On the Asymptotic Distortion Behavior of the Distributed Karhunen-Loève Transform

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Joint work with Martin Vetterli

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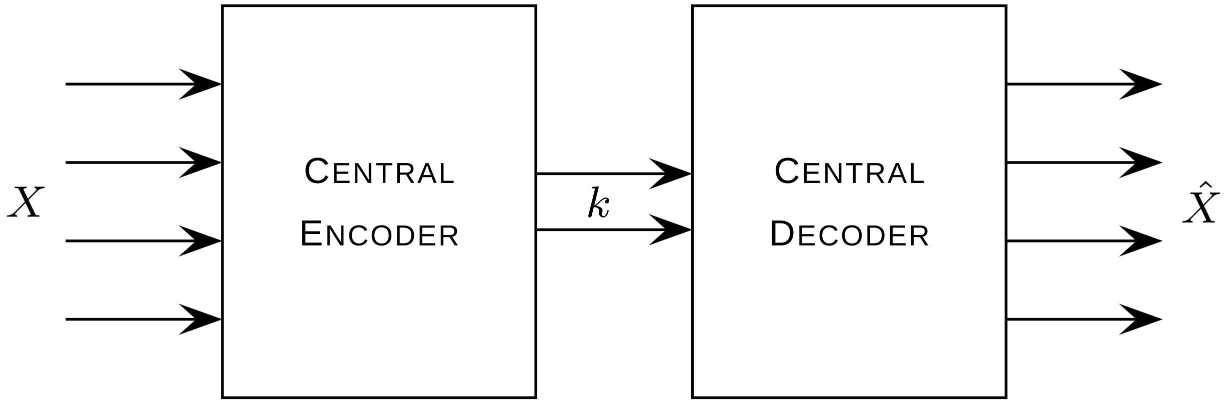
Outline

- The distributed Karhunen-Loève transform (dKLT)
- Theory of large Toeplitz matrices
- Distortion in the approximation framework
- Distortion in the compression framework
- Conclusions

The distributed Karhunen-Loève transform

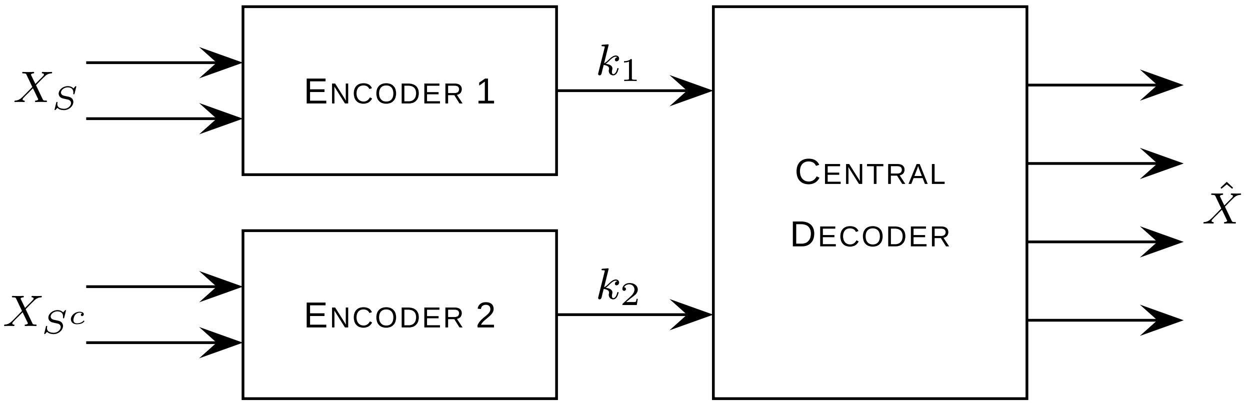
- Gaussian random source $X = (X_1, X_2, \ldots, X_{2n})$ to be encoded, covariance matrix Σ_X
- Centralized scenario:



- Central encoder provides a k-dimensional approximation vector $Y = (Y_1, Y_2, \ldots, Y_k)$
- Mean squared reconstruction error (distortion) $\mathrm{E}[\|X - \hat{X}\|^2]$ depends on the eigenvalues of Σ_X

The distributed Karhunen-Loève transform

- Distributed scenario:



where

$$X = (X_S, X_{S^c})$$
$$X_S = (X_1, X_2, \ldots, X_n)$$
$$X_{S^c} = (X_{n+1}, X_{n+2}, \ldots, X_{2n})$$

- What are the optimal transforms at Encoder 1 and 2?

