SOME APPLICATIONS OF SYMMETRIES IN DIFFERENTIAL GEOMETRY AND DYNAMICAL SYSTEMS

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Mes recherches se situent à l'interface de la géométrie Riemannienne et des géométries de contact et symplectique et portent sur la construction des métriques Kähler ou Sasakie-Einstein, sur l'étude des systèmes Hamiltonians conformes, la géométrie des fibrés cosphériques et les groupoïdes de Lie propres. Le thème principal de cette thèse est l'étude des applications des symétries Lie en géométrie différentielle et système dynamiques.

Le premier chapitre de cette thèse étudie la réduction singulière des symétries du fibré cosphérique, les propriétés conservatives des systèmes de contact et leurs réduction. Le fibré cosphérique d'une variété différentiable M (dénoté par $S^*(M)$) est le quotient de son fibré cotangent sans la section nulle par rapport à l'action par multiplication de $\mathbb{R}^+$ qui couvre l'identité sur M. C'est une variété de contact qui détient en géométrie de contact la position analogue du fibré cotangent en géométrie symplectique. En utilisant une métrique Riemannienne sur M, on peut identifier $S^*(M)$ avec son fibré tangent unitaire et son champ de Reeb avec le champ géodésique de M. Si M est munie de l'action propre d'un groupe de Lie G, le relèvement de cette action à $S^*(M)$ respecte la structure de contact et admet une application moment équivariante J. Nous étudions les propriétés topologiques et géométriques de l'espace réduit à moment zéro de $S^*(M)$, i.e. $(S^*(M))_0 := J^{-1}(0)/G$. Ainsi, nous généralisons les résultats de [16] au cas singulier. Appliquant la théorie générale de réduction de contact, théorie dévéloppée par Lerman et Willett dans [34] et [58], on obtient des espaces qui perdent toute information sur la structure interne du fibré cosphérique. En plus, la projection du fibré cosphérique sur sa base descend à une surjection continue de $(S^*(M))_0$ à M/G, mais qui n'est pas un morphisme d'espaces stratifiés si on munit l'espace réduit avec sa stratification de contact et l'espace de base avec la stratification standarde de type orbitale définie par l'action du groupe de Lie. Compte tenu des théorèmes de réduction du fibré cotangent (cas régulier et singulier) et du fibré cosphérique (cas régulier), on s'attend à ce que les strates de contact aient une structure fibrée additionnelle. Pour résoudre ces problèmes, nous introduisons une nouvelle stratification de $(S^*(M))_0$, nommée la *stratification C-L* (les deux majuscules symbolisent la nature coisotrope ou Legendréenne de leurs strates). Elle est compatible avec la stratification de contact de $(S^*(M))_0$ et la stratification de type orbital de M/G. Aussi, elle est plus fine que la stratification de contact et rend la projection de $(S^*(M))_0$ sur M/G un morphism d'espaces stratifiés. Chaque strate C-L est un fibré sur une strate de type orbital de M/G et elle peut être vue comme une union de strates C-L, une d'entre elles étant ouverte et dense dans la strate de contact correspondante et difféomorphe à un fibré cosphérique. Ainsi, nous avons identifié les strates maximales munies de structure de fibrés cosférique. Les autres strates sont des sous-variétés coisotropes ou Legendre dans les composantes de contact qui les contiennent. Par conséquant nous faison une analyse géométrique et topologique complète de l'espace réduit. Nous analysons aussi le comportement de la projection sur $(S^*(M))_0$ du flot de Reeb (flot géodésique).

L'ensemble de champs de vecteurs de contact (les analogues des champs de vecteurs Hamiltonians en géométrie symplectique) forment le "groupe de Lie" de l'algèbre des transformations de contact. Dans le premier chapitre nous présentons aussi la réduction des systèmes de contact (qui, localement, sont en correspondence bijective avec les équations non-autonomes de Hamilton-Jacobi) et les systèmes Hamiltonians dépendants de temps.

Dans le deuxième chapitre nous étudions les propriétés géométriques des quotients de variétés Sasaki et Kähler. Nous construisons une procédure de réduction pour les variétés symplectiques et Kähler (munies de symétries générées par un groupe de Lie) qui utilise les préimages rayon de l'application moment. Précisémmant, au lieu de considérer comme dans la réduction de Marsden-Weinstein (ponctuelle) la préimage d'une valeur moment μ, nous utilisons la préimage de $\mathbb{R}^+\mu$, le rayon positif de μ. Nous avons trois motivations pour développer cette construction.

Une est géométrique: la construction des espaces réduits de variétés Kähler correspondant á un moment non nulle qui soient canoniques dans le sense que la structure Kähler réduite est la projection de la structure Kähler initiale. La réduction ponctuelle (Marsden-Weinstein) donnée par $M_\mu := J^{-1}(\mu)/G_\mu$ où μ est une valeur de l'application moment J et G_μ est le sous-groupe d'isotropie de μ par rapport à l'action coadjointe de G n'est pas toujours bien définie dans le cas Kähler (si $G \neq G_\mu$). Le problème est causé par le fait que la structure complexe de M ne préserve pas la distribution horizontale de la submersion Riemannienne qui projète $J^{-1}(\mu)$ sur M_μ. La solution proposée dans la litterature utilise l'espace réduit à moment zéro de la difference symplectique de M avec l'orbite coadjointe de μ munie d'une forme Kähler-Einstein unique (construite par exemple dans [7], Chapitre 8) et différente de la forme de Kostant-Kirillov-Sternberg. L'unicité de la forme sur l'orbite coadjointe garantit un espace réduit bien défini. Par contre, ne plus utiliser la forme de Kostant-Kirillov-Sternberg entraîne le fait que l'espace réduit n'est plus canonique. L'espace réduit rayon que nous construisons est canonique et peut être défini pour tout moment. Il est le quotient de $J^{-1}(\mathbb{R}^+\mu)$ par rapport à un certain sous-groupe normal de G_μ.

La deuxième raison est une application à l'étude des systèmes Hamiltonians conformes (voir [40]). Ce sont des systèmes mécaniques non-autonomes, avec friction dont les courves intégrales préservent, dans le cas des symétries, les préimages rayons de l'application moment. Nous extendons la notion de champ Hamiltonian conforme, en montrant qu'on peut ainsi inclure dans cet étude de nouveaux systèmes mécaniques. Également, nous présentons la réduction de systèmes Hamiltonians conformes.

La troisième raison consiste à trouver des conditions necéssaires et suffisantes pour que les espaces réduits (rayons) des variétés Kähler (Sasakian)-Einstein soient aussi Kähler (Sasakian)-Einstein. Nous nous occupons de cela dans le deuxième chapitre de la thèse, dans [15] et dans [14] où nous utilisons des techniques de A. Futaki. Ainsi, nous pouvons construire de nouvelles structures de Sasaki-Einstein. Comme exemples de réductions rayon symplectic (Kähler) et contact (Sasaki) nous traitons le cas des fibrés cotangent et cosphérique. Nous montrons qu'ils sont des espaces universels pour la réduction rayon. Des exemples d'actions toriques sur des sphères sont aussi décrits.

Le troisième chapitre de cette thèse traite l'étude de l'espace des orbites d'un groupoïde propre.

Dans [63], [64] A. Weinstein a partiellement résolu le problème de la linéarisation des groupoïdes propres. En [60], N. T. Zung l'a achevé en démontrant un théorème de type Bochner pour les groupoïdes propres. Nous prouvons un théorème de stratification de l'espace d'orbites d'un groupoïde propre en utilisant des idées de la théorie des foliations et le théorème de "slice" (linéarisation) de Weinstein et Zung. Nous montrons explicitement que le feuilletage orbital d'un groupoïde propre est un feuilletage Riemannien singulier dans le sense de Molino. Pour cela nous avons deux motivations. D'un côté nous voulons montrer qu'il y ait une équivalence entre groupoïdes propres et "orbispaces" (des espaces qui sont localement des quotiens par rapport à l'action d'un groupe de Lie compact) et d'un autre nous voulons étudier la réduction des actions infinitésimales (actions d'algèbres de Lie) qui ne sont pas intégrables à l'action d'un groupe de Lie. Ces actions et leur intégrabilité ont été étudiées, entre autres, par Palais ([46]), Michor, Alekseevsky.

Mots cléfs: Variétés de contact et Sasakienne, variétés symplectique et Kähler, application moment, courbure Ricci, fibré cosphérique, réduction rayon, systèmes Hamiltoniens conformes, groupoïde Lie, stratification.

Abstract

My research lies at the interface of Riemannian, contact, and symplectic geometry. It deals with the construction of Kähler and Sasaki-Einstein metrics, with the study of conformal Hamiltonian systems, the geometry of cosphere bundles, and proper Lie groupoids. The main theme of this thesis is the study of applications of Lie symmetries in differential geometry and dynamical systems.

The first chapter of the thesis studies the singular reduction of cosphere fiber bundles. The copshere bundle of a differentiable manifold M (denoted by $S^*(M)$) is the quotient of its cotangent bundle without the zero section with respect to the action by multiplications of $\mathbb{R}^+$ which covers the identity on M. It is a contact manifold which has the same privileged position in contact geometry that cotangent bundles have in symplectic geometry. Using a Riemannian metric on M, we can identify $S^*(M)$ with its unitary tangent bundle and its Reeb vector field with the geodesic field on M. If M is endowed with the proper action of a Lie group G, the lift of this action on $S^*(M)$ respects the contact structure and admits an equivariant momentum map J. We study the topological and geometrical properties of the reduced space of $S^*(M)$ at zero momentum, i.e. $(S^*(M))_0 := J^{-1}(0)/G$. Thus, we generalize the results of [16] to the singular case. Applying the general theory of contact reduction developed by Lerman and Willett in [34] and [58], one obtains contact stratified spaces that lose all information of the internal structure of the cosphere bundle. Even more, the cosphere bundle projection to the base manifold descends to a continuous surjective map from $(S^*(M))_0$ to M/G, but it fails to be a morphism of stratified spaces if we endow $(S^*(M))_0$ with its contact stratification and M/G with the customary orbit type stratification defined by the Lie group action. Based on the cotangent bundle reduction theorems, both in the regular and singular case, as well as regular cosphere bundle reduction, one expects additional bundle-like structure for the contact strata. To solve these problems, we introduce a new stratification of the contact quotient at zero, called the *C-L stratification* (standing for the coisotropic or Legendrian nature of its pieces). It is compatible with the contact stratification of $(S^*(M))_0$ and the orbit type stratification of M/G. It is also finer than the contact stratification. Also, the natural projection of the C-L stratified quotient space $(S^*(M))_0$ to its base space, stratified by orbit types, is a morphism of stratified spaces. Each C-L stratum is a bundle over an orbit type stratum of the base and it can be seen as a union of C-L pieces, one of them being open and dense in its corresponding contact stratum and contactomorphic to a cosphere bundle. Hence we have identified the maximal strata endowed with cosphere bundle structure. The other strata are coisotropic or Legendrian submanifolds in the contact components that contain them. Consequently, we can perform a complete geometric and topological analysis of the reduced space.

We also study the behaviour of the projection on $(S^*(M))_0$ of the Reeb flow (geodesic flow).

The set of contact Hamiltonian vector fields (the analogous of Hamiltonian vector fields in symplectic geometry) form the "Lie" group of the algebra of contact transformations. In the first chapter we also present the reduction of contact systems (which locally are in bijective correspondence with the non-autonomus Hamilton-Jacobi equations) and time dependent Hamiltonian systems.

In the second chapter of this thesis we study quotients of Kähler and Sasaki-Einstein manifolds. We construct a reduction procedure for symplectic and Kähler manifolds (endowed with symmetries generated by a Lie group) which uses the ray pre-images of the associated momentum map. More precisely, instead of considering as in the Marsden- Weinstein reduction (point reduction) the pre-image of a momentum value μ, we use the pre-image of $\mathbb{R}^+\mu$, its positive ray. We have three reasons to develop this construction.

One is geometric: the construction of canonical reduced spaces of Kähler manifolds corresponding to a non zero momentum. By canonical we mean that the reduced Kähler structure is the projection of the initial Kähler structure. The point reduction (Marsden-Weinstein) given by $M_\mu := \frac{J^{-1}(\mu)}{G_\mu}$, where μ is a value of the momentum map J and G_μ the isotropy subgroup of μ with respect to the coadjoint action of G is not always well defined in the Kähler case (if $G \neq G_\mu$). The problem is caused by the fact that the complex structure of M does not leave invariant the horizontal distribution of the Riemannian submersion which projects $J^{-1}(\mu)$ on M_μ. The solution proposed in the literature uses the reduced space at zero momentum of the symplectic difference of M with the coadjoint orbit of μ endowed with a unique Kähler-Einstein form (constructed, for insatnce, in [7], Chapter 8) and different from the Kostant-Kirillov-Sternberg form. The uniqueness of the form on the coadjoint orbit ensures that the reduced space is well defined. On the other hand, not using the Kostant-Kirillov-Sternberg form implies the fact that the reduced space is no longer canonical. The ray reduced space that we construct is canonical and can be defined for any momentum. It is the quotient of $J^{-1}(\mathbb{R}^+\mu)$ with respect to a certain normal subgroup of G_μ.

The second reason is an application to the study of conformal Hamiltonian systems (see [40]). They are mechanical, non-autonomous systems with friction whose integral curves preserve, in the case of symmetries, the ray pre-images of the momentum map, but not the point (momentum) preimages of the Marsden-Weinstein quotient. We extend the notion of conformal Hamiltonian vector field by showing that one can thus include in this study new mechanical systems. Also, we present the reduction of conformal Hamiltonian systems.

The third reason consists of finding the necessary and sufficient conditions for the ray reduced spaces of Kähler (Sasakian)-Einstein manifolds to be also Kähler (Sasakian)-Einstein. We deal with this problem in the second chapter of the thesis, in [15], and in [14] where we use techniques of A. Futaki. Thus, we can construct new Sasaki-Einstein structures. As examples of symplectic (Kähler) and contact (Sasakian) ray quotients we treat the case of cotangent and cosphere bundles and show that they are universal spaces for ray reductions. Examples of toric actions on spheres are also described.

The third chapter of my thesis studies the space of orbits of a proper Lie groupoid. In [63], [64] A. Weinstein has partially solved the problem of linearization of proper groupoids. In [60], N. T. Zung has completed it by showing a theorem of Bochner type for proper groupoids. Using ideas from foliation theory and the slice (linearization) theorem of Weinstein and Zung, we prove a stratification theorem for the orbit space of a proper groupoid. We show explicitely that the orbital foliation of a proper Lie groupoid is a Riemannian singular foliation in the sense of Molino. For all these we have two motivations. On one hand we want to prove that there is an equivalence between proper groupoids and orbispaces (the spaces which are locally quotients with respect to an action of a compact Lie group). On the other hand we would like to study the reduction of infinitesimal actions (actions of Lie algebras) which are not integrable to Lie group actions. These actions and their integrability have been studied, among others, by Palais ([46]), Michor, Alekseevsky.

Key words: contact and Sasakian manifolds, symplectic and Kähler manifolds, momentum map, Lie symmetries, Ricci curvature, cosphere bundle, ray reduction, conformal Hamiltonian systems, stratifications.

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Contents

Chapter 1

Singular Cosphere Bundle Reduction

1.1 Introduction

The main goal of this chapter is to carry out the singular reduction of cosphere bundles at the zero value of the contact momentum map. This presents interest because cosphere bundles carry considerably more structure than a general contact manifold and have the same privileged position in contact geometry that cotangent bundles have in symplectic geometry. They have been intensively used in topological problems dealing with the classification of immersions and embeddings. Associating to each immersion (embedding) of a smooth manifold a Legendrian submanifold in its cosphere bundle one can use Legendrian contact homology to construct topological invariants. A beautiful introduction to the applications of these contact constructions is [18].

Contact reduction appears for the first time in the work of Guillemin and Sternberg [24] in the context of reducing symplectic cones. Albert [3] and, several years later, Geiges [21] and Loose [35] independently defined and studied contact reduction at the zero value of the contact momentum map for free proper contact actions of Lie groups. Reduction at a general value of the momentum map was studied by both Albert [3] and Willett [58] who proposed two different versions of dealing with it. It turns out that Willett's method is the one that naturally parallels the symplectic reduction theory, even in the singular case as shown by Lerman and Willett [34]. They prove that the resulting contact quotient depends only on the contact structure, that it is independent of any contact form defining the contact foliation, and that it is a stratified space, more precisely, a cone space. For an extension of Willett's method where the contact space can be defined at any value of the momentum map see [59].

The case of cosphere bundle reduction for proper free lifted Lie group actions was studied in [16] with a view of comparing the theory to that for cotangent bundle reduction. It turns out that in regular contact reduction of cosphere bundles there are no analogues of magnetic terms. In parallel, in [48] the authors have developed the theory of singular cotangent bundle reduction at the zero value of the momentum map and have found a finer stratification than that given by the general theory. This is due to the additional structure of the cotangent bundle and the fact that the Lie group action is a cotangent lifted action. A similar phenomenon occurs in contact reduction of cosphere bundles.

Applying the general theory of singular contact reduction due to Lerman and Willett [34] yields contact stratified spaces that, however, lose all information of the internal structure of the cosphere bundle. Based on the cotangent bundle reduction theorems, both in the regular and singular case, as well as regular cosphere bundle reduction, one expects additional bundle-like structure for the contact strata. The cosphere bundle projection to the base manifold descends to a continuous surjective map from the reduced space at zero to the orbit quotient of the configuration space, but it fails to be a morphism of stratified spaces if we endow the reduced space with its contact stratification and the base space with the customary orbit type stratification defined by the Lie group action. The present paper introduces a new stratification of the contact quotient at zero, called in what follows the *C-L stratification* (standing for the coisotropic or Legendrian nature of its pieces) which solves the above mentioned two problems. Its main features are the following. First, it is compatible with the contact stratification of the quotient and the orbit type stratification of the configuration orbit space. It is also finer than the contact stratification. Unlike the cotangent bundle case, the isotropy lattice of the group action on the base manifold Q no longer suffices for the description of this new stratification. In fact, this lattice I_Q indexes a new decomposition of each contact stratum of the reduced space, but the isotropy lattice of the zero level set of the momentum map is given by I_Q without those elements corresponding to orbit type submanifolds of dimension equal to that of their orbits. Second, the natural projection of the C-L stratified quotient space to its base space, stratified by orbit types, is a morphism of stratified spaces. Third, each C-L stratum is a bundle over an orbit type stratum of the base and each contact stratum can be seen as a union of C-L pieces, one of them being open and dense in its corresponding contact stratum and contactomorphic to a cosphere bundle. The other strata are coisotropic or Legendrian submanifolds in the contact components that contain them.

This chapter is structured as follows. Section 1.2 presents the definitions, conventions, and results on stratified spaces and contact reduction (regular and singular) that are used throughout this chapter. It quickly reviews the relevant results on regular contact cosphere reduction. Section 1.3 describes, for a general contact manifold, the relation between contact vector fields and the non autonomous Hamilton-Jacobi equation. In the case of symmetries we prove a Noether type theorem which gives invariant submanifolds of contact vector fields. Their reduction is studied and a simple characterization of relative equilibria is given. Section 1.4 deals with the singular cosphere bundle reduction at zero momentum. It begins the work on the stratification of the quotient by studying the case of one single orbit type (Theorem 1.4.1). The contact stratification and contact geometry of the reduced space are studied in Subsection 1.4.2, having as main results Theorems 1.4.2 and 1.4.3. The new *C-L stratification* is also introduced here and its properties are investigated. Theorem 1.4.4 presents a complete description of its frontier conditions. It is also explained what is the tool needed for an analysis of Withney or local triviality conditions for this new stratification. Section 1.5 studies the singular cosphere bundle reduction for almost semifree actions, that is, actions that are in bijective correspondence with free lifted actions on the cosphere bundle. The stratification is computed explicitly and the particular case of the circle acting on the cosphere bundle of the plane is carried out in detail. Section 1.6 studies the example of the diagonal action of the two-torus on two copies of the plane, lifted to the cosphere bundle. This example is rich enough to illustrate the relationships between the various stratifications and the strata are computed explicitly.

1.2 Preliminaries

In this section we will survey the main results of several topics that will be needed in the subsequent development of the thesis. We will assume that all topological spaces are paracompact. In addition, manifolds will be real, smooth and finite-dimensional. By group we will mean a finite-dimensional Lie group. Every action of a group G on a manifold M is supposed to be smooth and the usual notation $g \cdot m$ for $g \in G$ and $m \in M$ will be employed. The natural pairing between a vector space and its dual will be denoted by $\langle \cdot, \cdot \rangle$. By submanifold, we will always mean an embedded submanifold.

1.2.1 Stratified spaces and proper group actions

The natural framework for singular reduction is the category of stratified spaces. We briefly recall here the basic concepts (see [49]). Let X be a topological space and $\mathcal{Z}_X = \{S_i : i \in I\}$ a locally finite partition of X into locally closed disjoint subspaces $S_i \subset X$, where I is some index set. We say that $(X, \mathcal{Z}_X)$ is a *decomposed space* if every S_i is a manifold whose topology coincides with the induced one from X and if the *frontier condition* holds: $S_i \cap \overline{S_j} \neq \varnothing$ implies $S_i \subset \overline{S_j}$, whence $S_i \subset \partial S_j$, where $\partial S_j := \overline{S_j} \backslash S_j$. In this case, the elements of $\mathcal{Z}_X$ are called *pieces* of the decomposition.

In a topological space X, two subsets A and B are said to be *equivalent at x* if there exists an open neighborhood U of x such that $A \cap U = B \cap U$. These equivalence classes are called *set germs at x*. Let $\mathcal{S}$ be the map that associates to each point $x \in X$ the set germ $\mathcal{S}_x = [O]_x$ of a locally closed subset O of X. We say that $(X, \mathcal{S})$ is a *stratified space* if, for every point $x \in X$ there exists a neighborhood U of x endowed with a decomposition $\mathcal{Z}_U$ such that for every $y \in U$, $\mathcal{S}_y = [Z(y)]_y$, where $Z(y) \in \mathcal{Z}_U$ denotes the piece containing y. In this case we say that the decomposition $\mathcal{Z}_U$ *locally induces* $\mathcal{S}$.

Given two stratified spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ and a continuous map $f : X \to Y$, we say that f is a *morphism of stratified spaces* (or shorter, a *morphism*) if for every $x \in X$ there exist neighborhoods V of $f(x)$ and $U \subset f^{-1}(V)$ of x such that

(i) there exist decompositions $\mathcal{Z}_U$ and $\mathcal{Z}_V$ locally inducing the stratifications $\mathcal{S}$ and $\mathcal{T}$ respectively, with the property that for every $y \in U$ contained in a piece $S \in \mathcal{Z}_U$ there is an open neighborhood $y \in W \subset U$ such that $f\big|_W(S \cap W)$ is contained in the unique piece $R \in \mathcal{Z}_V$ that contains $f(y)$, and

(ii) $f\big|_{S \cap W} : S \cap W \to R$ is smooth.

In addition, we will say that f is a *stratified immersion* (resp. *submersion, diffeomorphism*, etc...) if so are all the maps $f\big|_{S \cap W}$ for every point $x \in X$. Given two different stratifications $\mathcal{S}$ and $\mathcal{S}'$ on the same topological space X, we say that $\mathcal{S}$ is *finer* than $\mathcal{S}'$ if the identity map 1_X, viewed as a map between stratified spaces $(X, \mathcal{S}) \to (X, \mathcal{S}')$, is a morphism.

Smooth manifolds are trivially stratified spaces and smooth maps between manifolds are their morphisms. Note that a decomposed space $(X, \mathcal{Z}_X)$ induces naturally a stratification $(X, \mathcal{S})$ by just taking S_x to be the set germ of the piece containing x, for every $x \in X$. In this case, we call the pieces $S_i \in \mathcal{Z}_X$ the *strata* of $(X, \mathcal{S})$ and say that they satisfy the frontier conditions defined by the underlying decomposition. In this paper the stratifications that will appear will be of this form and thus, for the sake of simplicity, when this is the case we will work most of the time with the decompositions inducing these stratifications.

Let $\phi : G \times M \to M$ be a smooth action of the Lie group G on the manifold M. Since M is paracompact it admits a Riemannian metric so if it is connected, M is second countable. The action is called *proper* if $\phi \times \mathrm{id}_M$ is a proper map. In this paper we only work with proper actions. For instance, every action of a compact group is automatically proper. The main properties of a proper action of G on M are:

(i) For each $m \in M$, its stabilizer (or isotropy group) G_m is compact.

(ii) The manifold structure of the orbit $G \cdot m$ is the one that makes the natural bijection $G/G_m \to G \cdot m$ a diffeomorphism. The inclusion $G \cdot m \hookrightarrow M$ is an injective immersion. In addition, the orbit is a closed subset of M. If M is connected, then the orbit is an embedded submanifold of M.

(iii) The quotient space equipped with the quotient topology is paracompact and the orbit map $\pi : M \to M/G$ is open and closed.

(iv) M admits a G-invariant Riemannian metric.

(v) If all the stabilizer groups are conjugate to a given subgroup $H \subset G$, then M/G is a smooth manifold, the orbit map $\pi : M \to M/G$ is a smooth locally trivial fiber bundle whose fibers are diffeomorphic to G/H, and the structure group of this locally trivial fiber bundle is $N(H)/H$, where $N(H)$ is the normalizer of H in G.

We now quote Palais' Tube Theorem [46] in a form adapted to our needs, which is of great importance in the local study of proper actions. Let $m \in M$. Choose an invariant Riemannian metric on M and use it to decompose $T_mM = \mathfrak{g} \cdot m \oplus S_m$, where $\mathfrak{g} \cdot m = \{\xi_M(m) : \xi \in \mathfrak{g}\}$. This splitting is G_m-invariant for the linear action of G_m on T_mM. The twisted action of G_m on $G \times S_m$ is defined by

$$h \cdot (g, s) = (gh^{-1}, h \cdot s) \tag{1.2.1}$$

for $h \in G_m, g \in G$ and $s \in S_m$. Since G_m acts freely on the right on G, the twisted action is free. In addition, G_m is compact by property (i) of proper group actions, so the quotient space, denoted by $G \times_{G_m} S_m$, is a manifold. The Tube Theorem implies the existence of a G_m-invariant open ball U around the origin in S_m such that the map $\psi : G \times_{G_m} S_m \to M$ defined by

$$\psi([g, s]) = g \cdot \exp_m(s) \tag{1.2.2}$$

maps $G \times_{G_m} U$ diffeomorphically and equivariantly onto a G-invariant neighborhood U' of $G \cdot m$ in M. Here, $\exp_m$ is the exponential map at m associated to the chosen Riemannian metric. The map ψ is called a *tube* for the action and S_m is called a *linear slice*, or simply a *slice* of the action at m.

Let I_M be the isotropy lattice of M, i.e. the set of conjugacy classes of subgroups of G which appear as stabilizers for the action of G on M. Such classes, called *orbit types*, are denoted by (H) . For each element $(H) \in I_M$ the (H)-orbit type manifold is defined by

$$M_{(H)} = \{m \in M \mid (G_m) = (H)\}. \tag{1.2.3}$$

In the same way, for any subset A of M one defines the orbit type sets of A by $A_{(H)} = A \cap M_{(H)}$ and the isotropy lattice of A by restriction. For a proper G-action on a manifold M such that M/G is connected, there is always a subgroup $H_0 \subset G$ such that $M_{(H_0)}$ is open and dense in M and H_0 is conjugate to a proper subgroup of any other stabilizer. This orbit type (H_0) is called the *principal orbit type* of I_M.

Obviously, the collection of orbit type manifolds forms a partition of M. For simplicity, we will make from now on the following important assumption: for every $(H) \in I_M$, all the connected components of $M_{(H)}$ have the same dimension and M is second countable. Hence we have:

(i) For every $(H) \in I_M$, $M_{(H)}$ is a G-invariant submanifold of M, and

(ii) M and M/G are stratified spaces with strata $M_{(H)}$ and $M^{(H)} := M_{(H)}/G$ respectively. Their frontier conditions are:
$$M^{(H)} \subset \partial M^{(L)} \Longleftrightarrow (L) \prec (H),$$
and correspondingly for M, where $(L) \prec (H)$ means that L is conjugate to a proper subgroup of H. Since $\prec$ defines a partial ordering in I_M we say that the frontier conditions of the stratification of M/G are induced by the isotropy lattice I_M.

Remark 1.2.1. *If one allows the connected components of the orbit type manifolds to have different dimensions, then one needs to work in the larger category of Σ-manifolds and Σ-decompositions. A Σ-manifold is a countable topological sum of connected smooth manifolds having possibly different dimensions (see [49] for more details). However, our results on the stratified nature of the studied quotient spaces remain valid.*

1.2.2 Contact Manifolds.

Recall that a *contact structure* on a smooth $(2n + 1)$-dimensional manifold C is a codimension one smooth distribution $\mathcal{H} \subset TC$ maximally non-integrable in the sense that it is locally given by the kernel of a one-form η with $\eta \wedge (d\eta)^n \neq 0$. Intuitively, this means that the contact distribution is as non integrable as possible, since the integrability condition reduces to $\eta \wedge d\eta = 0$. Such an η is called a (local) *contact form.* Any two proportional contact forms define the same contact structure. A contact structure which is the kernel of a global contact form is called *exact.* In the case of exact contact manifolds, $d\eta$ has rank n implying the existence of the Reeb vector field $\mathcal{R}$ uniquely defined by
$$i_{\mathcal{R}} d\,\eta = 0 \quad \text{and} \quad \eta(\mathcal{R}) = 1.$$

In the following we will consider only exact orientable contact manifolds. The theorem of Darboux (see [39]) states that, locally, every contact structure is diffeomorphic to $(\mathbb{R}^{2n+1}, dz - \Sigma_j y_j dx_j)$.

When studying the geometry of the singular reduced spaces of cosphere bundles one needs the notions of coisotropic and isotropic submanifolds in the contact context. Any integral submanifold N of $\mathcal{H}$ has the property that its tangent space at every point is an isotropic subspace of the symplectic vector space $(\ker \eta_x, d\eta_x)$ and that's why, sometimes, they are also called *isotropic* submanifolds. In particular, $\dim N \leq n$; if $\dim N = n$, then N is called a *Legendrian* submanifold. A submanifold N of the contact manifold $(C, \eta, \mathcal{R})$ is *coisotropic* if for any $x \in N$ the subspace $T_xN \cap \ker \eta_x$ is coisotropic in the symplectic vector space $(\ker \eta_x, d\eta_x)$.

A group G is said to act by *contactomorphisms* on a contact manifold if it preserves the contact structure $\mathcal{H}$. For an exact contact manifold (C, η), this means that $g^*\eta = f_g\eta$ for a smooth, real-valued, nowhere zero function f_g. G acts by *strong contactomorphisms* on C, if $g^*\eta = \eta$, i.e. G preserves the contact form, not only the contact structure. A G–action by strong contactomorphisms on (C, η) admits an equivariant momentum map $J : C \to \mathfrak{g}^*$ given by evaluating the contact form on the infinitesimal generators of the action: $\langle J(x), \xi\rangle := \eta(\xi_C)(x)$. Note the main difference with respect to the symplectic case: any action by strong contactomorphisms automatically admits an equivariant momentum map. Note also that orbits which lie in the zero level set of the contact momentum map are examples of isotropic submanifolds. The momentum map J is constant on the flow of the Reeb vector field. In addition,

$$\langle T_xJ(v), \xi\rangle = d\eta(x)(v, \xi_C(n))$$

for any $x \in C$, $v \in T_xC$, and $\xi \in \mathfrak{g}$. This immediately implies

$$[\mathrm{im}(T_xJ)]^\circ = \{\xi \in \mathfrak{g} \mid d\eta(x)(\xi_C(x), \cdot) = 0\},$$

which is the contact analogue of the bifurcation lemma from the usual theory of momentum maps on Poisson manifolds; the term on the left is the annihilator of the subspace in parentheses. For this (contact) momentum map, $0 \in \mathfrak{g}^*$ is a regular value if and only if the infinitesimal isometries induced by the action do not vanish on the zero level set of J. Moreover, if this is the case, the pull back of the contact form to $J^{-1}(0)$ is basic. For more details on contact manifolds and their associated momentum maps see [8], [22], and [58].

1.2.3 Regular contact reduction

Suppose G acts freely, properly, and by strong contactomorphisms on the exact contact manifold (C, η). Denote by $i_0 : J^{-1}(0) \hookrightarrow C$ and $\pi_0 : J^{-1}(0) \to C_0 := \frac{J^{-1}(0)}{G}$ the canonical inclusion and projection respectively. C_0 is called the *zero contact reduced space*. The reduction at zero for contact manifolds was done simultaneously by Albert([3]), Geiges([21]), and Loose([35]).

Theorem 1.2.1. *Let G be a Lie group acting properly, freely and by strong contactomorphisms on the contact manifold (C, η). Then there is a unique exact contact form on C_0, η_0 defined by $\pi_0^*\eta_0 = i_0^*\eta$.*

Regarding contact reduction at $\mu \neq 0$, up to now there are two versions available: one due Albert [3] and a more recent one due to Willett [58].

Albert's method [3]. Let (C, η) be an exact contact manifold with Reeb vector field $\mathcal{R}$ and let Φ be a "good" action of a Lie group by strong contactomorphisms. For $\mu \in \mathfrak{g}^*$, denote by G_μ the isotropy group at μ of the coadjoint action and by $\mathfrak{g}_\mu$ its Lie algebra. If $\mu \neq 0$ is a regular value of J the restriction of the contact form to $J^{-1}(\mu)$ is not basic. This problem is overcome by Albert by changing the infinitesimal action of $\mathfrak{g}_\mu$ on $J^{-1}(\mu)$ as follows: $\xi \mapsto \xi_C - \langle \mu, \xi \rangle \mathcal{R}$, where $\mathcal{R}$ is the Reeb vector field. In general, this infinitesimal action cannot be integrated to an action of G_μ. However, if R is complete, this $\mathfrak{g}_\mu$–action is induced by an action of the universal covering group $\widehat{G}_\mu$ (if G_μ is connected) given by

$$(e^{t\xi}, n) \mapsto \phi_{e^{t\xi}}(\rho^{-1}_{t\langle \mu, \xi \rangle}(n)),$$

where ρ_t is the flow of the Reeb vector field. Albert defines the reduced space as $J^{-1}(\mu)/\widehat{G}_\mu$ *via* this new action and shows that it is naturally a contact manifold. Unfortunately, as it was explained in [58], this contact quotient depends on the chosen contact form and not only on the contact structure. This disadvantage was corrected by Willett in [58].

Willett's method [58]. The idea is to expand μ and to shrink G_μ. As above, G is a Lie group that acts smoothly on an exact contact manifold (C, η) preserving the contact form η. Let $\mu \in \mathfrak{g}^*$. Willett calls the *kernel group of* μ, the connected Lie subgroup K_μ of G_μ with Lie algebra $\mathfrak{k}_\mu = \ker\,(\mu|_{\mathfrak{g}_\mu})$. It is easy to see that $\mathfrak{k}_\mu$ is an ideal in $\mathfrak{g}_\mu$ and therefore K_μ is a connected normal subgroup of G_μ. Contact reduction (or the contact quotient) of C by G at μ is defined by Willett as

$$C_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu.$$

Assume that K_μ acts freely and properly on $J^{-1}(\mathbb{R}^+\mu)$. Then J is transversal to $\mathbb{R}^+\mu$ and the pull back of η to $J^{-1}(\mathbb{R}^+\mu)$ is basic relative to the K_μ–action on $J^{-1}(\mathbb{R}^+\mu)$ and thus induces a one form $\eta_{\mathbb{R}^+\mu}$ on the quotient $C_{\mathbb{R}^+\mu}$. If, in addition, $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$ then the form η_μ is also a contact form. It is characterized, as usual, by the identity $\pi_\mu^* \eta_{\mathbb{R}^+\mu} = i_\mu^* \eta$, where $\pi_\mu : J^{-1}(\mathbb{R}^+\mu) \to C_{\mathbb{R}^+\mu}$ is the canonical projection and $i_\mu : J^{-1}(\mathbb{R}^+\mu) \hookrightarrow C$ is the canonical inclusion.

It is to be noted that for $\mu = 0$, Albert's and Willett's quotients coincide.

1.2.4 Singular contact reduction at zero momentum

Reduction theory for co-oriented contact manifolds in the singular context was introduced by Willett and Lerman in [**?**] and [58]. We now review briefly this construction at zero momentum, since it will be used in our refinement to the cosphere bundle case. Let G be a Lie group that acts by strong contactomorphisms on an exact contact manifold (C, η). Denote by $J : C \to \mathfrak{g}^*$ the associated momentum map. By the definition of J, its zero level set is a G-space.

Theorem 1.2.2. *Let (C, η) be an exact contact manifold and G a Lie group acting smoothly on C by strong contactomorphisms with momentum map $J : C \to \mathfrak{g}$. Then for every stabilizer subgroup H of G the set*

$$C_0^{(H)} := (J^{-1}(0))_{(H)}/G = (C_{(H)} \cap J^{-1}(0))/G$$

is a smooth manifold and the partition of the contact quotient

$$C_0 := \left(J^{-1}(0)\right)/G$$

into these manifolds is a stratification with frontier conditions induced by the partial order of $I_{J^{-1}(0)}$*. Moreover, there is a reduced exact contact structure on* $C_0^{(H)}$ *generated by the one-form* $\eta_0^{(H)}$ *characterized by*

$$(\pi_G^{(H)})^*\eta_0^{(H)} = (\tilde{i}_{(H)})^*\eta,$$

where $\pi_G^{(H)} : (J^{-1}(0))_{(H)} \to C_0^{(H)}$ *is the projection on the orbit space and* $\tilde{i}_{(H)} : (J^{-1}(0))_{(H)} \hookrightarrow C$ *is the inclusion.*

In what follows this stratification will be referred to as the *contact stratification* of C_0.

1.2.5 Regular cosphere bundle reduction

Cosphere bundles are the odd dimensional analogs of cotangent bundles in contact geometry. In the following, we will briefly recall their construction and their equivariant regular contact reduction, referring to [16], [18], [5], and [50] for more details.

Let Q be a n-dimensional manifold and θ the Liouville one-form on T^*Q, defined by $\theta(X_{p_x}) = \langle p_x, T_{p_x}\tau X_{p_x}\rangle$, where $p_x \in T_x^*Q$, $X \in T_{p_x}(T^*Q)$, and $\tau : T^*Q \to Q$ is the canonical projection. Let $\Phi : G \times Q \to Q$ be an action of G on Q. Denote by

$$\Phi_* : G \times T^*Q \to T^*Q$$

its natural (left) lift to the cotangent bundle. Consider the action of the multiplicative group $\mathbb{R}^+$ by dilations on the fibers of $T^*Q \setminus \{0_{T^*Q}\}$.

Definition 1.2.1. *The* cosphere bundle S^*Q *of* Q *is the quotient manifold* $(T^*Q \setminus \{0_{T^*Q}\})/\mathbb{R}^+$*.*

Let $\pi^+ : T^*Q \setminus \{0_{T^*Q}\} \to S^*Q$ and $\kappa : [\alpha_q] \in S^*Q \mapsto q \in Q$ be the canonical projections. Denote by $[\alpha_q]$ the elements of the cosphere bundle. Of course, $(\pi^+, \mathbb{R}^+, T^*Q \setminus \{0_{T^*Q}\}, S^*Q)$ is a $\mathbb{R}^+$-principal bundle. Also, we will use the π^+ notation for any $\mathbb{R}^+$ projection. The exact contact structure of S^*Q is given by the kernel of any one form θ_σ satisfying $\theta_\sigma = \sigma^*\theta$ for $\sigma : S^*Q \to T^*Q \setminus \{0_{T^*Q}\}$ a global section. Such σ always exists and, even more, the set of global sections of this principal bundle is in bijective correspondence with the set of C^∞ functions $f : T^*Q \setminus \{0_{T^*Q}\} \to \mathbb{R}^+$ satisfying

$$f_\sigma(r\alpha_q) = \frac{1}{r} f_\sigma(\alpha_q), \quad r \in \mathbb{R}^+,\ \alpha_q \in T^*Q \setminus \{0_{T^*Q}\}.$$

(See [16] for details).

Remark 1.2.2. *1. Let* $\mathcal{C}(S^*Q) = S^*Q \times \mathbb{R}^+$ *be the symplectic cone over* S^*Q*, endowed with the symplectic form* $d(t\theta_\sigma)$*. Then one can easily see that* $T_\sigma : \mathcal{C}(S^*Q) \to T^*Q \setminus \{0_{T^*Q}\}$ *given by* $T_\sigma([\alpha_q], t) = tf_\sigma(\alpha_q)\alpha_q$ *is a well defined symplectic diffeomorphism, that is, a symplectomorphism.*
2. If Q *is zero-dimensional, we set, by convention,* $S^*Q = \varnothing$*.*

The action Φ lifts to the cosphere bundle yielding a proper action

$$\widehat{\Phi}_* : G \times S^*Q \to S^*Q, \quad \widehat{\Phi}_*(g, [\alpha_q]) = [\Phi_*(g, \alpha_q)]$$

by contactomorphisms with all scale factors positive. In [34] it has been proved that for any proper action which preserves an exact contact structure, there exists a G-invariant contact form. As every contact form on the cosphere bundle is obtained *via* a global section as above, we shall chose once and for all a section σ for which $(\widehat{\Phi}_{*g})^*\theta_\sigma = \theta_\sigma$. Relative to this contact form the induced action on the cosphere bundle is by strong contactomorphisms. The associated momentum map, which depends on the section σ, will be denoted by J for simplicity, since in what follows no other contact form different from θ_σ will be used. As above, the exact contact structure of $S^*(Q/G)$ can be described as the kernel of a global contact form of type Θ_Σ, where

$$\Sigma : S^*(Q/G) \to T^*(Q/G) \setminus \{0_{T^*(Q/G)}\}$$

is a global section, and Θ is the Liouville one-form of $T^*(Q/G)$.

Regular reduction of cosphere bundles was done in [16]. Its main result at zero momentum is

Theorem 1.2.3. *Let Q be a differentiable manifold of real dimension n, G a finite dimensional Lie subgroup of* $\mathrm{Diff}(Q)$ *and $\Phi : G \times Q \to Q$ a smooth action of G on Q. Assume that K_μ acts freely and properly on $J^{-1}(\mathbb{R}^+\mu)$ and that* $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$*. Then the contact reduction*

$$(S^*Q)_{\mathbb{R}^+\mu} = J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is embedded by a map preserving the contact structures onto a subbundle of $S^*(Q/K_\mu)$. For zero momentum $\mu = 0$, the above embedding is actually a contactomorphic diffeomorphism.

For examples, see [16]. In the last 3 sections of this chapter, we will generalize Theorem 1.2.3 for zero momentum to non-free actions, within the framework of stratified spaces, relating our results to the contact stratification defined in Theorem 1.2.2.

1.3 Contact geometry and the Hamilton-Jacobi equation

In this section we will present a nice connection between contact geometry and the non-autonomus Hamilton-Jacobi equation. We will also present the contact version of Noether conservation law.

Lemma 1.3.1. *Let $(C, \mathcal{H})$ be a contact manifold with contact 1-form η and Reeb vector field $\mathcal{R}$. Then:*

1° *for any smooth function $H : C \to \mathbb{R}$, there is a unique vector field X_H such that*

$$i_{X_H}\eta = -H, \quad i_{X_H}d\eta = dH - \mathcal{R}(H)\eta. \tag{1.3.1}$$

2° *a vector field X verifies* (1.3.1) *if and only if $L_X\eta = g\eta$ for g some smooth function on C.*

Proof. To prove 1°, let H be a smooth function on N and $X_H := X_0 - \mathcal{R}(H)$, where X_0 is the unique vector field belonging to the contact distribution and defined by the relation $i_{X_0} d\eta|_{\mathcal{H}} = dH|_{\mathcal{H}}$. 2° If X verifies (1.3.1) it is easy to see that $L_X\eta = g\eta$ for $g := -\mathcal{R}(H)$. On the other hand, if one has $L_X\eta = g\eta$, let $H := -i_X\eta$. Applying Cartan's magic formula, one obtains $i_X d\eta = g\eta + dH$ and evaluating this 1-form on R, we find $g = \mathcal{R}(H)$ thus finishing the proof. □

Remark 1.3.1. *Vector fields verifying (1.3.1) are called* ***contact vector fields*** *and the above lemma shows that there is a one-to-one correspondance between the set of contact vector fields and the set of smooth functions on a contact manifold. More over, the Lie algebra of contactomorphisms is given by the set of contact vector fields with Lie bracket defined by* $\{f, g\} = -\eta([X_f, X_g]) = df(X_g) + dg(\mathcal{R})f$.

