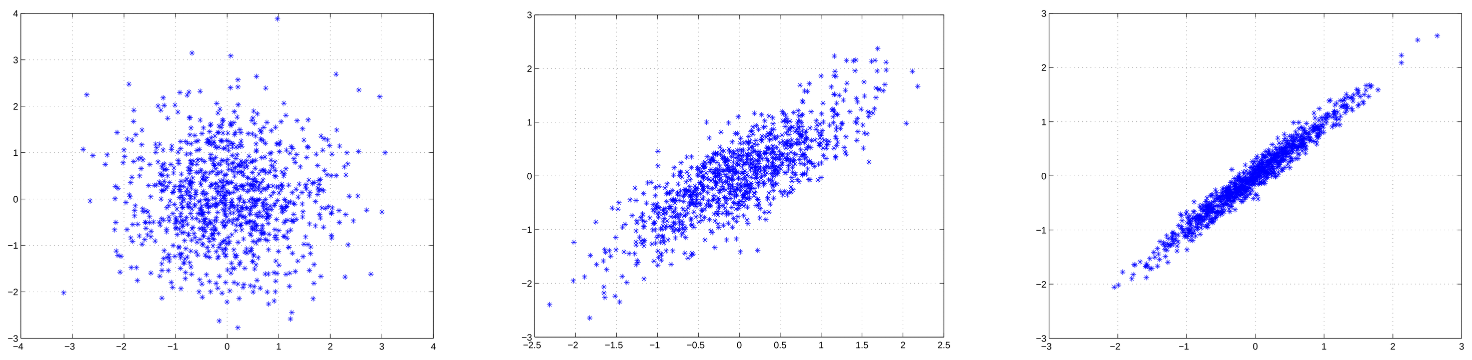
The Effective Rank: A Measure of Effective Dimensionality

Olivier Roy and Martin Vetterli

Motivations

- Rank-based optimization methods are unpractical
  - Discrete optimization
  - Smallest non-zero singular value minimization is sensitive to noise
  - A-priori knowledge of the minimum rank required
- Transform coding
  - Compressibility of multivariate random sources
  - Example: bivariate Gaussian distributions



  - Any measure of "effective dimensionality"?

Definition

- We consider a non all-zero matrix A of size $M \times N$
- Singular value decomposition $A = UDV$ where
  - U is a $M \times M$ unitary matrix with columns u_k for $k = 1, 2, \ldots, M-1$
  - V is a $N \times N$ unitary matrix with columns v_l for $l = 1, 2, \ldots, N-1$
  - D is a $M \times N$ diagonal matrix that contains the singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_Q \geq 0 \quad \text{where} \quad Q = \min\{M, N\}$$

- Spectral distribution

$$p_k = \frac{\sigma_k}{\sum_k \sigma_k} \quad \text{for } k = 1, 2, \ldots, Q$$

- Spectral entropy

$$H(p_1, p_2, \ldots, p_Q) = -\sum_{k=1}^{Q} p_k \log p_k$$

- We define the effective rank as

$$\boxed{\operatorname{erank}(A) = \exp\left\{H\left(p_1, p_2, \ldots, p_Q\right)\right\}}$$

Interpretation

- A is a linear mapping from $\mathbb{C}^N$ to $\mathbb{C}^M$ (geometrical shaping)
- The range of A, denoted by $\mathcal{R}$, is given by

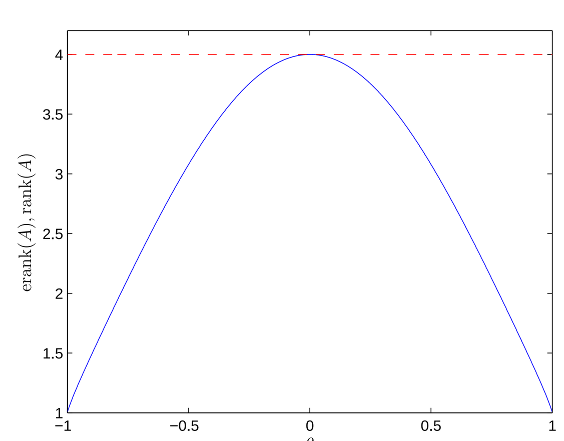
$$\mathcal{R} = \text{span}\,\{w_k\} \quad \text{where} \quad w_k \triangleq Av_k = \begin{cases} \sigma_k u_k & \text{for } k = 1, 2, \ldots, Q \\ 0 & \text{otherwise} \end{cases}$$