The distributed Karhunen-Loève transform

- Expected distortion behavior
  - Lower bound: joint KLT (jKLT)
  - Upper bound: marginal KLT (mKLT)
- General solution
  - Unknown
  - Distributed algorithm to find locally optimal solutions
- Particular cases
  - Conditional KLT (cKLT): $k_1 = n$ or $k_2 = n$ (side information)
  - Partial KLT (pKLT): $k_1 = 0$ or $k_2 = 0$ (hidden part)
- Questions
  - What is the gain/loss in having side information/hidden part?
  - How do these scenarios relate to the general distortion problem?

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Need for a better understanding of the distortion behavior

Large Toeplitz matrices

- Stationary processes have Toeplitz covariance matrices

$$T = \begin{pmatrix} t_0 & t_{-1} & 0 & 0 & 0 \\ t_1 & t_0 & t_{-1} & 0 & 0 \\ 0 & t_1 & t_0 & t_{-1} & 0 \\ 0 & 0 & t_1 & t_0 & t_{-1} \\ 0 & 0 & 0 & t_1 & t_0 \end{pmatrix}$$

- Circulant matrices have known eigenvalues

$$C = \begin{pmatrix} t_0 & t_{-1} & 0 & 0 & t_1 \\ t_1 & t_0 & t_{-1} & 0 & 0 \\ 0 & t_1 & t_0 & t_{-1} & 0 \\ 0 & 0 & t_1 & t_0 & t_{-1} \\ t_{-1} & 0 & 0 & t_1 & t_0 \end{pmatrix}$$

Large Toeplitz matrices

- Fundamental eigenvalue distribution theorem of Szegö:

  As the size of the matrix becomes large, the eigenvalues of T and C behave the same (provided that the sequence t_k decreases fast enough)

- Goals:
  - Use the above result to provide closed-form formulas of the distortion in the approximation setup
  - Relate this approach to the compression (rate-distortion) framework and derive the corresponding $D(R)$ curves

Distortion in the approximation framework

- Case at hand: first-order Gauss-Markov process

$$X_n = \rho X_{n-1} + Z_n$$

where $\rho \in [0, 1]$ and Z_n are i.i.d. Gaussian random variables with mean 0 and variance $1 - \rho^2$

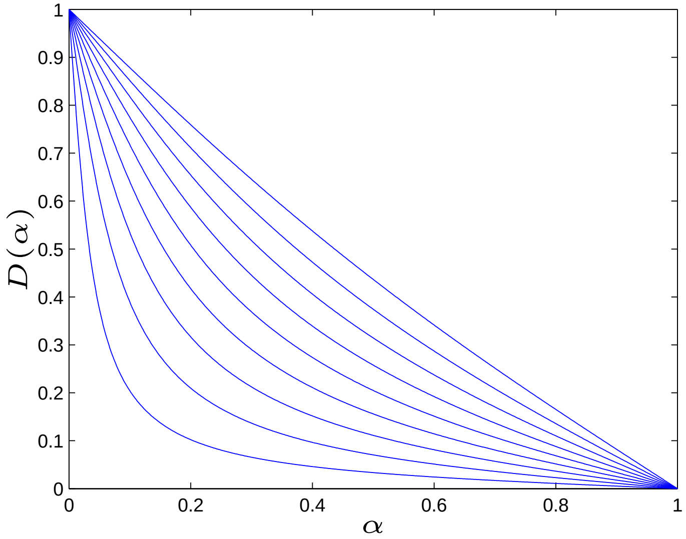
- Setup under consideration
  - Encoder 1: odd coefficients, sends a fraction $\alpha_1 \sim k_1/n \in [0, 1]$ of transformed coefficients
  - Encoder 2: even coefficients, sends a fraction $\alpha_2 \sim k_2/n \in [0, 1]$ of transformed coefficients
  - We compute the asymptotic normalized mean squared distortion $D(\alpha_1, \alpha_2)$

$$D(\alpha_1, \alpha_2) = \lim_{n \to \infty} \frac{\mathrm{E}[\|X - \hat{X}(\alpha_1, \alpha_2)\|^2]}{2n}$$

Distortion in the approximation framework

- Joint KLT: central encoding ($\alpha = (\alpha_1 + \alpha_2)/2$)

$$D(\alpha) = 1 - \frac{2}{\pi} \arctan\left(\frac{1+\rho}{1-\rho} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