Remark 1.3.2. *Notice that using the Darboux theorem, for each smooth function* H*, the vector field* X_H *is defined in canonical coordinates by the following contact differential equation*

$$\dot{x_j} = \frac{\partial H}{\partial y_j}, \quad \dot{y_j} = -\frac{\partial H}{\partial x_j} - y_j\frac{\partial H}{\partial z}, \quad \dot{z} = \sum_j y_j\dot{x_j} - H. \tag{1.3.2}$$

and, locally, the Lie bracket has the following form:

$$\{f, g\} = (\partial_{x_i} f\partial_{y_i} g + \partial_{y_i} f\partial_{x_i} g) - (x_i\partial_{x_i} f\partial_t g - x_i\partial_{x_i}\partial_z f) - (f\partial_z g - g\partial_z f).$$

Proposition 1.3.1. *Let* $H = H(x_1, ..., x_n, y_1, ..., y_n, z)$ *be a smooth function on the contact manifold* $\mathbb{R}^{2n+1}$ *and consider the Hamilton- Jacobi equation:*

$$\partial_t S + H(x, \partial_x S, S) = 0 \tag{1.3.3}$$

for $S = S(t, x)$ *a function on* $\mathbb{R}^{n+1}$. *There is a one to one correspondence between solutions of the above equation and equation* (1.3.2). *Namely, if* S *is a solution of* (1.3.3) *and* $x(t)$ *is a solution of* $\dot{x} = \partial_y H(x, \partial_x S, S)$, *then*

$$x(t), \quad y(t) = \partial_x S(t, x(t)), \quad z(t) = S(t, x(t))$$

represents a solution of (1.3.2). *Conversely, given an initial function* $S(0, x) = S_0(x)$ *one can construct a solution of the Hamilton-Jacobi equation using solutions of* (1.3.2) *with initial conditions of the form* $(x(0) = x_0, y(0) = \partial_x S_0(x_0), z(0) = S_0(x_0))$.

Proof. First, let's consider S and $x(t)$ solutions of the Hamilton-Jacobi equation (1.3.3) and $\dot{x} = \partial_y H(x, \partial_x S, S)$ respectively. Denote by $\gamma(t) := (x(t), y(t), z(t))$. Deriving (1.3.3) with respect to x_j, we obtain:

$$\begin{aligned} &\partial_{x_j}\partial_t S(t, x(t)) + \partial_{x_j} H(\gamma(t)) + \sum_i \partial_{y_i} H(\gamma(t))\partial_{x_j}\partial_{x_i} S(t, x(t)) \\ &\qquad + \partial_z H(\gamma(t))\partial_{x_j} S(t, x(t)) = 0 \\ \iff\ &\partial_{x_j}\partial_t S + \partial_{x_j} H + \sum_i \dot{x}_i\partial_{x_j}\partial_{x_i} S + y_j\partial_z H = 0 \\ \iff\ &\dot{y_j} = -y_j\partial_z H - \partial_{x_j} H. \end{aligned}$$

We have thus proved that $\gamma(t)$ is a solution of (1.3.2).

To show the converse, fix the initial function S_0 and consider

$$F(t, \tilde{x}, \partial_x S_0(\tilde{x}), \partial_t S_0(\tilde{x}), S_0(\tilde{x})) := (x(t), y(t), y_{n+1}(t), z(t)),$$

where $(x(t), y(t), z(t))$ is a solution of (1.3.2) with initial conditions given by $(x(0) = \tilde{x}, y(0) = \partial_x S_0(\tilde{x}), z(0) = S_0(\tilde{x}))$ and $y_{n+1}(t)$ is a solution of

$$\begin{cases} \dot{y}_{n+1} & = H(x(t), y(t), z(t))\partial_z H(x(t), y(t), z(t)) \\ y_{n+1}(0) & = -H(\tilde{x}, \partial_x S_0(\tilde{x}), S_0(\tilde{x})). \end{cases}$$

If $\mathcal{H} = \mathcal{H}(x, y, y_{n+1}, z) := H(x, y, z) + y_{n+1}$, then

$$\frac{d}{dt}\mathcal{H}(F(t)) = [\partial_{x_j} H \partial_{y_j} H + \partial_{y_j} H(-\partial_{x_j} H - y_j \partial_z H) + \partial_z H(y_j \partial_{y_j} H - H) \\ + H \partial_z H](F(t)) = 0$$

and hence, $\mathcal{H}(F(t)) = 0$ since the initial conditions imply $\mathcal{H}(F(0)) = 0$. As we only work with smooth functions, there is a $T_0 > 0$ such that the map X which associates $x(t, \tilde{x}) = x(t)$ to $(\tilde{x}, t)$ for $t < T_0$ is a $\mathcal{C}^1$ diffeomorphism onto its range. Therefore, we can define the smooth map $S(t, x) := z(X^{-1}(x)) = z(t, \tilde{x})$ which verifies the required initial condition. The proof will be completed by showing that

$$\partial_t S(t, x) = y_{n+1}(X^{-1}(x)) \quad \text{and} \quad \partial_x S(t, x) = y(X^{-1}(x)).$$

Thus, consider

$$T(t, \tilde{x}) := \partial_t z - \partial_t x \; y - y_{n+1} \qquad R(t, \tilde{x}) := \partial_{\tilde{x}} z - y \partial_{\tilde{x}} x.$$

$T = 0$ identically and hence,

$$\partial_t R = \partial_t R - \partial_{\tilde{x}} T = \partial_t x \, y_{\tilde{x}} - \partial_t y \, \partial_{\tilde{x}} x + \partial_{\tilde{x}} y_{n+1} \\ = \partial_y H \, y_{\tilde{x}} + (\partial_{\tilde{x}} H + y \partial_z H)\partial_{\tilde{x}} x + \partial_{\tilde{x}} y_{n+1} = -\partial_z H(\partial_{\tilde{x}} z + y \, \partial_z H),$$

since $\partial_{\tilde{x}}\mathcal{H}$. We have thus obtained that R is a solution of the liniar problem $\partial_z H R$ with initial condition $R(0, \tilde{x}) = 0$. This implies that $\partial_{\tilde{x}} z = y \partial_{\tilde{x}} x$. Using $\partial_{\tilde{x}} z = \partial_x S \partial_{\tilde{x}} x$ and the fact that T is the null function, it results that

$$\begin{cases} (y - \partial_x S)\partial_{\tilde{x}} x & = 0 \\ (y - \partial_x S)\partial_t x + (\partial_t S - y_{n+1}) & = 0. \end{cases}$$

and as X is a diffeomorphism, our proof is complete. □

Corollary 1.3.1. *$S(t, x)$ is a solution of* (1.3.3) *if and only if the associated Legendrian submanifolds*

$$L_t = \{(x, \partial_x S(t, x(t)), S(t, x(t)) | x \in \mathbb{R}^n\}$$

are related by $L_t = \Psi_t(L_0)$, where $\Psi_t : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the flow of the contact differential equation (1.3.2).

Remark 1.3.3. *For the Cauchy problem of equation* (1.3.3)

$$\begin{cases} \partial_t S + H(x, \partial_x S, S) &= 0 \quad \textit{on} \quad \Omega \quad \textit{an open domain in} \mathbb{R}^n \\ S(0,x) &= S_0(x) \\ S|_{\partial\Omega} &= \varphi, \end{cases}$$

where Ω is a domain in $\mathbb{R}$ one could construct the same correspondence as in the proof of Proposition 1.3.1.

In the context of contact geometry, the following theorem illustrates once more the principle that behind any symmetry there should be a conservation law.

Theorem 1.3.1. *Let (C, η) be an exact contact manifold, G a Lie Group acting properly on M and $H \in \mathcal{C}^\infty(C)^G$. Then the flow of the contact vector field associated to H, φ_t, invaries $J^{-1}(\mathbb{R}^+\mu)$ for any $\mu \in \mathfrak{g}^*$. In other words, modulo $\mathbb{R}^+$ the momentum map is constant on every integral curve of X_H, or φ_t invaries the level sets of the type $J^{-1}(\mathbb{R}^+\mu)$. Similarly, H is preserved by φ_t modulo $\mathbb{R}^+$.*

Proof. Let $t \mapsto x(t)$ be an integral curve of X_H, $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ such that $x(0) \in J^{-1}(\mu)$. Then

$$\begin{aligned} \frac{d}{dt}\langle J(x(t)), \xi\rangle = i_{X_H} d\langle J, \xi\rangle(x(t)) &= d\eta(x(t))(X_H(x(t)), \xi_M(x(t))) \\ &= dH(\xi_M)(x(t)) - R_{x(t)}(H)\langle \eta, \xi_M\rangle(x(t)) = \\ \left.\frac{d}{ds}\right|_{s=0} H(exp(-s\xi)\cdot x(t)) - R_{x(t)}(H)\langle J(x(t)), \xi\rangle &= -R_{x(t)}(H)\langle J(x(t)), \xi\rangle. \end{aligned}$$

Consider the real function G given by $G(t) := -R_{x(t)}(H)$. Then, the previous calculation implies that

$$\langle J(x(t)), \xi\rangle = e^{\int_0^t G(s)\mathrm{d}s}\langle \mu, \xi\rangle.$$

Since this is true for any $\xi \in \mathfrak{g}^*$, the conclusion follows. □

An immediate consequence of Theorem 1.3.1 is that the ray preimages of the momentum map are invariant submanifolds of the contact vector fields.

Remark 1.3.4. *Unlike the symplectic case, the level subsets $J^{-1}(\mu)$ are not conserved quantities. This would only happen if H were a first integral of the Reeb vector field R which locally means that $\frac{\partial H}{\partial z} = 0$ thus reducing everything to the symplectic case. Notice that the contact equations do not coincide with the time dependent Hamilton equations.*

Theorem 1.3.2. ***Reduction of contact dynamics in the regular case*** *Consider H a smooth G-invariant function on C. Then, the flow F_t of the contact vector field X_H induces a flow $F_t^{\mathbb{R}^+\mu}$ on the reduced space $C_{\mathbb{R}^+\mu}$ which satisfies*

$$\pi_\mu \circ F_t = F_t^{\mathbb{R}^+\mu} \circ \pi_\mu.$$

Even more, this flow is a contact flow associated to the smooth function $H_{\mathbb{R}^+\mu} : C_{\mathbb{R}^+\mu} \to \mathbb{R}$ defined by $H_{\mathbb{R}^+\mu} \circ \pi_\mu = H \circ i_\mu$ and the vector fields X_H, $X_{H_{\mathbb{R}^+\mu}}$ are π_μ- related.

The proof is similar to the one in the symplectic case, using that $F_t^*\eta = exp(h_t)\eta$ and the fact that the Reeb vector field of $C_{\mathbb{R}^+\mu}$ is given by $Ti_\mu(\mathcal{R})$.

Corollary 1.3.2. *If H,K are G-invariant smooth functions on C, then H, K is also G-invariant and*

$$\{H, K\}_{\mathbb{R}^+\mu} = \{H_{\mathbb{R}^+\mu}.K_{\mathbb{R}^+\mu}\}_{C_{\mathbb{R}^+\mu}},$$

where $\{\,,\}_{C_{\mathbb{R}^+\mu}}$ denotes the Poisson bracket on the reduced manifold and $\{H, K\}_{\mathbb{R}^+\mu}$ is the function induced on $C_{\mathbb{R}^+\mu}$ by $\{H, K\}$.

Remark 1.3.5. *Time dependent Hamiltonians, as defined in [1] live on the exact contact manifold $T^*Q \times \mathbb{R}$. And even if they are not contact vector fields, it is easy to see that in the presence of symmetries(not on the time, though), their flow leaves invariant the ray pre-images of the momentum map. Hence one can speak of reduced time dependent Hamiltonians. Under good conditions the reduced Hamiltonian lives on a subbundle of $T^*(Q/K_\mu)$. This can be easily verified for the damped harmonic oscillator with rotational symmetries. See, for instance [42].*

1.4 Singular cosphere bundle reduction at zero momentum

1.4.1 The decomposition of $J^{-1}(0)$

The geometric study of the contact reduced space $(S^*Q)_0$ passes through the analysis of the level set $J^{-1}(0)$ and, in particular, of its isotropy lattice $I_{J^{-1}(0)}$. We shall use the fact that both the cosphere bundle S^*Q and the lifted action of G on it are completely determined by the differential structure of Q and its supported G-action. This will allow us to obtain our first main result, Proposition 1.4.1, which describes this isotropy lattice, and hence the topology of the contact stratification of $(S^*Q)_0$, in terms of the isotropy lattice of Q without those elements corresponding to zero dimensional orbit types in Q/G. Also, as a preliminary result, and a "building block" for the general construction, we state an intermediary cosphere reduction result, Theorem 1.4.1, which applies to base manifolds Q on which the group action is not free but exhibits a single orbit type, that is, I_Q consists of only one element.

Lemma 1.4.1. *The isotropy lattice of the cosphere bundle coincides with the isotropy lattice of the cotangent bundle without the zero section*

$$I_{S^*Q} = I_{T^*Q\setminus\{0_{T^*Q}\}}.$$

Proof. It is enough to show that $G_{\alpha_q} = G_{[\alpha_q]}$ for any $\alpha_q \in T^*Q \setminus \{0_{T^*Q}\}$. Thus let $g \in G_{[\alpha_q]}$. This implies that $g[\alpha_q] = [g\alpha_q] = [\alpha_q] \iff g\alpha_q = r\alpha_q$ for $r > 0$. Since the action of G on Q is proper, there is a G-invariant Riemannian metric on Q and hence $\|g\alpha_q\| = \|\alpha_q\| = r\|\alpha_q\|$. It follows that $r = 1$ and $G_{[\alpha_q]} \subset G_{\alpha_q}$. The other inclusion being obvious, the proof is now complete. □

Remark 1.4.1. *We will write $J_{ct} : T^*Q \to \mathfrak{g}^*$ for the canonical momentum map for the cotangent-lifted action of G on T^*Q endowed with the canonical symplectic form. As $J^{-1}(0) = \pi_+(J_{ct}^{-1}(0) \setminus \{0_{T^*Q}\})$ note that*

$$(J^{-1}(0))_{(L)} = \pi_+\left((J_{ct}^{-1}(0))_{(L)} \setminus [(J_{ct}^{-1}(0))_{(L)} \cap \{0_{T^*Q}\}]\right)$$

since $(J_{ct}^{-1}(0) \setminus \{0_{T^*Q}\})_{(L)} = (J_{ct}^{-1}(0))_{(L)} \setminus [(J_{ct}^{-1}(0))_{(L)} \cap \{0_{T^*Q}\}]$.

The following theorem is an immediate consequence of Theorems 1.2.3 and 1.2.2.

Theorem 1.4.1. *Let G be a finite dimensional Lie group acting properly on the differentiable manifold Q such that all the points in Q have stabilizers conjugate to some K (that is, $Q = Q_{(K)}$). Then $J^{-1}(0)$ is a submanifold of $(S^*Q)_{(K)}$ and $(S^*Q)_0$, the reduced space at zero, is contact-diffeomorphic to $S^*(Q/G)$.*

In the following proposition we give the decomposition of $J^{-1}(0)$ and show how the topology of the contact quotient at zero is completely determined by the isotropy lattice of Q. For that, we will use the following partition of T^*Q. We fix once and for all a G-invariant Riemannian metric on Q. Then, for any $(H) \in I_Q$, the restriction of TQ to the submanifold $Q_{(H)}$ can be decomposed as the Whitney sum $T_{Q_{(H)}}Q = TQ_{(H)} \oplus NQ_{(H)}$, where, for every $q \in Q_{(H)}$, $N_qQ_{(H)} = T_qQ_{(H))}^{\perp}$. Note that each of the elements of the Whitney sum are G-invariant vector bundles over $Q_{(H)}$. Dualizing this splitting over each orbit type submanifold in Q, we obtain the following G-invariant partition of T^*Q:

$$T^*Q = \coprod_{(H)\in I_Q} T^*Q_{(H)} \oplus N^*Q_{(H)}.$$

Now, the restriction of this partition to $T^*Q\backslash\{0_{T^*Q}\}$ and afterwards its quotient by the action of $\mathbb{R}_+$, induces a G-invariant partition of S^*Q.

Let I_Q^* denote the isotropy lattice of Q without those elements (H) corresponding to orbit type submanifolds $Q_{(H)}$ for which the orbits of the restricted G-action have the same dimension as $Q_{(H)}$. At this moment, we will need some results on cotangent-lifted actions, which were proved in [48].

Lemma 1.4.2. *If G acts on Q and on T^*Q by cotangent lifts with momentum map $J_{ct} : T^*Q \to \mathfrak{g}^*$. Let (L), $(H) \in I_Q$ be arbitrary.*

(i) $(N^*Q_{(H)})_{(H)}$ *is the zero section of* $N^*Q_{(H)}$.

(ii) Let $J_{ct(H)}$ denote the canonical momentum map on $T^*Q_{(H)}$ associated to the lift of the action on $Q_{(H)}$ obtained by restriction from Q. Then

$$(J_{ct}^{-1}(0))_{(L)} = J_{ct(L)}^{-1}(0) \coprod_{(H)\succ(L)} \left(J_{ct(H)}^{-1}(0) \times (N^*Q_{(H)})_{(L)}\right) \tag{1.4.1}$$

(iii) If $(L) \neq (H)$, then $(N^*Q_{(H)})_{(L)} \neq \varnothing$ if and only if $(H) \succ (L)$.

Proposition 1.4.1. *Suppose G acts properly on the manifold Q. Then we have:*

(i) For $q \in Q_{(H)}$ such that $G_q = H$ and $(L) \in I_{S^*Q}$,

$$(J^{-1}(0))_{(L)} \cap S_q^*Q \neq \varnothing \iff (L) \in I_Q \text{ and } \big((H) \in I_Q^* \text{ or } (L) \prec (H)\big);$$

(ii) $(L) \in I_{J^{-1}(0)} \iff (L) \in I_Q^*$ and hence $\mathcal{C}_0^{(L)} \neq \varnothing \iff (L) \in I_Q^* \iff \dim Q^{(L)} \geq 1$;

(iii) The cosphere bundle projection k restricts to the G-equivariant continuous surjection $k_{(L)}$: $(J^{-1}(0))_{(L)} \to \overline{Q_{(L)}}$ which is also an open map;

(iv) For a fixed orbit type (L) in the zero momentum level set of the lifted G-action to S^*Q the corresponding orbit type submanifold admits the following G-invariant partition:

$$(J^{-1}(0))_{(L)} = J^{-1}_{(L)}(0) \coprod_{(H)\succ(L)} \pi^+ \left(J^{-1}_{ct(H)}(0) \times \left(N^*Q_{(H)}\right)_{(L)}\right), \tag{1.4.2}$$

where $(H) \in I_Q$;

(v) For every $(H) \succ (L)$ with $(L) \in I_Q^*$ and $(H) \in I_Q$ the restrictions

$$\widetilde{t}_{(L)} := k_{(L)}|_{J^{-1}_{(L)}(0)} \quad \text{and} \quad \widetilde{t}_{(H)\succ(L)} := k_{(L)}|_{\pi_+\left(J^{-1}_{ct(H)}(0)\times(N^*Q_{(H)})_{(L)}\right)}$$

are G-equivariant smooth surjective submersions onto $Q_{(L)}$ and $Q_{(H)}$ respectively. The mappings $J_{ct(H)}$ and $J_{(H)}$ denote the momentum maps of the restricted actions of G to $T^*Q_{(H)}$ and $S^*Q_{(H)}$ respectively (which are the same as the canonical momentum maps for the restricted G-action on $Q_{(H)}$).

Proof. To prove (i), let $(L) \in I_{S^*Q}$ and $q \in Q_{(H)}$ with $G_q = H$. Then

$$(J_{ct}^{-1}(0))_{(L)} \cap T_q^*Q = (S_q^H)^* \oplus (N_q^*Q_{(H)})_{(L)}, \tag{1.4.3}$$

where S_q^H is the linear slice for the G-action on $Q_{(H)}$(see section 3 of [48]). Since $(J^{-1}(0))_{(L)} \cap S_q^*Q = \varnothing \iff (J_{ct}^{-1}(0))_{(L)} \cap T_q^*Q = \{0\}$, then $(J^{-1}(0))_{(L)} \cap S_q^*Q = \varnothing$ only when $(S_q^H)^*$ and $(N_q^*Q_{(H)})_{(L)}$ are simultaneously zero. This amounts to $(L) = (H) \in I_Q \backslash I_Q^*$, (see Lemma 1.4.2) from where the result follows.

(ii) is a forward consequence of (i). The rest of this statement and the G-equivariant continuous surjectivity of $k_{(L)}$ are direct consequences of the fact that $I_Q = I_{J_{ct}^{-1}(0)}$. To prove the openness of $k_{(L)}$ it suffices to observe that for any open subset $\mathcal{U}$ of $(J^{-1}(0))_{(L)}$, $k_{(L)}(\mathcal{U}) = \tau_{(L)}(\pi^{-1}(\mathcal{U}))$, where $\tau_{(L)} : (J_{ct}^{-1}(0))_{(L)} \to \overline{Q_{(L)}}$ is the open canonical cotangent projection map.

Applying (1.4.1) and the fact that $(N^*Q_{(H)})_{(L)}$ does not contain the zero section when $(H) \neq (L)$ we have

$$(J_{ct}^{-1}(0) \setminus \{0_{T^*Q}\})_{(L)} = \left(J^{-1}_{ct(L)}(0)\right) \setminus \{0_{T^*Q_{(L)}}\} \coprod_{(H)\succ(L)} \left[J^{-1}_{ct(H)}(0) \times \left(N^*Q_{(H)}\right)_{(L)}\right].$$

Hence, applying π_+ to this relation, we get

$$(J^{-1}(0))_{(L)} = J^{-1}_{(L)}(0) \coprod_{(H)\succ(L)} \pi^+ \left(J^{-1}_{ct(H)}(0) \times \left((N^*Q_{(H)}\right)_{(L)}\right)$$

which proves statement (iv).

As for the proof of (v), it is enough to notice that

$$J^{-1}_{(L)}(0) \quad \text{and} \quad \pi^+ \left(J^{-1}_{ct(H)}(0) \times (N^*Q_{(H)})_{(L)}\right)$$

are bundles over $Q_{(L)}$ and $Q_{(H)}$ respectively. □

Remark 1.4.2. *Notice that for the description of orbit types in $J^{-1}(0)$, we need not only I^*_Q, but also the lattice I_Q since each $(J^{-1}(0))_{(L)}$ is written as a union with index (H) in I_Q, but (L) belongs to I^*_Q.*

1.4.2 The topology and contact geometry of $\mathcal{C}_0$

The secondary decomposition of $\mathcal{C}_0^{(L)}$

Define the fiber bundles:

$$s_{(H)\succ(L)} := J^{-1}_{ct(H)}(0) \times (N^*Q_{(H)})_{(L)} \to Q_{(H)}$$

$$s_{(L)} := J^{-1}_{ct(L)}(0) \to Q_{(L)}.$$

Taking into account that $\pi^+(s_{(H)\succ(L)})$ are G-invariant pieces of the partition (2.6.9) of $(J^{-1}(0))_{(L)}$ and that the actions of G and $\mathbb{R}^+$ commute, we can define:

$$CS_{(H)\succ(L)} := \frac{\pi^+(s_{(H)\succ(L)})}{G}$$

$$CC_{(L)} := \frac{J^{-1}_{(L)}(0)}{G} = \frac{\pi^+\left(s_{(L)} \setminus \{0_{T^*Q_{(L)}}\}\right)}{G} \simeq S^*\left(\frac{Q_{(L)}}{G}\right).$$

Notice that for the above equivalence we have applied Theorem 1.4.1 and that each contact stratum admits the following partition, which is the quotient of (2.6.9):

$$\begin{aligned} \mathcal{C}_0^{(L)} = \frac{\left(J^{-1}(0)\right)_{(L)}}{G} &= CC_{(L)} \coprod_{(H)\succ(L)} CS_{(H)\succ(L)} \\ &\simeq S^*\left(\frac{Q_{(L)}}{G}\right) \coprod_{(H)\succ(L)} CS_{(H)\succ(L)}. \end{aligned} \qquad (1.4.4)$$

Remark 1.4.3. *In the notations of the previous section, the maps $k_{(L)}$, $\widetilde{t}_{(L)}$, and $\widetilde{t}_{(H)\succ(L)}$ descend to*

$$k^{(L)} : \mathcal{C}_0^{(L)} \to \overline{Q^{(L)}}, \quad \widetilde{t}^{(L)} : CC_{(L)} \to Q^{(L)}, \text{ and } \quad \widetilde{t}^{(H)\succ(L)} : CS_{(H)\succ(L)} \to Q^{(H)};$$

$k^{(L)}$ is an open continuous surjection and the other two are smooth surjective submersions.

Theorem 1.4.2. *With the above notations, we obtain the following:*

(i) *$\overline{Q^{(L)}}$ is a stratified space with strata $Q^{(H)}$, for all $(L) \preccurlyeq (H)$ and with frontier conditions given by*

$$Q^{(K)} \cap \overline{Q^{(H)}} \neq \varnothing \iff (H) \preccurlyeq (K).$$

Moreover, $Q^{(L)}$ is open and dense in $\overline{Q^{(L)}}$.

(ii) *For every $(L) \in I_Q^*$ and $(H) \in I_Q$, the partition (1.4.4) is a stratification of the corresponding contact stratum $\mathcal{C}_0^{(L)}$, called the **secondary stratification**. The frontier conditions are given by:*

$$CS_{(H)\succ(L)} \subset \partial CC_{(L)} \quad \text{for all} \quad (H) \succ (L);$$

$$CS_{(H')\succ(L)} \subset \partial CS_{(H)\succ(L)} \iff (H') \succ (H) \succ (L).$$

Moreover, the piece $CC_{(L)}$ is diffeomorphic to $S^*(Q^{(L)})$, is open and dense in $\mathcal{C}_0^{(L)}$, and the map $k^{(L)}$ is a surjective submersion of stratified spaces.

Proof. Since the G-action is proper, the orbit type decomposition of Q induces a stratification of Q/G and the first part of the theorem follows immediately considering the relative topology of $\overline{Q^{(L)}}$ in Q/G. Also, (1.4.4) is a locally finite partition and its pieces are obviously submanifolds of $\mathcal{C}_0^{(L)}$. As $k^{(L)}$ is a continuous map and $(k^{(L)})^{-1}(Q^{(L)}) = CC_{(L)}$, it follows that $CC_{(L)}$ is open in $\mathcal{C}_0^{(L)}$. In order to prove the density, let $x \in \mathcal{C}_0^{(L)}$ and $\mathcal{U}$ be any open neighborhood of x. Hence, $\mathcal{V} = k^{(L)}(\mathcal{U})$ is an open subset of $\overline{Q^{(L)}}$ and, since $Q^{(L)}$ is dense in $\overline{Q^{(L)}}$, there is at least one element $y \in \mathcal{V} \cap Q^{(L)}$. Notice that $(k^{(L)})^{-1}(y) = (\widetilde{t}^{(L)})^{-1}(y) \subset CC_{(L)}$ and that there is at least an element in $(\widetilde{t}^{(L)})^{-1}(y)$ which is in $\mathcal{U}$. This means that $\mathcal{U} \cap CC_{(L)} \neq \varnothing$ which proves the density of $CC_{(L)}$.

Using the density of $CC_{(L)}$, the first frontier condition for the secondary stratification becomes obvious. For the second one, consider in $\mathcal{C}_0^{(L)}$ an arbitrary open neighborhood $\mathcal{U}$ of a point $x \in CS_{(H')\succ(L)}$. By the openness property of $k^{(L)}$, we obtain that $\mathcal{O} = k^{(L)}(\mathcal{U})$ is an open neighborhood of $k^{(L)}(x)$ in $\overline{Q^{(L)}}$. Applying (i), we have that $\mathcal{O} \cap Q^{(H)} \neq \varnothing \iff (H') \succ (H) \succ (L)$. Furthermore, the surjectivity of $\widetilde{t}^{(H)\succ(L)}$ implies $(\widetilde{t}^{(H)\succ(L)})^{-1}(z) \cap \mathcal{U} \neq \varnothing$ for any $z \in \mathcal{O} \cap Q^{(H)}$, proving that $CS_{(H')\succ(L)} \subset \partial CS_{(H)\succ(L)} \iff (H') \succ (H) \succ (L)$.

As $k^{(L)}$ restricted to each piece of the secondary decomposition is surjective, Remark 1.4.3 immediately implies that this map is a stratified surjective submersion. □

We will refer to the strata of the form $CS_{(H)\succ(L)}$ as *contact seams* due to their stitching role that will be explained later in Remark 1.4.5.

This theorem completes the topological description of each contact stratum $\mathcal{C}_0^{(L)}$ in terms of its secondary stratification. We shall now begin the investigation of geometrical aspects, namely to what extent the strata of this secondary stratification admit canonical contact structures in the sense that the 1-forms generating them are induced by some cosphere bundle structures compatible with the reduced contact form on the contact stratum. Thus, denote by

$$\widetilde{\Psi}^{(H)} : CC_{(H)} \to \left(S^*(Q^{(H)}), \Theta_{\Sigma}^{(H)} \right)$$

the bundle isomorphism given by Theorem 1.4.1, where $\Theta_{\Sigma}^{(H)}$ is a contact form on the cosphere bundle of $Q^{(H)}$. Observe that the restricted projection onto the first factor

$$\mathrm{p}_{1(H)\succ(L)} : \left(J_{ct(H)}^{-1}(0) \setminus \{0_{T^*Q_{(H)}}\} \right) \times (N^*Q_{(H)})_{(L)} \to J_{ct(H)}^{-1}(0) \setminus \{0_{T^*Q_{(H)}}\}$$

is $\mathbb{R}^+$ and G-equivariant so it descends to the surjective submersion

$$\widetilde{\mathrm{p}}_1^{(H)\succ(L)} : CS^{\circ}_{(H)\succ(L)} \to CC_{(H)},$$

where

$$CS^{\circ}_{(H)\succ(L)} := \frac{\pi^+ \left(J_{ct(H)}^{-1}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)} \right)}{G}$$

is an open and dense submanifold of the contact seam $CS_{(H)\succ(L)}$. Then, for any pair $(H) \succ (L)$, we have the following bundle map covering the identity on $Q^{(H)}$

$$\widetilde{\Psi}^{(H)\succ(L)} := \widetilde{\Psi}^{(H)} \circ \widetilde{\mathrm{p}}_1^{(H)\succ(L)} : CS^{\circ}_{(H)\succ(L)} \to S^*(Q^{(H)})$$

which is also a surjective submersion. We are now able to endow each cosphere-like stratum $CC_{(H)}$ and each $CS^{\circ}_{(H)\succ(L)}$ with 1-forms given by:

$$\left(CC_{(H)}, \eta_{(H)} := (\widetilde{\Psi}^{(H)})^* \Theta_{\Sigma}^{(H)} \right) \quad \text{and} \tag{1.4.5}$$

$$\left(CS^{\circ}_{(H)\succ(L)}, \eta_{(H)\succ(L)} := (\widetilde{\Psi}^{(H)\succ(L)})^* \Theta_{\Sigma}^{(H)} \right). \tag{1.4.6}$$

It is impossible to induce in this way a 1-form on the whole piece $CS_{(H)\succ(L)}$ and hence we are forced to restrict ourselves, for the time being, to $CS^{\circ}_{(H)\succ(L)}$. However, we will show later how to extend this form to the whole $CS_{(H)\succ(L)}$.

Theorem 1.2.2 gives the existence of an abstractly defined contact structure on each contact piece $C_0^{(L)}$ generated by a 1-form $\theta_{\sigma_0}^{(L)}$. One of the aims of this section is to investigate the compatibility of the previously defined forms $\eta_{(H)}$ and $\eta_{(H)\succ(L)}$ with the reduced contact form $\theta_{\sigma_0}^{(L)}$ and to describe as much as possible this abstract contact structure.

Theorem 1.4.3. *The strata $CC_{(L)}$ and $CS^{\circ}_{(H)\succ(L)}$ within the contact stratum $\mathcal{C}_0^{(L)}$ satisfy the following properties:*

(i) $(CC_{(L)}, \eta_{(L)})$ *is an open dense contact submanifold of the contact stratum $\mathcal{C}_0^{(L)}$ contactomorphic to $(S^*(Q^{(L)}), \Theta_{\Sigma}^{(L)})$.*

(ii) Using the above notations, the conformal classes of $\eta_{(L)}$ and $\eta_{(H)\succ(L)}$ admit smooth extensions to $\mathcal{C}_0^{(L)}$ equivalent to ${\theta_{\sigma}}_0^{(L)}$, namely

$$ {\theta_{\sigma}}_0^{(L)}\big|_{CC_{(L)}} \simeq \eta_{(L)} \quad \text{and} \quad {\theta_{\sigma}}_0^{(L)}\big|_{CS^{\circ}_{(H)\succ(L)}} \simeq \eta_{(H)\succ(L)}. $$

The extension of $\eta_{(L)}$ is unique.

(iii) The conformal class of $\eta_{(H)\succ(L)}$ can be smoothly and uniquely extended to the whole stratum $CS_{(H)\succ(L)}$. If $(H) \in I_Q^*$ then $CS_{(H)\succ(L)}$ is a coisotropic submanifold of the contact stratum $\mathcal{C}_0^{(L)}$. When $(H) \in I_Q \setminus I_Q^*$ then $CS_{(H)\succ(L)}$ is a Legendrian submanifold of the contact stratum $\mathcal{C}_0^{(L)}$.

Proof. (i) is a simple consequence of Theorem 1.4.1.

For (ii), let (L) and (H) be two fixed elements of I_Q^* and I_Q respectively and $i_0^{(H)\succ(L)} : CS^{\circ}_{(H)\succ(L)} \to \mathcal{C}_0^{(L)}$ the inclusion map. By definition,

$$ \begin{array}{lcl} {\theta_{\sigma}}_0^{(L)}\big|_{CS^{\circ}_{(H)}\succ(L)} \simeq \eta_{(H)\succ(L)} & \iff & \exists f > 0 \quad \text{in} \quad C^{\infty}(CS_{(H)\succ(L)}) \quad \text{such that} \\ {\theta_{\sigma}}_0^{(L)}\big|_{CS^{\circ}_{(H)\succ(L)}} = f\,\eta_{(H)\succ(L)} & \iff & (i_{\circ}^{(H)\succ(L)}\,)^* {\theta_{\sigma}}_0^{(L)} \simeq (\widetilde{\Psi}^{(H)\succ(L)})^* \Theta_{\Sigma}^{(H)}. \end{array} $$

To simplify the reading of the proof, consider the two figures 2.7 and 1.4.2 where $\pi_G^{(H)\succ(L)}$ and $\bar{\pi}_G^{(H)}$ denote the canonical G-projections and all the horizontal arrows in the first and second diagram are injections and projections respectively.

As $\pi_G^{(H)\succ(L)} \circ \pi_+$ is a submersion, it suffices to prove that

$$ (i_0^{(H)\succ(L)} \circ \pi_G^{(H)\succ(L)} \circ \pi_+\,)^* {\theta_{\sigma}}_0^{(L)} \simeq (\widetilde{\Psi}^{(H)\succ(L)} \circ \pi_G^{(H)\succ(L)} \circ \pi_+\,)^* \Theta_{\Sigma}^{(H)}. \tag{1.4.7} $$

Observe that $i_0^{(H)\succ(L)} \circ \pi_G^{(H)\succ(L)} \circ \pi_+ = \pi_G^{(L)} \circ \pi_+ \circ i_{(H)\succ(L)}$ and the first term of (1.4.7) becomes

$$ \begin{aligned} (\pi_G^{(L)} \circ \pi_+ \circ i_{(H)\succ(L)}\,)^* {\theta_{\sigma}}_0^{(L)} &= i^*_{(H)\succ(L)} \circ \pi_+^*\, ((\pi_G^{(L)})^* {\theta_{\sigma}}_0^{(L)}\,) \\ &= i^*_{(H)\succ(L)} \circ \pi_+^*\, (\widetilde{i}^*_{(L)} \theta_{\sigma}\,) = (\pi_+ \circ j_{(H)} \circ \Phi\,)^* \theta_{\sigma}, \end{aligned} $$

where in the last line we have used Theorem 1.2.2 together with the equality $\pi_+ \circ j_{(H)} \circ \Phi = \widetilde{i}_{(L)} \circ \pi_+ \circ i_{(H)\succ(L)}$, with $j_{(H)}$ and Φ inclusions defined by:

$$ \Phi : \left(J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)} \right) \hookrightarrow T^*Q|_{Q_{(H)}} \setminus \{0_{T^*Q_{(H)}}\} $$

$$
\begin{array}{ccccc}
J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)} & \xhookrightarrow{i_{(H)\succ(L)}} & (J^{-1}_{ct}(0) \setminus \{0_{T^*Q}\})_{(L)} & \xhookrightarrow{i_{(L)}} & T^*Q\setminus\{0_{T^*Q}\} \\
\downarrow \pi_+ & & \downarrow \pi_+ & & \downarrow \pi_+ \\
\pi_+\left(J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)}\right) & \xhookrightarrow{\widetilde{i}_{(H)\succ(L)}} & (J^{-1}(0))_{(L)} & \xhookrightarrow{\widetilde{i}_{(L)}} & S^*Q \\
\downarrow \pi_G^{(H)\succ(L)} & & \downarrow \pi_G^{(L)} & & \\
CS^\circ_{(H)\succ(L)} & \xhookrightarrow{i_0^{(H)\succ(L)}} & \mathcal{C}_0^{(L)} & & \\
\downarrow \widetilde{\Psi}^{(H)\succ(L)} & & & & \\
S^*(Q^{(H)}) & & & &
\end{array}
$$

Figure 1.4.1: Diagram defining $\eta_{(H)}$

and

$$j_{(H)} : T^*Q|_{Q_{(H)}} \setminus \{0_{T^*Q_{(H)}}\} \hookrightarrow T^*Q \setminus \{0_{T^*Q}\}.$$

Using this time $\bar{\pi}_G^{(H)} \circ \pi_+ \circ p_{1(H)\succ(L)} = \widetilde{p}_1^{(H)\succ(L)} \circ \pi_G^{(H)\succ(L)} \circ \pi_+$, we can write the second term of (1.4.7) as:

$$
\begin{aligned}
(\widetilde{\Psi}^{(H)} \circ \widetilde{p}_1^{(H)\succ(L)} \circ \pi_G^{(H)\succ(L)} \circ \pi_+)^*\Theta_\Sigma^{(H)} &= (\widetilde{\Psi}^{(H)} \circ \bar{\pi}_G^{(H)} \circ \pi_+ \circ p_{1(H)\succ(L)})^*\Theta_\Sigma^{(H)} \\
&= (\pi_+ \circ p_{1(H)\succ(L)})^* (\widetilde{\Psi}^{(H)} \circ \bar{\pi}_G^{(H)})^*\Theta_\Sigma^{(H)} \simeq (\pi_+ \circ p_{1(H)\succ(L)})^* l_{(H)}^* \theta_{(H)\Sigma},
\end{aligned}
$$

where $\theta_{(H)\Sigma}$ is a contact form on $S^*Q_{(H)}$. Let $p_{(H)} : T^*Q|_{Q_{(H)}} \setminus\{0_{T^*Q_{(H)}}\} \to T^*Q_{(H)}$ be the projection map. Since $l_{(H)} \circ \pi_+ \circ p_{1(H)\succ(L)} = \pi_+ \circ p_{(H)} \circ \Phi$, the second term is in the same conformal class as $\Phi^* p_{(H)}^* \pi_+^* \theta_{(H)\Sigma}$ and, hence, equation (1.4.7) is equivalent to

$$
\begin{aligned}
\Phi^* p_{(H)}^* \pi_+^* \theta_{(H)\Sigma} \simeq \Phi^* j_{(H)}^* \pi_+^* \theta_\sigma &\iff \Phi^* p_{(H)}^* (\Sigma \circ \pi_+)^* \theta_{(H)} \simeq \Phi^* j_{(H)}^* (\sigma \circ \pi_+)^* \theta \\
&\iff \Phi^* p_{(H)}^* f_\Sigma \theta_{(H)} \simeq \Phi^* j_{(H)}^* f_\sigma \theta,
\end{aligned}
$$

where θ and $\theta_{(H)}$ are the canonical one-forms on T^*Q and $T^*Q_{(H)}$ respectively, and σ, Σ are sections in the associated cosphere bundles. But $p_{(H)}^* \theta_{(H)} = j_{(H)}^* \theta$, as it can be easily seen in local coordinates, which proves (1.4.7).

As for the extension of the conformal class of $\eta_{(L)}$, an analogous proof can be developed just by considering the limit case $(H) = (L)$, when $CS^0_{(H)\succ(L)}$ degenerates in $CC_{(L)}$. In order to prove the uniqueness of this extension, let us consider a point $x \in \mathcal{C}_0^{(L)}$ and one tangent vector $v_x \in T_x\mathcal{C}_0^{(L)}$. As $CC_{(L)}$ is open and dense in $\mathcal{C}_0^{(L)}$, there is a sequence of points $x_k \in CC_{(L)}$ and one of vectors $v_{x_k} \in T_{x_k} CC_{(L)} \simeq T_{x_k}\mathcal{C}_0^{(L)}$ such that

$$\lim_{k\to\infty} x_k = x, \quad \lim_{k\to\infty} v_{x_k} = v_x.$$

$$
\begin{array}{ccccc}
\left(J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)}\right) & \xrightarrow{p_{1(H)\succ(L)}} & J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} & & \\
\Big\downarrow \pi_+ & & \Big\downarrow \pi_+ & & \\
\pi_+\left(J^{-1}_{ct(H)}(0) \setminus \{0_{T^*Q_{(H)}}\} \times (N^*Q_{(H)})_{(L)}\right) & \xrightarrow{\widetilde{p}_{1(H)\succ(L)}} & J^{-1}_{(H)}(0) & \xrightarrow{l_{(H)}} & S^*(Q_{(H)}) \\
\Big\downarrow \pi_G^{(H)\succ(L)} & & \Big\downarrow \bar{\pi}_G^{(H)} & & \\
CS^{\circ}_{(H)\succ(L)} & \xrightarrow{\widetilde{p_1}^{(H)\succ(L)}} & CC_{(H)} & \xrightarrow{\widetilde{\psi}^{(H)}} & S^*(Q^{(H)})
\end{array}
$$

Figure 1.4.2: Diagram defining $\eta_{(H)\succ(L)}$

From the above arguments and using the continuity of $\theta_{\sigma_0}^{(L)}$, we have that

$$\lim_{k\to\infty} \frac{\eta_{(L)}(x_k)(v_{x_k})}{g(x_k)} = \lim_{k\to\infty} \theta_{\sigma_0}^{(L)}(x_k)(v_{x_k}) = \theta_{\sigma_0}^{(L)}(x)(v_x),$$

with $g \in C^\infty(CC_{(L)})$ a positive function such that $\eta_{(L)} = g\,\theta_{\sigma_0}^{(L)}|_{CC_{(L)}}$. We have thus proved that the class of $\theta_{\sigma_0}^{(L)}$ is the unique smooth extension of the class of $\eta_{(L)}$ to $\mathcal{C}_0^{(L)}$.

(iii) To extend the class of $\eta_{(H)\succ(L)}$ from $CS^{\circ}_{(H)\succ(L)}$ to the whole piece $CS_{(H)\succ(L)}$, we will apply the same type of arguments as before, using this time that $CS^{\circ}_{(H)\succ(L)}$ is open and dense in $CS_{(H)\succ(L)}$. Namely, for any point $x \in CS_{(H)\succ(L)}$ and any $v_x \in T_xCS_{(H)\succ(L)}$, there is a sequence of points $x_k \in CS^{\circ}_{(H)\succ(L)}$ and one of vectors $v_{x_k} \in T_{x_k}CS^{\circ}_{(H)\succ(L)} \simeq T_{x_k}CS_{(H)\succ(L)}$ such that

$$\lim_{k\to\infty} x_k = x, \quad \lim_{k\to\infty} v_{x_k} = v_x.$$

Observe that

$$\lim_{k\to\infty} \frac{\eta_{(H)\succ(L)}(x_k)(v_{x_k})}{f(x_k)} = \theta_{\sigma_0}^{(L)}(x)(v_x)$$

and notice that this extension is also unique and given by the conformal class of $\theta_{\sigma_0}^{(L)}|_{CS_{(H)\succ(L)}}$.