- $\operatorname{rank}(A)$ is the dimension of the range
- $\operatorname{erank}(A)$ endows the range with an "effective dimension" computed from the spectral entropy

Example

- First-order autoregressive correlation matrix with parameter $\rho \in (-1, 1)$
- Rank (dashed) vs. effective rank (solid)

$$A = \begin{pmatrix} 1 & \rho & \rho^2 & \rho \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho & \rho^2 & \rho & 1 \end{pmatrix}$$



- As $|\rho| \to 0, \operatorname{erank}(A) \to \operatorname{rank}(A) = 4$. As $|\rho| \to 1, \operatorname{erank}(A) \to 1$.

Operational Meaning

- Stochastic operational meaning
  - Similar interpretation as the entropy but in an approximation context
  - Coefficient rate in discrete form [1]: fraction of significant coefficients in an infinite product expansion
- Deterministic operational meaning?

Properties

Property 1 *It holds that*

$$1 \leq \operatorname{erank}(A) \leq \operatorname{rank}(A) \leq Q\,.$$

Property 2 *For all $c \neq 0$,*

$$\operatorname{erank}(A) = \operatorname{erank}(A^*) = \operatorname{erank}(A^T) = \operatorname{erank}(\bar{A}) = \operatorname{erank}(cA)\,.$$

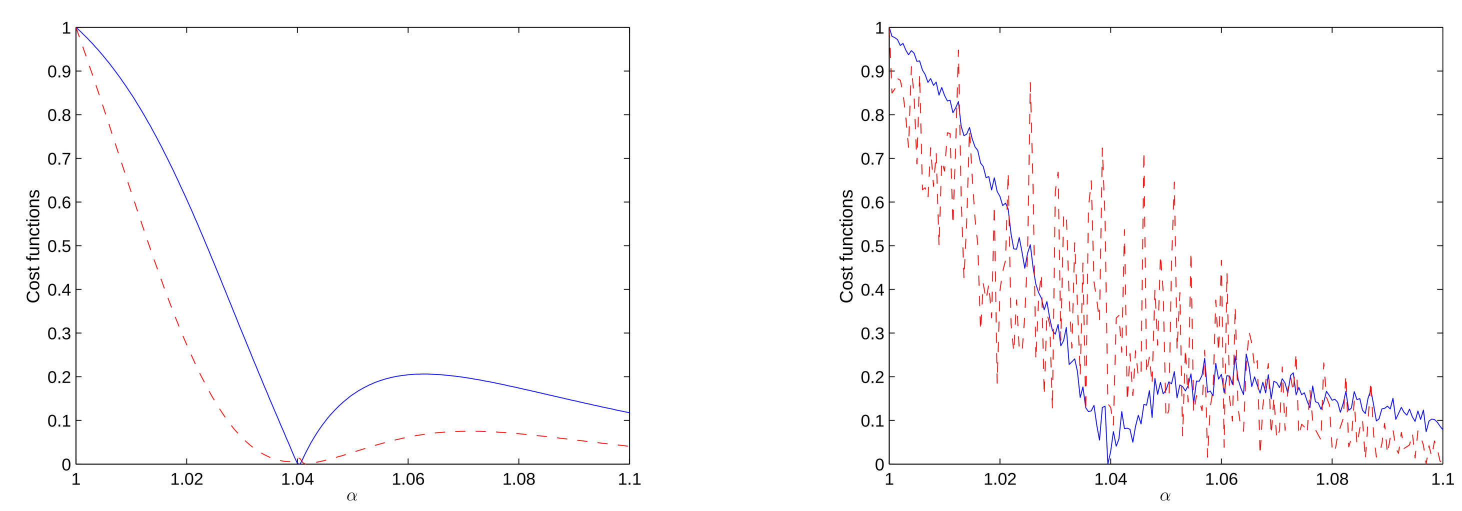
Property 3 *A unitary transformation on A does not change its effective rank.*