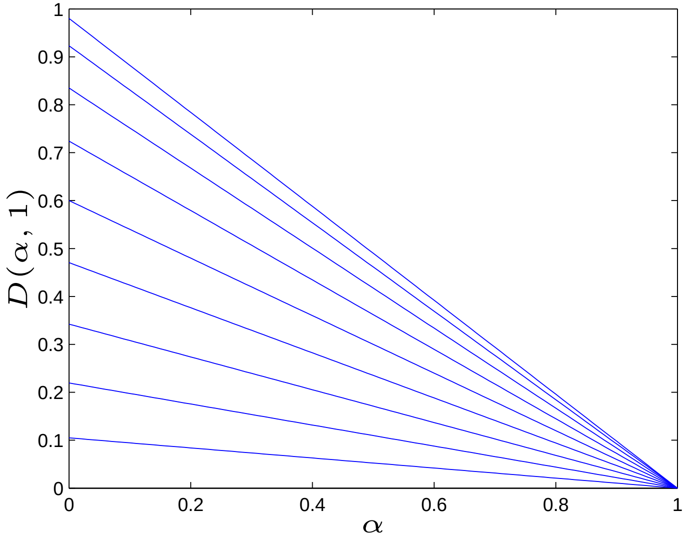


for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom)

Distortion in the approximation framework

- Conditional KLT: encoding with side information ($\alpha_1 = \alpha$, $\alpha_2 = 1$)

$$D(\alpha, 1) = \frac{1-\rho^2}{1+\rho^2}(1-\alpha)$$

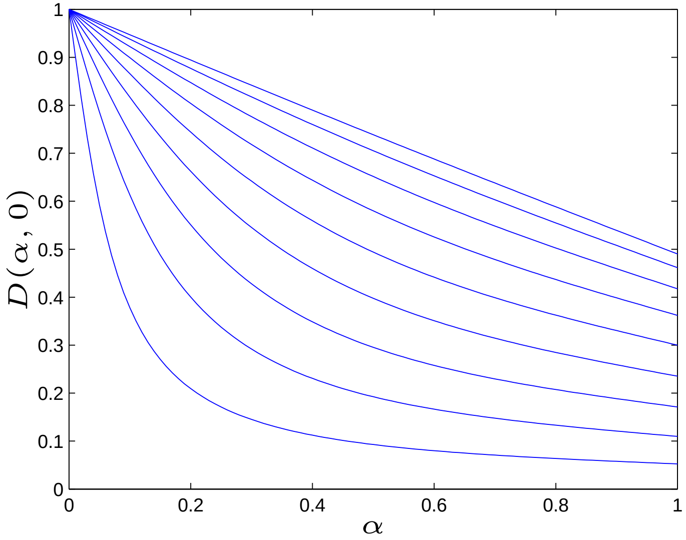


for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom)

Distortion in the approximation framework

- Partial KLT: encoding with hidden part ($\alpha_1 = \alpha$, $\alpha_2 = 0$)

$$D(\alpha, 0) = 1 + \frac{\alpha(1-\rho^2)}{2(1+\rho^2)} - \frac{2}{\pi} \arctan\left(\frac{1+\rho^2}{1-\rho^2} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

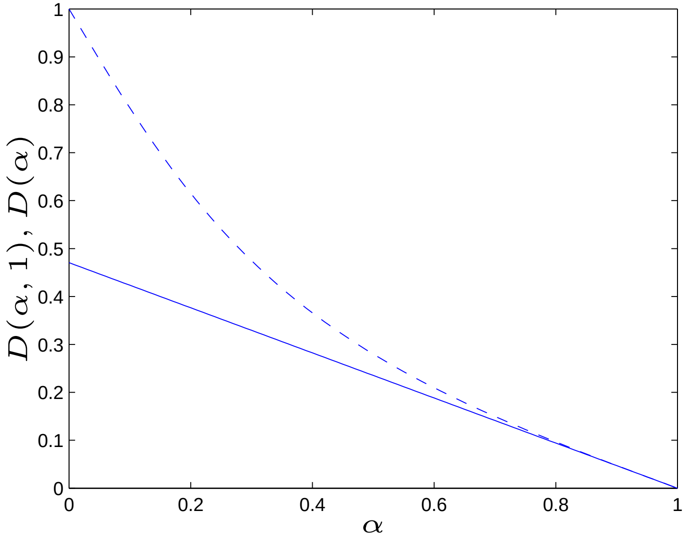


for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom)