To check the coisotropy and Legendrian submanifold conditions, let $x \in CS^{\circ}_{(H)\succ(L)}$. A direct count of dimensions gives:

$$\dim\ker\theta_{\sigma_0}^{(L)}(x) = \dim\mathcal{C}_0^{(L)} - 1 = 2(\dim Q_{(L)} - \dim G + \dim L - 1)$$

since $S^*Q^{(L)}$ is open in the corresponding contact stratum. At this point we need the following intermediate result.

Lemma 1.4.3. *The dimension of the tangent space to a contact seam is*

$$\dim T_xCS_{(H)\succ(L)} = \dim Q_{(H)} + \dim Q_{(L)} - 2\dim G + \dim H + \dim L - 1. \qquad (1.4.8)$$

Proof. We want to compute $\dim T_x CS_{(H)\succ(L)} = \dim CS_{(H)\succ(L)}$. For this, let $\pi(z) = k^0(x)$ be the base point of x, where $z \in Q_{(H)}$ with $G_z = H$ and note that $\dim CS_{(H)\succ(L)} = \dim(J_{ct}^{-1}(0) \cap T_z^*Q)_{(L)} + \dim Q_{(H)} - \dim G + \dim L - 1$, where the class (L) refers to the linear H-action on the vector space $J_{ct}^{-1}(0) \cap T_z^*Q$. On the other hand, the inverse of the Riemannian bundle isomorphism $TQ \to T^*Q$ maps $(J_{ct}^{-1}(0) \cap T_z^*Q)_{(L)}$ H-equivariantly isomorphically to $(S_z)_{(L)}$. Now, if ψ, U, and U' are like in the Tube Theorem (1.2.2), then ψ restricts to a diffeomorphism between $G \times_H \left((S_z)_{(L)} \cap U\right)$ and $U \cap Q_{(L)}$. Since $\dim G \times_H (S_z)_{(L)} = \dim G + \dim(S_z)_{(L)} - \dim H$, we can compute

$$\dim(S_z)_{(L)} = \dim Q_{(L)} - \dim G + \dim H.$$

Finally we obtain $\dim T_x CS_{(H)\succ(L)} = \dim Q_{(H)} + \dim Q_{(L)} - 2\dim G + \dim H + \dim L - 1$. □

Consequently, a simple dimension count gives

$$\begin{aligned}\dim T_x CS_{(H)\succ(L)} - \frac{1}{2}\dim\ker\theta_{\sigma 0}^{(L)}(x) &= \dim Q_{(H)} - \dim G + \dim H \\ &= \dim(S_z)_{(H)} \geq 0,\end{aligned}$$

where $z \in Q_{(H)}$ is the base point of x and S_z is the associated linear slice. Suppose first that $(H) \in I_Q^*$ and so $\dim T_x CS_{(H)\succ(L)} - \frac{1}{2}\dim\ker\theta_{\sigma 0}^{(L)}(x) \gneqq 0$. This implies that $CS^\circ_{(H)\succ(L)}$ and $CS_{(H)\succ(L)}$ can be neither isotropic nor Legendrian submanifolds of $\mathcal{C}_0^{(L)}$ and that $T_x CS^\circ_{(H)\succ(L)} \nsubseteq \ker\theta_{\sigma 0}^{(L)}(x)$ for any $x \in CS^\circ_{(H)\succ(L)}$.

Now let

$$W_x := T_x CS^\circ_{(H)\succ(L)} \cap \ker\theta_{\sigma 0}^{(L)}(x) = T_x CS_{(H)\succ(L)} \cap \ker\theta_{\sigma 0}^{(L)}(x)$$

and

$$V_x := \left\{v \in T_x CS^\circ_{(H)\succ(L)} \setminus \ker\theta_{\sigma 0}^{(L)}(x) : v = v_0 \oplus kR(x), k \in \mathbb{R}\,, v_0 \in \ker\theta_{\sigma 0}^{(L)}(x)\right\}.$$

One can easily check that V_x is a one dimensional vector space and that for any $x \in CS^\circ_{(H)\succ(L)}$, we have $T_x CS^\circ_{(H)\succ(L)} = W_x \oplus V_x$. As $\widetilde{\Psi}^{(H)\succ(L)}$ is a surjective submersion and $\theta_{\sigma 0}^{(L)}|_{CS^\circ_{(H)\succ(L)}} \simeq \eta_{(H)\succ(L)}$, it follows that $T_x\widetilde{\Psi}^{(H)\succ(L)}(W_x) = \ker\Theta_\Sigma^{(H)}(y)$ and $T_x\widetilde{\Psi}^{(H)\succ(L)}(V_x)=$ span $\{R_\Sigma(y)\}$, where $y = \widetilde{\Psi}^{(H)\succ(L)}(x)$ and $R_\Sigma(y)$ is the Reeb vector field of $\left(S^*(Q^{(H)}), \Theta_\Sigma^{(H)}\right)$. Therefore, we obtain

$$\begin{aligned}\operatorname{rank} d\eta_{(H)\succ(L)}(x)|_{W_x} &= \dim W_x - \dim\ker d\eta_{(H)\succ(L)}(x)|_{W_x} \\ &= \dim W_x - \dim\{v \in W_x : d\Theta_\Sigma^{(H)}(y)(T_x\widetilde{\Psi}^{(H)\succ(L)}v, T_x\widetilde{\Psi}^{(H)\succ(L)}w) = 0, \forall w \in W_x\} \\ &= \dim W_x - \dim\ker T_x\widetilde{\Psi}^{(H)\succ(L)}\big|_{W_x} = \dim S^*(Q^{(H)}) - 1.\end{aligned}$$

This shows that $\operatorname{rank} d\eta_{(H)\succ(L)}(x)\big|_W = 2\dim W - (\dim\mathcal{C}_0^{(L)} - 1)$ proving that $CS^\circ_{(H)\succ(L)}$ is a coisotropic submanifold. Since $CS^\circ_{(H)\succ(L)}$ is dense in $CS_{(H)\succ(L)}$, by an extension argument similar to the one used before, we have that $CS_{(H)\succ(L)}$ is also a coisotropic submanifold of the corresponding contact stratum.

If $(H) \in I_Q \backslash I_Q^*$, then $\dim T_x CS_{(H)\succ(L)} = \frac{1}{2} \dim \ker \theta_{\sigma_0}^{(L)}(x)$ and by the definition (1.4.6) $\eta_{(H)\succ(L)} = 0$ since $S^*(Q^{(H)})$ is the trivial bundle, proving thus that the $CS_{(H)\succ(L)}$ is a Legendrian submanifold of $\mathcal{C}_0^{(L)}$. □

Remark 1.4.4. *Note that the contact seams $CS_{(H)\succ(L)}$ can never be contact submanifolds of $\mathcal{C}_0^{(L)}$.*

The C-L stratification of $\mathcal{C}_0$

In this subsection we prove the existence of a new stratification of the contact reduced space $\mathcal{C}_0$, different from the contact stratification in Theorem 1.2.2. The existence of this new stratification, that we call the C-L stratification since its strata are coisotropic or Legendrian submanifolds of the corresponding contact stratum, is due to the bundle structure of the contact manifold that we start with. We will see that the C-L stratification is strictly finer than the contact one, if the base manifold Q has more than one orbit type. In principle, this is not an advantage since the contact stratification partitions the singular contact quotient in less and larger smooth components. However, if we take into account the bundle structure of the problem we can see why this new stratification is more appropriate.

The most important feature of regular cosphere bundle reduction, Theorem 1.2.3, is that if we start with the cosphere bundle of a manifold Q, we end up again with a cosphere bundle, this time over Q/G. Furthermore, the reduced contact structure on $S^*(Q/G)$ equals the canonical cosphere contact structure. In the singular setting however, the lack of smoothness of the quotient spaces involved forces us to choose another definition of fibration. The most natural one when working with decomposed or stratified spaces is the following: if A and B are decomposed spaces together with a continuous surjection $f : A \to B$, we say that $f : A \to B$ defines a *stratified bundle* over B if f is a morphism of decomposed spaces. In our case, there is a natural projection $k^0 : \mathcal{C}_0 \to Q/G$ induced from the cosphere bundle projection $k : S^*Q \to Q$. If we consider the natural orbit type stratification of Q/G and the contact one of $\mathcal{C}_0$, then the projection does not define a stratified bundle over Q/G since the image of a contact stratum $\mathcal{C}_0^{(L)}$ under the projection is $\overline{Q^{(L)}}$ which includes several orbit type strata of Q/G. We will prove that the choice of the coisotropic stratification for the contact quotient $\mathcal{C}_0$ solves this problem.

Consider the partition of $\mathcal{C}_0$ obtained by putting together all the secondary strata found in every contact stratum:

$$\mathcal{C}_0 = \coprod_{(L)} CC_{(L)} \coprod_{(K')\succ(K)} CS_{(K')\succ(K)} \tag{1.4.9}$$

for every pair of classes $(L), (K) \in I_Q^*$ and every $(K') \in I_Q$.

Theorem 1.4.4. *The partition* (1.4.9) *is a decomposition of $\mathcal{C}_0$ inducing a stratification, called the C-L stratification, that satisfies the following properties:*

(1) If Q/G is connected and (L_0) is the principal orbit type in Q, then $CC_{(L_0)}$ is open and dense in $\mathcal{C}_0$.

(2) $k^0 : \mathcal{C}_0 \to Q/G$ is a stratified bundle with respect to the C-L stratification of $\mathcal{C}_0$ and the orbit type stratification of Q/G.

(3) If I_Q consists of more than one class, the C-L stratification is strictly finer than the contact one, and they are identical otherwise.

(4) The frontier conditions for the C-L stratification of $\mathcal{C}_0$ are:

(i) $CC_{(K)} \subset \partial CC_{(H)} \Longleftrightarrow (H) \prec (K)$

(ii) $CS_{(K)\succ(H)} \subset \partial CC_{(H)} \Longleftrightarrow (H) \prec (K)$

(iii) $C_{(K)} \subset \partial CS_{(K)\succ(H)} \Longleftrightarrow (H) \prec (K)$

(iv) $CS_{(K')\succ(H)} \subset \partial CS_{(K)\succ(H)} \Longleftrightarrow (H) \prec (K) \prec (K')$

(v) $CS_{(K)\succ(H')} \subset \partial CS_{(K)\succ(H)} \Longleftrightarrow (H) \prec (H') \prec (K)$.

Proof. For (1), recall by Proposition 1.4.1 that $I_{J^{-1}(0)} = I_Q^*$. The principal orbit type of the isotropy lattice corresponds to an open and dense piece, so $(J^{-1}(0))_{(L_0)}$ is open and dense in $J^{-1}(0)$, since (L_0) is by hypothesis the principal orbit type in I_Q^* (assuming that $\dim Q \neq 0$) and hence in $I_{J^{-1}(0)}$. Consequently, as the orbit map $J^{-1}(0) \to \mathcal{C}_0$ is continuous and open, $\mathcal{C}_0^{(L_0)}$ is open and dense in $\mathcal{C}_0$. Now, since $\mathcal{C}_0^{(L_0)}$ is equipped with the relative topology with respect to $\mathcal{C}_0$ and $CC_{(L_0)}$ is open and dense in it (Theorem 1.4.2), it follows that $CC_{(L_0)}$ is also open and dense in $\mathcal{C}_0$. For (2), note that the restrictions of k^0 to $CC_{(L)}$ and $CS_{(H)\succ(L)}$ coincide with the corresponding restrictions of $k^{(L)}$, which, by Remark 1.4.3, are smooth surjective submersions over $Q^{(L)}$ and $Q^{(H)}$ respectively for every $(L) \in I_Q^*$ and $(H) \in I_Q$. This shows that these restrictions map each C-L stratum of $\mathcal{C}_0$ to an orbit type stratum of Q/G. Therefore, k^0 is a morphism of stratified spaces. To prove (3), recall from Theorem 1.4.1, that if I_Q consists of a single orbit type (H), then $\mathcal{C}_0 = \mathcal{C}_0^{(H)} = CC_{(H)}$ (assuming $\dim Q \neq 0$) and its contact and C-L stratifications are both trivial and identical. If there is more than one orbit type in the base, the number of C-L strata is strictly higher than the number of contact strata (which is equal to the number of orbit types of I_Q^*). The identity map in $\mathcal{C}_0$ injects each C-L stratum in the unique contact stratum to which it belongs and is hence a morphism of stratified spaces. Therefore, the C-L stratification is finer than the contact one. For (4), relations (ii) and (iv) follow from the frontier conditions of the secondary stratum $\mathcal{C}_0^{(H)}$. To prove (i), it suffices to recall from the general theory of singular contact reduction that $\mathcal{C}_0^{(K)} \subset \partial \mathcal{C}_0^{(H)}$ if and only if $(H) \prec (K)$. Using the density of any maximal secondary stratum $CC_{(L)}$ in the corresponding contact piece $\mathcal{C}_0^{(L)}$, (i) follows. (iii) is a consequence of (v) if one considers the limit case $CC_{(K)} = CS_{(K)\succ(K)}$.

Finally, to prove (v), choose a point $[x] \in CS_{(K)\succ(H')} \subset \mathcal{C}_0$ and an open neighborhood $[x] \in O \subset \mathcal{C}_0$. We shall show that $O \cap CS_{(K)\succ(H)} \neq \varnothing$ if $(H) \prec (H') \prec (K)$. Let $x \in J^{-1}(0)$ be a preimage of $[x]$. We can assume without loss of generality that $G_x = H'$ and that the projection of x, i.e. the point $z = k(x) \in Q$, satisfies $G_z = K$. Let U be the only open G-saturated set in $J^{-1}(0)$ such that $U/G = O$. Then, identifying S^*Q with the unit bundle in T^*Q via a G-invariant metric on Q, we have that x is a unit covector lying in the subset of the cotangent fiber at z given by $(S_z^K)^* \oplus (N_z^*Q_{(K)})_{(H')}$. By the general properties of linear representations of compact groups on vector spaces and the property (iii) of cotangent-lifted actions in Lemma 1.4.2, it follows that $p_2(U \cap T_z^*Q) \cap (N_z^*Q_{(K)})_{(H)} \neq \varnothing$ for every compact subgroup H of K such that $H \prec H'$ and $(N_z^*Q_{(K)})_{(H)} \neq \varnothing$, i.e., $(H) \in I_Q$.

Here, p_2 is the linear projection $(S_z^K)^* \oplus N_z^* Q_{(K)} \to N_z^* Q_{(K)}$. From this, it follows that if $x' \in p_2(U \cap T_z^* Q)) \cap (N_z^* Q_{(K)})_{(H)}$, then $[x'] \in O \cap CS_{(K)\succ(H)}$. □

Remark 1.4.5. *The previous result shows that, identifying a stratum $CC_{(H)}$ with $S^*(Q_{(H)})$ as shown in Theorem 1.4.3, the reduced space $\mathcal{C}_0$ is almost everywhere a collection of cosphere bundles, one for each orbit type stratum of positive dimension in Q/G. These cosphere bundles satisfy the same frontier conditions as their bases, i.e., $S^*(Q^{(K)}) \subset \partial S^*(Q^{(H)})$ if and only if $Q^{(K)} \subset \partial Q^{(H)}$ (condition (i)), but in this case there is always a contact seam $CS_{(K)\succ(H)}$ between them, which "glues together" these two cosphere bundles, as reflected in conditions (ii) and (iii).*

A remark on the local properties of the C-L stratification of $\mathcal{C}_0$.

Throughout this paper we have used a purely topological concept of stratification (see subsection 1.2.1). However, in the literature most of the time the notion of stratification is a finer one, in a sense incorporating some sort of smooth structure not confined to each stratum. Namely, the additional condition usually imposed to a stratified space X is that of being a locally trivial cone space (which together with a smooth structure of degree ≥ 2 implies that X is a Whitney space, see [49] for details).

According to [34], the contact quotient $\mathcal{C}_0$ together with the stratification given by Theorem 1.2.2 is a locally trivial cone space. In that paper, the authors prove this fact using a contact analogue of the equivariant symplectic tubular neighborhood of Marle, Guillemin and Sternberg. They study the conical properties of the stratification in the local model provided by the corresponding equivariant tube $\phi : \mathcal{C} \to U$. This is possible since the basic ingredients to construct the strata, the orbit types $\mathcal{C}_{(H)}$, are mapped in the local model to $U_{(H)}$. However, this is not the case for the cosphere bundle. The building blocks of the secondary and C-L stratifications of a cosphere bundle quotient are $\pi^+\left(s_{(L)} \setminus \{0_{T^*Q_{(L)}}\}\right)$ and $\pi^+\left(s_{(H)\succ(L)}\right)$. In order to express them in the tubular neighborhood, one would need the tube ϕ to be explicitly defined or at least adapted to the cosphere bundle category in a way that reflects the fibrated nature of $\mathcal{C}$. Consequently, the problem of studying the local triviality of the secondary or C-L stratifications implies finding such an adapted normal form for cosphere bundles, which is yet unknown.

1.5 Singular base actions with cosphere regular lifts

In the following definition we introduce a class of actions which may have singularities on Q but that will be proven to yield regular lifted actions on S^*Q.

Definition 1.5.1. *An almost semifree action of G on Q is a smooth action such that a) it is free almost everywhere, b) the connected components of every orbit of non-maximal dimension are isolated, and c) for every non-trivial isotropy subgroup $H \in I_Q$ with Lie algebra $\mathfrak{h}$, its induced adjoint representation on $(\mathfrak{g}/\mathfrak{h}) \setminus \{0\}$ given by $h \cdot [\xi] = [\mathrm{Ad}_h \xi]$ is free.*

Note that for any almost semifree action, the quotient space Q/G consists on an open and dense stratum $Q^{(e)}$, except possibly for a set of isolated singular points. The next proposition shows that the class of almost semifree actions is in one-to-one correspondence with the class of free actions on S^*Q.

Proposition 1.5.1. *Let S^*Q be the cosphere bundle of Q endowed with the lift of a proper action of a Lie group G on Q. This lifted action is free if and only if the action on Q is almost semifree.*

Proof. Recall that, identifying with the help of a G-invariant Riemannian metric S^*Q with the unit bundle $SQ \subset TQ$ and TQ with T^*Q, G acts freely on S^*Q if and only if its tangent-lifted action on TQ is free on the unit bundle, and hence if it is free away from the zero section (since by linearity the lifted action intertwines the fiber rescaling by non-zero factors). Let $q \in Q$ with stabilizer $G_q = H \neq \{e\}$, $S \subset T_qQ$ a linear slice for the G-action at q and $v = \xi_Q(q) + s \in T_qQ \setminus \{0\}$. Note that all the admissible ξ's differ by an element of the Lie algebra of H. Then $U = G \cdot \exp_q(S)$ is a G-invariant neighborhood of the orbit $G \cdot q = G \cdot \exp_q(0)$ and there is an H-isomorphism $f : T_qQ \to \mathfrak{g}/\mathfrak{h} \times S$ given by $f(\xi_Q(q) + s) = ([\xi], s)$, where the H-invariance is with respect to the linear action on T_qQ and the diagonal action on $\mathfrak{g}/\mathfrak{h} \times S$ given by $h \cdot ([\xi], s) = ([\mathrm{Ad}_h \xi], h \cdot s)$. Consequently, $G_v = H_v = H_s \cap H_{[\xi]}$.

Suppose first that the lifted action of G on S^*Q is free. Then any point $q' \in U \setminus G \cdot q$ can be written as $q' = g \cdot \exp_q(s)$ for some $0 \neq s \in S$ with $g \in G$ and $G_{q'} = gH_sg^{-1} = \{e\}$, since $G_s = H_s = \{e\}$ as assumed above. Hence the G-action on Q is almost semifree.

For $v = \xi_Q(q_0) \in T_{q_0}Q \setminus \{0\}$ with $\xi \in \mathfrak{g}$, $\xi \notin \mathfrak{h}$ we obtain that $G_v = \{e\} = H_v = H_{[\xi]}$, thus proving that the induced adjoint representation on $(\mathfrak{g}/\mathfrak{h}) \setminus \{0\}$ is free.

To prove the converse implication, let $v \in T_qQ \setminus \{0\}$ as before, with $v = s + \xi_Q(q_0)$, where $s \in S$ and $\xi \in \mathfrak{g}$. If s is different from zero, multiplying it if necessary by a positive scalar smaller than one, we can guarantee that $G \cdot \exp_q(s) \subset U \setminus G \cdot q$. Shrinking U if necessary, we can guarantee that all of the points in $U \setminus G \cdot q_0$ have trivial isotropy, since the orbits of non maximal dimension are isolated by hypothesis. Using again the Tube Theorem, the isotropy groups of these points are $gH_sg^{-1} = \{e\}$, for every $g \in G$, which forces $H_s = \{e\}$ and hence $H_v = \{e\}$. In the case when $s = 0$, we have that $G_v = H_{[\xi]} = \{e\}$, thus completing the proof. □

Remark 1.5.1. *To geometrically express the third condition in Definition 1.5.1, notice that every non trivial isotropy subgroup $H = G_q \in I_Q$ acts freely on $(\mathfrak{g}/\mathfrak{h}) \setminus \{0\}$ if and only if for any element $h \in H$ the associated diffeomorphism of Q maps bijectively $\{\exp(t\xi) \cdot q \,:\, t \in \mathbb{R}\}$ to $\{\exp(t\,\mathrm{Ad}_h \xi) \cdot q \,:\, t \in \mathbb{R}\}$ for every $\xi \in \mathfrak{g}$ with $[\xi] \neq 0$ in $\mathfrak{g}/\mathfrak{h}$.*

Notice that this is a major difference with the cotangent bundle case, where the cotangent-lifted action is free if and only if the base action is free as well. In the context of cosphere bundle reduction the reason for the special interest in semifree actions and in finding necessary and sufficient conditions for the freeness of the lifted cosphere action is the following. Given a cosphere bundle $\mathcal{C} = S^*Q$ with the lift of a proper almost semifree action on Q, if we ignore the bundle structure of the contact manifold $\mathcal{C}$ we are in the hypothesis of regular contact reduction, since G acts freely, properly, and by strong contactomorphisms on $\mathcal{C}$. Therefore, the contact reduced space $\mathcal{C}_0$ is a well defined smooth contact manifold.

On the other hand, since the action on Q is not free in general, we cannot apply the main result on regular cosphere bundle reduction of [16] (see Theorem 1.2.3) because in that case the quotient

Q/G will not be a smooth manifold. In fact, one expects $\mathcal{C}_0$ to be a smooth reduced manifold fibrating continuously over the topological stratified space Q/G, but this bundle description cannot be achieved by only applying the scheme of regular cosphere bundle reduction. However, the results of the previous section will allow us to provide such a "stratified bundle" picture of the contact quotient $\mathcal{C}_0$. Indeed, we have the following result.

Theorem 1.5.1. *Let G be a Lie group acting properly and almost semifreely on Q and by lifts on the cosphere bundle S^*Q with contact momentum map $J : S^*Q \to \mathfrak{g}^*$. Write the orbit type decomposition of Q/G as*

$$Q/G = Q^{(e)} \coprod_{(H)\in I_Q \setminus I_Q^*} *^{(H)},$$

where $Q^{(e)} = Q_{(e)}/G$ is open and dense in Q/G and each $*^{(H)}$ with $(H) \in I_Q \setminus I_Q^*$ is an isolated point of some lower dimensional stratum $Q^{(H)}$ with $(H) \succ (e)$, lying in the boundary of $Q^{(e)}$. Then the quotient $\mathcal{C}_0 = J^{-1}(0)/G$ is a smooth manifold which can be decomposed as

$$\mathcal{C}_0 \simeq S^*(Q^{(e)}) \coprod_{(H)\in I_Q \setminus I_Q^*} CS^{(H)} \tag{1.5.1}$$

where each $CS^{(H)}$ is a trivial bundle over $*^{(H)}$ and a connected submanifold of $\mathcal{C}_0$ lying in the boundary of $S^*(Q^{(e)})$. Moreover, the manifolds $CS^{(H)}$ are Legendrian submanifolds of $\mathcal{C}_0$ in one-to-one correspondence with the singular orbits of the G-action on Q and have dimension $\dim Q - \dim G - 1$.

Proof. Since $J^{-1}(0)$ consists of a single orbit type (e), due to the fact that the lifted action to S^*Q is free, the secondary and C-L stratifications coincide with the partition (1.5.1). As for every $(H) \in I_Q$ different from (e) we have $(H) \in I_Q \setminus I_Q^*$, the contact seams $CS^{(H)} := CS_{(H)\succ(e)}$ are Legendrian submanifolds of $\mathcal{C}_0$. The dimension of each connected component is then given by formula (1.4.8) noting that $\dim Q_{(e)} = \dim Q$ and $\dim Q^{(H)} = \dim Q_{(H)} - \dim G + \dim H = 0$ for every $(H) \in I_Q \setminus I_Q^*$, since the action on Q is almost semifree. □

Recall that a group action is called *semifree* if it is free everywhere except for a set of isolated fixed points. Semifree actions are important particular cases of almost semifree actions and they are commonly found in examples. The following example explicitly illustrates the geometric constructions of this paper in that situation.

Example: S^1 acting on $S^*\mathbb{R}^2$. Consider $Q = \mathbb{R}^2$ with Euclidean coordinates (x_1, x_2) and its cotangent bundle $T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x_1, x_2, y_1, y_2). The action of S^1 by rotations on $\mathbb{R}^2$ (a semifree action with $\mathbb{R}^2{}_{(S^1)} = \{(0,0)\}$) lifts to $T^*\mathbb{R}^2$ by the induced diagonal action. A Hilbert basis for the ring of S^1-invariant polynomials for this cotangent lifted action is given by (see [13], §1.4)

$$\begin{aligned}
\sigma_1 &= x_1^2 + x_2^2 + y_1^2 + y_2^2, \\
\sigma_2 &= 2(x_1 y_1 + x_2 y_2), \\
\sigma_3 &= y_1^2 + y_2^2 - x_1^2 - x_2^2, \\
\sigma_4 &= x_1 y_2 - x_2 y_1.
\end{aligned}$$

These polynomials satisfy the semialgebraic relations

$$\sigma_1 \geq 0, \quad \sigma_1^2 = \sigma_2^2 + \sigma_3^2 + 4\sigma_4^2.$$

We can identify the cosphere bundle $S^*\mathbb{R}^2$ with the subset of $T^*\mathbb{R}^2$ given by the constraint

$$\sigma_1 + \sigma_3 = 2.$$

The cotangent lifted action restricts to $S^*\mathbb{R}^2$ giving the free lifted action by contactomorphisms. Its associated momentum map is given by

$$J(x_1, x_2, y_1, y_2) = \sigma_4$$

for $(x_1, x_2, y_1, y_2) \in S^*\mathbb{R}^2$. Consequently, using invariant theory, the contact reduced space $J^{-1}(0)/S^1$ is identified with the semialgebraic variety of $\mathbb{R}^3 = \{\sigma_2, \sigma_3, \sigma_1\}$ defined by

$$\mathcal{C}_0 \simeq \left\{(\sigma_2, \sigma_3, \sigma_1) \in \mathbb{R}^3 \,:\, \sigma_1 \geq 0,\, \sigma_1^2 = \sigma_2^2 + \sigma_3^2,\, \sigma_1 + \sigma_3 = 2\right\}.$$

This contact reduced space is in fact a smooth manifold since it is the parabola obtained intersecting the plane $P = \{\sigma_1 + \sigma_3 = 2\}$ with the upper half of the cone $\sigma_1^2 = \sigma_2^2 + \sigma_3^2$. Its smooth structure is induced from the ambient space $\mathbb{R}^3$. This was to be expected since the action on the contact manifold $S^*\mathbb{R}^2$ is free.

However, this reduced space is no longer a cosphere bundle since the action on the base is semifree. We investigate now how the stratified bundle structure of $\mathcal{C}_0$ obtained in the previous sections arises here. Note that $Q/G = \mathbb{R}^2/S^1$ can be identified with the subset of $\mathbb{R}^3$ given by

$$Q/G = \{(0, -t, t) \,:\, t \geq 0\},$$

which is a half-open line parallel to the plane P containing $\mathcal{C}_0$. According to the notation employed in this section, Q/G is a stratified space with strata $Q^{(e)}$ and $* = (0,0,0)$. The continuous fibration $k^0 : \mathcal{C}_0 \to Q/G$ is given by $k^0(\sigma_2, \sigma_3, \sigma_1) = (0, 1-\sigma_1, \sigma_1 - 1)$. Note that $(k^0)^{-1}(Q^{(e)}) = L \coprod R$ and $(k^0)^{-1}(*) = (0,1,1)$ (see figure 1.5), where $\mathcal{C}_0 = L \coprod R \coprod \{(0,1,1)\}$. In addition, recall that $Q^{(e)} \simeq \mathbb{R}$ and that $S^*\mathbb{R} = \mathbb{R} \sqcup \mathbb{R}$.

So $(k^0)^{-1}(Q^{(e)}) = L \coprod R$ is diffeomorphic to the cosphere bundle $S^*Q^{(e)}$. The fiber over a point $(0, -t, t) \in Q^{(e)}$ is the pair of points $(2\sqrt{t}, 1-t, 1+t)$ and $(-2\sqrt{t}, 1-t, 1+t)$ which lie in L and R respectively. Finally, the point $(0,1,1)$, the minimum of the parabola $\mathcal{C}_0$, is the seam $CS_{(S^1)\succ(e)}$ lying in the boundary of $S^*(Q^{(e)})$. Finally, since both $\mathcal{C}_0$ and $S^*(Q^{(e)})$ are one-dimensional, their contact structures are trivial, due to the fact that the corresponding contact distributions must be zero-dimensional.

1.6 Example: diagonal toral action on $\mathbb{R}^2 \times \mathbb{R}^2$

We illustrate the main results obtained in this paper with one more example rich enough to show all the extra structure appearing in the cosphere bundle singular reduction. This time, the reduced contact space $\mathcal{C}_0$ will have dimension bigger than one and will have hence a non-trivial contact structure.

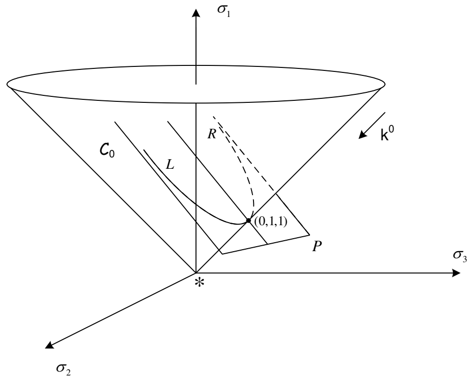

Figure 1.5.1: The contact reduced space as a parabola fibrating over a half-closed line.

Consider the proper action of $G = \mathbb{T}^2$ on $Q = \mathbb{R}^2 \times \mathbb{R}^2$, where each S^1 factor acts by rotations on the corresponding $\mathbb{R}^2$ factor. The isotropy lattice for this action is shown in Figure 1.6.1, where the subconjugation partial order is represented by arrows. Also, the corresponding stratification lattice is shown. A stratification lattice is a graphical arrangement of all the strata of a stratified space where for any two strata A, B with $A \subseteq \overline{B}$ and such that there is no other stratum C with the properties $A \subseteq \overline{C}$ and $C \subseteq \overline{B}$ we write $A \to B$. For the action under study, we have $I_Q \setminus I_Q^* = \{(\mathbb{T}^2)\}$.

Let $(x, y) = (x_1, x_2, y_1, y_2)$ be the Euclidean coordinates of a point in Q and $z = (x, y, u, v) = (x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2)$ the ones of a covector in $T^*Q \simeq \mathbb{R}^4 \times \mathbb{R}^4$. The ring of G-invariant polynomials on T^*Q is generated by

$$\begin{aligned} \rho_1 &= \|x\|^2 + \|u\|^2 & \sigma_1 &= \|y\|^2 + \|v\|^2 \\ \rho_2 &= 2(x \cdot u) & \sigma_2 &= 2(y \cdot v) \\ \rho_3 &= \|u\|^2 - \|x\|^2 & \sigma_3 &= \|v\|^2 - \|y\|^2 \\ \rho_4 &= x_1 u_2 - x_2 u_1 & \sigma_4 &= y_1 v_2 - y_2 v_1. \end{aligned}$$

These polynomials, which form a Hilbert basis, are subject to the following semi-algebraic relations

$$\rho_1 \geq 0, \quad \sigma_1 \geq 0, \quad \rho_1^2 = \rho_2^2 + \rho_3^2 + 4\rho_4^2, \quad \sigma_1^2 = \sigma_2^2 + \sigma_3^2 + 4\sigma_4^2.$$

Identifying the cosphere bundle $S^*\mathbb{R}^4$ with $\mathbb{R}^4 \times S^3 \subset \mathbb{R}^4 \times \mathbb{R}^4$, where $S^3 = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|u\|^2 + \|v\|^2 = 1\}$, it is easy to see that its contact structure is given by the kernel of the restriction of the Liouville one-form $\theta = udx + vdy$ and that the associated momentum map $J : S^*\mathbb{R}^4 \to \mathbb{R}^2$ is given by $J(x, y, u, v) = (\rho_4, \sigma_4) \in \mathbb{R}^2$. Consequently, we still have two more constraints to describe the zero-momentum level set:

$$\rho_4 = 0 \quad \text{and} \quad \sigma_4 = 0.$$

Notice that we can also see $S^*\mathbb{R}^4$ as the subset of $\mathbb{R}^8$ defined by the additional constraint

$$\rho_1 + \rho_3 + \sigma_1 + \sigma_3 = 2.$$

The associated G-invariant Hilbert map is defined by

$$\gamma : J^{-1}(0) \to \mathbb{R}^3 \times \mathbb{R}^3, \gamma(z) = (\rho_1(z), \rho_2(z), \rho_3(z); \sigma_1(z), \sigma_2(z), \sigma_3(z)),$$

and we can identify the reduced contact space with the image of γ, i.e. with the semialgebraic variety of $\mathbb{R}^6$ defined by

$$\mathcal{C}_0 \simeq \{(\rho;\sigma) \in \mathbb{R}^6 \,:\, \rho_1, \sigma_1 \geq 0,\, \rho_1^2 = \rho_2^2 + \rho_3^2,\, \sigma_1^2 = \sigma_2^2 + \sigma_3^2,\, \rho_1 + \rho_3 + \sigma_1 + \sigma_3 = 2\}$$

which is the intersection between the product of two cones, $C_1 \times C_2$ and the hypersurface $H := \{(\rho_1, \rho_3, \sigma_1, \sigma_3) \in \mathbb{R}^4 \,:\, \rho_1 + \rho_3 + \sigma_1 + \sigma_3 = 2\}$. (see Figure 1.6.2).

The Reeb vector field on $S^*\mathbb{R}^4$ is given by $R(x, y, u, v) = (u, v, 0, 0)$ for any $(x, y, u, v) \in \mathbb{R}^4 \times S^3$ and the flow of the corresponding reduced Reeb vector field on $\mathcal{C}_0$ at a point $(\rho_{\mathbf{0}}; \sigma_{\mathbf{0}})$ is easily computed as

$$\begin{aligned}
\rho_1(t) &= \rho_{01} + \rho_{02}t + \tfrac{1}{2}(\rho_{01} + \rho_{03})t^2 \\
\rho_2(t) &= \rho_{02} + (\rho_{01} + \rho_{03})t \\
\rho_3(t) &= \rho_{03} - \rho_{02}t - \tfrac{1}{2}(\rho_{01} + \rho_{03})t^2 \\
\sigma_1(t) &= \sigma_{01} + \sigma_{02}t + \tfrac{1}{2}(\sigma_{01} + \sigma_{03})t^2 \\
\sigma_2(t) &= \sigma_{02} + (\sigma_{01} + \sigma_{03})t \\
\sigma_3(t) &= \sigma_{03} - \sigma_{02}t - \tfrac{1}{2}(\sigma_{01} + \sigma_{03})t^2.
\end{aligned}$$

Applying Proposition 1.4.1, we know that the orbit types of $J^{-1}(0)$ are exactly those given by I_Q^* and hence the contact strata of $\mathcal{C}_0$ are in bijective correspondence with the strata of Q given by I_Q^*. We then have

$$\begin{array}{rcl}
T^*Q_{\mathbf{e}} &=& \{(x, y, u, v) \in \mathbb{R}^8 \,:\, (x, y) \neq \mathbf{0}\} \\
T^*Q_{(S^1 \times e)} &=& \{(\mathbf{0}, y, \mathbf{0}, v) \in \mathbb{R}^8 \,:\, y \neq \mathbf{0}\} \\
T^*Q_{e \times S^1} &=& \{(x, \mathbf{0}, u, \mathbf{0}) \in \mathbb{R}^8 \,:\, x \neq \mathbf{0}\} \\
N^*Q_{\mathbf{e}} &=& \{(x, y, \mathbf{0}, \mathbf{0}) \,:\, x \neq \mathbf{0}\,, y \neq \mathbf{0}\} \\
N^*Q_{S^1 \times e} &=& \{(\mathbf{0}, y, u, \mathbf{0}) \,:\, y \neq \mathbf{0}\} \\
N^*Q_{e \times S^1} &=& \{(x, \mathbf{0}, \mathbf{0}, v) \,:\, x \neq \mathbf{0}\}.
\end{array}$$

Consequently, a direct computation gives the following orbit types for the zero momentum map

$$\begin{array}{rcl}
J^{-1}(0) &=& \{z \in \mathbb{R}^4 \times S^3 : \rho_4(z) = \sigma_4(z) = 0\} \\
(J^{-1}(0))_{(\mathbf{e})} &=& \left\{z \in J^{-1}(0) : x \neq 0,\, y \neq 0\right\} \\
&& \coprod \left\{z \in J^{-1}(0) \,:\, x = 0\,, y \neq 0,\, u \neq 0\right\} \\
&& \coprod \left\{z \in J^{-1}(0) \,:\, x \neq 0, y = 0,\, v \neq 0\right\} \\
&& \coprod \left\{z \in \{0_{\mathbb{R}^4}\} \times S^3 \,:\, u \neq 0,\, v \neq 0\right\} \\
(J^{-1}(0))_{(e \times S^1)} &=& \left\{z \in J^{-1}(0) \,:\, y = v = 0,\, x \neq 0\right\} \\
&& \coprod \{z \in J^{-1}(0) \,:\, x = y = v = 0,\, \|u\| = 1\} \\
(J^{-1}(0))_{(S^1 \times e)} &=& \left\{z \in J^{-1}(0) \,:\, x = u = 0,\, y \neq 0\right\} \\
&& \coprod \{z \in J^{-1}(0) \,:\, x = y = u = 0,\, \|v\| = 1\}.
\end{array}$$

Using the image of the Hilbert map γ we can realize the contact strata given by Theorems 1.2.2, 1.4.2, and 1.4.4 as:

$$
\begin{aligned}
\mathcal{C}_0^{(\mathbf{e})} &= CC_{(\mathbf{e})} \coprod CS_{(S^1\times e)\succ(\mathbf{e})} \coprod CS_{(e\times S^1)\succ(\mathbf{e})} \coprod CS_{(\mathbb{T}^2)\succ(\mathbf{e})} \\
CC_{(\mathbf{e})} &= \big\{(\rho;\sigma) : \rho_1,\sigma_1 > 0,\ \rho_1 \neq \rho_3,\ \sigma_1 \neq \sigma_3,\ \rho_1^2 = \rho_2^2 + \rho_3^2, \\
&\quad\ \sigma_1^2 = \sigma_2^2 + \sigma_3^2,\ \rho_1 + \rho_3 + \sigma_1 + \sigma_3 = 2\big\} \\
CS_{(S^1\times e)\succ(\mathbf{e})} &= \{(\rho;\sigma) : \rho_1,\sigma_1 > 0,\ \sigma_1 \neq \sigma_3,\ \rho_1 = \rho_3,\ \rho_2 = 0, \\
&\quad\ 2\rho_1 + \sigma_1 + \sigma_3 = 2,\ \sigma_1^2 = \sigma_2^2 + \sigma_3^2\} \\
&= (\mathbb{R}_+ \times C_2) \cap \{2\rho_1 + \sigma_1 + \sigma_3 = 2,\ \sigma_1 \neq \sigma_3\} \\
CS_{(e\times S^1)\succ(\mathbf{e})} &= \{(\rho;\sigma) : \rho_1,\sigma_1 > 0,\ \rho_1 \neq \rho_3,\ \sigma_1 = \sigma_3,\ \sigma_2 = 0, \\
&\quad\ 2\sigma_1 + \rho_1 + \rho_3 = 2,\ \rho_1^2 = \rho_2^2 + \rho_3^2\} \\
&= (C_1 \times \mathbb{R}_+) \cap \{2\sigma_1 + \rho_1 + \rho_3 = 2,\ \rho_1 \neq \rho_3\} \\
CS_{(\mathbb{T}^2)\succ(\mathbf{e})} &= \{(\rho;\sigma) : \rho_1,\sigma_1 > 0,\ \rho_1 = \rho_3,\ \sigma_1 = \sigma_3,\ \rho_2 = \sigma_2 = 0, \\
&\quad\ \rho_1 + \sigma_1 = 1\} \\
\mathcal{C}_0^{(e\times S^1)} &= CC_{(e\times S^1)} \coprod CS_{(\mathbb{T}^2)\succ(e\times S^1)} \\
CC_{(e\times S^1)} &= \big\{(\rho;\mathbf{0}) : \rho_1 > 0,\ \rho_1 + \rho_3 = 2,\ \rho_1^2 = \rho_2^2 + \rho_3^2\big\} \setminus \{(1,0,1;\mathbf{0})\} \\
CS_{(\mathbb{T}^2)\succ(e\times S^1)} &= \{(1,0,1;0,0,0)\} \\
\mathcal{C}_0^{(S^1\times e)} &= CC_{(S^1\times e)} \coprod CS_{(\mathbb{T}^2)\succ(S^1\times e)} \\
CC_{(S^1\times e)} &= \big\{(\mathbf{0};\sigma) : \sigma_1 > 0,\ \sigma_1 + \sigma_3 = 2,\ \sigma_1^2 = \sigma_2^2 + \sigma_3^2\big\} \setminus \{(\mathbf{0};1,0,1)\} \\
CS_{(\mathbb{T}^2)\succ(S^1\times e)} &= \{(0,0,0;1,0,1)\}\,.
\end{aligned}
$$

The corresponding contact, secondary and C-L stratification lattices in $\mathcal{C}_0$ are shown in Figure 1.6.3. Notice that $(\mathbf{e})$ is the principal orbit type in Q and, therefore, $CC_{(\mathbf{e})}$ is open and dense in the reduced space $\mathcal{C}_0$. The contact seams $CS_{(\mathbb{T}^2)\succ(S^1\times e)}$, $CS_{(\mathbb{T}^2)\succ(e\times S^1)}$, and $CS_{(\mathbb{T}^2)\succ(\mathbf{e})}$ are Legendrian submanifolds of their contact strata, while the rest are coisotropic. Every contact seam is mapped by the flow of the reduced Reeb vector field into the CC-secondary stratum of its corresponding contact stratum as it can be easily checked.

In order to understand the bundle structure of these stratifications, we embed Q in T^*Q as the zero section and we identify Q/G with the subset of the image of γ given by

$$Q/G = \{(t_1,0,-t_1;t_2,0,-t_2) : t_1,t_2 \geq 0\} \simeq \mathbb{R}_+ \times \mathbb{R}_+,$$

a half-plane parallel to H. The strata of its orbit stratification are

$$
\begin{aligned}
Q^{(e\times S^1)} &= \{(t_1,0,-t_1;\mathbf{0}) : t_1 > 0\} \\
Q^{(S^1\times e)} &= \{(\mathbf{0};t_2,0,-t_2) : t_2 > 0\} \\
Q^{(\mathbf{e})} &= \{(t_1,0,-t_1;t_2,;0,-t_2) : t_1,t_2 > 0\}
\end{aligned}
$$

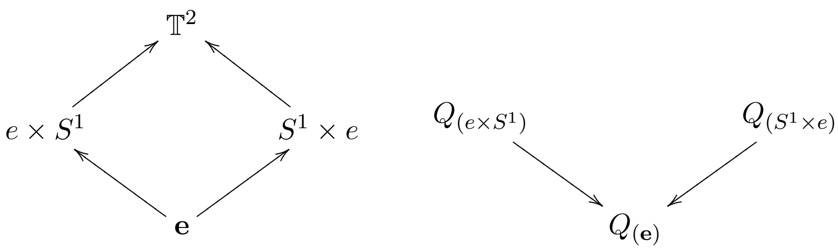

Figure 1.6.1: Isotropy and stratification lattices for the $\mathbb{T}^2$ action on $\mathbb{R}^4$.