Property 4 *For two positive semidefinite Hermitian matrices A and B,*

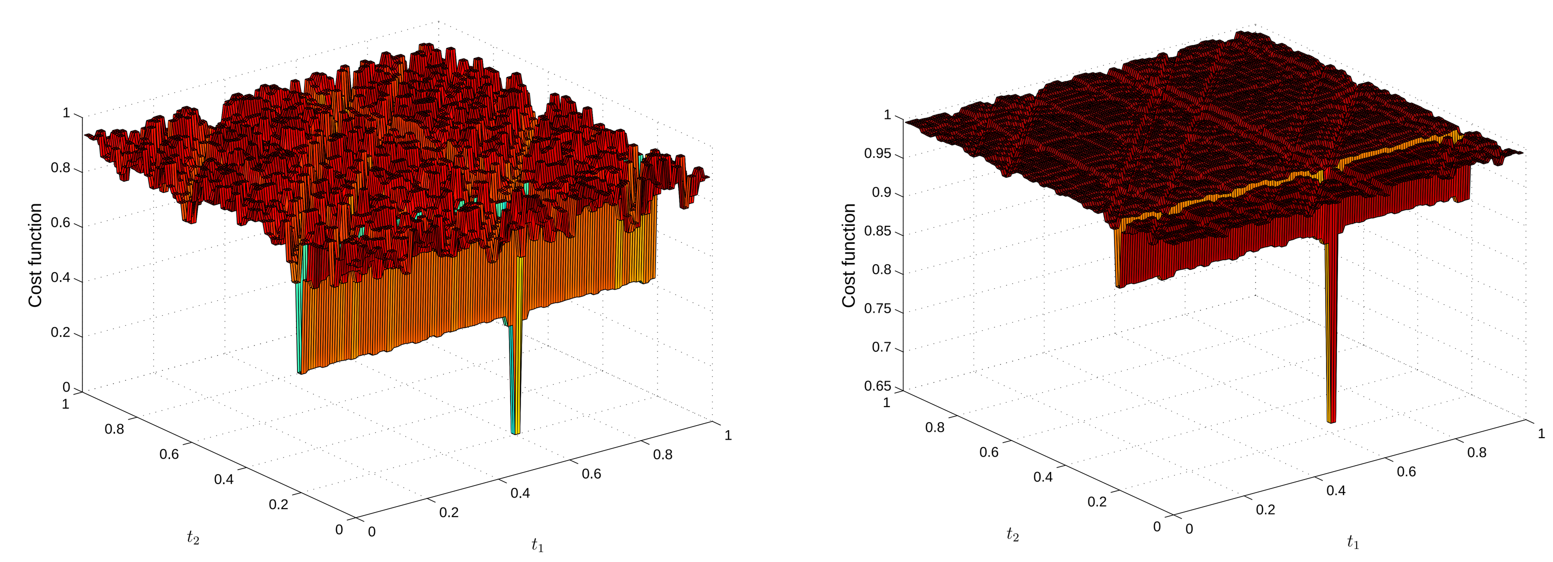
$$\operatorname{erank}(A + B) \leq \operatorname{erank}(A) + \operatorname{erank}(B)\,.$$

Applications

- Subspace-based analysis methods
- Parametric estimation from tomographic sampling [2]
  - Localization of diffusive sources relies on the estimation of an unknown parameter α
  - Estimation of α in noise-free (left) and noisy (right) conditions using minimization of the smallest non-zero singular value (dashed) and minimization of the effective rank (solid)



- Superresolution [3]
  - Unregistered set of samples with unknown offsets t_1 and t_2
  - Estimation of t_1 and t_2 using minimization of the smallest non-zero singular value (left) and minimization of the effective rank (right)



References

[1] L. L. Campbell, "Minimum coefficient rate for stationary random processes," *Information and Control*, vol. 3, no. 4, pp. 360–371, 1960.

[2] I. Jovanović, L. Sbaiz, and M. Vetterli, "Tomographic approach for parametric estimation of local diffusive sources and application to heat diffusion," *Proc. IEEE Int. Conf. Image Processing*, September 2007.

[3] P. Vandewalle, L. Sbaiz, J. Vandewalle, and M. Vetterli, "Super-resolution from unregistered and totally aliased signals using subspace methods," *IEEE Trans. Signal Processing*, vol. 55, pp. 3687–3703, 2007.

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Audiovisual Communications Laboratory (LCAV), École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne

email: {olivier.roy,martin.vetterli}@epfl.ch web: http://lcavwww.epfl.ch/~{oroy,vetterli}