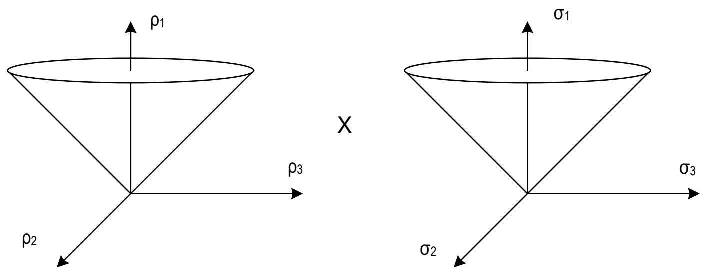

Figure 1.6.2: The ambient space of $\mathcal{C}_0$

and we obtain that the corresponding cosphere-like strata of $\mathcal{C}_0$ are diffeomorphic to the cosphere bundles

$$S^*(Q^{(e\times S^1)}) \simeq S^*(Q^{(S^1\times e)}) \simeq \mathbb{R} \sqcup \mathbb{R} \quad \text{and} \quad S^*(Q^{(\mathbf{e})}) \simeq \mathbb{R}^2 \times S^1.$$

The continuous fibration $k^0 : \mathcal{C}_0 \to Q/G$ is given by $k^0(\rho_1, \rho_2, \rho_3; \sigma_1, \sigma_2, \sigma_3) = (\rho_1 - 1, 0, 1 - \rho_1; \sigma_1 - 1, 0, 1 - \sigma_1)$.

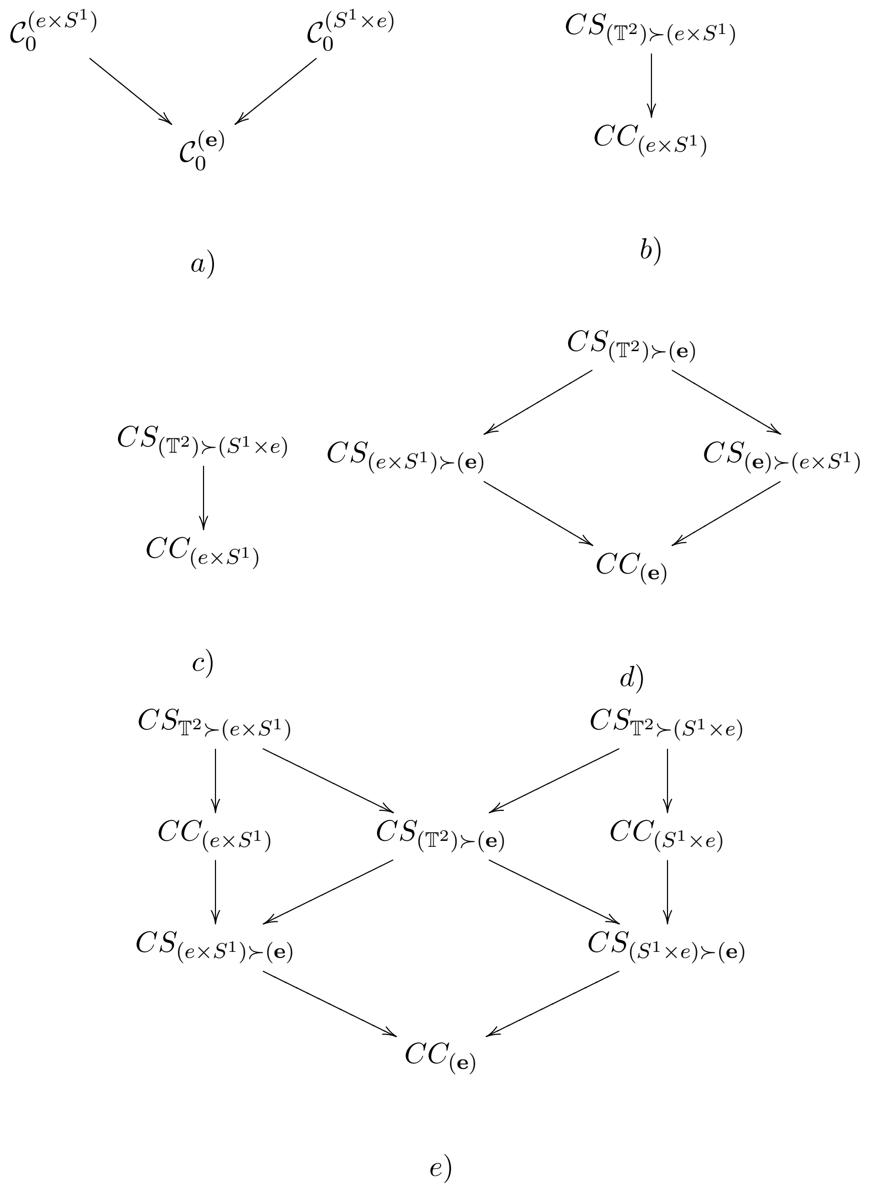

Figure 1.6.3: a) Contact stratification of $\mathcal{C}_0$. Secondary stratifications of: b) $\mathcal{C}_0^{(e\times S^1)}$, c) $\mathcal{C}_0^{(S^1\times e)}$ and d) $\mathcal{C}_0^{(\mathbf{e})}$. e) Coisotropic-Legendrian stratification of $\mathcal{C}_0$.

Chapter 2

Kähler and Sasakian ray reductions

2.1 Introduction

In this chapter we study geometric properties of Sasakian and Kähler quotients. We construct a reduction procedure for symplectic and Kähler manifolds using the ray preimages of the momentum map. More precisely, instead of taking as in point reduction the preimage of a momentum value μ, we take the preimage of $\mathbb{R}^+\mu$, the positive ray of μ. We have two reasons to develop this construction. One is geometric: non zero Kähler point reduction is not always well defined. The problem is that the complex structure may not leave invariant the horizontal distribution of the Riemannian submersion $\pi_\mu : J^{-1}(\mu) \to M_\mu$. The solution proposed in the literature, is based on the Shifting Theorem (see Theorem 6.5.2 in [57]). More precisely, one endowes the coadjoint orbit of μ, $\mathcal{O}_\mu$ with a unique up to homotheties Kähler-Einstein metric of positive Ricci curvature. This uniqness modulo homotheties is guaranteed by the choice of a Ad^*-invariant scalar product on $\mathfrak{g}^*$. Then, one performs the zero reduction of the Kähler difference of the base manifold M and $\mathcal{O}_\mu$. Unfortunately, the use of the Shifting Theorem is correct only in the case of totally isotropic momentum (i.e. $G_\mu = G$) since the Kostant-Kirillov-Sternberg form is replaced by another Kähler metric.

Namely, one performs zero reduction to the Kähler difference of the base manifold with $\mathcal{O}_\mu$. The uniqueness of the metric on the coadjoint orbit makes the reduced Kähler structure canonical. On the other hand, the ray Kähler reduction always exists and is canonical in the sense that its pull-back through the quotient projection is the initial Kähler form(this is not true for point reduction). [1]

The other reason is that it provides invariant submanifolds for conformal Hamiltonian systems. They are usually non-autonomous mechanical systems with friction whose integral curves preserve, in the case of symmetries, the ray pre-images of the momentum map. We extend the class of conformal

[1] We believe that the literature on non zero Kähler point reduction is not very accurate. Namely, using the unique Kähler-Einstein form on the coadjoint orbit, instead of the KKS-form makes impossible the use of the Shifting Theorem since the momentum map of the orbit will no longer be the inclusion. Therefore, one cannot connect the symplectic quotient M_μ in a canonical way to the zero quotient of the Kähler diffrence.

Hamiltonian systems already studied in the literature and complete the existing Lie Poisson reduction with the general ray one.

As examples of symplectic (Kähler) and contact (Sasakian) ray reductions we treat the case of cotangent and cosphere bundles. We show, proving a shifting type theorem that, theoretically, $(T^*Q)_{\mathbb{R}^+\mu}$ and $(S^*Q)_{\mathbb{R}^+\mu}$ are universal ray reduced spaces. Concrete examples of toric actions on spheres are also computed.

Using techniques of A. Futaki, we prove that, under appropiate hypothesis, ray quotients of Kähler-Einstein or Sasaki-Einstein manifolds remain Kähler or Sasaki-Einstein. Note that it suffices to prove the Kähler case and the compatibility of ray reduction with the Boothby-Wang fibration.

2.2 Sasaki-Einstein geometry

Traditionally, Ssakian manifolds were defined via contact structures by adding a Riemannian metric with certain compatibility conditions.

Definition 2.2.1. *A Sasakian structure on an exact contact manifold* $(S, \eta, \mathcal{R})$ *is a Riemannian metric* g *on* S *such that there is a* $(1,1)$*-tensor field* φ *with*

$$\varphi^2 = -Id + \eta \otimes \mathcal{R} \qquad \eta(X) = \mathrm{g}(X, \mathcal{R}) \qquad d\eta(X, Y) = \mathrm{g}(X, \varphi Y), \tag{2.2.1}$$

for any vector fields X, Y.

A good reference for this point of view is the book of D. E. Blair, [8].

There are other equivalent definitions of a Sasakian manifold and in the following proposition we present four of them. The first one is most in the spirit of the original definition of Sasaki(see [51]). The most geometric approach is highlighted in the second definition. It only uses the holonomy reduction of the associated cone metric and it was introduced by C. P. Boyer and K. Galicki in [10].

Proposition 2.2.1. *Let* (S, g) *be a Riemannian manifold of dimension* m*,* ∇ *the associated Levi-Civita connection, and* R *the Riemannian curvature tensor of* ∇*. Then, the followings are equivalent:*

- *there exists a unitary Killing vector field* $\mathcal{R}$ *on* S *so that the tensor field* φ *of type* $(1,1)$*, defined by* $\varphi(X) = \nabla_X \mathcal{R}$*, satisfies the condition*

  $$(\nabla_X \varphi)(Y) = \mathrm{g}(\mathcal{R}, Y)X - \mathrm{g}(X, Y)\mathcal{R},$$

  for any pair of vector fields X *and* Y *on* S*;*

- *the holonomy group of the cone metric on* S*,* $(\mathcal{C}(S), \mathcal{C}(\mathrm{g})) := (S \times \mathbb{R}^+, r^2\,\mathrm{g} + dr^2)$ *reduces to a subgroup of* $U(\frac{m+1}{2})$*. In particular,* $m = 2n + 1$*, for a* $n \geq 1$ *and* $(\mathcal{C}(S), \mathcal{C}(\mathrm{g}))$ *is Kähler;*

- *there exists a unitary Killing vector field* $\mathcal{R}$ *on* S *so that the Riemannian curvature satisfies the condition*

  $$R(X, \mathcal{R})Y = \mathrm{g}(\mathcal{R}, Y)X - \mathrm{g}(X, Y)\mathcal{R},$$

  for any pair of vector fields X *and* Y *on* S*;*

- *there exists a unitary Killing vector field $\mathcal{R}$ on S so that the sectional curvature of every section containing $\mathcal{R}$ equals one;*
- (S, g) *is a Sasakian manifold.*

For the proof, see [10].

Example: Sasakian spheres. One of the simplest compact examples of Sasakian manifolds is the standard sphere $S^{2n+1} \subset \mathbb{C}^n$ with the metric induced by the flat one on $\mathbb{C}^n$. The characteristic Killing vector field(i.e. the associated Reeb vector field) is given by $\mathcal{R}(\overrightarrow{p}) = -i\overrightarrow{p}$, i being the imaginary unit, and the contact form by the dual 1-form to $\mathcal{R}$. More general Sasakian structures on the sphere can be obtained by deforming this standard structure as follows. Let $\eta_A = \frac{1}{\sum a_j|z_j|^2}\eta$, for $0 < a_1 \leq a_2 \cdots \leq a_n$. Its Reeb field is $\mathcal{R}_A = \sum a_j(x_j\partial y_j - y_j\partial x_j)$. Clearly, η and η_A underly the same contact structure. Define the metric g_A by the conditions:

- $\mathrm{g}_A(X, Y) = \frac{1}{2}d\eta_A(IX, Y)$ on the contact distribution (here I is the standard complex structure of $\mathbb{C}^n$);
- $\mathcal{R}_A$ is normal to the contact distribution and has unit length.

It can be seen that $S_A^{2n-1} := (S^{2n-1}, \mathrm{g}_A)$ is a Sasakian manifold (cf. [27]). These Sasakian spheres are important since it was shown in [43] that each compact Sasakian manifold admits a CR-immersion in a S_A^{2N+1}.

Contact reduction at zero momentum was extended to the metric context (Sasakian manifolds) in [23]. Namely, they have constructed reduction at zero momentum of Sasakian manifolds. For Sasaki-Einstein base manifolds there were also given sufficient conditions for the reduced space to be Sasaki-Einstein.

Theorem 2.2.1. *Let $(S, \mathrm{g}, \mathcal{R})$ be a Sasakian manifold and G a compact Lie group acting on S by contact isometries. If the zero contact reduced space is well defined, then it is a Sasakian manifold.*

Theorem 2.2.2. *In the hypothesis of the above theorem, if S is Sasaki-Einstein and the length of the multi vector field $\xi_{1S} \wedge ... \wedge \xi_{\dim(G)S}$ is constant on the preimage of zero of the momentum map, then S_0 is also Sasaki-Einstein. Here, $\{\xi_j\}_{j=1,\dim(G)}$ denote a basis of $\mathfrak{g}$.*

2.3 Non zero momentum Sasakian reduction

2.3.1 The reduction theorem

Theorem 2.3.1. *Let $(S, g, \mathcal{R}, \eta)$ be a $(2n-1)$ dimensional Sasakian manifold, let G be a Lie group of dimension d acting on S by strong contactomorphisms. Let $J : S \to \mathfrak{g}^*$ be the momentum map associated to the action of G and let μ be an element of the dual $\mathfrak{g}^*$. We assume that:*

1. Ker $\mu + \mathfrak{g}_\mu = \mathfrak{g}$.
2. *The action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries.*
3. *J is transverse to $\mathbb{R}^+\mu$.*

Then the contact quotient

$$S_{\mathbb{R}^+\mu} = J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is a Sasakian orbifold with respect to the projected metric and the Reeb vector field.

Proof. We already know that the reduced space $S_{\mathbb{R}^+\mu}$ is a contact manifold (see [58]). What is left to be proved is that the metric g and the Reeb field $\mathcal{R}$ project on $S_{\mathbb{R}^+\mu}$, the latter onto a Killing field such that the curvature tensor of the projected metric satisfies formula (2.2.1).

From the transversality condition satisfied by the momentum map one knows that $J^{-1}(\mathbb{R}^+\mu)$ is an isometric Riemannian submanifold of S (which induced metric we also denote by g). As the flow of the Reeb field leaves invariant the level sets of the momentum J, one derives that the restriction of $\mathcal{R}$ is still a unit Killing field on $J^{-1}(\mathbb{R}^+\mu)$.

In order to establish the metric properties of the canonical projection $\pi_\mu : J^{-1}(\mathbb{R}^+\mu) \to S_{\mathbb{R}^+\mu}$, we have to understand the extrinsic geometry of the submanifold $J^{-1}(\mathbb{R}^+\mu) \subset S$. The first step is to find a basis of the normal bundle of $J^{-1}(\mathbb{R}^+\mu)$. To this end we look at the direct sum $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$ where $\mu \mid_{\mathfrak{m}} = 0$ (such a decomposition exists, because Ker $\mu + \mathfrak{g}_\mu = \mathfrak{g}$). Let $\mathfrak{m}_M = \{\xi_M | \xi \in \mathfrak{m}\}$ and recall that (see [58, Theorem 1]):

$$(T_x J^{-1}(\mathbb{R}^+\mu) \cap \operatorname{Ker} \eta_x) \oplus \mathbb{R}\mathcal{R}_x \oplus \mathfrak{m}_M(x) = (T_x \Phi^{-1}(0) \cap \operatorname{Ker} \eta_x) \oplus \mathbb{R}\mathcal{R}_x, \tag{2.3.1}$$

for any $x \in J^{-1}(\mathbb{R}^+\mu)$, where Φ is the momentum map associated to the action of K_μ on S.

Let now $\{\xi_1, \ldots, \xi_k\}$ and $\{\eta_1, \ldots, \eta_m\}$ be two bases in $\mathfrak{k}_\mu$ and, respectively, $\mathfrak{m}$. Without loss of generality, one may suppose that the fundamental fields $\{\eta_{jS}\}_{j=1,m}$ form an orthogonal basis of $\mathfrak{m}_S$, g-orthogonal on $TJ^{-1}(\mathbb{R}^+\mu) \cap \operatorname{Ker} \eta$ and that $\{\xi_{iS}\}_{i=1,k}$ are mutually orthogonal.

With these hypotheses, one derives that $\{\varphi\xi_{iS}, \varphi\eta_{jS}\}$ are linearly independent in each $x \in S$ and

$$g(\varphi\eta_{jS}, W) = g(\varphi\xi_{iS}, W) = d\eta(W, \xi_{iS}) = -\langle dJ(W), \xi_i\rangle = \langle r\mu, \xi_i\rangle = 0$$

for any vector field W tangent to $J^{-1}(\mathbb{R}^+\mu)$. Therefore, for any i, j, the fields $\{\varphi\xi_{iS}, \varphi\eta_{jS}\}$ belong to the normal bundle of $J^{-1}(\mathbb{R}^+\mu)$. A simple counting of the dimensions in the relation (2.3.1), together with the fact that $\{\varphi\xi_{iS}\}$ is a basis in the normal bundle of $T\Phi^{-1}(0)$ (see the proof of [23, Theorem 3.1]), imply that $\{\varphi\xi_{iS}, \varphi\eta_{jS}\}$ is indeed a basis of the normal bundle of $J^{-1}(\mathbb{R}_+\mu)$[2].

Let ∇, ∇^S be the Levi-Civita covariant derivatives of $J^{-1}(\mathbb{R}^+\mu)$ and S respectively and let A_i, A_j be the Weingarten operators associated to the unitary normal sections $\varphi\xi_{iS}/\|\xi_{iS}\|$, $1 \le i \le k$, $\varphi\eta_{jS}/\|\eta_{jS}\|$, $1 \le j \le m$. By applying the Weingarten formula and the relation (2.6.9), one obtains, for any X, Y, Z tangent to $J^{-1}(\mathbb{R}^+\mu)$:

$$\begin{aligned} g(A_i Y, Z) &= \|\xi_{iS}\|^{-1}\{g(\xi_{iS}, Y)\eta(Z) - g(\varphi\nabla^S_Y \xi_{iS}, Z)\}, \\ g(A_j Y, Z) &= \|\eta_{jS}\|^{-1}\{g(\eta_{jS}, Y)\eta(Z) - g(\varphi\nabla^S_Y \eta_{jS}, Z)\}. \end{aligned}$$

As K_μ acts by strong contact isometries, the metric g projects on a metric $g^{S_{\mathbb{R}^+\mu}}$ on $S_{\mathbb{R}^+\mu}$ with respect to which the canonical projection π_μ becomes a Riemannian submersion. We now show that

[2] $\{\varphi\xi_{iS}, \eta_{jS}\}$ is also a basis for $T^\perp J^{-1}(\mathbb{R}^+\mu)$ Our choice is only technically motivated.

the vertical distribution $\mathcal{V}$ is locally generated by the vector fields $\{\xi_{iS}\}$. We have indeed:

$$T_x\pi_\mu(\xi_{iS}(x)) = T_x\pi_\mu\,(\dot{c}(0)) = (\pi_\mu \dot{\circ}\, c)(0)$$

where $c(t) = \Phi(\exp t\xi_{iS}, x)$.

But $(\pi_\mu \circ c)(t) = \pi_\mu(x)$ for any t and then

$$T_x\pi_\mu(\xi_{iS}(x)) = 0 \quad \text{for any } x \in J^{-1}(\mathbb{R}^+\mu).$$

This proves that $\{\xi_{iS}\}_{1\le i\le k} \subset \mathcal{V}_x$ and, as $\dim \mathcal{V}_x = k$, it implies that $\{\xi_{iS}\}$ generate $\mathcal{V}$.

The formulae $\mathcal{L}_{\xi_{iS}}\mathcal{R} = 0$ for $i = 1, \dots, k$ prove that $\mathcal{R}$ is a projectable vector field and its projection $\mathcal{R}^{\mathbb{R}^+\mu}$ is a unit Killing field on the reduced space $S_{\mathbb{R}^+\mu}$.

Let X, Y, Z be vector fields orthogonal to $\mathcal{R}^{\mathbb{R}^+\mu}$. Using O'Neill's formulae (see [7, (9.28f)]) we derive:

$$\begin{aligned} \mathrm{g}^{S_{\mathbb{R}^+\mu}}(R^{S_{\mathbb{R}^+\mu}}(X, \mathcal{R}^{\mathbb{R}^+\mu})Y, Z) &= \mathrm{g}(R(X^h, \xi)Y^h, Z^h) + 2\,\mathrm{g}(A(X^h, \xi), A(Y^h, Z^h) \\ &\quad - \mathrm{g}(A(\mathcal{R}, Y^h), A(X^h, Z^h)) + \mathrm{g}(A(X^h, Y^h), A(\mathcal{R}, Z^h)), \end{aligned}$$

where X^h denotes the horizontal lift of the vector field X, A is O'Neill's (1,2) tensor field given by the relation: $A(Z^h, X^h) =$ *vertical part of* $\nabla^S_{Z^h}X^h$ and R the curvature tensor of the connection ∇ on $J^{-1}(\mathbb{R}^+\mu)$. On the other hand:

$$\begin{aligned} \mathrm{g}(\nabla_{Z^h}\mathcal{R}, \xi_{iS}) &= g(\varphi Z^h, \xi_{iS}) = d\eta(\xi_{iS}, Z^h) = \langle dJ(Z^h), \xi_{iS}\rangle \\ &= r\langle \mu, \xi_{iS}\rangle = 0, \end{aligned}$$

and hence:

$$\mathcal{R}^{S_{\mathbb{R}^+\mu}}(X, \mathcal{R}^{\mathbb{R}^+\mu})Y = R(X^h, \mathcal{R})Y^h.$$

This completes the proof.

□

Remark 2.3.1. *Under the hypothesis of the above theorem, the dimension of the reduced space is* $2n - (d + \mu) + 3 = 2n + 1 - 2k - m$, *where* $\mu = \dim(G_\mu)$.

2.3.2 Examples: actions of tori on spheres.

In Willett's reduction scheme, the smallest dimension of G which produces non-trivial examples is 2. We here present some complete computations for various actions of $G = T^2$ on $M = S^7$ with the standard Sasakian structure given by the contact form $\eta = \sum(x_j dy_j - y_j dx_j)$. When possible, we briefly discuss also the reduction at zero with the same group (the notations for the momentum maps will be the ones used in the previous section). Generalizations to S^{2n-1} are also indicated.

Note that our examples show the dependence of the dimension of the quotient on the choice of μ.

Example 2.3.1. *Let first T^2 act on S^7 by*

$$((e^{it_0}, e^{it_1}), (z_0, ..., z_3)) \mapsto (e^{it_0}z_0, e^{it_0}z_1, e^{it_1}z_2, e^{it_1}z_3).$$

Since G is commutative, $\mathfrak{g}_\mu = \mathfrak{g} = \mathbb{R}^2$.

For any $(r_1, r_2) \in \mathfrak{g}$ the associated infinitesimal generator is given by

$$\begin{aligned}(r_1, r_2)_{S^7}(z) &= r_1(-y_0\partial_{x_0} + x_0\partial_{y_0}) + r_1(-y_1\partial_{x_1} + x_1\partial_{y_1}) \\ &+ r_2(-y_2\partial_{x_2} + x_2\partial_{y_2}) + r_2(-y_3\partial_{x_3} + x_3\partial_{y_3})\end{aligned}$$

and the momentum map $J : S^7 \to (\mathbb{R}^2)^*$ reads $J(z) = \langle(|z_0|^2 + |z_1|^2, |z_2|^2 + |z_3|^2), \cdot\rangle$.

Let $\mu : \mathbb{R}^2 \to \mathbb{R}$, $\mu = \langle v, \cdot\rangle$, $v \in \mathbb{R}^2 \setminus \{0\}$ fixed. Then:

$$J^{-1}(\mathbb{R}^+\mu) = \begin{cases} S^3(\sqrt{\frac{v_1}{v_1+v_2}}) \times S^3(\sqrt{\frac{v_2}{v_1+v_2}}), & \textit{if } v_1, v_2 > 0 \\ S^3(\sqrt{\frac{v_1}{v_1+v_2}}), & \textit{if } v_1 > 0, v_2 = 0 \\ S^3(\sqrt{\frac{v_2}{v_1+v_2}}), & \textit{if } v_1 = 0, v_2 > 0 \end{cases}$$

For $v = (1, 0)$ $J^{-1}(\mathbb{R}^+\mu) = S^3$, Ker μ *$= \mathfrak{k}_\mu = \{0\} \times \mathbb{R}$, $K_\mu = \{e\} \times S^1$. The action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is trivial and hence $S_{\mathbb{R}^+\mu} = S^3$. In this case 0 is not a regular value of Φ-the momentum map associated to the K_μ action but, nevertheless, $\Phi^{-1}(0)$ is a submanifold of S^7 and hence the reduced space at zero, $\Phi^{-1}(0)/K_\mu$ is a Sasaki manifold. As $\Phi^{-1}(0) = S^3$ and $\mathcal{C}(S^n) = \mathbb{R}^{n+1} \setminus \{0\}$, we obtain that $(\mathcal{C}(S^7))_0 = \mathbb{R}^4 \setminus \{0\}$. Note that for this choice of μ reducing and taking the cone are commuting operations exactly as in the zero case.*

For $v = (1, 1)$ we obtain: $J^{-1}(\mathbb{R}^+\mu) = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}})$, $\mathfrak{k}_\mu = \{(-x, x)|x \in \mathbb{R}\}$, $K_\mu = \{(e^{-it}, e^{it})|e^{it} \in S^1\}$. The action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is given by

$$((e^{-it}, e^{it}), z) \mapsto (e^{-it}z_0, e^{-it}z_1, e^{it}z_2, e^{it}z_3),$$

thus $S_{\mathbb{R}^+\mu} = S^2 \times S^3$.

We can generalize this example for $S = S^{2n+1}$ by considering the action

$$((e^{it_0}, e^{it_1}), z) = (e^{it_0}z_0, e^{it_0}z_1, e^{it_1}z_2, ..., e^{it_1}z_n).$$

Now the momentum map is $J(z) = \langle(|z_0|^2 + |z_1|^2, \sum |z_k|^2), \cdot\rangle$. For μ as above, we have:

$$J^{-1}(\mathbb{R}^+\mu) = \begin{cases} S^3(\sqrt{\frac{v_1}{v_1+v_2}}) \times S^{2n-3}(\sqrt{\frac{v_2}{v_1+v_2}}), & \textit{if } v_1, v_2 > 0 \\ S^3(\sqrt{\frac{v_1}{v_1+v_2}}), & \textit{if } v_1 > 0, v_2 = 0 \\ S^{2n-3}(\sqrt{\frac{v_2}{v_1+v_2}}), & \textit{if } v_1 = 0, v_2 > 0 \end{cases}$$

For the same particular choices of μ as above, we obtain as reduced spaces respectively S^3, S^{2n-3} or $S^3 \times \mathbb{C}P^{n-2}$.

Example 2.3.2. *Let now the action be given by*

$$((e^{it_0}, e^{it_1}), z) \mapsto (e^{-it_0} z_0, e^{it_0} z_1, e^{it_1} z_2, e^{it_1} z_3).$$

The infinitesimal generator of the action will be

$$\begin{aligned}(r_1, r_2)_{S^7}(z) &= r_1(y_0\partial_{x_0} - x_0\partial_{y_0}) + r_1(-y_1\partial_{x_1} + x_1\partial_{y_1}) \\ &\quad + r_2(-y_2\partial_{x_2} + x_2\partial_{y_2}) + r_2(-y_3\partial_{x_3} + x_3\partial_{y_3}).\end{aligned}$$

The momentum map is $J(z) = \langle(|z_1|^2 - |z_0|^2, |z_2|^2 + |z_3|^2), \cdot\rangle$ *and*

$$J^{-1}(\mathbb{R}^+\mu) = \left\{ z \in S^7 \mid \exists s > 0 \text{ such that} \begin{cases} |z_1|^2 - |z_0|^2 - sv_1 = 0, \\ |z_2|^2 + |z_3|^2 - sv_2 = 0. \end{cases} \right\} \tag{2.3.2}$$

For $v = (1, 0)$ *we obtain*

$$J^{-1}(\mathbb{R}^+\mu) = \{z \in S^7 | z_2 = z_3 = 0, |z_1| > |z_0|\} = S^3 \setminus \{|z_1| \leq |z_0|\}.$$

The action of $K_\mu = \{e\} \times S^1$ *on* $J^{-1}(\mathbb{R}^+\mu)$ *is trivial, thus* $M_{\mathbb{R}^+\mu} = S^3 \setminus \{|z_1| \leq |z_0|\}$*, an open submanifold of* S^3*. For* $v = (1, 1)$*, solving for* s *the equations in* (2.3.2) *gives* $s \in (0, \frac{1}{2}]$*. Hence:*

$$\begin{aligned}J^{-1}(\mathbb{R}^+\mu) &\simeq \left(S^1(\frac{1}{\sqrt{2}}) \times S^5(\frac{1}{\sqrt{2}})\right) \setminus \left\{z \in S^7 \mid |z_0|^2 = \frac{1}{2}\right\} \\ &\simeq S^1(\frac{1}{\sqrt{2}}) \times (S^5(\frac{1}{\sqrt{2}}) \setminus S^1(\frac{1}{\sqrt{2}}))\end{aligned}$$

an open submanifold of the product of spheres.

The action of K_μ *on* $J^{-1}(\mathbb{R}^+\mu)$ *is given by*

$$((e^{-it}, e^{it}), z) \mapsto (e^{it} z_0, e^{-it} z_1, e^{it} z_2, e^{it} z_3).$$

Let A *denote the set* $\left\{z \in S^7(\sqrt{2}) \mid 0 < |z_2|^2 + |z_3|^2 \leq 1\right\}$*. Obviously, the above action of* K_μ *can be understood on the whole* $\mathbb{C}^4$ *and, as such, restricts to an action on* A*. Then* $S_{\mathbb{R}^+\mu}$ *is diffeomorphic with* $(S^1 \times S^5) \cap A/K_\mu$*. To identify the quotient, let* $g : \left(S^1 \times S^5\right) \cap A \to \left(S^1 \times S^5\right) \cap A$ *be given by*

$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_1^{-1}, z_2, z_3).$$

g *induces a map from* $\left(\left(S^1 \times S^5\right) \cap A\right) S^1$ *(with respect to the diagonal action of* S^1*) to* $\left(\left(S^1 \times S^5\right) \cap A\right) / K_\mu$*. The map*

$$(z_0, ..., z_3) \mapsto (\bar{z}_1 z_0, z_1, \bar{z}_1 z_2, \bar{z}_1 z_3)$$

is a diffeomorphism of $\left(S^1 \times S^5\right) \cap A$ *equivariant with respect to the diagonal action of* S^1 *and the action of* S^1 *on the first factor. Hence* $S_{\mathbb{R}^+\mu}$ *is diffeomorphic to* $S^5(\frac{1}{\sqrt{2}}) \setminus \mathrm{pr}\left\{z \in S^7 \mid |z_0|^2 = \frac{1}{2}\right\} \simeq S^5(\frac{1}{\sqrt{2}}) \setminus S^1(\frac{1}{\sqrt{2}})$*, where* $\mathrm{pr} : \mathbb{C}^4 \to \mathbb{C}^3$*,* $\mathrm{pr}(z_0, \ldots, z_3) = (z_0, z_2, z_3)$*.*

If we change the action on z_0 *with* $e^{-ikt} z_0$*, the reduced space will be the above one quotiented by* $\mathbb{Z}^k$ *(see also [23, Example 4.2]).*

Example 2.3.3. *Let us take this time:*

$$((e^{it_0}, e^{it_1}), z) \mapsto (e^{it_0} z_0, e^{it_1} z_1, e^{it_1} z_2, e^{it_1} z_3),$$

whose infinitesimal generator is

$$(r_1, r_2)_{S^7}(z) = r_1(-y_0 \partial x_0 + x_0 \partial y_0) + r_2 \sum_{j=1}^{3} (-y_j \partial x_j + x_j \partial y_j).$$

The momentum map is:

$$J(z) = \langle (|z_0|^2, |z_1|^2 + |z_2|^2 + |z_3|^2), \cdot \rangle.$$

For $J^{-1}(\mathbb{R}^+\mu)$ we obtain the following possibilities:

$$J^{-1}(\mathbb{R}^+\mu) = \begin{cases} S^1(\sqrt{\frac{v_1}{v_1+v_2}}) \times S^5(\sqrt{\frac{v_2}{v_1+v_2}}), & \text{if } v_1, v_2 > 0 \\ S^5(\sqrt{\frac{v_2}{v_1+v_2}}), & \text{if } v_1 = 0, v_2 > 0 \\ S^1(\sqrt{\frac{v_1}{v_1+v_2}}), & \text{if } v_2 = 0, v_1 > 0 \end{cases} \tag{2.3.3}$$

In particular, for $v = (1,0)$, $S_{\mathbb{R}^+\mu} = S^1$, for $v = (0,1)$, $S_{\mathbb{R}^+\mu} = S^5$ and for $v = (1,1)$ one obtains the same quotient as in the preceding example.

Example 2.3.4. *Considering the weighted action of T^2 on S^7 given this time by*

$$((e^{it_0}, e^{it_1}), z) \mapsto (e^{it_0 \lambda_0} z_0, e^{it_1 \lambda_1} z_1, z_2, z_3),$$

one obtains the momentum map

$$J(z) = \langle (\lambda_0 |z_0|^2, \lambda_1 |z_1|^2), \cdot \rangle.$$

For $v = (0,1)$ and λ_1 strictly positive, the reduced space is $S^5 \setminus S^3$ if $\lambda_0 \neq 0$ and $S^7 \setminus S^5$ if $\lambda_0 = 0$.

The cone construction is verified in this case. Indeed, $J^{-1}(0) = S^3$ and

$$(\mathcal{C}(S^7))_0 \simeq \mathcal{C}(S^5) = \mathcal{C}(S^3) \cup \mathcal{C}(S^5 \setminus S^3).$$

If $v = (1,1)$ and $\lambda_0, \lambda_1 > 0$,

$$J^{-1}(\mathbb{R}^+\mu) = \left\{ z \in S^7 \mid |z_1| = \sqrt{\frac{\lambda_0}{\lambda_1}} |z_0|, z_0 \neq 0 \right\} = S^7 \cap (\mathbb{C}^* \times A) \tag{2.3.4}$$

where A is the ellipsoid of equation

$$|z_1|^2 (1 + \frac{\lambda_1}{\lambda_0}) + |z_2|^2 + |z_3|^2 = 1.$$

The action of K_μ *on* $J^{-1}(\mathbb{R}^+\mu)$ *is given by*

$$((e^{-it}, e^{it}), z) \mapsto (e^{-it\lambda_0} z_0, e^{it\lambda_1} z_1, z_2, z_3)$$

and the reduced space

$$S_{\mathbb{R}^+\mu} = \bigcup_{(z_2,z_3)\in \mathrm{pr}(J^{-1}(\mathbb{R}^+\mu))} S^1(\beta^{-\lambda_0}\alpha^{\lambda_1}) \times \{(z_2, z_3)\}$$

where $\mathrm{pr} : \mathbb{C}^4 \to \mathbb{C}^2$, $\mathrm{pr}(z_0, \ldots, z_3) = (z_2, z_3)$, $\beta = \sqrt{\frac{\lambda_0(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$ *and* $\alpha = \sqrt{\frac{\lambda_1(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$.

If $[z] = [z']$ *in the reduced space then* $z_2 = z_2'$ *and* $z_3 = z_3'$. *So let* (z_2, z_3) *be fixed in* $pr(J^{-1}(\mathbb{R}^+\mu))$. $z \in J^{-1}(\mathbb{R}^+\mu)$ *and* $\mathrm{pr}(z) = (z_2, z_3)$ *imply* $|z_0| = \alpha$ *and* $|z_1| = \beta$. *The action of* K_μ *on* $J^{-1}(\mathbb{R}^+\mu)$ *is in fact the diagonal action of* S^1 *on the first two coordinates. Let* $f : (S^1(\alpha) \times S^1(\beta) \times \{(z_2, z_3)\})/S^1 \to S^1(\alpha^{\lambda_1}\beta^{-\lambda_0})$ *be the map given by*

$$[z] \mapsto z_0^{\lambda_1} z_1^{\lambda_0}.$$

One can easily check that f *is a diffeomorphism.*

In the previous examples, the Reeb flow on the reduced space is the restriction of the canonical one of the standard sphere. In this latter case, we obtain a non-standard Reeb flow.

We now write the flow of the Reeb field of the reduced contact form on $M_{\mathbb{R}^+\mu}$ *(for* $v = (1, 1)$*). Let* $r(t) = \left(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right)$, $Z = (z_0^2, z_0^3)^t$. *Then the flow is written as where*

$$\begin{aligned} A &= \|z_0^0\|^{\lambda_1} \|z_0^1\|^{\lambda_0}, \\ a &= \lambda_1 v_0 + \lambda_0 v_1, \ \textit{with}\, v_0 = \arg(z_0^0), v_1 = \arg(z_0^1), \\ b &= \lambda_1 + \lambda_0, \\ \mathcal{R}^{\mathbb{R}^+\mu}(t) &= \mathrm{diag}(r(t), r(t)). \end{aligned}$$

2.3.3 The sectional curvature of the quotient

Contact CR submanifolds

In this section we want to evaluate the sectional curvature of the Sasakian reduced space, both at zero and non zero momentum. For technical reasons, it will be convenient to place ourselves in a slightly more general situation. We first recall the following definition (see e.g. [6]):

Definition 2.3.1. *Let* $(S, g_S, \mathcal{R})$ *be a Sasakian manifold. An isometric submanifold* N *is called* contact CR *or* semi-invariant *if it admits two mutually orthogonal distributions* D *and* $D^\perp$*, such that:*

1. TN *decomposes orthogonally as:* $TN = D \oplus D^\perp \oplus \langle \mathcal{R} \rangle$ *and*

2. $\varphi D = D, \quad \varphi D^\perp \subseteq T^\perp N.$

We see that, in general, the normal bundle of the submanifold also splits into two orthogonal distributions: $\varphi D^{\perp}$ and its orthogonal complement that we denote by ν and which is invariant to the action of φ. We then have:

$$TM_{|N} = D \oplus D^{\perp} \oplus \langle \mathcal{R} \rangle \oplus \varphi D^{\perp} \oplus \nu.$$

For a vector field V normal to N we shall denote $\bar{V}$, respectively $\tilde{V}$ its component in $\varphi D^{\perp}$, respectively in ν. Such submanifolds have been extensively studied in the last thirty years.

Obviously, very natural examples are the level sets of Sasakian momentum maps. To better mimic our situation, we moreover make the following:

Assumption. There exists a Riemannian submersion $\pi : N \longrightarrow P$ over a Sasakian manifold $(P, g_P, \mathcal{R}_P)$ such that:

1. $D \oplus \langle \mathcal{R} \rangle$ represents the horizontal distribution of the submersion; (and hence $D^{\perp}$ represents the vertical distribution of the submersion);

2. The two Reeb fields are π-related: $\mathcal{R}$ is basic and projects over $\mathcal{R}_P$.

This situation was already considered by Papaghiuc in [47], on the model of Kobayashi's paper [30] where the similar setting was discussed in Kählerian context.

Let $\phi := \nabla^P \mathcal{R}_P$ and observe that in our assumption we have $(\phi X)^h = \varphi X^h$.

We want to relate the sectional curvature of planes generated by orthonormal pairs $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. This is usually known as φ-sectional curvature, the analogue in Sasakian geometry of holomorphic sectional curvature; it completely determines the curvature tensor, cf. [8], so it is worth having information about it.

We first apply (as in the proof of Theorem 2.3.1) O'Neill's formula to relate the curvatures of N and P. For X tangent to P and orthogonal to ζ, (this is not restrictive, as the planes passing through the Reeb field have sectional curvature 1 on a Sasakian manifold), using the anti-symmetry of the tensor A, we obtain:

$$R^N(X^h, \varphi X^h, X^h, \varphi X^h) - R^P(X, \phi X, X, \phi X) = -3\|A(X^h, \varphi X^h)\|_N^2, \tag{2.3.5}$$

where the sub-index refers to the norm with respect to g^N.

The next step is to apply the Gauss equation to the Riemannian submanifold N of S:

$$\begin{aligned} R^S(X^h, \varphi X^h, X^h, \varphi X^h) &- R^N(X^h, \varphi X^h, X^h, \varphi X^h) \\ &= \|h(X^h, \varphi X^h)\|_S^2 - \mathrm{g}_S(h(X^h, X^h), h(\varphi X^h, \varphi X^h)). \end{aligned} \tag{2.3.6}$$

We now need to relate the tensors A and h. To this end, we write hE, respectively vE for the horizontal, respectively vertical part of a tangent (to N) vector field E and we first decompose

$$\nabla^S_{X^h}(\varphi Y^h) = h\nabla^S_{X^h}(\varphi Y^h) + A(X^h, \varphi Y^h) + h(X^h, \varphi Y^h).$$

Then we use the formulae $(\nabla^S_E \varphi)F = \eta(F)E - \mathrm{g}_S(E, F)\xi$ (see [8]) and $(\nabla^S_E \varphi)F = \nabla^S_E(\varphi F) - \varphi \nabla^S_E F$ to express $\nabla^S_{X^h}(\varphi Y^h)$. Finally, equaling the tangent and normal parts in the equation we obtain this way, we arrive at the following relations:

$$\begin{aligned} A(X^h, \varphi Y^h) &= v\varphi h(X^h, Y^h), \\ h(X^h, \varphi Y^h) &= \varphi A(X^h, Y^h) + \varphi h(\widetilde{X^h, Y^h}). \end{aligned} \tag{2.3.7}$$

Note that if $\varphi D^\perp = T^\perp N$ ($i.e.$ $\nu = \{0\}$), and this is the case when N is the *zero* level set of a Sasakian momentum map, the above relations simplify to:

$$\begin{aligned} A(X^h, \varphi Y^h) &= \varphi h(X^h, Y^h), \\ h(X^h, \varphi Y^h) &= \varphi A(X^h, Y^h). \end{aligned} \tag{2.3.8}$$

In the general case, from (2.3.7) we easily derive:

$$h(\varphi X^h, \varphi Y^h) = \overline{h(X^h, Y^h)} - h(\widetilde{X^h, Y^h}),$$

and hence

$$\mathrm{g}_S(h(\varphi X^h, \varphi Y^h), h(X^h, Y^h)) = \|\overline{h(X^h, Y^h)}\|_S^2 - \|h(\widetilde{X^h, Y^h})\|_S^2. \tag{2.3.9}$$

From equation (2.6.10) it follows that on the orthogonal complement of ξ, the tensor φ acts like an isometry. Therefore, using again (2.3.7), we derive:

$$\begin{aligned} \|h(X^h, \varphi Y^h)\|_S^2 &= \|A(X^h, Y^h)\|_S^2 + \|h(\widetilde{X^h, Y^h})\|_S^2, \\ \|A(X^h, \varphi Y^h)\|_S^2 &= \|\overline{h(X^h, Y^h)}\|_S^2. \end{aligned} \tag{2.3.10}$$

Let us denote $K^P_\phi(X)$, respectively $K^S_\varphi(X^h)$ the sectional curvature of the plane $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. Adding equations (2.3.5), (2.3.6) and using (2.3.9), (2.3.10), we finally obtain (taking again into account the anti-symmetry of A):

$$K^P_\phi(X) = K^S_\varphi(X^h) + 4\|\overline{h(X^h, X^h)}\|_S^2 - 2\|h(\widetilde{X^h, X^h})\|_S^2. \tag{2.3.11}$$

The curvature of the quotient

In general, from equation (2.3.11) one hopes to deduce the positivity of the φ-sectional curvature of the quotient. This depends on the extrinsic geometry of the level set, which is a data additional to the reduction scheme: the second fundamental form of the level set cannot be entirely expressed in terms of the action. But in some particular cases, one is able to derive a conclusion.

Obviously the simplest situation occurs when $J^{-1}(\mathbb{R}^+\mu)$ is totally geodesic in S: then the φ-sectional curvatures of S and $_{\mathbb{R}^+\mu}$ are equal. In fact, one is only interested in the vanishing of $h(X^h, Y^h)$, which, by the first equation in (2.3.7), is implied by the vanishing of O'Neill's integrability tensor A. This is a rather strong condition, implying that $J^{-1}(\mathbb{R}^+\mu)$ is a locally a (not necessarily Riemannian) product and cannot be predicted by the action. Other conditions on the second fundamental form which are common in Riemannian and Cauchy-Riemann submanifold theory, see e.g. [7], (mixed totally geodesic, (contact)-totally umbilical, extrinsic sphere etc.) and permit some speculations in (2.3.11) or even the computation of the Ricci curvature of the quotient, seem to be artificial in this context, as not directly expressible in terms of the action.

We apply the above computation for N being $J^{-1}(0)$ and for P being the respective reduced space. Then equation (2.3.11) implies:

Proposition 2.3.1. *The reduced space at* 0 *of a Sasakian manifold with positive* φ*-sectional curvature (in particular of an odd sphere with the standard Sasakian structure) has strictly positive* φ*-sectional curvature.*

2.4 Symplectic and Kähler ray-reductions

Let G be a Lie group acting smoothly, properly, by symplectomorphisms and in a Hamiltonian way on a symplectic manifold (M, ω). Denote by $J : M \to \mathfrak{g}^*$ the associated momentum map and recall that it is G-equivariant. For any element $\mu \in \mathfrak{g}^*$, let K_μ be the its kernel group.

Definition 2.4.1. *We define the quotient of* M *by* G *at* $(\mathbb{R}^+)\mu$ *to be* $M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$. $M_{\mathbb{R}^+\mu}$ *will be called the* ***ray reduced space at*** μ.

In this section we will show that, under certain hypothesis, the ray quotient admits a natural symplectic or Kähler structure, once the initial manifold is symplectic or Kähler. The proof of the next theorem is an analogous of the proof given in [58] for the contact case (see Theorem 1).

For the two results of this section we will need three lemmas. The first is a characterization of a locally free action and the last two are classical results of symplectic linear algebra.

Lemma 2.4.1. J *is transverse to* $\mathbb{R}^+\mu$ *if and only if* K_μ *acts locally freely on* $J^{-1}(\mathbb{R}^+\mu)$.

Lemma 2.4.2. *Consider a symplectic vector space* (V, Ω) *and* $W \subset V$ *an isotropic subspace. Then,* $\ker \Omega \mid_{W^\Omega} = W$*, where* W^Ω *is the symplectic perpendicular of* W.

Lemma 2.4.3. *Let* V *be a vector space and* $\Omega : V \times V \to \mathbb{R}$ *an antisymmetric and bilinear two-form. If* V *admits the direct decomposition* $V = X \oplus V$ *with respect to* Ω *and* $\ker \Omega \subseteq \ker \Omega \mid_X$*, then* $\ker \Omega = \ker \Omega \mid_X$.

We are now ready to prove the first theorem of this section.

Theorem 2.4.1. *Suppose* (M, ω) *is a symplectic manifold endowed with a Hamiltonian action of the Lie group* G*. Let* $\mu \in \mathfrak{g}^*$ *and* K_μ *its kernel group. Denote by* $J : M \to \mathfrak{g}^*$ *the associated momentum map and assume that the following hypothesis are verified:*

1° K_μ *acts properly on* $J^{-1}(\mathbb{R}^+\mu)$;

1° J *is transverse to* $\mathbb{R}^+\mu$;

3° $\mathfrak{g} = \ker \mu + \mathfrak{g}_\mu$.

Then the ray quotient at μ

$$M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is a naturally symplectic orbifold, i.e. its symplectic structure $\omega_{\mathbb{R}^+\mu}$ *is given by*

$$\pi_\mu^* \omega_{\mathbb{R}^+\mu} = i_\mu^* \omega,$$

where

$$\pi_\mu : J^{-1}(\mathbb{R}^+\mu) \to M_{\mathbb{R}^+\mu} \qquad \textit{and} \qquad i_\mu : J^{-1}(\mathbb{R}^+\mu) \hookrightarrow M$$

are the canonical projection and immersion respectively.

Proof. The transversality of the momentum map with respect to $\mathbb{R}^+\mu$, ensures that $J^{-1}(\mathbb{R}^+\mu)$ is a submanifold of M. Lemma 2.4.1 implies that the quotient $M_{\mathbb{R}^+\mu}$ is an orbifold and that π_μ is a surjective submersion in the category of orbifolds.

The first step is to see that the restriction of the symplectic form on $J^{-1}(\mathbb{R}^+\mu)$ is projectable on the quotient $M_{\mathbb{R}^+\mu}$ to a canonical 2-form denoted $\omega_{\mathbb{R}^+\mu}$. For any $\xi \in \mathfrak{k}_\mu$ and any x in M, we have that

$$T_x\pi_\mu(\xi_M(x)) = \left.\frac{d}{dt}\right|_{t=0} \pi_\mu(\exp t\xi \cdot x) = \left.\frac{d}{dt}\right|_{t=0} \pi_\mu(x) = 0.$$

Hence, $\langle\{\xi_{J^{-1}(\mathbb{R}^+\mu)} \,|\, \xi \in \mathfrak{k}_\mu\}\rangle \subset \ker(T\pi_\mu)$. A count of dimensions shows that, in fact, the vertical distribution of π_μ is generated by all the infinitezimal isometries associated to the elements of $\mathfrak{k}_\mu$. Since $\omega \,|_{J^{-1}(\mathbb{R}^+\mu)} = i_\mu^*\omega$ is K_μ-invariant, it follows that its Lie derivative with respect to all vector fields $\{\xi_{J^{-1}(\mathbb{R}^+\mu)} \,|\, \xi \in \mathfrak{k}_\mu\}$ is zero. Let $x \in J^{-1}(\mathbb{R}^+\mu)$ with $J(x) = r\mu$ and $v \in T_x(J^{-1}(\mathbb{R}^+\mu))$. Then, identifying $T_{J(x)}\mathbb{R}^+\mu$ with $\mathbb{R}\mu$, we obtain

$$\begin{aligned} \omega(i_\mu(x))(\xi_M(x), T_x i_\mu v) = T_{i_\mu(x)} J \,|_{J^{-1}(\mathbb{R}^+\mu)} (v)(\xi) = \\ i_\mu^*(TJ \,|_{J^{-1}(\mathbb{R}^+\mu)})(v)(\xi) = \mu(\xi) = 0. \end{aligned}$$

It follows that $i_\mu^*\omega$ is a basic one-form which projects on $M_{\mathbb{R}^+\mu}$ to the closed form $\omega_{\mathbb{R}^+\mu} \in \Lambda^2(T^*M_{\mathbb{R}^+\mu})$ with the property that $\pi_\mu^*\omega_{\mathbb{R}^+\mu} = i_\mu^*\omega$.

Since $\omega_{\mathbb{R}^+\mu}$ is a closed form, it remains to prove that it is also non-degenerate. For this, we will show that $T_x(K_\mu \cdot x) = \ker(i_\mu^*\omega)(x)$, for any $x \in J^{-1}(\mathbb{R}^+\mu)$. Fix $x \in J^{-1}(\mathbb{R}^+\mu)$ with $J(x) = t\mu$ and denote by $\Psi : M \to \mathfrak{k}_\mu^*$ the momentum map associated to the action of the kernel group of μ on M. Let $i^T : \mathfrak{g}^* \hookrightarrow \mathfrak{k}_\mu^*$ be the canonical inclusion. Then, $\Psi = i^T \circ J$ and $J^{-1}(\mathbb{R}^+\mu) \subset J^{-1}(\mathfrak{k}_\mu^\circ) = \Psi^{-1}(0)$. Notice that $J^{-1}(\mathbb{R}^+\mu) \cap G \cdot x = G_{\mathbb{R}^+\mu} \cdot x$, where $G_{\mathbb{R}^+\mu} = \{g \in G \mid \mathrm{Ad}_g^*\,\mu = r\mu, r > 0\}$ is the ray isotropy group of μ. This Lie group has many interesting properties for which we refer the reader to Section 2.7. For any $v \in (T_x(K_\mu \cdot x))^{\omega_x}$, $\omega_x(v, \xi_M(x)) = 0$, $\forall \xi \in \mathfrak{k}_\mu$ if and only if $T_xJ(v)(\xi) = 0$, $\forall \xi \in \mathfrak{k}_\mu$. Therefore, $(T_x(K_\mu \cdot x))^{\omega_x} = T_xU$, where $U := J^{-1}(\mathfrak{k}_\mu^\circ) = \Psi^{-1}(0)$. We can assume U to be a submanifold of M because the transversality condition satisfied by the momentum map implies that K_μ acts locally freely at least on a neighborhood of $J^{-1}(\mathbb{R}^+\mu)$ in U, if not on the whole U.

Applying Lemma 2.4.2 for $(V, \Omega) := (T_xM, \omega_x)$ and $W := T_x(K_\mu \cdot x)$, we obtain that $\ker \omega_x \,|_{T_xU} = T_x(K_\mu \cdot x)$. We have already seen that $T_x(K_\mu \cdot x) \subset \ker i_\mu^*\omega_x$. It follows that

$$\ker \omega_x \,|_{T_xU} \subset \ker \omega_x \,|_{T_xJ^{-1}(\mathbb{R}^+\mu)} . \tag{2.4.1}$$

Since $\mathfrak{g} = \ker\mu + \mathfrak{g}_\mu$, we can chose a decomposition $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$, where $\mu\mid_{\mathfrak{m}}= 0$. Let $\mathfrak{m}_M := \{\xi_M(x) \mid \xi \in M\}$. For any $\xi \in \mathfrak{m}$ and $\eta \in \mathfrak{k}_\mu$, the equivariance of the momentum map implies that

$$T_xJ(\xi_M(x))(\eta) = \xi_{\mathfrak{g}^*}(t\mu)(\eta) = -t\langle\mu, [\xi, \eta]\rangle = t\eta_{\mathfrak{g}^*}(\mu)(\xi) = 0.$$

Therefore, $\mathfrak{m}_M(x) \subset T_xU$ and $T_xJ(\mathfrak{m}_M(x)) \subset T_{t\mu}(G \cdot t\mu)$. It is easy to see that

$$T_xJ\mid_{\mathfrak{m}_M(x)} : \mathfrak{m}_M(x) \to T_{t\mu}(G \cdot t\mu)$$

is a linear isomorphism and, hence,

$$T_xJ(\mathfrak{m}_M(x)) = T_{t\mu}(G \cdot t\mu). \tag{2.4.2}$$

Notice that equation 2.4.2, the third hypothesis which can equivalently be expressed as $\{0\} = (\ker\mu)^\circ \cap (\mathfrak{g}_\mu)^\circ = \mathbb{R}\mu \cap T_{t\mu}(G \cdot t\mu)$, and the fact that $T_xJ(T_xJ^{-1}(\mathbb{R}^+\mu)) \subset \mathbb{R}\mu$ imply that

$$\mathfrak{m}_M(x) \cap T_xJ^{-1}(\mathbb{R}^+\mu) = \{0\}. \tag{2.4.3}$$

A simple dimension calculus shows that $\mathfrak{m}_M(x)$ and $T_xJ^{-1}(\mathbb{R}^+\mu)$ are complementary subspaces of T_xU. We have also seen that they are perpendicular with respect to $\omega_x\mid_{T_xU}$. Using relation 2.4.1, we can now apply Lemma 2.4.3 for $V := T_xM$, $W := \mathfrak{m}_M(x)$, and $X := T_xJ^{-1}(\mathbb{R}^+\mu)$. Thus, we obtain that $\ker\omega_x\mid_{T_xU}= T_x(K_\mu \cdot x) = \ker\omega_x\mid_{T_xJ^{-1}(\mathbb{R}^+\mu)}$, for any $x \in J^{-1}(\mathbb{R}^+\mu)$. This shows that $\omega_{\mathbb{R}^+\mu}$ is a non-degenerate form, completing thus our proof. □

Notice that in the case $\mu = 0$ we recover the reduced symplectic space at zero. Without the hypothesis that K_μ acts properly on $J^{-1}(\mathbb{R}^+\mu)$, the quotient $M_{\mathbb{R}^+\mu}$ may not be Hausdorff. As the Lemma 2.4.1 proves, the second hypothesis of this theorem ensures that $M_{\mathbb{R}^+\mu}$ is an orbifold. If μ is non-zero and the kernel and isotropy groups of μ coincide, then the quotient may fail to be symplectic. For an example, take the symplectization of the contact manifold used for instance in 3.7 from [58]. One will obtain a quotient of odd dimension and, hence, the necessity of the last condition in Theorem 2.4.1.

Corollary 2.4.1. *In the hypothesis of Theorem 2.4.1, if the dimension of M is $2n$ and the Lie group G is d-dimensional, then the dimension of the symplectic quotient is $2n - 2k - m = 2n - p - d + 2$, where $p = \dim(G_\mu) = k + 1$.*

A large class of examples can be obtained in the case when (M, ω) is the cotangent bundle of a manifold Q endowed with the canonical syplectic form $\omega_0 = -d\theta_0$.

Theorem 2.4.2. *Let Q be a differentiable manifold of real dimesnion n, G a finite dimensional Lie group acting smoothly on Q. Denote by μ an element of the dual Lie algebra $\mathfrak{g}^*$ and by K_μ its kernel group. Assume that k_μ acts freely and properly on $J^{-1}(\mathbb{R}^+\mu)$, with $J : T^*Q \to \mathfrak{g}^*$ the canonical momentum map associated to the G-action. Then the ray reduced space $(T^*(Q))_{\mathbb{R}^+\mu}$ is embedded by a map preserving the symplectic structures onto a subbundle of $T^*(Q/K_\mu)$.*

We do not give the proof here since it is similar to the analogous result for ray reduction of cosphere bundles (see Theorem 4.2 in [16]).

We will now extend this reduction procedure to the metric context, i.e. for Kähler manifolds.

Theorem 2.4.3. *Let (M, g, ω) be a Kähler manifold and G a Lie group acting on M by Hamiltonian symplectomorphisms. If $J : M \to \mathfrak{g}^*$ is the momentum map associated to the action of G and μ an element of $\mathfrak{g}^*$, assume that:*

$1°$ Ker $\mu + \mathfrak{g}_\mu = \mathfrak{g}$;

$2°$ *the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries;*

$3°$ *J is transverse to $\mathbb{R}^+\mu$.*

Then the ray quotient at μ

$$M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is a Kähler orbifold with respect to the projection of the metric g.

Proof. From Theorem 2.4.1, we already know that $(M_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu})$ is a symplectic orbifold. It remains to show that the symplectic structure is also a Kähler one with corresponding metric given by the projection of g. The second hypothesis ensures that $(J^{-1}(\mathbb{R}^+\mu)\,,\, i_\mu^*\,\mathrm{g})$ is an isometric Riemannian submanifold of M.

Again, we will use a decomposition $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$, where $\mu\mid_{\mathfrak{m}} = 0$. Let $\Psi : M \to \mathfrak{k}_\mu^*$ be the momentum map associated to the action of K_μ on M and $\mathfrak{m}_M := \{\xi_M(x) \mid \xi \in M\}$. In the proof above we have already seen that

$$T_x J^{-1}(\mathbb{R}^+\mu) \oplus \mathfrak{m}_M(x) = T_x \Psi^{-1}(0), \tag{2.4.4}$$

for any $x \in J^{-1}(\mathbb{R}^+\mu)$. Let $\{\xi_1, \cdots \xi_k\}$ and $\{\eta_1, \cdots \eta_m\}$ be basis in $\mathfrak{k}_\mu$ and $\mathfrak{m}$ respectively, where $m = \dim \mathfrak{m}$ and $k = \dim \mathfrak{k}_\mu$. Without loss of generality, we can assume that the infinitesimal isometries $\{\xi_{iM}\}_{i=1,m}$ and $\{\eta_{jM}\}_{j=1,m}$ are g-orthogonal basis of $\mathfrak{m}$ and $\mathfrak{k}_\mu$. Thus, $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ are linearly independent in each point of $J^{-1}(\mathbb{R}^+\mu)$. Even more, $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ belong to the normal fiber bundle to $J^{-1}(\mathbb{R}^+\mu)$ since

$$\mathrm{g}(J\eta_{jM}, V) = \mathrm{g}(J\xi_{iM}, V) = \omega(\xi_{iM}, V) = -TJ(V)(\xi) = -r\mu(\xi_i) = -r\mu(\eta_j) = 0$$

for any V vector field on $J^{-1}(\mathbb{R}^+\mu)$. The next step is to show that $\{J\xi_{iM}\}_{i=1,m}$ is a basis in the normal bundle of $T\Psi^{-1}(0)$. Notice that $\{\xi_{iM}\mid_{J^{-1}(\mathbb{R}_+\mu)}\}_{i=1,m}$ are tangent to $J^{-1}(\mathbb{R}^+\mu)$ and

$$\mathrm{g}(J\xi_{iM}, V) = \omega(\xi_{iM}, V) = T\Psi(V)(\xi_i) = Ti^T(TJ(V))(\xi_i) = 0,$$

for any V differentiable section of $T\Psi^{-1}(0)$. Here, we have used that $\Psi = i_T^* \circ J$, where $i_T^* : \mathfrak{g}^* \to \mathfrak{k}_\mu^*$ is the canonical projection. Therefore, $\{J\xi_{iM}\}_{i=1,m}$ are vector fields normal to TU, where $U = J^{-1}(k_\mu^\circ) = \Psi^{-1}(0)$. As $\dim TU = \dim M - \dim \mathfrak{k}_\mu$, these vector fields form a basis of the normal fiber bundle to TU. Equation 2.4.4 implies that $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ form a basis of the normal bundle to $J^{-1}(\mathbb{R}^+\mu)$. Since the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is isometric, $i_\mu^*\,\mathrm{g}$ projects on $M_{\mathbb{R}^+\mu}$ in $\mathrm{g}_{\mathbb{R}^+\mu}$ and the projection π_μ becomes thus a Riemannian submersion. Obviously, the vertical distribution of this Riemannian submersion is given by $\{\xi_{iM}\}_{i=1,m}$. Then, $T_x J^{-1}(\mathbb{R}^+\mu) = \{\xi_{iM}\}(x) \oplus \mathcal{H}_x$, where $\mathcal{H}_x$ is the horizontal distribution at x associated to the Riemannian submersion π_μ. To see that $(\omega_{\mathbb{R}^+\mu}, \mathrm{g}_{\mathbb{R}^+\mu})$ is an almost Kähler structure, we need to check that

$$\omega_{\mathbb{R}^+\mu}([x])(T_x\pi_\mu v, T_x\pi_\mu w) = \mathrm{g}_{\mathbb{R}^+\mu}([x])(I_{\mathbb{R}^+\mu}T_x\pi_\mu v, T_x\pi_\mu w),$$

for any $[x] = \pi_\mu(x) \in J^{-1}(\mathbb{R}^+\mu)$ and $v, w \in \mathcal{H}_x$. Here, $I_{\mathbb{R}^+\mu}$ denotes the projection of the complex structure I of ω. Since $T_x\pi_\mu$ is an isomorphism from the horizontal space at x onto $T_{[x]}M_{\mathbb{R}^+\mu}$ which identifies $(\omega_{\mathbb{R}^+\mu}, \mathrm{g}_{\mathbb{R}^+\mu})([x])$ with $(i_\mu^*\omega, i_\mu^* g) \mid_{\mathcal{H}_x}$ suffices to show that the horizontal distribution is I-invariant. Let $v \in \mathcal{H}_x$. Then $\omega(Iv, \xi_{iM}) = g(v, \xi_{iM}) = 0$, for any $\xi_i \in \mathfrak{k}_\mu$. Also $\mathrm{g}(Iv, \mathcal{C}\xi_{iM}) = \mathrm{g}(v, \xi_{iM}) = 0$ and $\mathrm{g}(Iv, I\eta_{jM}) = \mathrm{g}(v, \eta_{jM}) = 0$, for all $i = \overline{1,k}$ and $j = \overline{1,m}$. It follows that Iv is also a horizontal vector. To show that $I_{\mathbb{R}^+\mu}$ is integrable we will evaluate the Nijenhuis tensor. Thus,

$$\begin{aligned} N_{\mathbb{R}^+\mu}(T_x\pi_\mu(v), T_x\pi_\mu(w)) &= [T_x\pi_\mu(v), T_x\pi_\mu(w)] - [I_{\mathbb{R}^+\mu}T_x\pi_\mu(v), I_{\mathbb{R}^+\mu}T_x\pi_\mu(w)] \\ &\quad + I_{\mathbb{R}^+\mu}([I_{\mathbb{R}^+\mu}T_x\pi_\mu(v), T_x\pi_\mu(w)]) + I_{\mathbb{R}^+\mu}([T_x\pi_\mu(v), I_{\mathbb{R}^+\mu}T_x\pi_\mu(w)]) \\ &= T_x\pi_\mu([v,w]) - T_x\pi_\mu([Iv, Iw]) + I_{\mathbb{R}^+\mu}(T_x\pi_\mu([Iv,w])) + I_{\mathbb{R}^+\mu}(T_x\pi_\mu([v, Iw])) \\ &= T_x\pi_\mu([v,w] - [Iv, Iw]) + T_x\pi_\mu(I([Iv,w])) + T_x\pi_m u(I([v, Iw])) \\ &= T_x\pi_\mu(N(v,w)) = 0, \end{aligned}$$

where N is the Nijenhuis tensor of (ω, g). Thus, $I_{\mathbb{R}^+\mu}$ is integrable and $(M_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu}, \mathrm{g}_{\mathbb{R}^+\mu})$ a Kähler manifold. □

Remark 2.4.1. *Unfortunately, non zero Kähler point reduction is not canonical. As it is very well explained in [11](see Exercise 3), the complex structure does not leave invariant the horizontal distribution of the Riemannian submersion given by the quotient projection($\pi_{\mathbb{R}^+\mu}$). Therefore, it is not projectable on $M_{\mathbb{R}^+\mu}$. In the literature, non zero Kähler quotients are defined using a unique up to homotheties Kähler-Einstein metric of positive Ricci curvature on the coadjoint orbit of μ, $\mathcal{O}_\mu$. For the construction of this metric and apllications, see [32], Chapter 8 in [7], and [29]. This uniqueness modulo homotheties is guaranteed by the choice of a Ad^*-invariant scalar product on $\mathfrak{g}^*$. Namely, one performs zero reduction to the Kähler difference of the base manifold with $\mathcal{O}_\mu$. The uniqueness of the metric on the coadjoint orbit makes the reduced Kähler structure canonical. If G is a compact group and an Ad^*-invariant scalar product on $\mathfrak{g}^*$ is fixed, then there is a unique up to homotheties Kähler-Einstein structure on $\mathcal{O}_\mu$. Even if this way one can endow $M_{\mathbb{R}^+\mu}$ with a unique (up to homotheties) Kähler-Einstein structure, the relation (2.4.1) is no longer verified. Note, that the canonical(i.e. relation (2.4.1)is satisfied) ray reduced Kähler space is always well defined. As always, this is guaranteed by the kernel group which is taken using the kernel of the momentum value.*

2.5 Cone and Boothby-Wang compatibilities

In this section (M, η) is a $2n+1$-dimensional exact contact manifold and G a Lie group acting on it by strong contactomorphisms. Denote by $\mathcal{C}(M) := (M \times \mathbb{R}^+, dr^2 \wedge \eta + r^2 d\eta)$ the symplectic cone associated to M. We imbed M in the cone as $M \times \{1\}$. Recall that if (M, η, g) is a Sasaki manifold, then the associated symplectic cone admits a canonical Kähler structure given by $\mathcal{C}(\mathrm{g}) := r^2\,\mathrm{g} + dr^2$. Consider the lift of the G-action on the cone $\mathcal{C}(M)$ given by: $g \cdot (x, r) := (g \cdot x, r)$, for any $g \in G$ and $(x, r) \in \mathcal{C}(M)$. This action commutes with the translations on the $\mathbb{R}^+$ component and, in the Sasaki case, it is by holomorphic isometries. In the Sasakian case, we can also define a complex structure given as follows:

$$JY := \varphi Y - \eta(Y)R, , JR := \mathcal{R},$$

where $R = r\partial_r$ is the vector field generated by the 1-group of transformations $\rho_t : (x, r) \to (x, tr)$ and $\varphi := \nabla\mathcal{R}$, with ∇ the Levi-Civita connection associated to g. It is easy to see that (M, η, g) is Einstein if and only if the cone metric $\mathcal{C}(\mathrm{g})$ is Ricci flat, i. e., $(\mathcal{C}(M), \mathcal{C}(\mathrm{g}))$ is Calabi-Yau (i. e. Kähler Ricci-flat).

Let $\Phi : M \to \mathfrak{g}^*$ be the contact momentum map associated to the G-action on M. The lifted action on the cone is Hamiltonian and a corresponding equivariant symplectic momentum map is given by

$$\Phi_s : \mathcal{C}(M) \to \mathfrak{g}^*\,, \Phi_s(x, r) := e^s J(x)\,, \text{ for any } (x, r) \in \mathcal{C}(M).$$

Having established the above notations, we are ready to prove that reduction and the cone construction are commuting operations.

Lemma 2.5.1. *Let $(M, \eta, \mathrm{g}, \mathcal{R})$ be a Sasakian manifold and $(\mathcal{C}(M), \mathcal{C}(\mathrm{g}), I)$ its Kähler cone. Suppose a Lie group G acts on M by strong contactomorphisms and commuting with the action of the 1-parameter group generated be the field R. Let μ be an element of the dual of the Lie algebra of G. Then the Kähler cone of the reduced contact space at μ is the reduced space at μ for the lifted action on $\mathcal{C}(M)$.*

Proof. Let K_μ be the kernel group of μ, $M_{\mathbb{R}^+\mu}$ the corresponding contact reduced space, and $\mathcal{C}(M_{\mathbb{R}^+\mu})$ the rduced space for the lift of the action on the cone. Since the K_μ-action commutes with homotheties on the $\mathbb{R}^+$ component, there is a natural diffeomorphism between $\mathcal{C}(M_{\mathbb{R}^+\mu})$ and $\mathcal{C}(M)_{\mathbb{R}^+\mu}$:

$$\Psi : \mathcal{C}(M)_{\mathbb{R}^+\mu} \to \mathcal{C}(M_{\mathbb{R}^+\mu})\ ,\ \Psi([x, r]) := ([x], r),\ \forall [x, r] \in \mathcal{C}(M)_{\mathbb{R}^+\mu}.$$

Using the commutativity of the diagram in figure 2.7, it is easy to see that Ψ is also a symplectomporphic isometry. Namely,

$$(\Psi \circ \pi_{1\mu})^*(\eta_\mu \wedge dr^2 + r^2 d\eta_\mu) = i^*_{1\mu}(\eta \wedge dr^2 + r^2 d\eta),$$

and

$$\Psi^*(\mathcal{C}(\mathrm{g}_{\mathbb{R}^+\mu})) = \mathcal{C}(g)_{\mathbb{R}^+\mu},$$

where $i_{1\mu} : \Phi_s^{-1}(\mathbb{R}_+\mu) \to \mathcal{C}(M)$, $\pi_{1\mu} : \Phi_s^{-1}(\mathbb{R}_+\mu) \to \mathcal{C}(M)_{\mathbb{R}^+\mu}$, and, $\pi_\mu : \Phi^{-1}(\mathbb{R}_+\mu) \to (M)_{\mathbb{R}^+\mu}$ are the canonical inclusion and K_μ-projections, respectivelly. □

Recall that a celebrated theorem of Boothby and Wang (see Section 3.3 in [8]) states that if M is also compact and regular, then it admits a contact form whose Reeb vector field generates a free, effective S^1-action on it. Even more, M is the bundle space of a principal circle bundle $\pi : M \to N$

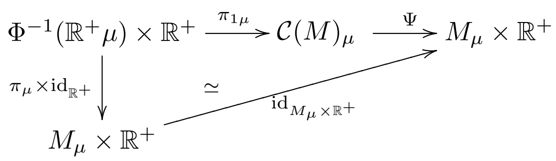

Figure 2.5.1: Commutative diagram used in the proof of Lemma 2.5.1

over a symplectic manifold of dimension $2n$ with symplectic form ω determining an integer cocycle. In this case, this contact form, η is a connection form on the bundle $\pi : M \to N$ with curvature form $d\eta = \pi^*\omega$. N is actually the space of leaves of the characteristic foliation on M (i.e. the 1-dimesional foliation defined by the Reeb vector field of η). If M is a Sasaki manifold, then N becomes a Hodge manifold and the fibers of π are totally geodesic. This case was treated by Y. Hatakeyama in [25]. Even more, in [9], Theorem 2.4 it was proven that M is Sasaki-Einstein if and only if N is Kähler-Einstein with scalar curvature $4n(n+1)$ and that all the above still holds in the category of orbifolds if M is quasi-regular, i.e. all the leaves of the characteristic foliation are compact.

Proposition 2.5.1. *Let $\pi : (M, \mathrm{g}) \to (N, \mathrm{h})$ be a the Boothby-Wang fibration associated to the quasi-regular, compact, Sasaki manifold M. Suppose a connected Lie group G acts by strong contactomorphisms on (M, g) with momentum map $J_M : M \to \mathfrak{g}^*$. Let μ be an element of $\mathfrak{g}^*$, with kernel group K_μ. Assume that the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries and that* $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$. *Then, the reduced space of N at μ is well defined and there is a canonical Boothby-Wang fibration of the reduced spaces:*

$$\tilde{\pi} : M_{\mathbb{R}^+\mu} \to N_{\mathbb{R}^+\mu}.$$

Proof. Denote by η the contact form of the Boothby-Wang fibration and by $\mathcal{R}$ its Reeb vector field. Since $[\mathfrak{R}, \xi_M] = 0$ for any $\xi \in \mathfrak{g}$ and G is connected, the action generated by the Reeb vector field commutes with the action of G. Hence there is a well defined action of G on N. Even more, this action is by symplectomorphisms. If $J_M : M \to \mathfrak{g}^*$ is the equivariant momentum map associated to the G-action on M, the induced application

$$J_N : N \to \mathfrak{g}^* \, , J_N(\pi(x)) := J_M(x),$$

for any $x \in M$. Indeed, if $\Phi^t_{\mathcal{R}}$ is the flow of the Reeb vector field, we have

$$\begin{aligned} J_M(\Phi^t_{\mathcal{R}}(x))(\xi) = \eta(\Phi^t_{\mathcal{R}}(x))(\xi_M(\Phi^t_{\mathcal{R}}(x)) = ((\Phi^t_{\mathcal{R}})^*\eta)(x)(\xi_M(x)) = \eta(x)(\xi_M(x)) \\ = J_M(x)(\xi), \end{aligned}$$

for any $\xi \in \mathfrak{g}$ and any $x \in M$. This proves that J_N is well defined. Using the fact that $\pi^*\omega = d\eta$, it is easy to see that J_N is an equivariant momentum map associated to the G-action on N. We also have that $\pi(J_M^{-1}(\mathbb{R}^+\mu)) = J_N^{-1}(\mathbb{R}^+\mu)$ and obviously the action of K_μ on $J_N^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries. Therefore, the quotient space $N_{\mathbb{R}^+\mu}$ is a well defined symplectic orbifold and the induced projection $\tilde{\pi} : M_{\mathbb{R}^+\mu} \to N_{\mathbb{R}^+\mu}$ becomes a Boothby-Wang fibration. □

2.6 Conformal Hamiltonian vector fields

In this section we will study the dynamical behavior of a class of conformal Hamiltonian systems which include simple mechanical systems with friction or systems with Rayleigh dissipation. We will see how and why in the presence of symmetries the right tool for the study of these systems is the ray reduction and not the point one. In this case, the invariant submanifolds of the dynamical system are no longer the point pre-images of the associated momentum map, but the ray pre-images. In fact, we will extend the work of McLachlan and Perlmutter in [40]. Namely, we will enlarge the class of conformal Hamiltonian systems defined by them and we will complete the Lie-Poisson reduction of these systems with the general ray reduction.

Recall that an autonomus Hamiltonian system with an appropriate symmetry group obeys certain conservation laws. Namely, if J is an associated equivariant momentum map, $J : M \to \mathfrak{g}^*$, the preimages $\{J^{-1}(\mu) | \mu \in \mathfrak{g}^*\}$ are invariant submanifolds of the Hamiltonian vector field. In symplectic geometry this conservation property is known as the *Noether theorem* and it states that if $t \to c(t)$ is a solution of the Hamiltonian system starting at the point x_0 with momentum $J(x_0) = \mu$, then the image of c is entirely included in $J^{-1}(\mu)$. Hence, the Hamiltonian system can be reduced to an other Hamiltonian on the point reduced space $M_\mu = \frac{J^{-1}(\mu)}{G_\mu}$. The second conservation property concerns the energy of the system. More precisely, the Hamiltonian is preserved by any integral curve of the system. To summarize, we have:

Theorem 2.6.1. ***(Symplectic point reduction)*** *Let G be a Lie group acting on the connected symplectic manifold (M, ω) by a free, proper and canonical action. Suppose this action has an associated G-equivariant momentum map $J : M \to \mathfrak{g}^*$. Denote by μ an arbitrary element of $\mathfrak{g}^*$ and by G_μ the isotropy group of μ under the coadjoint action of G on $\mathfrak{g}^*$. Then:*

- *The point reduced space $M_\mu := \frac{J^{-1}(\mu)}{G_\mu}$ is a well defined symplectic manifold with symplectic structure ω_μ uniquely characterized by the relation $\pi_\mu^* \omega_\mu = i_\mu^* \omega$. The maps $i_\mu : J^{-1}(\mu) \hookrightarrow M$ and $\pi_\mu : J^{-1}(\mu) \to M_\mu$ denote the inclusion and projection, respectively. The pair (M_μ, ω_μ) is called the* symplectic point reduced space*.*

- *If $H \in \mathcal{C}^\infty(M)^G$ is a G-invariant Hamiltonian, then the flow Φ_t of the hamiltonian vector field X_H leaves the connected componenets of $J^{-1}(\mu)$ invariant and commutes with the G-action. Thus, it induces a flow Φ_t^μ on M_μ defined by*

$$\pi_\mu \circ \Phi_t \circ i_\mu = \Phi_t^\mu \circ \pi_\mu.$$

- *The vector field on M_μ generated by Φ_t^μ is Hamiltonian with asscoiated reduced Hamiltonian function $H_\mu \in \mathcal{C}^\infty(M_\mu)$ defined by*

$$H_\mu \circ \pi_\mu = H \circ i_\mu.$$

The vector fields X_H and X_{H_μ} are π_μ-related and the triple $(M_\mu, \omega_\mu, X_{H_\mu})$ is called the reduced Hamiltonian system.

This theorem is a calssical result of J. Marsden and A. Weinstein. For the proof and physical examples, see [37] and [38].

However, in physics there are a lot of systems whose energy is not conserved, but dissipated.

Example 2.6.1. ***Simple mechanical systems with friction-unforced Duffing oscillators.*** *The equation of motion of an unforced Duffing oscillator is described by:*

$$\ddot{x} + k\dot{x} + (x^2 - 1)x = 0, \tag{2.6.1}$$

with k a real constant. Equation 2.6.1 is equivalent to the system

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} - ky, \end{cases} \tag{2.6.2}$$

where H is the Hamiltonian given by

$$H(x, y) := \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}. \tag{2.6.3}$$

Lemma 2.6.1. *Note that the vector field X_H^k on the symplectic manifold $(T^*\mathbb{R}, \omega_0) \simeq (\mathbb{R}^2, \omega_0 = dx \wedge dy)$ described by the system 2.6.2 can be written as $X_H + Z$, where Z is the unique vector field defined by $i_Z\omega_0 = -k\theta_0$ with $\theta_0 = -ydx$ the canonical* 1*-form of $T^*\mathbb{R}$. Even more, $i_{X_H^k}\omega_0 = dH - k\theta_0$.*

Example 2.6.2. ***Simple mechanical systems with friction which dissipate energy and symplectic area-conformal Hamiltonians on $\mathbb{R}^{2n}$.*** *More general, on the canonical symplectic manifold $(\mathbb{R}^{2n}, \omega_0 = dq \wedge dp)$ with coordinates (q, p) consider the set of vector fields X_H^k defined by*

$$\begin{cases} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - kp. \end{cases} \tag{2.6.4}$$

and with $H = T + V(q)\,, T = \frac{1}{2}p^T M(q)p$, T positive definite and $k < 0$ describe systems with friction which dissipate energy and symplectic area, and verify the condition $i_{X_H^k}\omega_0 = dH - k\theta_0$.

Example 2.6.3. ***A special case of mechanical systems with Rayleigh dissipation.*** *All dynamical systems*

$$\begin{cases} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - R(q)\frac{\partial H}{\partial p}, \end{cases} \tag{2.6.5}$$

for which $\dot{H} = -R(q)\langle \frac{\partial H}{\partial p}, \frac{\partial H}{\partial p} \rangle \leq 0$ and which are subject to the condition

$$R(q) = kM(q) \tag{2.6.6}$$

are precisely of the form (2.6.4). Here $R(q)$ defines a positive metric.

All the above examples, were treated in [40]. In this article the authors study conformal Hamiltonian systems on symplectic exact manifolds. In the following paragraph we will briefly recall their definition and properties as studied in [40]. From now on $(M, \omega = -d\theta)$ is an exact symplectic manifold.

Definition 2.6.1. *The vector field* X_H^k *on* M *is* conformal with real parameter k *if* $i_{X_H^k} = dH - k\theta$ *for a* $H \in \mathcal{C}(M)$*. This condition is equivalent to* $L_{X_H^k} = -k\omega$.

Note that the hypothesis of exact symplectic manifold does not restrain the generality since a symplectic manifold admits a vector field X_H^k with $L_{X_H^k} = -k\omega$ if and only if it is exact. If, in addition, $H^1(M) = 0$, then all the conformal vector fields on M are given by

$$\{X_H + kZ | H \in \mathcal{C}^\infty(M)\},$$

where Z is defined by $i_Z\omega = -\theta$. For the proof, see Proposition 1 in [40]. It was noticed by the authors that, in the case of Lie symmetries, the conformal Hamiltonian vector fields have a special behaviour with respect to the associated momentum map. Namely,

Proposition 2.6.1. *Let* G *be a Lie group which acts on* $(M, \omega = -d\theta)$ *leaving the* 1*-form* θ *invariant and* H *a smooth,* G*-invariant function on* M*. Denote by* $J : M \to \mathfrak{g}^*$ *the associated* G*-equivariant momentum map. Then,* X_H^k *is a* G*-invariant vector field for any real* k *and its flows preserves the ray pre-images of the associated momentum map as follows:*

$$J(x(t)) = e^{-kt} J(x(0)),$$

for any x *integral curve of* X_H^k *and any time* t.

Remark 2.6.1. *In other words, the motion is constrained to a ray of momentum values entirely determined by the initial momentum. Hence, the ray pre-images of the momentum map are invariant submanifolds for the conformal Hamiltonian vector fields.*

In the hypothesis of Proposition 2.6.1, using the G-invariance of the conformal Hamiltonian vector field, the authors have performed the conformal Lie Poisson reduction and reconstruction of solutions for conformal Hamiltonian vector fields on T^*G. However, they could not exploit the ray momentum conservation, nor perform a reduction which uses not only the group invariance, but also the ray-momentum one. Proposition 2.6.1 and Theorem 2.4.1, immediately suggest that the appropriate method of reduction for conformal Hamiltonian vector fields is the ray reduction constructed in Section 2.4.

But before passing to details, let us come back to the examples given at the beginning of this section. Recall that a Rayleigh system(Example 2.6.3) is a conformal Hamiltonian if and only if the condition 2.6.6 is satisfied. However, any Rayleigh system on $\mathbb{R}^{2n}$ with vector field X has the following property

$$i_X\omega_0 = dH - f\theta_0,$$

where $f(q,p) := \frac{R(q)\frac{\partial H}{\partial p}}{p}$. Of course, f may not be well defined at points on the q-axis, and , if necessary, we restrict X to $\mathbb{R}^{2n} \setminus U$, with U a small closed cylinder with axis $(\mathbb{R}^n, q)$.

The forced Duffing oscillator is the dynamical system with equation of motion

$$\ddot{x} + k\dot{x} + (x^2 - 1)x = \gamma \cos \omega t. \tag{2.6.7}$$

It is equivalent to the following system

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} - ky + \gamma \cos \omega t, \end{cases} \tag{2.6.8}$$

where the Hamiltonian is again $H(x,y) := \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$. Denote by X_D the vector field defined by 2.6.8 and note that $i_{X_D}\omega_0 = dH - f_D\theta_0$, with $f_D(t,x,y) := -(k + \frac{\gamma \cos \omega t}{y})$, for any real pair (x,y) with $y \neq 0$ and any time t.

These examples suggest the following enlarged definition of a conformal Hamiltonian vector field on an exact symplectic manifold.

Definition 2.6.2. *The vector field X_H^f on the exact symplectic manifold $(M, \omega = -d\theta)$ is* conformal Hamiltonian *with conformal parameter the smooth function f and smooth Hamiltonian H if $i_{X_H^f} = dH - f\theta$.*

Remark 2.6.2. *Observe that if $H^1(M) = 0$, X_H^f is conformal Hamiltonian if and only if $L_{X_H^f}\omega = -d(f\theta)$.*

Remark 2.6.3. *The conformal Hamiltonian $X_H^f = X_H + Z_f$ is the summ of the Hamiltonian vector field determined by H and Z which is uniquely determined by the relation $i_Z\omega = -f\theta$. In local coordinates (q,p), Z is given by $fp\frac{\partial}{\partial p}$.*

The next proposition shows that this enlarged class of conformal Hamiltonians behaves well in the presence of symmetries.

Proposition 2.6.2. *Let G be a Lie group which acts on $(M, \omega = -d\theta)$ leaving the 1-form θ invariant, H and f smooth, G-invariant functions on M. Denote by $J : M \to \mathfrak{g}^*$ the associated G-equivariant momentum map. Then, X_H^f is a G-invariant vector field and its flow preserves the ray pre-images of the associated momentum map as follows:*

$$J(x(t)) = e^{e^{\int_0^t -f(x(s))\mathrm{d}s}} J(x(0)),$$

for any x integral curve of X_H^f and any time t.

Proof. Denote by ϕ the action of G on M. Then, for any $g \in G$ we have

$$\phi_g^*(i_{X_H^f}\omega) = \phi_g^*(dH - f\theta) = dH - f\theta = i_{X_H^f}\omega, \tag{2.6.9}$$

since f and H are G-invariant. On the other hand,

$$\phi_g^*(i_{X_H^f}\omega) = i_{\phi_g^* X_H^F}\phi_g^*\omega = i_{\phi_g^* X_H^f}\omega. \tag{2.6.10}$$

Since ω is non-degenerate, (2.6.9) and (2.6.10) imply that X_H^f is G-invariant.

First recall that any exact synplectic manifold admits an equivariant momentum map given by $J : (M, \omega = d\theta) \to \mathfrak{g}^*$, $\langle J(x), \xi\rangle := \theta(\xi_M)(x)$, for any $x \in M$ and $\xi \in \mathfrak{g}$. Now, let $x(t)$ be an integral curve of X_H^f. Then,

$$\frac{d}{dt}\langle J(x(t)), \xi\langle = TJ_\xi(X_H^f(x(t))) = \omega(x(t))(X_H^f(x(t)), \xi_M(x(t))) = \\ dH(\xi_M(x(t)) - f(x(t))\theta(\xi_M(x(t))) = -f(x(t))J_\xi(x(t)).$$

Henece, $J(x(t)) = e^{e^{\int_0^t -f(x(s))\mathrm{d}s}} J(x(0))$ for any $\xi \in \mathfrak{g}$ and any time t. □

Remark 2.6.4. *Note that if f and H are K_μ-invriant, with K_μ the kernel group associated to $\mu \in \mathfrak{g}^*$, then the corresponding conformal Hamiltonian is also K_μ-invariant.*

Proposition 2.6.2 suggests that the ray reduction is natural for conformal Hamitlonian systems.

Proposition 2.6.3. *Consider $(M, \omega = -d\theta)$ an exact symplectic manifoldendowed with the smooth action of a lie group G. Choose an element μ in $\mathfrak{g}^*$ with kernel group K_μ. Denote by $J : M \to \mathfrak{g}^*$ the associated equivariant momentum map defined by $J(x)(\xi) := i_{\xi_M}\theta$, for any $x \in M$ and $\xi \in \mathfrak{g}$ with infinitezimal isometry ξ_M. Suppose all the hypothesis of Theorem 2.4.1 are fulfilled and X_H^f is a conformal Hamiltonian vector field with H and f K_μ-invariant functions. Then,*

- *the flow of X_H^f induces a flow on the ray reduced space $M_{\mathbb{R}}^+\mu$ defined by*

$$\pi_{\mathbb{R}^+\mu} \circ \Phi_t \circ i_{\mathbb{R}^+\mu} = \Phi_t^{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu}.$$

- *the vector field generated by the flow $\Phi_t^{\mathbb{R}^+\mu}$ is conformal Hamiltonian $X_{H_{\mathbb{R}^+\mu}}^{f_{\mathbb{R}^+\mu}}$ with*

$$f_{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu} = f \circ i_{\mathbb{R}^+\mu}\,, H_{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu} = H \circ i_{\mathbb{R}^+\mu}.$$

  the vector fields X_H^f and $X_{H_{\mathbb{R}^+\mu}}^{f_{\mathbb{R}^+\mu}}$ are $\pi_{\mathbb{R}^+\mu}$-related.

- *a point $x \in M$ is a relative equilibrium of X_H^f with respect to the G-symmetries if and only if there is an element $\xi \in k_\mu$ such that $X_H^f(x) = \xi_M(x)$ or, equivalently, $\Phi_t(x) = \exp t\xi \cdot x$, for any t. The relative equilibria of X_H^f coïncide via the $\pi_{\mathbb{R}^+\mu}$-projection with the equilibria of $X_{H_{\mathbb{R}^+\mu}}^{f_{\mathbb{R}^+\mu}}$, or, equivalently, with the points $x \in M$ for which there is a $\xi \in k_\mu$ such that*

$$d(J^\xi - H)(x) = f(x)\theta(x). \tag{2.6.11}$$

Remark 2.6.5. *Note that, in local copordinates (q,p), condition (2.6.11) is equivalent to*

$$\begin{cases} pf(q,p) & = \frac{\partial(J^\xi - H)}{\partial q}(q,p) = pf(q,p) \\ 0 & = -\frac{\partial(J^\xi - H)}{\partial p}. \end{cases} \tag{2.6.12}$$

Proof. The first two points of the theorem are a direct consequence of Proposition 2.6.2. For the rest, suffice it to use the definition of a conformal Hamiltonian vector field and the relation $\omega(\xi_M, \cdot) = dJ^\xi(\cdot)$. □

Example 2.6.4. *The reduction of a Rayleigh system on* $T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*})$**.** *On* $(T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*}), (q,p)) \simeq \big((\mathbb{R}^{2*} \times \mathbb{R}^{2*}) \times \mathbb{R}^4, (q,p)\big)$ *consider the Rayleigh system given by* $H(q,p) = \frac{1}{2}(\|q\|^2 + \|p\|^2)$ *and* $R(q) = \|q\|^2$. *Note that* $f = R$ *and it is well defined at any point in* $\mathbb{R}^4$. *Consider the cotangent lift of the rotation action of* $S^1 \times S^1$ *on* $\mathbb{R}^{2*} \times \mathbb{R}^{2*}$. *The reason for restricting* $\mathbb{R}^4$ *to* $(\mathbb{R}^{2*} \times \mathbb{R}^{2*})$ *is to have free symmetries. It is a free and proper action and* H *and* f *are* $S^1 \times S^1$*-invariant. Let* $\mu := \langle (0,1), \cdot \rangle$ *be an element of* $(\mathbb{R} \times \mathbb{R})^*$*, the dual of the Lie algebra of* $S^1 \times S^1$. *Then* $K_\mu = \{e\} \times S^1$ *and* $\mathfrak{k}_\mu = \{0\} \times \mathbb{R}$. *The momentum map asscoiated to the* $S^1 \times S^1$*-action is given by*

$$J : \mathbb{R}^{2*} \times \mathbb{R}^{2*} \times \mathbb{R}^4 \to (\mathbb{R} \times \mathbb{R})^*\, , J(q,p) = (q_1 \cdot \bar{p}_1^T, q_2 \cdot \bar{p}_2^T),$$

for any $(q,p) = (q_1, q_2, p_1, p_2) \in \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^4$ *with* $\bar{p}_i^T = (p_{i2}, -p_{i1})$, $i = 1,2$ *and* $J^{-1}(\mathbb{R}^+\mu) = \{(q,p) \in (\mathbb{R}^4 \setminus \{0\}) \times \mathbb{R}^4 | q_1 \cdot \bar{p}_1^T = 0\,, q_2 \cdot \bar{p}_2^T \in \mathbb{R}^+\}$. *By Theorem 2.4.2, the ray reduced space* $\big(T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*})\big)_{\mathbb{R}^+\mu}$ *is embedded in* $T^*(\frac{\mathbb{R}^{2*} \times \mathbb{R}^{2*}}{\{e\} \times S^1}) \simeq T^*(\mathbb{R}^2 \setminus \{0\} \times (0,\infty))$. *The reduced Rayleigh system is given by* $H_{\mathbb{R}^+\mu}(q,s) = \frac{1}{2}(\|q\|^2 + s^2)$ *and* $R_{\mathbb{R}^+\mu}(q) = \|q\|^2$.

Remark 2.6.6. *Note that all the conformal Hamiltonian vector fields on cotangent bundles are pullbacks of time-dependent Hamiltonians on the same cotangent bundle. Namely, for a conformal Hamiltonian with functional parameter* f *the pullback is defined by* $\Psi_t : T^*Q \to T^*Q\, , \Psi_t(\alpha_q) := e^{-ft}\alpha_q$, *for all* $\alpha_q \in T_q^*Q$.

2.7 Ray Reductions of Cotangent and Cosphere bundles of a Lie Group

In this section we will construct examples of ray reduced spaces for lifted actions on cotangent and cosphere bundles. We will also show that these ray reduced spaces are universal in the sense that any symplectic or contact ray reduced space can be recovered from the ray reduced space of a cotangent or cosphere bundle.

Let G denote a d-dimensional Lie group with Lie algebra $\mathfrak{g}$. G acts on itself by left translations. This action lifts canonically to an action on T^*G which admits an equivariant and right invariant momentum map

$$J_L : T^*G \to G \quad , \quad J_L(\alpha_g) := T_e^* R_g(\alpha_g).$$

Similarly, for right translations we can construct the equivariant and left invariant momentum map

$$J_R : T^*G \to G \quad , \quad J_R(\alpha_g) := T_e^* L_g(\alpha_g).$$

Denote by $\mathcal{O}_\mu$ the coadjoint orbit of an element μ of $\mathfrak{g}^*$ and by $\mathcal{O}_{\mathbb{R}_+\mu}$ its *coadjoint ray orbit*

$$\mathcal{O}_\mu := \{Ad^*_{g^{-1}}\mu = g\mu \,|\, g \in G\} \quad \text{and} \quad \mathcal{O}_{\mathbb{R}_+\mu} := \{Ad^*_{g^{-1}} r\mu \,|\, g \in G\,, r \in \mathbb{R}_+\}.$$

Recall that $\mathfrak{g}^*$ is a Poisson manifold with respect to the Lie-Poisson brackets $\{\cdot,\cdot\}_\pm$ defined by

$$\{f,g\}_\pm(\mu) := \pm\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle \quad , \quad f\,, g \in \mathcal{C}^\infty(\mathfrak{g}^*)\,, \mu \in \mathfrak{g}^*,$$

where the element $\frac{\delta f}{\delta\mu}$ of $\mathfrak{g}$ is defined by the equality $\langle \nu, \frac{\delta f}{\delta\mu}\rangle := \mathrm{D}\, f(\mu)\cdot\nu$, for any $\nu \in \mathfrak{g}^*$. For a direct proof of this see, for example, [36], Theorem 14.3.

Each coadjoint orbit admits a G-invariant symplectic structure described in the following theorem. This symplectic structure is associated to various names: Lie, Borel, Weil and, more recently, Arnold ([4]), Kirillov ([28]), Kostant ([31]), and Souriau ([54]).

Theorem 2.7.1. *Let G be a Lie group and μ an element of the dual of its Lie algebra $\mathfrak{g}^*$. Then the coadjoint orbit of μ, $\mathcal{O}_\mu$ is a symplectic manifold with G-invariant symplectic structure $\omega^\pm_{\mathcal{O}_\mu}$ given by*

$$\omega^\pm_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) := \pm\langle \nu, [\xi, \eta]\rangle,$$

*for arbitrary $\nu \in \mathcal{O}_\mu$, and ξ, $\eta \in \mathfrak{g}$. The symplectic structures $\omega^\pm_{\mathcal{O}_\mu}$ are usually called the **Kostant-Kirillov-Souriau** (KKS) symplectic forms. The connected components of $\{(\mathcal{O}_\mu, \omega^\pm_{\mathcal{O}_\mu})\}_{\mu\in\mathfrak{g}^*}$ are the symplectic leaves of the Poisson manifold $(\mathfrak{g}^*, \{\cdot,\cdot\}_\pm)$.*

An interesting feature of coadjoint orbits is that they can be regarded as point reduced spaces.

Theorem 2.7.2. *For the lifted left action of G on T^*G the reduced space at $\mu \in \mathfrak{g}^*$, $((T^*G)_\mu, \omega_\mu)$ is well defined and the momentum map for the lifted right action of G on T^*G induces a symplectic diffeomorphism $\bar{J}_R : ((T^*G)_\mu, \omega_\mu) \to (\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$ given by $\bar{J}_R(\left[T^*_g R_{g^{-1}}\mu\right]) := Ad^*_g\mu$.*

For a proof see, for instance, Theorem 6.2.2 in [57].

Since for the ray-reduction the role of the coadjoint orbit will be played by a diagonal product of the coadjoint ray orbit and the quotient of G by the kernel group, we will now describe their manifold structures. We will see that, in general, $\mathcal{O}_{\mathbb{R}_+\mu}$ is an initial smooth submanifold of $\mathfrak{g}^*$. However, if the coadjoint action is proper, its ray orbits will be closed embedded submanifolds.

Definition 2.7.1. *Let $G_{\mathbb{R}^+\mu}$ be the ray isotropy group of μ defined by $G_{\mathbb{R}^+\mu} := \{g \in G \,|\, Ad^*_g\mu = r_g\mu$, for a $r_g \in \mathbb{R}^+\}$.*

Lemma 2.7.1. *The ray isotropy group $G_{\mathbb{R}^+\mu}$ is a closed Lie subgroup of G. Its Lie algebra is given by*

$$\mathfrak{g}_{\mathbb{R}^+\mu} = \{\xi \in \mathfrak{g} \,|\, ad^*_\xi\mu = r_\xi\mu \quad \textit{for a} \quad r_\xi \in \mathbb{R}\}.$$

Proof. We have the following sequence of subgroups $G_\mu < G_{\mathbb{R}^+\mu} < G$. To prove that the ray isotropy group is closed in G, suppose $(g_n)_{n\in\mathbb{N}}$ is a convergent sequence in $G_{\mathbb{R}^+\mu}$ with $\lim_{n\to\infty} g_n = g \in G$. Then $\lim_{n\to\infty} Ad^*_{g_n}\mu = (\lim_{n\to\infty} r_{g_n})\mu = Ad^*_g\mu$, for $(r_{g_n})_{n\in\mathbb{N}}$ a convergent sequence of positive numbers. Since the coadjoint map is linear and $\mu \neq 0$, $\lim_{n\to\infty} r_{g_n}$ is a strictly positive number and hence $g \in G_{\mathbb{R}^+\mu}$. Thus, the ray isotropy group is closed. To determine its Lie algebra, let first ξ be an element of $\mathfrak{g}_{\mathbb{R}^+\mu}$. We want to show that $\exp(t\xi)$ belongs to $G_{\mathbb{R}^+\mu}$ for arbitrary $t \in \mathbb{R}$. Then

$$\frac{d}{dt}Ad^*_{\exp t\xi}\mu = Ad^*_{\exp t\xi}(ad^*_\xi\mu) = Ad^*_{\exp t\xi}(r_\xi\mu) = r_\xi Ad^*_{\exp t\xi}\mu.$$

We have used the following formula

$$\frac{d}{dt}Ad^*_{g(t)}\mu(t) = Ad^*_{g(t)}\left(ad^*_{\xi(t)}\mu(t) + \frac{d\mu}{dt}\right), \tag{2.7.1}$$

where $\xi(t) = T_{g(t)}R^{-1}_{g(t)}(\frac{dg}{dt})$ and $g(t)$, $\mu(t)$ are smooth curves in G and $\mathfrak{g}^*$, respectively. It follows that $Ad^*_{\exp t\xi}\mu = e^{r_\xi t}\mu$ and $\exp t\xi \in G_{\mathbb{R}^+\mu}$, for every real t. For the reverse inclusion, suppose ξ is an element of the Lie algebra of the ray isotropy group. Then we know that $\exp t\xi \in G_{\mathbb{R}^+\mu}$ and $Ad^*_{\exp t\xi}\mu = r_t\mu$ with r_t a positive real number for every $t \in \mathbb{R}$. Deriving at zero the above equality, we obtain that $ad^*_\xi\mu = \left(\frac{d}{dt}\big|_{t=0} r_t\right)\mu$, completing thus the proof of this lemma. □

Remark 2.7.1. *As we saw in the proof of Theorem 2.4.1, if the Lie group G acts in a Hamiltonian way on the manifold M and this action admits an equivariant momentum map $J : M \to \mathfrak{g}^*$, then for every $x \in J^{-1}(\mathbb{R}^+\mu)$ we have that*

$$J^{-1}(\mathbb{R}^+\mu) \cap (G \cdot x) = G_{\mathbb{R}^+\mu} \cdot x \quad , \quad T_x(G_{\mathbb{R}^+\mu} \cdot x) = T_x(G \cdot x) \cup T_x(J^{-1}(\mathbb{R}^+\mu)).$$

Lemma 2.7.2. *The ray isotropy group $G_{\mathbb{R}^+\mu}$ acts on $G \times \mathbb{R}^+$ by $g' \cdot (g, r) \to (gg', rr_{g'})$, where $Ad^*_{g'}\mu = r_{g'}\mu$. This action is free and proper and, therefore, the twisted product $G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$ is well defined. Even more, the surjective map*

$$f : G \times \mathbb{R}^+ \to \mathcal{O}_{\mathbb{R}^+\mu} \quad , \quad f(g, r) := Ad^*_{g^{-1}}(r\mu)$$

descends to a bijection on the twisted product $G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$ defining thus a smooth structure on the coadjoint ray orbit.

Proof. Since it consists of direct calculations, we skip the proof of this Lemma. □

Remark 2.7.2. *Note that the above Lemma implies that the dimension of the ray orbit at μ is given by* $\dim(\mathcal{O}_{\mathbb{R}^+\mu}) = \dim G + 1 - \dim(G_{\mathbb{R}^+\mu})$.

For technical reasons we need a precise description of the tangent space of the ray orbit.

Lemma 2.7.3. *Let $\mathcal{O}_{\mathbb{R}^+\mu}$ be the coadjoint ray orbit of $\mu \in \mathfrak{g}^*$. Then its tangent space at μ is given by*

$$T_\mu\mathcal{O}_{\mathbb{R}^+\mu} = \{ad^*_\xi\mu + r\mu \,|\, r \in \mathbb{R}\,, \xi \in \mathfrak{g}\}.$$

Proof. Consider the smooth curve in $\mathcal{O}_{\mathbb{R}^+\mu}$ given by $\mu(t) := Ad^*_{\exp(t\xi)}(e^{tr}\mu)$, where r is an arbitrary real number. Note that $\mu(0) = \mu$ and $\frac{d}{dt}\big|_{t=0}\mu(t) = \xi_{\mathfrak{g}^*}(\mu) + r\mu = ad^*_\xi\mu + r\mu$. Therefore, $A := \{ad^*_\xi\mu + r\mu \,|\, r \in \mathbb{R}\,, \xi \in \mathfrak{g}\} \subset T_\mu\mathcal{O}_{\mathbb{R}^+\mu}$.

Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}^+\mu} \oplus \mathfrak{m}_{\mathbb{R}^+\mu}$ be a splitting of $\mathfrak{g}$, and $\{\xi_1, \cdots \xi_k\}$, $\{\xi_{k+1}, \cdots \xi_d\}$ basis of $\mathfrak{g}_{\mathbb{R}^+\mu}$ and $\mathfrak{m}_{\mathbb{R}^+\mu}$, respectively. It is easy to see that the set $\{\xi_{k+1\,\mathfrak{g}^*}(\mu), \cdots, \xi_{d\,\mathfrak{g}^*}(\mu), \mu\}$ forms a basis of A. And since $\dim(\{\xi_{k+1\,\mathfrak{g}^*}(\mu), \cdots, \xi_{d\,\mathfrak{g}^*}(\mu), \mu\})$ is $d + 1 - \dim(G_{\mathbb{R}^+\mu})$, it follows that $A = T_\mu\mathcal{O}_{\mathbb{R}^+\mu}$. □

Proposition 2.7.1. *The coadjoint ray orbit $\mathcal{O}_{\mathbb{R}^+\mu}$ is an initial Poisson submanifold of $\mathfrak{g}^*$ and if the coadjoint action is proper, it is even a closed embedded submanifold.*

Proof. Denote by $\tilde{f} : G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+ \to \mathcal{O}_{\mathbb{R}^+\mu} \hookrightarrow \mathfrak{g}^*$, $\tilde{f}([g, r]) := Ad^*_{g^{-1}}(r\mu)$ the bijection of Lemma 2.7.2. Let $\pi_{\mathbb{R}^+\mu} : G \times \mathbb{R}^+ \to G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$ be the canonical projection. $\tilde{f}$ is a smooth, one-to -one map since $\tilde{f} \circ \pi_{\mathbb{R}^+\mu} = f : G \times \mathbb{R}^+ \hookrightarrow \mathfrak{g}^*$. To prove that it is also an immersion, we will show that $T_{[g,r]}\tilde{f}$ is injective for every $[g, r] \in G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$. Note that $\ker T_{[g,r]}\tilde{f} = T_{(g,r)}\pi_{\mathbb{R}^+\mu}(\ker T_{(g,r)}f)$, for each $[g, r]$ element of the twisted product $G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$.

We will first show that $\ker T_{(e,r)}f = \{(\xi, rr_\xi) \in \mathfrak{g} \times \mathbb{R} \,|\, \xi \in \mathfrak{g}_{\mathbb{R}^+\mu}\,, ad^*_\xi\mu = r_\xi\mu\}$. For the direct inclusion, just use formula 2.7.1 to obtain

$$T_{(e,r)}f(\xi, rr_\xi) = \frac{d}{dt}\bigg|_{t=0} Ad^*_{\exp(-t\xi)}(e^{r_\xi t}r)\mu = -ad^*_\xi(r\mu) + rr_\xi\mu = 0,$$

for every $\xi \in \mathfrak{g}_{\mathbb{R}^+\mu}$. The other inclusion can be proved in a similar way. In general, for any element $(g, r) \in G \times \mathbb{R}^+$, observe that

$$\ker T_{(g,r)}f = T_{(e,r)}L_g \ker T_{[e,r]}\tilde{f} = T_{(e,r)}L_g(\{(\xi, rr_\xi) \in \mathfrak{g} \times \mathbb{R} \,|\, \xi \in \mathfrak{g}_{\mathbb{R}^+\mu}\,, ad^*_\xi\mu = r_\xi\mu\}).$$

Applying to the above relation $T_{(g,r)}\pi_{\mathbb{R}^+\mu}$ and recalling that $Ad^*_{\exp t\xi}\mu = e^{r_\xi t}\mu$, for each $\xi \in \mathfrak{g}_{\mathbb{R}^+\mu}$ and t a real number, we obtain that $\ker T_{[g,r]}\tilde{f} = \{0_{[g,r]}\}$. It follows immediately that $\tilde{f}$ is an injective immersion. The fact that it is also a Poisson map is a simple consequence of Lemma 2.7.3 and of the formula for the Hamiltonian vector field associated to $H \in \mathcal{C}^\infty(\mathfrak{g}^*) : X_H(\mu') = ad^*_{\frac{\delta\mu}{\mu}}\mu'$, for any $\mu' \in \mathfrak{g}^*$.

To see that the ray orbit is also an initial submanifold of $\mathfrak{g}^*$, we will prove that its connected components are accessible sets of an integrable singular distribution. For this, let D be the singular distribution on $\mathfrak{g}^*$ defined by

$$D(\mu') := \{ad^*_\xi\mu' + r\mu' \,|\, r \in \mathbb{R}\,, \xi \in \mathfrak{g}\}, \text{ for every } \mu' \in \mathfrak{g}^*.$$

It is a smooth distribution since it is generated by the family of vector fields $(X^{\xi,r})_{\{\xi\in\mathfrak{g},r\in\mathbb{R}^+\}}$ with flows given by

$$\Phi^{\xi,r}(\mu') := Ad^*_{\exp t\xi}e^{tr}\mu', \text{ for every } \mu' \in \mathfrak{g}^*.$$

To show that D is an involutive distribution, let (ξ, r') and (η, r) be two arbitrary elements of $\mathfrak{g} \times \mathbb{R}$. We will prove that $T_{\mu'}\Phi_t^{\eta,r}(X^{\xi,r}(\mu')) = X^{Ad_{\exp(-t\eta)}\xi, r'}(Ad^*_{\exp t\eta}e^{tr}\mu')$ for any real t. Indeed,

$$X^{Ad_{\exp(-t\eta)}\xi,r'}(Ad^*_{\exp t\eta}e^{tr}\mu') = \left.\frac{d}{ds}\right|_{s=0}\left(Ad^*_{\exp(sAd_{-t\eta}\xi)}(e^{sr'}Ad^*_{\exp t\eta}e^{tr}\mu')\right) =$$
$$\left.\frac{d}{ds}\right|_{s=0}\left(Ad^*_{\exp -t\eta \exp s\xi \exp t\eta}(e^{sr'}Ad^*_{\exp t\eta}e^{tr}\mu')\right)$$
$$= \left.\frac{d}{ds}\right|_{s=0}\left(Ad^*_{\exp t\eta}Ad^*_{\exp s\xi}(e^{sr'}e^{tr}\mu')\right) = T_{\mu'}\Phi_t^{\eta,r}(X^{\xi,r}(\mu')).$$

Lemma 2.7.3 implies that $\mathcal{O}_{\mathbb{R}^+\mu}$ is an integral submanifold of D of maximal dimension. Recall that according to the Stefan-Sussmann theorem (see [55] and [56]) the maximal integral manifolds of D are the accessible sets in $\mathfrak{g}^*$ obtained by finite compositions of the flows $\Phi^{\xi,r}_{\{\xi\in\mathfrak{g}, r\in\mathbb{R}\}}$. In particular, they are initial submanifolds of $\mathfrak{g}^*$. Hence the maximal integral manifold of D through μ' is $G^0 \cdot \mu'$, where G^0 is the connected component of e in G and the dot stands for the coadjoint action. Now, it follows easily that the connected components of $\mathcal{O}_{\mathbb{R}^+\mu}$ are $(G^0 \cdot \mu')_{\mu' \in \mathcal{O}_{\mathbb{R}^+\mu}}$. Therefore, $\mathcal{O}_{\mathbb{R}^+\mu}$ is an initial submanifold of $\mathfrak{g}^*$.

Suppose now that the coadjoint action is proper. We want to prove that $\tilde{f}$ is a closed map, hence a homeomorphism onto its image. This will immediately imply that the coadjoint ray orbit is an embedded submanifold of $\mathfrak{g}^*$. For this, let $\tilde{F}$ be a closed subset of $G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$ and $(\tilde{f}([g_n, r_n]))_{n\in\mathbb{N}}$ a convergent sequence in $\tilde{f}(\tilde{F})$. Then we have that $(f((g, r_n)))_{n\in\mathbb{N}}$ is a convergent sequence in $f(F)$, where F is the closed set $= \pi^{-1}_{\mathbb{R}^+\mu}(\tilde{F})$. A result of Palais, see Theorem 4.3.1 in [46], guarantees the existence of a G-invariant metric on $\mathfrak{g}^*$. Using this metric, we obtain that $\| Ad^*_{g_n} r_n\mu \|_{n\in\mathbb{N}} = (r_n \| \mu \|)_{n\in\mathbb{N}}$ is also a convergent subsequence. Hence $\lim_{n\to\infty} r_n = r$ a strictly positive number since $\mu \neq 0$. It follows that $(Ad^*_{g_n}(r_n\mu))_{n\in\mathbb{N}}$ and $(r_n\mu)_{n\in\mathbb{N}}$ are two convergent sequences. Using the properness of the coadjoint action we obtain that $(g_n)_{n\in\mathbb{N}}$ admits a convergent subsequence. The fact that F is closed immediately implies that $\lim_{n\in\mathbb{N}}(\tilde{f}((g, r_n))) \in \tilde{f}(\tilde{F})$, completing thus the proof of this Proposition. □

Remark 2.7.3. *Notice that arguments similar to the ones used in the second part of the proof of Proposition 2.7.1, show that if the coadjoint action is proper any coadjoint ray orbit is a closed submanifold of* $\mathfrak{g}^*$.

Remark 2.7.4. *In general, for each* $\mu \in \mathfrak{g}^*$, *the coadjoint orbit is an immersed submanifold of the corresponding ray coadjoint orbit. If the coadjoint action is proper, then it is a closed embedded sumbanifold.*

Fix μ an element of $\mathfrak{g}^*$. Notice that the Lie algebra of the kernel group of μ, $\mathfrak{k}_\mu$ is closed in $\mathfrak{g}_\mu$. Let U be an open neighborhood of 0 in $\mathfrak{g}$ such that the exponential map $\exp : U \to \exp(U)$ is a diffeomorphism. Choose $V \subset U$ a closed neighborhood of 0. Then $\exp(V) \subset \exp(U)$ is a closed neighborhood of e. We want to show that $\exp(V) \cap K_\mu$ is closed in G. Thus, suppose $(k_n)_{n\in\mathbb{N}} = (\exp \xi_n)_{n\in\mathbb{N}}$ is a convergent sequence of $\exp(V) \cap K_\mu$ with $(\xi_n)_{n\in\mathbb{N}}$ a sequence in V. Since $\exp^{-1} k_n = \xi_n$

for every $n \in \mathbb{N}$, it follows that in fact $\xi_n \in \mathfrak{k}_\mu$. Using the continuity of the exponential map and the fact that the kernel algebra is closed in $\mathfrak{g}$, we have that $\lim_{n\to\infty} \exp^{-1} k_n = \lim_{n\to\infty} \xi_n = \xi \in \mathfrak{k}_\mu$. Therefore $\lim_{n\to\infty} \exp \xi_n = \exp \lim_{n\to\infty} \xi_n = \exp \xi \in K_\mu$ and $\exp(V) \cap K_\mu$ is closed in G. A standard result of Lie theory (see, for instance, [17]), Corollary 1.10.7) implies that the kernel group of μ is a closed regular Lie subgroup of G and the quotient $\frac{G}{K_\mu}$ is a smooth manifold.

Now we are ready to define the manifold which will play the role of the cotangent orbit for the ray reduction, namely the diagonal product of the ray orbit and the quotient of G by the corresponding kernel group

$$\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) := \{(Ad^*_g r\mu, \hat{g}) \,|\, g \in G \,\text{and}\, r \in \mathbb{R}^+\}.$$

Recall that given two surjective submersions $\pi_1 : M_1 \to E$ and $\pi_2 : M_2 \to E$, the diagonal of $M_1 \times M_2$ over (π_1, π_2), $\mathrm{Diag}\,(M_1 \times M_2) := \{(x_1, x_2) \in M_1 \times M_2 \,|\, \pi_1(x_1) = \pi_2(x_2)\}$ is a submanifold of $M_1 \times M_2$ and its tangent space is given by

$$\begin{aligned} T_{(x_1,x_2)}(\mathrm{Diag}(M_1 \times M_2)) \simeq &\{(v_1, v_2) \in T_{x_1}M_1 \times T_{x_2}M_2 \,|\, T_{x_1}\pi_1(v_1) = T_{x_2}\pi_2(v_2)\} = \\ &\mathrm{Diag}(T_{x_1}M_1 \times T_{x_2}M_2). \end{aligned}$$

In particular, for $\pi_1 : \mathcal{O}_{\mathbb{R}^+\mu} \to \frac{G}{K_\mu}$ defined by $\pi_1(Ad^*_g r\mu) := \hat{g}$ and $\pi_2 := Id_{\frac{G}{K_\mu}}$ we obtain that

$$T_{(Ad^*_g r\mu, \hat{g})}\,\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \simeq \mathrm{Diag}\left(T_{Ad^*_g r\mu}\mathcal{O}_{\mathbb{R}^+\mu}, T_{\hat{g}}\frac{G}{K_\mu}\right) \simeq T_{Ad^*_g r\mu}\mathcal{O}_{\mathbb{R}^+\mu}.$$

More precisely, we have that

$$T_{(Ad^*_g r\mu, \hat{g})}\,\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) = \{\left(ad^*_\xi(Ad^*_g r\mu) + r' Ad^*_g r\mu, \hat{\xi}_G(\hat{g})\right) \,|\, \xi \in \mathfrak{g}\,, r' \in \mathbb{R}\},$$

for any $(Ad^*_g r\mu, \hat{g}) \in \mathcal{O}_{\mathbb{R}^+\mu}$. Here $\hat{\xi}_G(\hat{g})$ denotes the projection on $\frac{G}{K_\mu}$ of the infinitesimal isometry associated to ξ with respect to the action by right translations of G on itself. Let $\omega^-_{\mathbb{R}^+\mu}$ be the two form on $\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ defined by

$$\begin{aligned} \omega^-_{\mathbb{R}^+\mu}(Ad^*_g r\mu, \hat{g})((ad^*_{\xi_1} Ad^*_g r\mu + r_1 Ad^*_g r\mu, \widehat{\xi_{1G}}(\hat{g})), (ad^*_{\xi_2} Ad^*_g r\mu + r_2 Ad^*_g r\mu, \widehat{\xi_{2G}}(\hat{g}))) \\ = -\langle Ad^*_g r\mu, [\xi_1, \xi_2]\rangle + r_2\langle Ad^*_g r\mu, \xi_1\rangle - r_1\langle Ad^*_g r\mu, \xi_2\rangle, \quad (2.7.2) \end{aligned}$$

for any $(Ad^*_g r\mu, \hat{g}) \in \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ and any tangent vectors $(ad^*_{\xi_i} Ad^*_g r\mu + r_i Ad^*_g r\mu, \widehat{\xi_{iG}})_{i=1,2} \in T_{(Ad^*_g r\mu, \hat{g})}\,\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$. In fact, as we will see from Theorem 2.7.3, $(\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right), \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}})$ is a well defined symplectic manifold. One could also prove this directly.

Theorem 2.7.3. *Consider the cotangent lift of the action by left translations of a Lie group G on itself. For every $\mu \in \mathfrak{g}^*$ with $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$, the ray reduced space $(T^*(G)_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu})$ is well defined and symplectomorphic to the diagonal manifold $\left(\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right), \omega^-_{\mathbb{R}^+\mu}\right)$ with symplectic form $\omega^-_{\mathbb{R}^+\mu}$ defined by 2.7.2.*

Proof. Let $J_R : T^*G \to \mathfrak{g}^*$ and $J_L : T^*G \to \mathfrak{g}^*$, $J_R(\alpha_g) := T_e^*L_g(\alpha_g)$, $J_L(\alpha_g) := T_e^*R_g(\alpha_g)$ be the equivariant momentum maps associated to the cotangent lifts of right and left translations of G on itself. These lifted actions are proper and free. Therefore, if μ is an element of $\mathfrak{g}^*$ such that $\ker\mu+\mathfrak{g}_\mu = \mathfrak{g}$ the ray reduced space at μ, $(T^*G)_{\mathbb{R}^+\mu} = \frac{J_L^{-1}(\mathbb{R}^+\mu)}{K_\mu}$ is well defined. Note that $J_L^{-1}(\mathbb{R}^+\mu) = \{T_g^*R_{g^{-1}}(r\mu)\,|\, g\in G\,,\, r\in\mathbb{R}^+\}$. The right momentum map induces the application $\bar{J}_R : (T^*G)_{\mathbb{R}^+\mu} \to \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)$ defined by $\bar{J}_R([\alpha_g]) := (J_R(\alpha_g),\hat{g}) = (Ad_g^*r\mu,\hat{g})$, for any $\alpha_g = T_g^*R_{g^{-1}}(r\mu)$. To see that $\bar{J}_R$ is well defined, fix an arbitrary $k\in K_\mu$. Then,

$$\begin{aligned}\bar{J}_R([k\cdot\alpha_g]) = \bar{J}_R([k\cdot T_g^*R_{g^{-1}}r\mu]) &= \bar{J}_R([T_{kg}^*L_{k^{-1}}T_g^*R_{g^{-1}}r\mu])\\ &= (T_e^*L_{kg}T_{kg}^*(R_{g^{-1}}\circ L_{k^{-1}})(r\mu),\hat{kg}) = (Ad_g^*(r\mu),\hat{g}),\end{aligned}$$

proving thus that $\bar{J}_R$ is indeed well-defined. Since the kernel group of μ is a subgroup of its isotropy group, $\bar{J}_R$ is also one to one. Surjectiveness is obvious and hence $\bar{J}_R$ is a bijection. Its inverse is given by $\bar{J}_{R^{-1}} : \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)\to(T^*G)_{\mathbb{R}^+\mu}$, $\bar{J}_{R^{-1}}(Ad_g^*r\mu,\hat{g}) = [T_g^*R_{g^{-1}}r\mu]$.

Let $\lambda\in\Lambda^1(G\times\mathbb{R}^+)$ be the right invariant 1-form given by $\lambda(g,r)(v_g,v_r) := T_g^*R_{g^{-1}}r\mu(v_g)$. Observe that the map $F : G\times\mathbb{R}^+ \to J_L^{-1}(\mathbb{R}^+\mu)$ defined by $F(g,r) := T_g^*R_{g^{-1}}r\mu$ is the graph of λ and hence a diffeomorphism. It follows that its K_μ-projection, $\bar{F} : \frac{G}{K_\mu}\times\mathbb{R}^+ \to (T^*G)_{\mathbb{R}^+\mu}$ is a diffeomorphism too. Lemma 2.7.2 implies that $\bar{f} : \mathrm{Diag}\left((G\times_{G_{\mathbb{R}^+\mu}}\mathbb{R}^+)\times\frac{G}{K_\mu}\right)\to\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)$, $f([g,r],\hat{g}) := (Ad_{g^{-1}}^*r\mu,\hat{g^{-1}})$ is also a diffeomorphism. The map F' in the above diagramm is a also a diffoemorphism defined by $F'(\hat{g},r) := ([g,r],\hat{g})$. The commutativity of this diagramm implies that $\bar{J}_R$ is a diffeomorphism too.

$$\begin{array}{ccc}(T^*G)_{\mathbb{R}^+\mu} & \xrightarrow{\bar{J}_R} & \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)\\ \bar{F}^{-1}\downarrow & \simeq & \downarrow\bar{f}^{-1}\\ (\frac{G}{K_\mu}\times\mathbb{R}^+) & \xrightarrow{\bar{F}'} & \mathrm{Diag}\left((G\times_{G_{\mathbb{R}^+\mu}}\mathbb{R}^+)\times\frac{G}{K_\mu}\right)\end{array}$$

Using the fact that $\bar{J}_R$ is a diffeomorphism with inverse given by

$$\bar{J}_{R^{-1}} : \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)\to(T^*G)_{\mathbb{R}^+\mu}\quad,\quad \bar{J}_{R^{-1}}(Ad_g^*r\mu,\hat{g}) = [T_g^*R_{g^{-1}}r\mu],$$

we can endow $\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)$ with the symplectic form $\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}} := \bar{J}^*_{R^{-1}}\omega_{\mathbb{R}^+\mu}$. In order to give the explicit description of $\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}$ fix $(Ad_g^*r\mu,\hat{g})\in\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right)$ and two tangent vectors $\{v_i = (ad^*_{\xi_i}Ad_g^*r\mu + r_iAd_g^*r\mu,\hat{\xi}_{iG}(\hat{g})\}_{i=1,2}$ in
$T_{(Ad_g^*r\mu,\hat{g})}\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu}\times\frac{G}{K_\mu}\right))$. It follows that

$$\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad_g^*r\mu,\hat{g})(v_1,v_2) = \omega_{\mathbb{R}^+\mu}([T_g^*R_{g^{-1}}r\mu])(T_{Ad_g^*r\mu,\hat{g}}\bar{J}_{R^{-1}}(v_1),T_{Ad_g^*r\mu,\hat{g}}\bar{J}_{R^{-1}}(v_2)).$$

Note that

$$
\begin{aligned}
&T_{(Ad^*_g r\mu,\hat{g})}\bar{J}_{R^{-1}}(v_i) = \\
&\qquad T_{(Ad^*_g r\mu,\hat{g})}\bar{J}_{R^{-1}}\Big(\frac{d}{dt}\Big|_{t=0}(Ad^*_{\exp t\xi_i}e^{tr_i}Ad^*_g r\mu, (\widehat{g\cdot\exp t\xi_i}))\Big) \\
&\qquad = \frac{d}{dt}\Big|_{t=0}(\bar{J}_{R^{-1}}(Ad^*_{\exp t\xi_i}e^{tr_i}Ad^*_g r\mu, (\widehat{g\cdot\exp t\xi_i}))) \\
&\qquad = \frac{d}{dt}\Big|_{t=0}(\pi_{K_\mu}(T^*_{g\exp t\xi_i}R_{g\exp t\xi_i^{-1}}re^{tr_i}\mu)) \\
&= T_{T^*_g R_{g^{-1}}r\mu}\pi_{K_\mu}\Big(\frac{d}{dt}\Big|_{t=0}e^{tr_i}T^*_g R_{g^{-1}}r\mu\cdot\exp t\xi_i\Big) = T_{T^*_g R_{g^{-1}}r\mu}\pi_{K_\mu}(X^{\xi_i}(T^*_g R_{g^{-1}}r\mu)),
\end{aligned}
$$

where X^{ξ_i} is the vector field on T^*G with flow given by
$\Phi^i(t,\alpha'_g) := T^*_{g\exp t\xi_i}R_{\exp -t\xi_i}e^{tr_i}\alpha'_g$, for any $\alpha'_g \in T^*_{g'}G$. In this proof all the dots denote the action Then, using the fact that $\pi^*_{K_\mu}\omega_{\mathbb{R}^+\mu} = i^*_{\mathbb{R}^+\mu}(-d\Theta)$ and the above calculus, we obtain

$$
\begin{aligned}
\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad^*_g r\mu,\hat{g})(v_1,v_2) &= \pi^*_{K_\mu}\omega_{\mathbb{R}^+\mu}(T^*_g R_{g^{-1}}r\mu)(X^{\xi_1}(T^*_g R_{g^{-1}}r\mu), X^{\xi_2}(T^*_g R_{g^{-1}}r\mu)) \\
&= -d\Theta(T^*_g R_{g^{-1}}r\mu)(X^{\xi_1}(T^*_g R_{g^{-1}}r\mu), X^{\xi_2}(T^*_g R_{g^{-1}}r\mu)) = \\
-X^{\xi_1}(\Theta(X^{\xi_2})(T^*_g R_{g^{-1}}r\mu) &+ X^{\xi_2}(\Theta(X^{\xi_1})(T^*_g R_{g^{-1}}r\mu) + \Theta([X^{\xi_1},X^{\xi_2}])(T^*_g R_{g^{-1}}r\mu).
\end{aligned}
$$

Next, we want to show that

$$
\Theta(X^{\xi_i}) = J^{\xi_i}_R \text{ and } X^{\xi_i}(J^{\xi_j}_R)(T^*_g R_{g^{-1}}r\mu) = \langle Ad^*_g r\mu, [\xi_i,\xi_j]\rangle + r_i\langle Ad^*_g r\mu, \xi_j\rangle, \tag{2.7.3}
$$

for $i = 1,2$. Indeed, for any $\alpha_g \in T^*_g G$ we have

$$
\begin{aligned}
\Theta(X^{\xi_i})(\alpha_g) = \langle\alpha_g, T_{\alpha_g}\pi(X^{\xi_i}(\alpha_g))\rangle &= \langle\alpha_g, \frac{d}{dt}\Big|_{t=0}\pi(e^{tr_i}\alpha_g\cdot\exp t\xi_i)\rangle \\
&= \langle\alpha_g, \xi_{iG}(g)\rangle = J^{\xi_i}_R(\alpha_g).
\end{aligned}
$$

This also implies that X^{ξ_i} and ξ_{iG} are π-related vector fields. And

$$
\begin{aligned}
&X^{\xi_i}(J^{\xi_j}_R)(T^*_g R_{g^{-1}}r\mu) = \frac{d}{dt}\Big|_{t=0}J^{\xi_j}_R(e^{tr_i}T^*_g R_{g^{-1}}r\mu\cdot\exp t\xi_i) = \\
&\frac{d}{dt}\Big|_{t=0}T^*_e L_{g\exp t\xi_i}(T^*_{g\exp t\xi_i}R_{\exp -t\xi_i}(e^{tr_i}T^*_g R_{g^{-1}}r\mu))(\xi_j) = \\
&\frac{d}{dt}\Big|_{t=0}Ad^*_{g\exp t\xi_i}(e^{tr_i}r\mu)(\xi_j) = Ad^*_g(ad^*_{Ad_g\xi_i}r\mu + r_i r\mu)(\xi_j) = \\
&\qquad Ad^*_g(ad^*_{Ad_g\xi_i}r\mu + r_i r\mu)(\xi_j) = (ad^*_{\xi_i}(Ad^*_g r\mu) + r_i Ad^*_g r\mu)(\xi_j).
\end{aligned}
$$

Note that in the above calculation we have again used formula 2.7.1. Applying 2.7.3, it follows that

$$\begin{aligned}\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad^*_g r\mu, \hat{g})(v_1, v_2) &= -X^{\xi_1}(\Theta(X^{\xi_2})(T^*_g R_{g^{-1}} r\mu) + X^{\xi_2}(\Theta(X^{\xi_1})(T^*_g R_{g^{-1}} r\mu) \\ &+\Theta(X^{[\xi_1,\xi_2]})(T^*_g R_{g^{-1}} r\mu) = -\langle Ad^*_g r\mu, [\xi_1, \xi_2]\rangle - r_1\langle Ad^*_g r\mu, \xi_2\rangle \\ &+\langle Ad^*_g r\mu, [\xi_2, \xi_1]\rangle + r_2\langle Ad^*_g r\mu, \xi_1\rangle + J_R^{[\xi_1,\xi_2]}(T^*_g R_{g^{-1}} r\mu) \\ &= -\langle Ad^*_g r\mu, [\xi_1, \xi_2]\rangle + r_2\langle Ad^*_g r\mu, \xi_1\rangle - r_1\rangle Ad^*_g r\mu, \xi_2\rangle.\end{aligned}$$

In particular, for $g = e$ and $r = 1$ we have that

$$\begin{aligned}\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(\mu, \hat{e})((ad^*_{\xi_1}\mu + r_1\mu, \hat{\xi}_1), (ad^*_{\xi_1}\mu + r_1\mu, \hat{\xi}_2)) &= -\langle \mu, [\xi_1, \xi_2]\rangle + r_2\langle \mu, \xi_1\rangle \\ &-r_1\langle \mu, \xi_2\rangle, \text{ for any } \xi_{i\, i=1,2} \in \mathfrak{g}.\end{aligned}$$

The first term in the above expression is precisely $\omega^-_{\mathcal{O}_\mu}(\mu)(ad^*_{\xi_1}\mu, ad^*_{\xi_2}\mu)$ and hence the minus sign in the notation of the symplectic form on $\operatorname{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$. □

Corollary 2.7.1. *In the hypothesis of Theorem 2.7.3, the symplectic form* $\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}$ *defined by 2.7.2 is* G*-invariant with respect to the following action*

$$g_1 \cdot (Ad^*_g r\mu, \hat{g}) := \left(Ad^*_{g_1^{-1}} Ad^*_g r\mu, \widehat{gg_1^{-1}}\right),$$

for each g_1 *in* G *and* $(Ad^*_g r\mu, \hat{g})$ *in* $\operatorname{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$.

Proof. Fix g_1 in G and $x := (Ad^*_g r\mu, \hat{g})$ in $\operatorname{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$. Let v_ξ be the tangent vector $(ad^*_\xi(Ad^*_g r\mu) + r_\xi Ad^*_g r\mu, \hat{\xi}_G(\hat{g})) \in T_{(Ad^*_g r\mu, \hat{g})} \operatorname{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$. Here ξ is an arbitrary element of $\mathfrak{g}$. Then, we have

$$\begin{aligned}&\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(g_1 \cdot x)(g_1 \cdot v_\xi, g_1 \cdot v_\eta) = \\ &\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(g_1 \cdot x)\left(\left.\frac{d}{dt}\right|_{t=0}(Ad^*_{\exp t\xi g_1^{-1}} e^{tr_\xi} Ad^*_g r\mu), T_g \widehat{R_{g_1^{-1}}} \xi_G(\hat{g}), \right. \\ &\left.\left.\frac{d}{dt}\right|_{t=0}(Ad^*_{\exp t\eta g_1^{-1}} e^{tr_\eta} Ad^*_g r\mu), T_g \widehat{R_{g_1^{-1}}} \eta_G(\hat{g})\right) = \\ &\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(g_1 \cdot x)\left(\left(Ad^*_{g_1^{-1}} v_\xi, (\widehat{Ad_{g_1}\xi})_G(\widehat{gg_1^{-1}})\right), \left(Ad^*_{g_1^{-1}} v_\eta, (\widehat{Ad_{g_1}\eta})_G(\widehat{gg_1^{-1}})\right)\right) = \\ &\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(g_1 \cdot x)\left(\left(v_{Ad_{g_1}\xi}, (\widehat{Ad_{g_1}\xi})_G(\widehat{gg_1^{-1}})\right), \left(v_{Ad_{g_1}\eta}, (\widehat{Ad_{g_1}\eta})_G(\widehat{gg_1^{-1}})\right)\right) = \\ &-\langle Ad^*_{gg_1^{-1}} r\mu, [Ad_{g_1}\xi, Ad_{g_1}\eta]\rangle + r_\eta\langle Ad^*_{gg_1^{-1}} r\mu, Ad_{g_1}\xi\rangle - r_\xi\langle Ad^*_{gg_1^{-1}} r\mu, Ad_{g_1}\eta\rangle = \\ &-\langle Ad^*_g r\mu, [\xi, \eta]\rangle + r_\eta\langle Ad^*_g r\mu, \xi\rangle - r_\xi\langle Ad^*_g r\mu, \eta\rangle = \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(x)(v_\xi, v_\eta).\end{aligned}$$

Therefore, $\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}$ is G-invariant. □

Proposition 2.7.2. *The symplectomorphic G-action on* $\left(\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right), \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}\right)$ *admits an equivariant momentum map*

$$-I_{\mathcal{O}_{\mathbb{R}^+\mu}} : \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \to \mathfrak{g}^*\,, I_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad^*_g r\mu, \hat{g}) := -Ad^*_g r\mu,$$

for each $(Ad^*_g r\mu, \hat{g})$ *in* $\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$.

Proof. Let ξ be an element of $\mathfrak{g}$ and denote by $I^{\xi}_{\mathcal{O}_{\mathbb{R}^+\mu}} : \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \to \mathbb{R}$ the map given by $(Ad^*_g r\mu, \hat{g}) \mapsto \langle Ad^*_g r\mu, \xi\rangle$. The infinitesimal generator associated to ξ on $\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ is

$$\xi_{\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)}(Ad^*_g r\mu, \hat{g}) = \left.\frac{d}{dt}\right|_{t=0} \left(Ad^*_{\exp -t\xi} Ad^*_g r\mu, \widehat{g \exp -t\xi}\right) =$$
$$(ad^*_{-\xi}(Ad^*_g r\mu), -\hat{\xi}_G(\hat{g})), \text{ for any } (Ad^*_g r\mu, \hat{g}) \in \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right).$$

Then, for all $\left(ad^*_\eta Ad^*_g r\mu + r_\eta Ad^*_g r\mu, \hat{\eta}_G(\hat{g})\right) \in T_{(Ad^*_g r\mu, \hat{g})} \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$, we obtain that

$$i_{\xi_{\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)}} \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad^*_g r\mu, \hat{g})(ad^*_\eta Ad^*_g r\mu + r_\eta Ad^*_g r\mu, \hat{\eta}_G(\hat{g})) = \tag{2.7.4}$$
$$\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad^*_g r\mu, \hat{g})\left(\left(ad^*_{-\xi}(Ad^*_g r\mu), -\hat{\xi}_G(\hat{g})\right), \left(ad^*_\eta Ad^*_g r\mu + r_\eta Ad^*_g r\mu, \hat{\eta}_G(\hat{g})\right)\right) =$$
$$\langle Ad^*_g r\mu, [\xi, \eta]\rangle - r_\eta \langle Ad^*_g r\mu, \xi\rangle.$$

On the other hand,

$$T_{(Ad^*_g r\mu, \hat{g})} I^{\xi}_{\mathcal{O}_{\mathbb{R}^+\mu}} \left(ad^*_\eta Ad^*_g r\mu + r_\eta Ad^*_g r\mu, \hat{\eta}_G(\hat{g})\right) = \left.\frac{d}{dt}\right|_{t=0} \langle Ad^*_{\exp t\eta} e^{tr_\eta} Ad^*_g r\mu, \xi\rangle \tag{2.7.5}$$
$$= -\langle Ad^*_g r\mu, [\xi, \eta]\rangle + r_\eta \langle Ad^*_g r\mu, \xi\rangle.$$

Equalities (2.7.4) and (2.7.5) imply that $X_{-I^{\xi}_{\mathcal{O}_{\mathbb{R}^+\mu}}} = \xi_{\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)}$ for all $\xi \in \mathfrak{g}$. Hence the proof of this proposition is complete. □

Recall that the symplectic difference of two symplectic manifolds $(M_i, \omega_i)_{i=1,2}$ is $M_1 \ominus M_2 := (M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$, where $(\pi_i : M_1 \times M_2 \to M_i)_{i=1,2}$ are the canonical projections. If the Lie group G acts on both M_1 and M_2 such that these actions admit equivariant momentum maps $(J_i : M_i \to \mathfrak{g}^*)_{i=1,2}$, then the diagonal action of G on the symplectic difference $M_1 \ominus M_2$ admits an equivariant momentum map given by $J_d := J_1 \circ \pi_1 - J_2 \circ \pi_2 : M_1 \ominus M_2 \to \mathfrak{g}^*$.

The following theorem illustrates the theoretical importance of the diagonal product $\mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ in the reduction procedure. Namely, any ray reduced space can be seen as the symplectic difference of the initial manifold and the diagonal product of the associated ray coadjoint orbit with the quotient of G by the kernel group.

Theorem 2.7.4. ***Shifting Theorem*** *Let the Lie group G act smoothly on the symplectic manifold (M, ω) such that it admits an equivariant momentum map $J : M \to \mathfrak{g}^*$. Fix μ an element of the dual Lie algebra of G and suppose that the hypothesis of Theorem 2.4.1 are fulfilled. Then G acts diagonaly on $M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ and its symplectic reduced space at zero is well defined. Even more, $\left(M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)\right)_0$ is symplectomorphic to $M_{\mathbb{R}^+\mu}$, the ray reduced space at μ of M.*

Proof. The symplectic difference $M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ has symplectic form $\pi_1^*\omega - \pi_2^*\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}$ and momentum map $J_d := J \circ \pi_1 + I_{\mathcal{O}_{\mathbb{R}^+\mu}} \circ \pi_2$. Of course, $\pi_1 : M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \to M$ and $\pi_2 : M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \to \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ are the canonical projections. It is easy to check that in the hypothesis of Theorem 2.4.1, the 0-symplectic reduced space is well defined.

Let $\phi : J^{-1}(\mathbb{R}^+\mu) \to M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ be the map defined by $x \in J^{-1}(\mathbb{R}^+\mu) \mapsto (x, (-J(x), \hat{e}))$. Denote by $[\phi]$ its (K_μ, G)-projection

$$[\phi] : M_{\mathbb{R}^+\mu} \to \left(M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)\right)_0, [\phi](\hat{x}) := [x, (-J(x), \hat{e})],$$

where $[,]$ and $\hat{}$ denote the G and K_μ-classes, respectively. This map is well defined. Indeed, let k be an element of the kernel group of μ. Then, $[\phi](\widehat{kx}) = [kx, (-J(kx), \hat{e})] = [kx, (-k \cdot J(x), \widehat{k^{-1}})] = [k \cdot (x, (-J(x), \hat{e}))] = [\phi](\hat{x})$, for any $\hat{x} \in M_{\mathbb{R}^+\mu}$. To see that $[\phi]$ is injective, let $\hat{x}_1, \hat{x}_2$ be elements of $M_{\mathbb{R}^+\mu}$ such that $[x_1, (-J(x_1), \hat{e})] = [x_2, (-J(x_2), \hat{e})]$. Then, there is g an element of G such that $(gx_1, (-gJ(x), \hat{g}^{-1}) = (x_2, (-J(x_2), \hat{e}))$. It follows that $g \in K_\mu$ and $gx_1 = x_2$. Hence $\hat{x}_1 = \hat{x}_2$ and $[\phi]$ is one-to-one. If $[x, (Ad_g^* r\mu, \hat{g})]$ is an element of $\left(M \ominus \mathrm{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)\right)_0$, then $J_d(x, (Ad_g^* r\mu, \hat{g})) = J(x) + Ad_g^* r\mu = 0$. Therefore, $gx \in J^{-1}(\mathbb{R}^+\mu)$,

$$[\phi](\hat{gx}) = [gx, (-J(gx), \hat{e})] = [gx, (-gAd_g^* r\mu, \widehat{gg^{-1}})] = [x, (-J(x), \hat{e})],$$

and $[\phi]$ is onto. As it is obviously a smooth map, we obtain that $[\phi]$ is in fact a diffeomorphism with inverse given by $[x, (Ad_g^* r\mu, \hat{g})] \mapsto [gx]$.

To show that $[\phi]$ is also a symplectic map, fix $\hat{x}$ in $M_{\mathbb{R}^+\mu}$ and $(v_i)_{i=1,2}$ in $T_x J^{-1}(\mathbb{R}^+\mu)$. Note that $T_x(\pi_2 \circ \phi)(v_i)$ belongs to $\mathbb{R}\mu \simeq T_{J(x)}(\mathbb{R}^+\mu)$ for each $i = 1, 2$. Suppose $J(x) = r\mu$ and $(T_x(\pi_2 \circ \phi)(v_i) = r_i\mu)_{i=1,2}$ with $(r_i)_{i=1,2}$ reals. Then, using the function equalities $[\phi] \circ \pi_{K_\mu} = \pi_G \circ \phi$ and

$\pi_1 \circ \phi = Id_{J^{-1}(\mathbb{R}^+\mu)}$, we obtain

$$
\begin{aligned}
[\phi]^*(\pi_1^*\omega - \pi_2^*\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}})_0(\hat{x})(T_x\pi_{K_\mu}v_1, T_x\pi_{K_\mu}v_2) = \\
(\pi_1^*\omega - \pi_2^*\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}})_0([x, (-J(x), \hat{e})])(T_x([\phi] \circ \pi_{K_\mu})v_1, T_x([\phi] \circ \pi_{K_\mu})v_2) = \\
(\pi_1^*\omega - \pi_2^*\omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}})(\phi(x))(T_x\phi v_1, T_x\phi v_2) = \\
\omega(x)(T_x(\pi_1 \circ \phi)v_1, T_x(\pi_1 \circ \phi)v_2) - \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(J(x), \hat{e})(T_x(\pi_2 \circ \phi)v_1, T_x(\pi_2 \circ \phi)v_2) = \\
i_\mu^*\omega(x)(T_x(i_\mu \circ \pi_1 \circ \phi)v_1, T_x(i_\mu \circ \pi_1 \circ \phi)v_2) - \omega^-_{\mathcal{O}_{\mathbb{R}^+\mu}}(r\mu, \hat{e})(r_1\mu, r_2\mu) = \\
i_\mu^*\omega(x)(T_x(i_\mu \circ \pi_1 \circ \phi)v_1, T_x(i_\mu \circ \pi_1 \circ \phi)v_2) = \omega_{\mathbb{R}^+\mu}(\hat{x})(T_x\pi_{K_\mu}v_1, T_x\pi_{K_\mu}v_2),
\end{aligned}
$$

completing thus the proof of this theorem. □

In the remaining of this section we will study the ray reduced spaces of the cosphere bundle of the Lie group G. Consider the action of the multiplicative group $\mathbb{R}^+$ by dilatations on the fibers of $T^*G \setminus \{0_{T^*G}\}$. The cosphere bundle of G, S^*G is the quotient manifold $(T^*G \setminus \{0_{T^*G}\})/\mathbb{R}^+$. Denote by $\pi_{\mathbb{R}^+} : T^*G \setminus \{0_{T^*G}\} \to S^*(G)$ the canonical projection. Then, $(\pi_{\mathbb{R}^+}, \mathbb{R}^+, T^*G \setminus \{0_{T^*G}\}, S^*G)$ is a $\mathbb{R}^+$-principal bundle. S^*G admits a canonical contact structure given by the kernel of any one form constructed as the pull-back of the Liouville form on T^*G through a global section of the $\mathbb{R}^+$-principal bundle $(\pi_{\mathbb{R}^+}, \mathbb{R}^+, T^*G \setminus \{0_{T^*G}\}, S^*G)$. Namely, for every global section $\sigma : S^*G \to T^*G \setminus \{0_{T^*G}\}$ the one-form $\Theta_\sigma = \sigma^*\Theta$ determines the same contact structure. Note that $\pi^*_{\mathbb{R}^+}\Theta_\sigma = f_\sigma\Theta$, where $f_\sigma : T^*G \setminus \{0_{T^*G}\} \to \mathbb{R}^+$ is a smooth function with the property that $f_\sigma(r\alpha_g) = \frac{1}{r} f\sigma(\alpha_g)$ for any $r \in \mathbb{R}^+$ and $\alpha_g \in T^*G$. The action by left translations of G on its cotangent bundle induces a free and proper action on the copshere bundle given by

$$g' \cdot \{\alpha_g\} := \{T^*_{g'g}L_{g^{-1}}\alpha_g\},$$

for all $\{\alpha_g\} \in S^*G$ and $g' \in G$. Since it is a proper action which preserves the contact structure, there is always a global section σ such that the action will preserve the associated contact form Θ_σ. Then, this action admits an equivariant momentum map defined by

$$\langle J_{sL}(\{\alpha_g\}), \xi\rangle := \Theta_\sigma(\{\alpha_g\}(\xi_{S^*G})(\{\alpha_g\}) = f_\sigma(\alpha_g)\alpha_g(\xi_G(g)),$$

where $\{\alpha_g\} \in S^*G$ and $\xi \in \mathfrak{g}$. That is, $J_{sL}(\alpha_g) = f_\sigma(\alpha_g)\alpha_g$, for any $\{\alpha_g\} \in S^*G$. Here we have briefly recalled the construction and some of the properties of the cosphere bundle of a Lie group. For more details the interesting reader is referred to [16], [18], and [50].

Denote by Diag $\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu}\right)$ the diagonal product of the $\pi_{\mathbb{R}^+}$-quotient of the ray orbit of μ and $\frac{G}{K_\mu}$. The quoteint space $S^*(\mathcal{O}_{\mathbb{R}^+\mu})$ is a smooth manifold since the $\mathbb{R}^+$-action on $\mathcal{O}_{\mathbb{R}^+\mu}$ is free and proper. The map

$$[g] \in \frac{G}{G_{\mathbb{R}^+\mu}} \longrightarrow Ad^*_g r\mu$$

is a diffeomorphism. Define the following one form on Diag $\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right)$

$$\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}(\{Ad^*_g r\mu\},\hat{g})(T_{Ad^*_g r\mu}\pi_{\mathbb{R}^+\mu}(ad^*_\xi Ad^*_g r\mu + r' Ad^*_g r\mu, \hat{\xi_G}(\hat{g})) := f_\sigma(T^*_g R_{g^{-1}} r\mu)\langle Ad^*_g r\mu, \xi\rangle, \quad (2.7.6)$$

for any $(\{Ad^*_g r\mu\},\hat{g}) \in \mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right)$ and any tangent vector

$$T_{Ad^*_g r\mu}\pi_{\mathbb{R}^+\mu}(ad^*_\xi Ad^*_g r\mu + r' Ad^*_g r\mu, \hat{\xi_G}(\hat{g}) \in T_{(\{Ad^*_g r\mu\},\hat{g})}\,\mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right).$$

As we will see in the proof of the following Theorem, the diagonal manifold $\left(\mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right),\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}\right)$ is a well defined exact contact manifold.

Theorem 2.7.5. *Let the Lie group G act on its cosphere bundle S^*G by the lift of left translations on itself. Suppose μ is an element of the dual of its Lie algebra with kernel group K_μ and the property that* $\ker\mu + \mathfrak{g}_\mu = \mathfrak{g}$*, where $\mathfrak{g}_\mu$ is the isotropy algebra of μ for the coadjoint action. Then the ray reduced space at μ, $(S^*G)_{\mathbb{R}^+\mu}$ is well defined and contactomorphic to $\left(\mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right),\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}\right)$, where $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ is the one form define by 2.7.6.*

Proof. First note that since the K_μ and $\mathbb{R}^+$-actions commute we have the equality $J^{-1}_{sL}(\mathbb{R}^+\mu) = \pi_{\mathbb{R}^+}(J^{-1}_L(\mathbb{R}^+\mu))$. Even more the maps,

$$\phi : (S^*G)_{\mathbb{R}^+\mu} \to \frac{(T^*G)_{\mathbb{R}^+\mu}}{\mathbb{R}^+} \quad \text{and} \quad \bar{J}_{R^+} : \frac{(T^*G)_{\mathbb{R}^+\mu}}{\mathbb{R}^+} \to \mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right)$$

defined by $\phi([\{\alpha_g\}]) := \{[\alpha_g]\}$ and $\bar{J}_{R^+}(\{[T^*_g R_{g^{-1}} r\mu := (\{Ad^*_g r\mu\},\hat{g})$, for any α_g in $J^{-1}_L(\mathbb{R}^+\mu)$ are diffeomorphisms. Let $\Psi : (S^*G)_{\mathbb{R}^+\mu} \to \mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right)$ be the map $\Psi := \bar{J}_{R^+}\circ\phi$. It is obviously a diffeomorphism with inverse given by

$$\Psi^{-1} : \mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu})\times\frac{G}{K_\mu}\right) \to (S^*G)_{\mathbb{R}^+\mu}\,, \Psi^{-1}(\{Ad^*_g r\mu\},\hat{g}) = [\{T^*_g R_{g^{-1}} r\mu\}],$$

for any $g \in G$ and $r \in \mathbb{R}^+$. Denote by $\eta_{\mathbb{R}^+\mu}$ the reduced contact form of $(S^*G)_{\mathbb{R}^+\mu}$. Then,

$$
\begin{aligned}
&(\Psi^{-1})^*(\eta_{\mathbb{R}^+\mu})(\{Ad^*_g r\mu\},\hat{g})(T_{Ad^*_g r\mu}\pi_{\mathbb{R}^+\mu}(ad^*_\xi Ad^*_g r\mu + r' Ad^*_g r\mu, \hat{\xi}_G(\hat{g})) = \\
&\quad (\pi^s_{\mathbb{R}^+\mu})^*(\{T^*_g R_{g^{-1}} r\mu\}) \left(\frac{d}{dt}\bigg|_{t=0} \pi_{\mathbb{R}^+}(Ad^*_{g\exp t\xi} e^{tr'} Ad^*_g r\mu, \widehat{g\cdot \exp t\xi}) \right) = \\
&\quad (\pi^s_{\mathbb{R}^+\mu})^*(\{T^*_g R_{g^{-1}} r\mu\}) \left(T_{T^*_g R_{g^{-1}} r\mu}\pi_{\mathbb{R}^+}(X^\xi(T^*_g R_{g^{-1}} r\mu)) \right) = \\
&\quad \Theta_\sigma(\{T^*_g R_{g^{-1}} r\mu\}) \left(T_{T^*_g R_{g^{-1}} r\mu}\pi_{\mathbb{R}^+}(X^\xi(T^*_g R_{g^{-1}} r\mu)) \right) = \\
&(\pi^*_{\mathbb{R}^+}\Theta)(T_{T^*_g R_{g^{-1}} r\mu})(X^\xi(T^*_g R_{g^{-1}} r\mu)) = f_\sigma(T^*_g R_{g^{-1}} r\mu)\Theta(T^*_g R_{g^{-1}} r\mu)(X^\xi(T^*_g R_{g^{-1}} r\mu)) \\
&\quad = f_\sigma(T^*_g R_{g^{-1}} r\mu) J^\xi_R(T^*_g R_{g^{-1}} r\mu) = f_\sigma(T^*_g R_{g^{-1}} r\mu)\langle Ad^*_g r\mu, \xi\rangle = \\
&\qquad \eta_{\mathcal{O}_{\mathbb{R}^+\mu}}(\{Ad^*_g r\mu\},\hat{g})(T_{Ad^*_g r\mu}\pi_{\mathbb{R}^+\mu}(ad^*_\xi Ad^*_g r\mu + r' Ad^*_g r\mu, \hat{\xi_G}(\hat{g})), \quad (2.7.7)
\end{aligned}
$$

for all $\xi \in \mathfrak{g}$ and $g \in G$. Hence, $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ is a contact form and Ψ the required contactomorphism. □

Corollary 2.7.2. *In the hypothesis of Theorem 2.7.5, the contact form $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ defined by 2.7.6 is G-invariant with respect to the following action*

$$g_1 \cdot (\{Ad^*_g r\mu\},\hat{g}) := \left(\{Ad^*_{g_1^{-1}} Ad^*_g r\mu\}, \widehat{gg_1^{-1}} \right),$$

for each g_1 in G and $(Ad^*_g r\mu, \hat{g}) \in \mathrm{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu}\right)$.

2.8 Ray quotients of Kähler-Einstein Fano manifolds

In this section we will study metric properties of ray reduced spaces of compact Kähler-Einstein manifolds of positive first Chern class and Ricci curvature.

Let M be a compact complex manifold of positive first Chern class. Choose a metric Kähler g such that its Kähler form ω_{g} represents the first Chern class of M. Denote by ρ_{g} the associated Ricci form. Then, since both ω_{g} and ρ_{g} represent the fisrt Chern class, there is a smooth function f such that $\rho_{\mathrm{g}} - \omega_{\mathrm{g}} = \frac{i}{2\pi}\partial\bar{\partial}f$. By Theorem 2.4.3 in [19] there is an isomorphism between the complex Lie algebra of holomorphic vector fields on M and the set of all complex-valued functions u satisfying $\Delta_f u - u = 0$. This isomorphism is given by $u \mapsto \operatorname{grad} u$. Here, Δ_f is the differential operator given by $\Delta - \nabla^i\nabla_i - Id$, with Δ the complex Laplacian and ∇ the covariant derivative associated to g. Suppose the compact connected Lie group G acts on (M, g) by holomorphic isometries. Then, the infinitezimal isometries associated to the elements of the Lie algebra $\mathfrak{g}$ embed in the space of holomorphic vector fields on M as follows: assign to each $\xi \in \mathfrak{g}$ the holomorphic vector field $\xi'_M := \frac{1}{2}(\xi_M - iI_{\mathrm{g}}\xi_M)$. I_{g} denotes the complex structure of (M, g). Therefore, applying Theorem 2.4.3 in [19], there is a smooth complex function $u_{\xi'_M}$ with $\operatorname{grad} u_{\xi'_M} = \xi'_M$.

Lemma 2.8.1. *For every ξ element of the Li algebra $\mathfrak{g}$, the above defined function $u_{\xi'_M}$ is pure imaginary.*

Proof. Since G acts by holomorphic isometries, $\xi_M(f) = 0$ and we have

$$\Delta_f u_{\xi'_M} = \Delta u_{\xi'_M} - \xi'_M(f) = \Delta u_{\xi'_M} + \frac{i}{2} I_{\mathrm{g}}(\xi_M) f \text{ and } \Delta_f \bar{u}_{\xi'_M} = \Delta \bar{u}_{\xi'_M} - \frac{i}{2} I_{\mathrm{g}}(\xi_M) f.$$

Using the fact that $\Delta_f u_{\xi'_M} - u_{\xi'_M} = 0$ it follows that

$$\Delta(u_{\xi'_M} + \bar{u}_{\xi'_M}) = u_{\xi'_M} + \bar{u}_{\xi'_M}. \tag{2.8.1}$$

On the other hand, it is well known that on a complex connected Riemannian manifold, if X, the gradient of a function u is a holomorphic vector field, then it is a Killing vector field if and only if $u + \bar{u}$ is constant. In particular, if u is pure imaginar, then the real part of X is a Killing vector field. For a proof of this, see for instance [12]. Applying this to $X = \mathrm{grad}(u_{\xi'_M} + \bar{u}_{\xi'_M})$ we obtain that $u_{\xi'_M} + \bar{u}_{\xi'_M}$ is a constant function. Hence, 2.8.1 implies that $u_{\xi'_M} + \bar{u}_{\xi'_M} = 0$. □

Proposition 2.8.1. *Let M be a compact complex manifold of positive first Chern class and dimension n. Choose any Kähler metric g which represents the first Chern class and suppose the Lie group G acts on (M, g) by holomorphic isometries. Then the map $J : M \to \mathfrak{g}^*$, $\langle J(x), \xi \rangle := \frac{i}{2\pi} u_{\xi'_M}$ defines an equivariant momentum map for the action of G on M.*

Proof. In a local holomorphic coordinate system $(z_1, ..., z_n)$, the Kähler form associated to g is given by $\omega_{\mathrm{g}} = \frac{i}{2\pi} \mathrm{g}_{\alpha\bar{\beta}} \, dz^\alpha \wedge dz^{\bar{\beta}}$. Then, for any ξ in $\mathfrak{g}$,

$$i_{\xi'_M} \omega_{\mathrm{g}} = i_{\mathrm{grad}\, u_{\xi'_M}} \omega_{\mathrm{g}} = \frac{i}{2\pi} \, \mathrm{g}^{\alpha\bar{\beta}} \, \nabla_{\bar{\beta}} u_{\xi'_M} \, \mathrm{g}_{\alpha\bar{\gamma}} \, d\bar{z}^\gamma = \bar{\partial} J_\xi.$$

and

$$i_{\xi_M} \omega_{\mathrm{g}} = i_{(\xi'_M + \bar{\xi}'_M)} \omega_{\mathrm{g}} = i_{\xi'_M} \omega_{\mathrm{g}} + i_{\bar{\xi_M}} \omega_{\mathrm{g}} = dJ_\xi,$$

proving thus that J is a momentum map. To show the equivariance of J, fix $g \in G$ and $\xi \in \mathfrak{g}$. Observe that the g-action commutes with the operator Δ_f and that for any vector field Y of type $(0,1)$ we have

$$\begin{aligned} \omega_{\mathrm{g}}(\mathrm{grad}\, g^* u_{\xi'_M}, Y) = Y g^* u_{\xi'_M} J_\xi = (g_* Y) j_\xi = \omega_{\mathrm{g}}(\mathrm{grad}\, g^* u_{\xi'_M}, g_* Y) = \\ \omega_{\mathrm{g}}(g_*^{-1} \xi'_M, Y) = \omega_{\mathrm{g}}((ad_{g^{-1}} \xi)'_M, Y) = \omega_{\mathrm{g}}(\mathrm{grad}\, u_{(ad_{g^{-1}} \xi)'_M}, Y). \end{aligned}$$

Hence, J is also G-equivariant. □

Recall that the notion of *Fano manifold.* In algebraic geometric a Fano manifold is a compact complex manifold of dimension n which can be embedded into a complex projective space by the sections of $(\Lambda^n T^{*,1,0} M)^m$ for m sufficiently large. The geometric formulation states that M is a Fano manifold if and only if it is of positive first Chern class. The simplest example is $\mathbb{P}^1(\mathbb{C})$. It is the only Fano manifold which is a Kähler-Einstein manifold with respect to to the Fubini-Study metric. The two-dimensional Fano manifolds are called del Pezzo surfaces and they have been completely classified. For more details and references, see [19] and [26].

Theorem 2.8.1. *Let $(M, \mathrm{g}, \omega_{\mathrm{g}})$ be a Fano manifold with $\omega_{\mathrm{g}} \in c_1(M)$, the first Chern class of M and G a Lie group acting on M by holomorphic isometries. Suppose that μ is an element of the dual of the Lie algebra of G such that the ray reduced space is a well defined Kähler orbifold $(M_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu})$ and denote by K_μ its kernel group. Assume that K_μ is compact. Consider f the G-invariant function with the property that the difference between the Ricci form of* g *and its Kähler form is $\frac{i}{2\pi}\partial\bar{\partial}f$. Choose any basis of $\mathfrak{g}$, $\{\xi_i\}_{i=1,k}$ and let $\{\xi'_{iM}\}_{i=1,k}$ be the set of holomorphic vector fields $\{\frac{1}{2}(\xi_M + iI_{\mathrm{g}}\xi_M)\}_{i=1,k}$. Then, the Ricci form of the ray reduced space is given by*

$$\rho_{\mathbb{R}^+\mu} = \frac{i}{2\pi}\partial\bar{\partial}(f_{\mathbb{R}^+\mu} + \log \|\xi\|^2_{\mathbb{R}^+\mu}), \tag{2.8.2}$$

where $f_{\mathbb{R}^+\mu}$ and $\|\xi\|_{\mathbb{R}^+\mu}$ are the K_μ-projections of f and the point-wise norm of the exterior product $\xi'_M := \xi'_{1M} \wedge ... \wedge \xi'_{kM}$. Consequently, $M_{\mathbb{R}^+\mu}$ is also Fano.

Proof. First note that any infinitesimal isometry ξ_M is K_μ-invariant. More precisely, $\xi_M(gx) = g_*(ad_{g^{-1}}\xi)_M(x)$ for any point x. Since the kernel group K_μ is compact, $\det(ad_{g^{-1}}|_{k_\mu}) = 1$ and

$$(\xi'_{1M} \wedge ... \wedge \xi'_{kM})_{gx} = (\det(ad_{g^{-1}}|_{k_\mu})g_*(\xi'_{1M} \wedge ... \wedge \xi'_{kM})_x) = g_*(\xi'_{1M} \wedge ... \wedge \xi'_{kM})_x).$$

The action of K_μ being by isometries it is clear that the point wise norm of $(\xi'_{1M} \wedge ... \wedge \xi'_{kM})$ is also K_μ-invariant. Recall from Theorem 2.4.3 that we have the following orthogonal decompostion:

$$T_xM = \mathcal{V}_x \oplus \mathcal{H}_x \oplus \mathfrak{m}_M(x) \oplus I_{\mathrm{g}}(\mathcal{V}_x). \tag{2.8.3}$$

$\mathcal{V}_x$ is the vertical space at x of the Riemannian submersion $\pi_{\mathbb{R}^+\mu} : J^{-1}(\mathbb{R}^+\mu) \to M_{\mathbb{R}^+\mu}$ and it is generated by $\{\xi_{iM}(x)\}_{i=1,k}$. The horizontal space at x is $\mathcal{H}_x$ and $\mathfrak{m}_M(x)$ is invariant with respect to the complex structure $\mathcal{C}_{\mathrm{g}}$. Let $\mathcal{V}_{\mathbb{C}}$ and $\mathfrak{M}$ be the distributions defined by $\{\mathcal{V}_x \oplus \mathcal{C}_{\mathrm{g}}(\mathcal{V}_x)\}_{x\in M}$ and $\{\mathfrak{m}_M(x)\}_{x\in M}$. Consider the following decompositions

$$\begin{aligned} \mathcal{V} \otimes \mathbb{C} &= \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1} \\ \mathfrak{M} \otimes \mathbb{C} &= \mathfrak{M}^{1,0} \oplus \mathfrak{M}^{0,1} \\ i^*_{\mathbb{R}^+\mu}T^{1,0}M &= \mathcal{H}^{1,0} \oplus \mathcal{V}^{1,0} \oplus \mathfrak{M}^{1,0}. \end{aligned}$$

Denote by ∇^h, ∇^v, ∇^m, and $\nabla^{\mathbb{R}^+\mu}$ the connections induced by the Levi-Civita connection of M on $\mathcal{H}^{1,0}$, $\mathcal{V}^{1,0}$, $\mathfrak{M}^{1,0}$, and $i^*_{\mathbb{R}^+\mu}T^{1,0}M$(or their determinant bundles). θ^h, θ^v, θ^m, and $\theta^{\mathbb{R}^+\mu}$ are the connections forms of the above defined connections with respect to the local, orthogonal and K_μ-invariant frames $Y_1 \wedge ... \wedge Y_s$, $\xi'_{1M} \wedge ... \wedge \xi'_{kM}$, $\eta'_{1M} \wedge ... \wedge \eta'_{mM}$, $Y_1 \wedge ... \wedge Y_s \wedge \xi'_{1M} \wedge ... \wedge \xi'_{kM}$, respectively. Then, $\theta^{\mathbb{R}^+\mu} = \theta^h + \theta^v + \theta^m$. Extend the connection forms by

$$\begin{array}{lllll} \theta^h_h(Y) = \theta^h(Y) & \theta^h_h(\xi_M) = 0 & \theta^h_h(\eta_M) = 0 & \theta^h_v(Y) = 0 & \theta^h_v(\xi_M) = \theta^h(\xi_M) \\ \theta^v_h(Y) = 0 & \theta^v_h(\xi_M) = 0 & \theta^v_h(\eta_M) = 0 & \theta^v_v(Y) = 0 & \theta^v_v(\xi_M) = \theta^v(\xi_M) \\ \theta^m_h(Y) = 0 & \theta^m_h(\xi_M) = 0 & \theta^v_h(\eta_M) = 0 & \theta^m_v(Y) = 0 & \theta^m_v(\xi_M) = \theta^m(\xi_M) \end{array}$$

$$\begin{array}{llll} \theta^h_v(\eta_M)=0 & \theta^h_m(\xi_M)=0 & \theta^h_m(\eta_M) & =\theta^h(\eta_M) \\ \theta^v_v(\eta_M)=0 & \theta^v_m(\xi_M)=0 & \theta^v_m(\eta_M) & =\theta^v(\eta_M) \\ \theta^m_v(\eta_M)=0 & \theta^m_m(\xi_M)=0 & \theta^v_h(\eta_M) & =\theta^m(\eta_M) \end{array},$$

for any $y\in\mathcal{H}$, $\xi_M\in\mathcal{V}$, and $\eta_M\in\mathcal{M}$. Then $\theta=\theta^h_h+B$, where $B=\theta^h_v+\theta^h_m+\theta^v_h+\theta^v_v+\theta^v_m+\theta^m_h+\theta^m_v+\theta^m_m$. Finally, let $\theta_{\mathbb{R}^+\mu}$ be the connection form of $\det T^{1,0}M_{\mathbb{R}^+\mu}$ with respect to the local orthogonal frame $\pi_{\mathbb{R}^+\mu *}Y_1\wedge...\wedge\pi_{\mathbb{R}^+\mu *}Y_s$.

Note that the Levi-Civita connection of $M_{\mathbb{R}^+\mu}$ is given by

$$\nabla_{\mathbb{R}^+\mu\hat{X}}\hat{Y}=\pi_{\mathbb{R}^+\mu *}((\nabla^h_X Y)),$$

for any $\hat{X}$, $\hat{Y}$ vector fields on the quotient. Here h denotes the horizontal projection and X, Y are the unique sections of the horizontal distribution which project onto $\hat{X}$ and $\hat{Y}$. Then, we obtain

$$\begin{aligned} \theta_{\mathbb{R}^+\mu}(\hat{X})(\pi_{\mathbb{R}^+\mu *}Y_1\wedge...\wedge\pi_{\mathbb{R}^+\mu *}Y_s) &= \\ \sum\nolimits_{i=1}^s \pi_{\mathbb{R}^+\mu *}(Y_1)\wedge...\wedge\pi_{\mathbb{R}^+\mu *}(\nabla^h_X Y_j)\wedge...\wedge\pi_{\mathbb{R}^+\mu *}(Y_s) \\ =\pi_{\mathbb{R}^+\mu *}(\nabla_X(Y_1\wedge...\wedge Y_s)=\pi_{\mathbb{R}^+\mu *}(\theta^h(X)Y_1\wedge...\wedge Y_s). \end{aligned}$$

Therefore, $\pi_{\mathbb{R}^+\mu *}\theta_{\mathbb{R}^+\mu *}=\theta^h_h$ and

$$\pi^*_{\mathbb{R}^+\mu}\rho_{\omega_{\mathbb{R}^+\mu}}=\frac{i}{2\pi}d\pi^*_{\mathbb{R}^+\mu}\theta_{\mathbb{R}^+\mu}=\frac{i}{2\pi}(d\theta^h_h-B)=i^*_{\mathbb{R}^+\mu}\rho_\omega-\frac{i}{2\pi}B. \tag{2.8.4}$$

Observe that

$$d\theta^v_h=d\pi^*_{\mathbb{R}^+\mu}(\partial\log\|\xi'\|^2)=\pi^*_{\mathbb{R}^+\mu}(\bar\partial\partial\log\|\xi'\|^2). \tag{2.8.5}$$

Indeed, sine $\nabla^v_{\bar{Y}}\xi'_M=0$ and $\nabla^v_Y\xi'_M=\frac{\langle\nabla_Y\xi'_M,\bar{\xi'_M}\rangle}{\|\xi'_M\|^2}\xi'_M$, for any Y section of $\mathcal{H}$. Applying Lemma 7.3.8 in [19], we know that for any γ, section of $\det T^{1,0}M$ and any ξ in k_μ, $L_{\xi_M}\gamma=\nabla_{\xi_M}\gamma-(2\pi i\Delta J^\xi)\gamma$. In particular, for $\gamma:=Y_1\wedge...\wedge Y_s\wedge\xi'_{1M}\wedge...\wedge\xi'_{kM}\wedge\eta'_{1M}\wedge...\wedge\eta'_{mM}$, along $J^{-1}(\mathbb{R}^+\mu)$ we get $\nabla_{\xi_M}\gamma=L_{\xi_M}\gamma+(2\pi i\Delta J^\xi)\gamma=-(\Delta J^\xi)\gamma-(\xi'_M f)\gamma$ and $\nabla_{\eta_M}\gamma=-(\eta'_M f)\gamma$. Recall that from the definition of J we have that $u_{\xi'_M}=0$ and $u_{\eta'_M}=0$, for all $\xi\in\mathfrak{k}_\mu$ and $\eta\in\mathfrak{m}$. Hence, $B(\xi_M)=-\xi'_M f$ and $B(\eta_M)=-\eta'_M f$. Then,

$$B=-i_{\mathbb{R}^+\mu *}\partial f+\pi_{\mathbb{R}^+\mu *}\partial f_{\mathbb{R}^+\mu}\quad\text{and}\quad dB=i_{\mathbb{R}^+\mu *}\partial\bar\partial f-\pi_{\mathbb{R}^+\mu *}\partial\bar\partial f_{\mathbb{R}^+\mu}. \tag{2.8.6}$$

From (2.8.4), (2.8.5), and (2.8.6), the conclusion of the theorem follows. □

Corollary 2.8.1. *In the hypothesis of Theorem 2.4.3, suppose M is also Kähler-Einstein of positive Ricci curvature. Then ray reduced space $M_{\mathbb{R}^+\mu}$ is Kähler-Einstein if and only if $\|\xi\|_{\mathbb{R}^+\mu}$ is constant on $J^{-1}(\mathbb{R}^+\mu)$.*

Proof. It is just a matter of definitions. □

Theorem 2.4.3, Proposition 2.5.1, and Theorem 2.8.1 entail the following corollary.

Corollary 2.8.2. *In the hypothesis of Theorem 2.4.3, assume that K_μ is compact and G connected. If $\|\xi\|_{\mathbb{R}^+\mu}$ is constant on $J^{-1}(\mathbb{R}^+\mu)$, then the ray reduced contact space is Sasaki-Einstein.*

Theorems 2.5 in [9] and 2.8.1 imply

Corollary 2.8.3. *In the hypothesis of Theorem 2.8.1, if M has Ricci curvature strictly bigger then -2, then so does $M_{\mathbb{R}^+\mu}$.*

Example 2.8.1. *All the reduced spaces of Examples* 3.1 *and* 3.2 *of [15] are Sasaki-Einstein.*

Chapter 3

The stratification of proper groupoids

3.1 Proper Lie groupoids

In this section we will briefly review the definition and properties of proper Lie groupoids. By convention, the base space of any Lie groupoid is assumed to be Hausdorff, but the space of arrows is not necessarily Hausdorff.

One of the fundamental examples of Lie groupoids is the one associated to a Lie group action and usually called the *action groupoid*. Let $G \times M \to M$ be a smooth action of the Lie group G on the manifold M. The associated action groupoid, denoted by $G \ltimes M$ is given by

$$G \ltimes M := G \times M \overset{s,t}{\rightrightarrows} M \quad , \quad s(g,m) := m \quad , \quad t(g,m) := g \cdot m.$$

Definition 3.1.1. *A Lie groupoid* $\mathcal{G} \overset{s,t}{\rightrightarrows} M$ *is called* proper *if* $\mathcal{G}$ *is a Hausdorff manifold and the source-target map (or the anchor map as it is sometimes called)* $(s,t) : \mathcal{G} \to M \times M$ *is a proper topological map.*

Example 3.1.1. An action groupoid is proper if and only if the group action is proper.

Example 3.1.2. ([64]) Let G be a semisimple non-compact Lie group. Let $E \subset \mathfrak{g}^*$ be the *elliptic subject* of G defined as the set of all the elements of $\mathfrak{g}^*$ who have compact coadjoint isotropy groups. Under the orbit method, E corresponds to the discrete series of representations of G. E is an open subset of $\mathfrak{g}^*$ and the restriction of the coadjoint action to E is proper. Therefore, the restriction of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ to E is a proper groupoid.

Definition 3.1.2. *Two Lie groupoids* $\mathcal{G}_1 \rightrightarrows M_1$ *and* $\mathcal{G}_2 \rightrightarrows M_2$ *are said to be* Morita equivalent (*see [61]*) *if there exists a manifold* X *endowed with a left and right action of* $\mathcal{G}_1 \rightrightarrows M_1$ *and* $\mathcal{G}_2 \rightrightarrows M_2$ *respectively and with momentum maps* $\rho : X \to M_1$ *and* $\sigma : X \to M_2$ *such that*

- *the two actions commute with each other;*
- X *is a* G_1*-principal bundle over* $X \overset{\sigma}{\to} M_2$*;*

- *X is a G_2-principal bundle over $X \xrightarrow{\rho} M_1$.*

(X, ρ, σ) is called an equivalence bimodule *between the Lie groupoids $\mathcal{G}_1$, $\mathcal{G}_2$.*

For the equivalence of this definition with others existing in the literature, see [**?**], Part 3. The following proposition shows how many of the topological properties of proper group actions still hold in the case of proper groupoids.

Proposition 3.1.1. *([41], [63], [60]) Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then,*

- *the isotropy group $\mathcal{G}_x := \{g \in \mathcal{G} | s(g) = t(g) = x\}$ of any base point x is a compact Lie group;*
- *each orbit of $\mathcal{G}$ is a closed embedded submanifold of M;*
- *the orbit space $M/\mathcal{G}$ with the induced quotient topology is a Hausdorff space;*
- *any Lie groupuoid Morita equivalent to $\mathcal{G}$ is also proper;*
- *if N is a submanifold of M which intersects a groupoid orbit $\mathcal{O}$ transversally at a point $x \in M$ and S is a sufficiently small open neighbourdhood of x in N, then the restriction groupoid $\mathcal{G}_S := s^{-1}(S) \cap t^{-1}(S) \rightrightarrows S$ is a proper Lie groupoid which has x as a fixed point.*

The groupoid $\mathcal{G}_S$ in the last item of the above proposition is called a *slice* of $\mathcal{G}$ at $x \in M$ and this notion makes sense even for non-proper groupoids. Note that two arbitrary slices of a groupoid at two points lying on the same orbit are locally isomorphic.

However, the properness of a Lie groupoid is not sufficient to imply all the nice properties of proper Lie group actions, such as the stability of fixed points or the existence of tubes (see [63]). That is why one has to impose the additional condition of source local triviality. Note that in the case of proper actions, the source and target maps are globally trivial fibrations. Recall that a base point of a groupoid is *fixed* if its orbit contains only one point. A fixed point is *stable* if each one of its neighborhoods contains a $\mathcal{G}$-invariant one. Every fixed point of a source locally trivial proper topological groupoid is stable. Note that the source local triviality hypothesis is necessary and it is not preserved under Morita equivalence.

The analogue of the Bochner's linearization theorem for groupoids was conjectured by A. Weinstein and proved by N. T. Zung in [60].

Theorem 3.1.1. *Any proper Lie groupoid $\mathcal{G} \rightrightarrows M$ with a fixed point $x \in M$ is locally isomorphic to a linear action groupoid, namely the action groupoid of the action of the compact isotropy group $\mathcal{G}_x$ on the tangent space T_xM.*

An immediate consequence of Theorem 3.1.1 is the following Tube Theorem for Lie groupoids. This theorem was obtained by Weinstein in [63] under the hypothesis that Theorem 3.1.1 holds. If $\mathcal{O}$ is an orbit of the groupoid $\mathcal{G} \rightrightarrows M$, then $\mathcal{G}_{\mathcal{O}}$ denotes the restriction of $\mathcal{G}$ to $\mathcal{O}$ and $\tau : N_{\mathcal{O}} \to \mathcal{O}$ the normal bundle of $\mathcal{O}$ in M. The structure of $\mathcal{G}$ induces a linear action of $\mathcal{G}_{\mathcal{O}}$ on $N_{\mathcal{O}}$. This action may be defined as follows. Let $g \in \mathcal{G}_{\mathcal{O}}$ and $v \in N_{\mathcal{O}}$ be such that $s(g) = \tau(v)$. Choose γ a parametrized curve on the base such that $\gamma(0) = s(g)$ and $\left[\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma(\varepsilon)\right] = v \in N_{\mathcal{O}}$. Let $\tilde{\gamma}$ be a parametrized curve in $\mathcal{G}$ with $\tilde{\gamma}(0) = g$ and $s \circ \tilde{\gamma} = \gamma$. Then we define $g \cdot x := \left[\frac{d}{d\varepsilon}\big|_{\varepsilon=0}(t \circ \tilde{\gamma})(\varepsilon)\right] \in N_{\mathcal{O}}$. It is easy to verify that this is a well defined action. We denote by $\mathcal{G}_{\mathcal{O}} \ltimes N_{\mathcal{O}}$ the corresponding action groupoid.

Recall that $\mathcal{O}$ is of finite type if there is a proper map $F : \mathcal{O} \to \mathbb{R}$ with a finite number of critical points. Now we are ready to state the Tube Theorem for proper Lie groupoids.

Theorem 3.1.2. **(*Tube Theorem [63], [60]*)** *Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper Lie groupoid and let $\mathcal{O}$ be an orbit of finite type. Then there is an invariant neighborhood $\mathcal{U}$ of $\mathcal{O}$ in M such that the restriction $\mathcal{G}_{\mathcal{U}}$ of $\mathcal{G}$ to $\mathcal{U}$ is isomorphic to the restriction of $\mathcal{G}_{\mathcal{O}} \ltimes N_{\mathcal{O}}$ to a tubular neighborhood of the zero section in $N_{\mathcal{O}}$ (and also isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N_{\mathcal{O}}$ itself).*

In fact, this Tube Theorem for groupoids represents the linearization of source locally trivial proper groupoids near orbits of finite type. Note that in the case of an action groupoid the groupoid linearization is slightly weaker then the linearizaton of the action since it only implies the orbital linearization of the action. More precisely, locally, the group orbits are those of a linear action, but the action on these orbits may still be non linear.

As a consequence of the Tube Theorem we have the following result which will be needed in Section 7.

Lemma 3.1.1. *Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid with paracompact base. Then, to any open and $\mathcal{G}$-invariant cover of M one can associate a $\mathcal{G}$-invariant partition of unity.*

Proof. Let $(U_i)_{i\in I}$ be an open, $\mathcal{G}$-invariant cover of M. The properness of the groupoid implies the closeness of the orbit projection map $\pi : M \to M/\mathcal{G}$. Since closed maps preserve paracompactness, it follows that the orbit space is also paracompact. Therefore, there is an open and locally finite subcover, $(\widetilde{V}_i)_{i\in I}$, of $(\widetilde{U}_i = \pi(U_i))_{i\in I}$ such that $(V_i := \pi^{-1}(\widetilde{V}_i))_{i\in I}$ is an open, $\mathcal{G}$-invariant, and locally finite subcover of M. For any $i \in I$, let $[x_i] := \pi(x_i)$ be a point of $\widetilde{V}_i$ such that $W_i := \mathcal{G} \cdot S_i \subset V_i$, where S_i is a slice of the groupoid at x_i. Obviously, $\tilde{W}_i := \pi(W_i)$ is contained in $\tilde{V}_i$ for any indice $i \in I$. The slice theorem for proper groupoids implies that the orbit space of the restricted groupoid to W_i is equivalent to the orbit space of the linear action of $\mathcal{G}_{x_i}$ on an open subset of zero in $T_{x_i}S_i$, B_i. Even more, this action is orthogonal with respect to a $\mathcal{G}_{x_i}$-invariant inner product on B_i. Using the fact that any ray function on B_i is $\mathcal{G}_{x_i}$-invariant, we can construct positive functions $f_i \in C^\infty(B_i)^{\mathcal{G}_{x_i}}$ with $f_i(0) = 0$ and $\mathrm{supp}(f_i)$ included in any given neighborhood of zero in B_i. Extend by zero each of these functions on the whole base and denote by $(\tilde{f}_i)_{i\in I}$ the induced functions on the orbit space $M/\mathcal{G}$. Since $(\tilde{V}_i)_{i\in I}$ is a locally finite family, so are the families $(\mathrm{supp}(\tilde{f}_i))_{i\in I}$ and $(\mathrm{supp}(f_i))_{i\in I}$. Consider $g_i := \frac{f_i}{\sum_{i\in I} f_i}$ and $\tilde{g}_i := \frac{\tilde{f}_i}{\sum_{i\in I}\tilde{f}_i}$. It is easy to see that $(g_i)_{i\in I}$ is the required $\mathcal{G}$-invariant partition of unity and $(\tilde{g}_i)_{i\in I}$ is a partition of unity of the orbit space associated to the open cover $(V_i)_{i\in I}$.

□

3.2 The Stratification of the Orbit Space of Proper Lie Group Actions

Recall that the local model of a proper Lie groupoid is an action groupoid. That is why, the purpose of this section is to review the stratification of the orbit space of a proper Lie group action and to reformulate it in a way which can be naturally generalized to proper Lie groupoids.

Let $G \times M \to M$ be a smooth proper action of the Lie group G on the manifold M. It is well known that the connected components of the orbit types

$$M_{(H)} = \{x \in M \,|\, G_x \text{ is conjugate to } H\}, \tag{3.2.1}$$

are smooth embedded submanifolds of M. Here, G_x denotes the stabilizer of x. The subgroups H labelling the orbit types of M are compact. The orbit space M/G is a Hausdorff paracompact space locally modeled as the linear quotient by G_x of a vector subspace of T_xM complementar to the tangent space to the orbit of x.

Denote by $\pi : M \to M/G$ the orbit projection of the G-action. It is a consequence of Palais' slice theorem for proper actions that the connected components of the sets $\pi(M_{(H)})$ are smooth manifolds, and that the partition of M/G into these connected manifolds is a smooth stratified cone space satisfying the Whitney conditions. This stratification is called the *orbit type stratification* of M/G. Since the connected component of x in the orbit type $M_{(H)}$ is given by $G_e \cdot M_H^x$, this stratification is also called the *isotropy type stratification*. Here, $M_H = \{x \in M \,|\, G_x = H\}$ is the isotropy type of H, M_H^x its connected component containing x, and G_e is the connected component of the identity in G. For further details, we refer the interested reader to [17], [57], or [49].

The main problem in trying to generalize this picture to the orbit space of a proper groupoid $\mathcal{G} \rightrightarrows M$ is that the characterization (3.2.1) of the pieces inducing the stratification makes no sense for a groupoid. Namely, the isotropy groups of a groupoid at points in different orbits cannot be compared by conjugation. In this section we adopt an equivalent approach to the construction of the orbit types and hence of the orbit type stratification of M/G. This approach is based on ideas coming from singular foliation theory, namely projectable vector fields. It has the advantage of admitting a natural generalization to proper Lie groupoids which will be presented in Section **??**.

The first step is to realize the connected components of the orbit types as the leaves of a singular integrable distribution.

Proposition 3.2.1. *If G is a Lie group acting properly on the manifold M, the family of (local) vector fields G-invariant or tangent to the orbits generates a singular integrable distribution whose leaves are the connected components of the orbit types. Therefore, the partition of the orbit space M/G into the G- projections of the leaves coincides with the stratification by orbit types.*

This was proved in [44] and [57]; see for instance Theorem 3.4.10 and Theorem 3.5.1 in [57]. The definition of a G-invariant vector field uses the fact that the action of G associates to any group element a diffeomorphism of M. In the case of a groupoid, this diffeomorphism is defined only on the orbit through the source of this element. Thus, this definition can not be generalized to groupoids.

However, the notion of $\mathcal{G}$-invariant vector fields makes sense only for smooth sections of the normal bundle to the orbits. For this, one should use the $\mathcal{G}$-action defined for the linear model of a proper groupoid. Note that since the normal bundle is singular, technically it is very difficult to determine the normal, $\mathcal{G}$-invariant vector fields. Therefore this formulation would be of no practical use for the groupoid case.

To overcome this difficulty we introduce another family of local vector fields which we call *G-derivations* and whose definition depends only on the properties of the orbit foliation on M, and no longer implicitly on the G-action itself. This new family of vector fields will generate exactly the same distribution, but the condition defining them is easy to check in concrete exemples even for grupoids.

Definition 3.2.1. *Let the Lie group G act properly on the smooth manifold M.*

1. *The local G-projectable vector fields are the local vector fields whose flow sends orbits to orbits. We will denote the family of local G-projectable vector fields by $\mathcal{P}(M,G)$.*
2. *The (local) G-derivations of M are those (local) vector fields which act as derivations on the ring of smooth G-invariant functions on M, $\mathcal{C}^\infty(M)^G$. We will use the following notation for the family of G-derivations on M*

$$\mathcal{D}(G,M) = \{X \in \mathfrak{X}(M)_{\mathrm{loc}} \,|\, X(f) \in C^\infty(M)^G,\ \forall f \in \mathcal{C}^\infty(M)^G\}.$$

3. *Let $IT(G,M)$ denote the sum of the family of local G-invariant vector fields and the family of local vector fields tangents to the orbits.*

Recall that in the theory of foliations, an important class of vector fields is the one of projectable (or foliate) vector fields (see, for instance, [**?**]). They are defined as those vector fields whose Lie bracket with any vector field tangent to the foliation is again tangent to the foliation and are used in the study of transverse parallelizability of foliations. In the case of regular foliations, the projectable vector fields are precisely those whose flow leaves the connected components of the orbits invariant. Hence, the name in Definition 3.2.1, (1). The same result also holds in the case of an orbit foliation given by the proper action of a Lie group.

Proposition 3.2.2. *Let the Lie group G act smoothly and properly on the manifold M. Then, the projectable vector fileds of the induced orbit foliation of M are those vector fields whose flow takes connected components of the orbits to connected components of the orbits.*

Proof. The proof is based on arguments similar to the ones used in the proof of Proposition 3.2.3 and, therefore, we will skip it. □

Remark 3.2.1. *Note that the above characterization is also true for singular Riemannian foliations admitting slices.*

The G-projectable vector fields form a subclass of the class of projectable ones and if G is connected they coïncide. Even if they provide a nice interpretation in the language of foliation theory of the orbit types, in practice, the condition defining them is very difficult to check since one should first integrate them. That is why, in the next two propositions we will show that in the case of proper group actions the G-projectable vector fields, G-derivations, and $IT(G,M)$ coïncide. Note that we could not find a direct way of proving that the G-derivations and $IT(G,M)$ coïncide. We use G-projectable vector fields as an intermediate.

Proposition 3.2.3. *Consider a proper smooth action of a Lie group G on a manifold M. Then the set of local G-projectable vector fields equals the set of G-derivations.*

Proof. To prove the first inclusion, consider a local vector field whose flow Φ_X^t takes orbits to orbits and a G-invariant smooth function f defined on the domain of X. Then Xf is also G-invariant since

$$\begin{aligned}(Xf)(gx) &= \left.\frac{d}{dt}\right|_{t=0}(f\circ\Phi_X^t)(gx) = \left.\frac{d}{dt}\right|_{t=0}(f(g'\Phi_X^t(x)))\\ &= \left.\frac{d}{dt}\right|_{t=0}(f\circ\Phi_X^t)(x) = (Xf)(x).\end{aligned}$$

For the reverse inclusion, consider X a G-derivation and let Φ_X^t be its flow. Since $\Phi_X^{t_1+t_2} = \Phi_X^{t_1}\circ\Phi_X^{t_2}$ for any t_1, t_2, and t_1+t_2 such that the previous expression is well defined, it suffices to prove that the flow of X takes orbits to orbits in a small neighborhood of $t=0$. For any $f\in C^G(M)$ denote by $\widetilde{f}$ the induced Withney smooth function on the orbit space. Let $x\in M$ a point in the domain of X. We will first study the case when x belongs to the principal orbit type of the connected component of M containing x. Therefore, there is an open G-invariant neighborhood of x in M such that the corresponding restricted action has a single orbit type. Consequently, we can suppose that, locally, the orbit space of the G-action is a smooth manifold and the orbit projection a surjective submersion. The vector field X defines a derivation $\overline{X}$ on the algebra of smooth functions $C^\infty(M/G)$ as follows:

$$\overline{X}(h) := \widetilde{X(h\circ\pi)},$$

for any $h\in C^\infty(M/G)$. The smooth vector field associated to this derivation will also be denoted by $\overline{X}$. Then, for any point y in the domain of X and any smooth function h defined on a neighborhood of $\pi(y)$, we have

$$\overline{X}(\pi(y))[h] = (\overline{X}h)(\pi(y)) = \widetilde{X(h\circ\pi)}(\pi(y)) = (X(h\circ\pi))(y) = T_y\pi(X(y))[h].$$

This proves that X is a projectable vector field on the space of orbits and that its projection is precisely the associated derivation $\overline{X}$, namely $T\pi\circ X = \overline{X}\circ\pi$. Since their flows are also π-related, we obtain that Φ_X^t sends orbits to orbits.

If there is a $x\in M$ in the domain of X, but not belonging to the principal orbit type of the connected component of M containing x, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in the principal orbit type with limit x. We know that such a sequence exists because the principal orbit type is dense in M. Then the limit of the sequence $(gx_n)_{n\in\mathbb{N}}$ is gx. Since the points gx_n belong to the principal orbit type, we have that $\Phi_X^t(gx_n) = g'_n\Phi_X^t(x_n)$ for $(g'_n)_{n\in\mathbb{N}}$ a sequence in G and $\lim\limits_{n\to\infty}\Phi_X^t(gx_n) = \Phi_X^t(gx)$. We also have that $\lim_{n\to\infty} \Phi_X^t(x_n) = \Phi_X^t(x)$. Then, the properness of the G-action guarantees the existence of a convergent subsequence of $(g_n')_{n\in\mathbb{N}}$, $(g_{n_k}')_{k\in\mathbb{N}}$ with limit g'. It follows that $(g_{n_k}'\Phi_X^t(x_{n_k}))_{k\in\mathbb{N}}$ converges to $g'\Phi_X^t(x)$. The uniqueness of the limit insures that $\Phi_X^t(gx) = g'\Phi_X^t(x)$ and the proof is thus complete. □

Proposition 3.2.4. *If the Lie group G acts properly on the smooth manifold M, then the generalized distribution generated by $IT(G, M)$ is exactly the distribution generated by the family of G-projectable vector fields.*

Proof. Let $Z = X + Y \in IT(G, M)$ with X tangent to the orbits and Y a G-equivariant local vector field. The Trotter formula for the flow of the sum of two vector fields gives

$$\Phi_Z^t(gx) = \lim_{n\to\infty} (\Phi_X^{t/n} \circ \Phi_Y^{t/n})^n(gx) = \lim_{n\to\infty} (g_n \Phi_Y^t(x)),$$

for some $g_n \in G$. Since the G-orbits are closed submanifolds of M, the above limit belongs to $G \cdot \Phi_Y^t(x)$ and the first inclusion follows.

To show that G-projectable vector fields actually belong to $IT(G, M)$ we will first show how to associate a G-equivariant vector field $\widetilde{V}$ to any arbitrary local vector field V on M. Since the G-action is proper there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in M with associated slices $\{S_n\}_{n\in\mathbb{N}}$ such that $\{\pi(S_n)\}_{n\in\mathbb{N}}$ is a locally finite open covering of the orbit space M/G (see Theorem 4.2.4 in [49]). Consequently, there is also a G-invariant partition of unity $\{f_n\}_{n\in\mathbb{N}}$ subordinated to the open covering $\{G \cdot S_n\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, define $\widetilde{V}_n$ to be the following section of the restricted tangent bundle $TM|_{S_n} \to S_n$

$$(\widetilde{V}_n f)(s) := \int_{G_{x_n}} \langle \mathbf{d}f(s), g^{-1} \cdot V(gs)\rangle \mu_n,$$

where μ_n is the Haar measure of the isotropy subgroup G_{x_n}, s is a point of the slice S_n, and f a smooth function defined on a neighborhood of s. Even more, we can extend $\widetilde{V}_n$ on the restricted bundle $TM|_{G\cdot S_n} \to G \cdot S_n$ as follows

$$(\widetilde{V}_n f)(g \cdot s) := (\widetilde{V}_n(f \circ \Phi_g))(s),$$

for any $g \in G$, $s \in S_n$, and f smooth function defined on a neighborhood of gs. The above expression is well defined. To see this, note that given $g_1, g_2 \in G$ satisfying $g_1 s = g_2 s$, then $g_1^{-1} g_2 \in G_{x_n}$ by the slice axioms for S_n. Now, using the G_{x_n}-equivariance of $\widetilde{V}_n$, we obtain

$$(\widetilde{V}_n f)(g_1 s) = (\widetilde{V}_n(f \circ \Phi_{g_1}))(s) = (\widetilde{V}_n(f \circ \Phi_{g_1} \circ \Phi_{g_1^{-1}g_2}))(s) = (\widetilde{V}_n(f \circ \Phi_{g_2}))(s) = (\widetilde{V}_n f)(g_2 s).$$

Now we can define a G-invariant section of the whole tangent bundle by

$$\widetilde{V} := \sum_{n\in\mathbb{N}} f_n \widetilde{V}_n.$$

Let X be a G-projectable field. Denote by $\widetilde{X}$ the associated G-invariant vector field constructed before. It remains to show that the vector field $X - \widetilde{X}$ is tangent to the orbits at every point. To see this, consider f a smooth G-invariant function. Then,

$$\begin{aligned}((X - \widetilde{X})f)(x) &= (Xf)(x) - \sum_{n\in\mathbb{N}}(f_n\widetilde{X}_n f)(x)\\ &= (Xf)(x) - \sum_{n\in\mathbb{N}} f_n(x) \int_{G_{xn}} \langle \mathbf{d}f(x), g^{-1}\cdot X(gx)\rangle \mathrm{d}\mu_n\\ &= \sum_{n\in\mathbb{N}} f_n(x) \int_{G_{xn}} \left(\langle \mathbf{d}f(x), X(x)\rangle - \langle \mathbf{d}(f\circ\Phi_{g^{-1}})(gx), X(gx)\rangle\right) \mathrm{d}\mu_n\\ &= \sum_{n\in\mathbb{N}} f_n(x) \int_{G_{xn}} \left((Xf)(x)) - (Xf)(gx))\right) \mathrm{d}\mu_n.\end{aligned}$$

But Proposition 3.2.3 implies that Xf is again a G-invariant function and, thus, $(X - \widetilde{X})f = 0$ for any $f \in C^G(M)$. Therefore, we obtain that $X = \widetilde{X} + (X - \widetilde{X}) \in IT(G, M)$.

□

As a consequence of Propositions 3.2.3 and 3.2.4, one can use the distribution generated by the family of G-derivations to describe the connected components of the orbit types, as well as the strata of the orbit space M/G.

Corollary 3.2.1. *For the proper action of a Lie group G on a smooth manifold M, the family of G-derivations is an integrable singular distribution whose leaves are the connected components of the orbit types $M_{(H)}$. Their projections under the orbit map $\pi : M \to M/G$ are the strata of the orbit type stratification of M/G.*

3.3 Free and proper groupoids

Definition 3.3.1. *A groupoid $\mathcal{G} \rightrightarrows M$ is a free groupoid if and only if all isotropy groups are trivial.*

Notice that in the case of an action groupoid, the above definition coincides with the one of a free action of a Lie group on a smooth manifold.

Theorem 3.3.1. *If $\mathcal{G} \overset{s,t}{\rightrightarrows} M$ is a free and proper groupoid, then the orbit space $M/\mathcal{G}$ is a smooth manifold and the orbit projection $\pi : \mathcal{G} \to M/\mathcal{G}$ is a submersion. Even more, if the groupoid is source locally trivial, then the orbit projection becomes a differentiable locallly trivial fibration.*

Proof. Since the groupoid $\mathcal{G} \rightrightarrows M$ is proper and free, the map $(s,t) : \mathcal{G} \to M \times M$ is a proper and injective submersion. Denote by R its image which is the graph of the equivalence relation on M induced by the groupoid orbits. Hence, by the local fibration theorem(see, for instance [1], [2]), R is an injectively immersed submanifold of $M \times M$. The properness hypothesis implies that R is also closed in $M \times M$ and (s, t) is a homeomorphism onto its image. Therefore R is an embedded submanifold of $M \times M$. Applying a theorem of Godement, see [2] or [53], [2, Part II, Chap. 3, 12, Theorem 2], we obtain that $M/\mathcal{G}$ is a smooth manifold such that $\pi : \mathcal{G} \to M/\mathcal{G}$ becomes a submersion. To see that it is a diffrential fibration, suffice to notice that the orbit si diffeomorphic to the corresponding s-fiber. $\square$

3.4 The orbit space as a metric space

Theorem 3.4.1. *Let $\mathcal{G} \rightrightarrows M$ a source-locally trivial and proper Lie groupoid with paracompact base. Then, there is a Riemannian metric* g *invariant with respect to the $\mathcal{G}$-action on the vectors normal to the orbits, i.e.,*

$$\mathrm{g}(g \cdot x)(g \cdot v_1, g \cdot v_2) = \mathrm{g}(x)(v_1, v_2),$$

for any v_1, v_2 normal vectors at x and any $g \in s^{-1}(x)$.

Proof. The slice theorem for proper groupoids implies that $M/\mathcal{G}$ is locally the quotient of a vector space by the action of a proper group. Hence $M/\mathcal{G}$ is locally compact. Applying Alexandroff's Theorem(see [1], page 29), we obtain that $M/\mathcal{G}$ is also a σ-compact space since it is second countable. Therefore, there is a sequence $(x_n)_n \subset M$ such that $(\pi(S_{x_n}))_n$ is a locally finite covering of the orbit space $M/\mathcal{G}$, with S_{x_n} a slice through x_n and π the orbit projection. Consequently, $(\mathcal{G} \cdot S_{x_n})_n$ is a locally finite covering of M. As the orbit space is also normal, every x_n has a neighborhood K_n with compact closure in S_{x_n} and such that $(\pi(K_n))_n$ is a covering of $M/\mathcal{G}$. Then, there exists a function $f_n : S_{x_n} \to [0, \infty)$ differentiable, with compact support and strictly positive on K_n. By Theorem 2.3 in [60], the slice groupoid $\mathcal{G}_{S_{x_n}}$ of x_n is isomorphic to the action groupoid $\mathcal{G}_{x_n} \times V_{x_n} \rightrightarrows V_{x_n}$, where V_{x_n} is a small neighborhood of zero in the normal space to the orbit $\mathcal{G} \cdot x_n$ at x_n. By averaging if necessary, we can assume that f_n is $\mathcal{G}_{S_{x_n}}$-invariant. Consequently, f_n can be extended to $\mathcal{G} \cdot S_{x_n}$ by defining $f_n(g \cdot z) := f_n(z)$ for any $z \in S_{x_n}$ and $g \in s^{-1}(z)$. Note that f_n is $\mathcal{G}$-invariant on $\mathcal{G} \cdot S_{x_n}$.

Let g_0 be a Riemannian metric on M. For any $z \in S_{x_n}$, consider the orthogonal decomposition $T_zM = T_z(\mathcal{G} \cdot z) \oplus T_z^{\perp}(\mathcal{G} \cdot z)$ with respect to g_0. Since x_n is a fixed point for the slice groupoid $\mathcal{G}_{S_{x_n}} \overset{s_n,t_n}{\rightrightarrows} S_{x_n}$, $s_n^{-1}(x_n) = \mathcal{G}_{x_n} \simeq s_n^{-1}(z)$ and therefore a compact Lie group for any $z \in S_{x_n}$. Define for each $n \in \mathbb{N}$ the following bundle metric on $TM|_{S_{x_n}} \to S_{x_n}$:

$$\begin{cases} \mathrm{g}_{0n}(z)(u_1, u_2) := \mathrm{g}_0(z)(u_1, u_2), & \text{if } (u_1, u_2) \in T_z(\mathcal{G} \cdot z) \\ \mathrm{g}_{0n}(z)(u, v) := 0, & \text{if } u \in T_z(\mathcal{G} \cdot z),\ v \in T_z^{\perp}(\mathcal{G} \cdot z) \\ \mathrm{g}_{0n}(z)(v_1, v_2) := \int\limits_{s_n^{-1}(z)} g_0(g \cdot z)(g \cdot v_1, g \cdot v_2) \mathrm{d}\mu_z, & \text{if } v_1, v_2 \in T_z^{\perp}(\mathcal{G} \cdot z), \end{cases}$$

where μ_z is the Haar measure induced by $\mathcal{G}_{x_n}$ on the compact Lie group $s_n^{-1}(z)$. It follows that g_{0n} is $\mathcal{G}_{S_{x_n}}$-invariant. Therefore, we can extend g_{0n} to the vector bundle $TM|_{\mathcal{G} \cdot S_{x_n}} \to \mathcal{G} \cdot S_{x_n}$ such that it becomes a $\mathcal{G}$-invariant metric. Indeed, for $z \in S_{x_n}$ and $g \in s^{-1}(z)$ define

$$\begin{cases} \mathrm{g}_{0n}(g\cdot z)(u_1,u_2) := \mathrm{g}_0(g\cdot z)(u_1,u_2), & \text{if } (u_1,u_2)\in T_{g\cdot z}(\mathcal{G}\cdot z)\\ \mathrm{g}_{0n}(g\cdot z)(u,v) := 0, & \text{if } u\in T_{g\cdot z}(\mathcal{G}\cdot z),\ v\in T^{\perp}_{g\cdot z}(\mathcal{G}\cdot z)\\ \mathrm{g}_{0n}(g\cdot z)(v_1,v_2) := \mathrm{g}_{0n}(z)(g^{-1}\cdot v_1, g^{-1}\cdot v_2), & \text{if } v_1,v_2\in T^{\perp}_{g\cdot z}(\mathcal{G}\cdot z).\end{cases}$$

To check that this metric is well defined, let $g_1\cdot z = g_2\cdot z$ and $v_1, v_2 \in T^{\perp}_{g_1\cdot z}(\mathcal{G}\cdot z)$. Using the $\mathcal{G}_{S_{x_n}}$-invariance of the metric, we have that $\mathrm{g}_{0n}(g_2\cdot z)(v_1,v_2) = \mathrm{g}_{0n}(z)(g_2^{-1}\cdot v_1, g_2^{-1}\cdot v_2) = \mathrm{g}_{0n}(g_1^{-1}g_2\cdot z)(g_2^{-1}\cdot v_1, g_2^{-1}\cdot v_2) = \mathrm{g}_{0n}(g_1\cdot z)(v_1,v_2)$. Taking $\mathrm{g} = \Sigma\, \mathrm{g}_{0n}\, f_n$, the conclusion follows immediately. □

Recall that a Riemannian submersion is a submersion between smooth Riemannian manifolds such that its tangent map restricted to the horizontal distribution on the total space becomes an isometry. Combining the above theorem and Theorem 3.3.1, we obtain that the orbit projection of a free and proper groupoid becomes a Riemannian submersion:

Corollary 3.4.1. *Let $\pi : \mathcal{G} \to M/\mathcal{G}$ be the orbit projection of a free and proper gropoid with paracompact base M endowed with a $\mathcal{G}$-invariant metric (in the sense of Theorem 3.4.1). Then, the projection of this metric on the orbit space is a well defined metric for which π becomes a Riemannian submersion.*

3.5 Local Properties of the Orbit Space of a Proper Lie Groupoid

In this section we prove that the orbit space of a proper Lie groupoid with all of its orbits of finite type is a smooth stratifed cone space satisfying the Whitney conditions. This fact seems to be well known, although we could not find a complete proof in the literature. The main tool used to this end is a slice theorem for proper Lie groupoids 3.1.2 which can be used to show that a proper Lie groupoid with all of its orbits of finite type is locally Morita equivalent to the action groupoid corresponding to a linear representation of a compact Lie group on a vector space. At this point, since a stratification and its properties are local in nature, the statement will follow by the situation already existing for orbit spaces of proper Lie group actions.

Let $s, t : \mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and assume that all its orbits are of finite type. There is a local model for the groupoid, due to Weinstein and Zung, that plays the role of Palais' tube theorem for proper Lie group actions. We briefly explain the construction of this local model, since it will be needed all throughout this chapter. See [60] for details.

Let $\mathcal{O} \subset M$ be an orbit and consider its normal bundle $N\mathcal{O} = TM\big|_{\mathcal{O}}/T\mathcal{O}$. The restricted groupoid $\mathcal{G}_{\mathcal{O}} = s^{-1}(\mathcal{O})\cap t^{-1}(\mathcal{O}) \rightrightarrows \mathcal{O}$ acts in a natural way on $N\mathcal{O}$ with momentum map the canonical projection $N\mathcal{O} \to \mathcal{O}$. The tube theorem states that there exist invariant neighborhoods U of $\mathcal{O}$ in M and V of the zero section in $N\mathcal{O}$ such that $\mathcal{G}_U \rightrightarrows U$ and $\mathcal{G}_{\mathcal{O}} \ltimes V \rightrightarrows V$ are isomorphic as Lie groupoids. In particular, there is an orbit preserving diffeomorphism between U and V. In the case $\mathcal{G}$ is the action groupoid for a proper action of a Lie group G on M we recover Palais' tube theorem.

We now use this model to proof that the orbit space for a proper groupoid with orbits of finite type is an orbispace, i.e. a topological space that locally looks like the quotient of a vector space by a linear representation of a compact Lie group. Let $x \in \mathcal{O}$ and let $\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$ be its isotropy group. By the properness hypothesis, $\mathcal{G}_x$ is a compact Lie group which acts linearly on $N_x\mathcal{O}$. We then have the following

Proposition 3.5.1. *$\mathcal{G}_\mathcal{O} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O}$ and $\mathcal{G}_x \times N_x\mathcal{O} \rightrightarrows N_x\mathcal{O}$ are Morita equivalent Lie groupoids.*

Proof. For the groupoids under consideration we choose

$$N\mathcal{O} \leftarrow^{\pi_2} s^{-1}(x) \times N_x\mathcal{O} \rightarrow^{\pi_1} N_x\mathcal{O}$$

with projections $\pi_1(g, v) = g \cdot v$ according to the induced action of $\mathcal{G}_\mathcal{O}$ on $N\mathcal{O}$, and $\pi_2(g, v) = v$. It is straightforward to check that the actions

$$\begin{array}{rcl}(\mathcal{G}_\mathcal{O} \ltimes N\mathcal{O}) \times (s^{-1}(x) \times N_x\mathcal{O}) & \to & (s^{-1}(x) \times N_x\mathcal{O}) \\ (g', v') \cdot (g, v) & \mapsto & (gg', v),\end{array}$$

and

$$\begin{array}{rcl}(\mathcal{G}_x \times N_x\mathcal{O}) \times (s^{-1}(x) \times N_x\mathcal{O}) & \to & (s^{-1}(x) \times N_x\mathcal{O}) \\ (g', v) \cdot (g, v) & \mapsto & (gg'^{-1}, g' \cdot v)\end{array}$$

are well defined and satisfy the required axioms. □

Since Morita equivalent Lie groupoids have homeomorphic orbit spaces, choosing invariant neighborhoods V and U as before and applying Proposition 3.5.1 we have

Corollary 3.5.1. *The orbit space of a source-locally trivial proper Lie groupoid with all of its orbits of finite type is a Whitney stratified space for which every point has a neighborhood homeomorphic to the quotient of an open invariant neighborhood of zero in a vector space acted upon linearly by a compact Lie group.*

Proof. We can choose V in the tube theorem as the $\mathcal{G}_\mathcal{O}$-saturation of some $\mathcal{G}_x$-invariant open neighborhood V' of zero in $N_x\mathcal{O}$. Therefore, a neighborhood O of $[x]$ in $M/\mathcal{G}$ is given by $O = U/\mathcal{G}_U \simeq V/(\mathcal{G}_\mathcal{O} \ltimes V) \simeq V'/\mathcal{G}_x$ where the first homeomorphism is due to the tube theorem and the second to the Morita equivalence of $\mathcal{G}_\mathcal{O} \ltimes V$ and $\mathcal{G}_x \times V'$, and the fact that $V'/(\mathcal{G}_x \times V') = V'/\mathcal{G}_x$, which express the equality between the orbit space for the groupoid $\mathcal{G}_x \times V'$ and the orbit space for the Lie group action $\mathcal{G}_x \times N_x\mathcal{O} \to N_x\mathcal{O}$ restricted to V'. The stated properties of $M/\mathcal{G}$ now follow just by noting that locally this orbit space has the same properties as an orbit space for a proper group action (see [17]), among which we find the Whitney condition. □

In the next section we study the global properties of this orbit space and identify the strata that induce this Whitney stratification.

Remark For the same reasons, the orbit space for a proper groupoid with orbits of finite type has additional properties. Namely, it is a cone space with a C^∞ smooth structure in the sense of [49].

3.6 The Strata of the Orbit Space

In this section we realize the strata of the orbit space of a proper Lie groupoid with the accessible sets of a family of vector fields defined on its base.

Definition 3.6.1. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The set of (local) $\mathcal{G}$-derivations vector fields is defined by*

$$\mathfrak{D}(\mathcal{G}, M) = \{X \in \mathfrak{X}(M)_{\text{loc}} : X(f) \in C^{\mathcal{G}}(M), \forall f \in C^{\mathcal{G}}(M)\},$$

where $C^{\mathcal{G}}(M)$ is the set of smooth functions constants on the orbits of $\mathcal{G}$.

Recall from Section 2.4 that in the case of a proper Lie group action the set of G-derivations corresponds to the set of G-projectable vector fields (those whose flow sends orbits to orbits). This is an integrable generalized distribution with leaves the connected components of the orbit type submanifolds $M_{(H)}$. Then, the projection of these leaves to the orbit space M/G are its smooth strata. The main result of this section is the following theorem, which provides an analogous description for the strata of the orbit space of a proper Lie groupoid.

Theorem 3.6.1. *For a source-locally trivial proper Lie groupoid $\mathcal{G} \rightrightarrows M$ with all of its orbit of finite type, the set of the local $\mathcal{G}$-derivations generates a generalized integrable distribution. Furthermore, the projection of its leaves onto the orbit space $M/\mathcal{G}$ are the smooth strata of the Whitney stratification referred to in Corollary 3.5.1.*

In the proof of this theorem we will need a technical result about the existence of a partition of unit on M compatible with $\mathcal{G}$.

Lemma 3.6.1. *Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid with paracompact base. Then, to any open and $\mathcal{G}$-invariant cover of M one can associate a $\mathcal{G}$-invariant partition of unity.*

Proof. Let $(U_i)_{i\in I}$ be an open, $\mathcal{G}$-invariant cover of M. The properness of the groupoid implies the closeness of the orbit projection map $\pi : M \to M/\mathcal{G}$. Since closed maps preserve paracompactness, it follows that the orbit space is also paracompact. Therefore, there is an open and locally finite subcover, $(\widetilde{V}_i)_{i\in I}$, of $(\widetilde{U}_i = \pi(U_i))_{i\in I}$ such that $(V_i := \pi^{-1}(\widetilde{V}_i))_{i\in I}$ is an open, $\mathcal{G}$-invariant, and locally finite subcover of M. For any $i \in I$, let $\overline{x}_i := \pi(x_i)$ be a point of $\widetilde{V}_i$ such that $W_i := \mathcal{G} \cdot S_i \subset V_i$, where S_i is a slice of the groupoid at x_i. Obviously, $\overline{W}_i := \pi(W_i)$ is contained in $\overline{V}_i$ for any indice $i \in I$. The slice theorem for proper groupoids implies that the orbit space of the restriction groupoid to W_i is equivalent to the orbit space of the linear action of $\mathcal{G}_{x_i}$ on an open subset of zero in $T_{x_i}S_i$, B_i. Even more, this action is orthogonal with respect to a $\mathcal{G}_{x_i}$-invariant inner product on B_i. Using the fact that any ray function on B_i is $\mathcal{G}_{x_i}$-invariant, we can construct positive functions $f_i \in C^{\mathcal{G}_{x_i}}(B_i)$ with $f_i(o) = 0$ and $\text{supp}(f_i)$ included in any given neighborhood of zero in B_i. Extend by zero each of these functions on the whole base and denote by $(\overline{f}_i)_{i\in I}$ the induced functions on the orbit space $M/\mathcal{G}$. Since $(\overline{V}_i)_{i\in I}$ is a locally finite family, so are the families $(\text{supp}(\overline{f}_i))_{i\in I}$ and $(\text{supp}(f_i))_{i\in I}$. Consider $g_i := \frac{f_i}{\sum_{i\in I} f_i}$ and $\overline{g}_i := \frac{\overline{f}_i}{\sum_{i\in I} \overline{f}_i}$. It is easy to see that $(g_i)_{i\in I}$ is the required $\mathcal{G}$-invariant partition of unity and $(\overline{g}_i)_{i\in I}$ is a partition of unity of the orbit space associated to the open cover $(V_i)_{i\in I}$.

□

Proof. (Of Theorem 3.6.1)

□

The family of local $\mathcal{G}$-derivations generate a smooth distribution on M that we will show to be Sussmann integrable. Let $x \in M$, $\mathcal{O}$ the orbit through x and $U \ni x$ an invariant open neighborhood. Using the tube theorem we can substitute $\mathcal{G}_U \rightrightarrows U$ by $\mathcal{G}_{\mathcal{O}} \ltimes V \rightrightarrows V$, where V is a neighborhood of the zero section in $N\mathcal{O}$. By Proposition 3.5.1 this action groupoid is Morita equivalent to $\mathcal{G}_x \times O \rightrightarrows O$, with O a $\mathcal{G}_x$-invariant neighborhood of 0 in $N_x\mathcal{O}$. In particular, we can consider the bi-bundle $V \longleftarrow^{\pi_2} \mathcal{S} \rightarrow^{\pi_1} O$ and the family of vector fields

$$\mathfrak{D}(\mathcal{S}) = \{X \in \mathfrak{X}(\mathcal{S}) \,:\, X(f) \in C^{\mathrm{inv}}(\mathcal{S})\,, \forall f \in C^{\mathrm{inf}}(\mathcal{S})\},$$

where $C^{\mathrm{inv}}(\mathcal{S})$ is the set of smooth functions on $\mathcal{S}$ invariant under the commuting actions of $\mathcal{G}_{\mathcal{O}} \ltimes V$ and $\mathcal{G}_x \times O$.

By construction, functions in $C^{\mathcal{G}_{\mathcal{O}} \ltimes V}(V)$ lift by π_2 to functions in $C^{\mathrm{inv}}(\mathcal{S})$. Analogously, functions in $C^{\mathcal{G}_x \times O}(O)$ lift by π_1 to functions in $C^{\mathrm{inv}}(\mathcal{S})$. Therefore, vector fields in $\mathfrak{D}(\mathcal{S})$ are π_1 and π_2 projectable, and since these two projections are surjective submersions, $\mathfrak{D}(\mathcal{S})$ generates the families of basic vector fields on V and O. This induces a bijection between $\mathfrak{D}(\mathcal{G}_{\mathcal{O}} \ltimes V, V)$ and $\mathfrak{D}(\mathcal{G}_x \times O, O)$.

The previous argument shows that $\mathfrak{D}(\mathcal{G}, M)$ is everywhere defined, since any $x \in M$ has a neighborhood V as before on which $\mathfrak{D}(\mathcal{G}, M)\big|_U$ is in one to one correspondence to $\mathfrak{D}(\mathcal{G}_x \times O, O)$, which according to Propositions 3.2.3 and 3.2.4 equals $L^{\mathcal{G}_x}(\mathcal{G}_x, O) \neq \varnothing$.

In addition,$\mathfrak{D}(\mathcal{G}, M)$ is invariant under the flows of its vector fields. To see this, recall that using Proposition 3.2.3 together with the previous discussion, $\mathfrak{D}(\mathcal{G}, M)\big|_U$ is in one to one correspondence with the family of projectable vector fields on O. Let Y and Z be projectable vector fields on O and let F_t^Y, F_t^Z be their flows. Then the vector field ${F_T^Y}_* Z$ has as flow $F_t^Y \circ F_t^Z \circ F_{-t}^Y$. By definition F_t^Y and F_t^Z send $\mathcal{G}_x$-orbits to $\mathcal{G}_x$-orbits and hence so does the above composition. This proves that the family of projectable vector fields on O is invariant under its flows. Using backwards the one to one correspondence we have the same behavior for $\mathfrak{X}(M)_{\mathcal{G},\mathrm{bas}}\big|_U$.

Now, since $\mathfrak{X}(M)_{\mathcal{G},\mathrm{bas}}$ is an everywhere defined family of vector fields invariant by its flows generating a smooth distribution, the Stefan-Sussmann theorem implies that it is integrable, and that its leaves are the accessible sets of vector fields in $\mathfrak{X}(M)_{\mathcal{G},\mathrm{bas}}$. Again by using Morita equivalence, for any point in the orbit space one can find an open neighborhood $U/\mathcal{G}_U$ for which the decomposition of $U/\mathcal{G}_U$ in the sets $L \cap U$ for L a leaf of the generalized foliation of $\mathfrak{X}(M)_{\mathcal{G},\mathrm{bas}}$ equals the decomposition of $U/\mathcal{G}_U = O/\mathcal{G}_x$ into orbit types. Since the latter is the decomposition locally inducing the stratification of $M/\mathcal{G}$ referred to in Corollary 3.5.1, it follows that the leaves of $\mathfrak{X}(M)_{\mathcal{G},\mathrm{bas}}$ are the global strata of this stratification.

Corollary 3.6.1. *In the hypothesis of Theorem 3.6.1, the stratification of the space of orbits $M/\mathcal{G}$ induces in a natural way a stratification of the space of objects $\mathcal{G}$. This stratification has the same properties as the base stratification.*

Proof. If $M = \bigcup_{\alpha \in I} M_\alpha$ is the foliation described in Theorem 3.6.1, then $\mathcal{G} = \bigcup_{\alpha \in I} \mathcal{G}_\alpha$, with $\mathcal{G}_\alpha := s^{-1}(M_\alpha) \cap t^{-1}(M_\alpha)$ is the required stratification of the space of object. □

Ways of continuing

- Prove that orbispaces and proper groupoids are Morita equivalent.
- Dynamical behaviour, equilibria for time dependent Hamiltonians, conformal Hamiltonians. Try the study of eyefish lens using contact equations and prefered optical axis.
- The reduction of cosphere bundle can be easily extended to jet bundles. Reduce Monges-Ampere equations (contact and symplectic reduction simultaneously) for classifications.
- Cosphere bundles have been intensively used in topological problems dealing with the classification of immersions and embeddings. The first one to use these ideas was V. I. Arnold (see [4]) who studied the structure of the space of immersed plane curves using contact invariants of associated Legendrian knots in the cosphere bundle of $\mathbb{R}^2$. Study the reduction of all topological invariants associated to cosphere (contact homology). Then, maybe one should be able to use the symmeties when classifying immersions or embeddings.

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