Distortion in the approximation framework

- Gain due to side information

$$D(\alpha) - D(\alpha, 1) = 1 - \frac{(1-\rho^2)}{(1+\rho^2)}(1-\alpha) - \frac{2}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right)$$

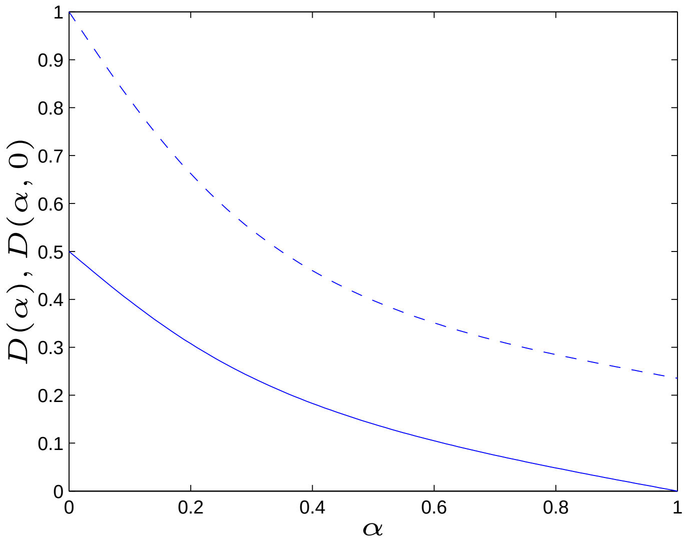


for $\rho = 0.6$

Distortion in the approximation framework

- Loss due to hidden part

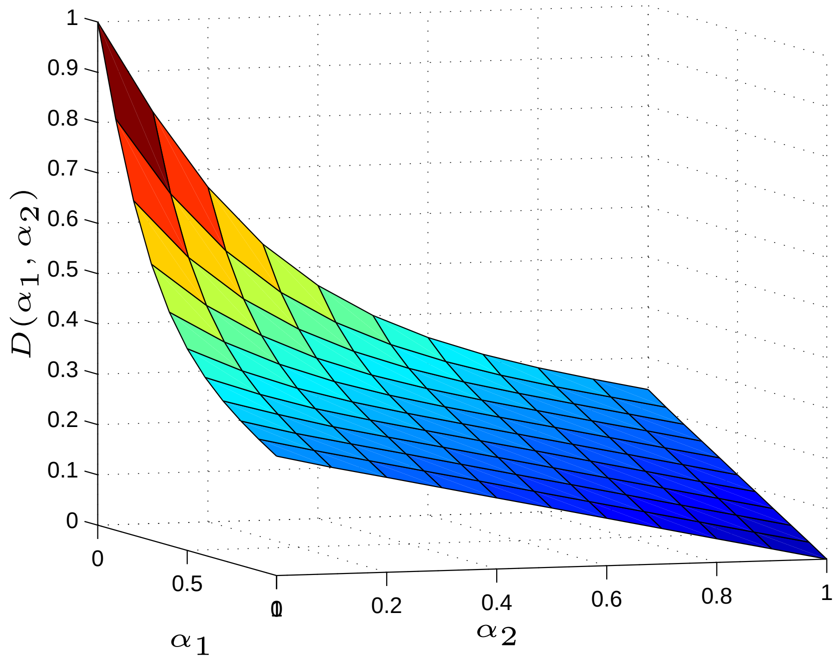
$$D(\alpha, 0) - D(\alpha) = \frac{1}{2} + \frac{\alpha(1-\rho^2)}{2(1+\rho^2)} - \frac{1}{\pi} \arctan\left(\frac{1+\rho^2}{1-\rho^2} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$



for $\rho = 0.6$

Distortion in the approximation framework

- General two-encoder setup



for $\rho = 0.6$

- Here the dKLT algorithm is conjectured to converge to the global minimum

Distortion in the compression framework

- Approximation framework
  - Convenient representation over finite intervals
  - Easier to derive closed-form fomulas (no parametric representation)
  - More relevant from a signal processing point of view
- Compression framework
  - Parametric representation in general
  - Closed-form formulas at high rates
  - For jointly Gaussian sources, approximation followed by compression is optimal (transform coding)

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Let’s go back to the rate-distortion framework!!

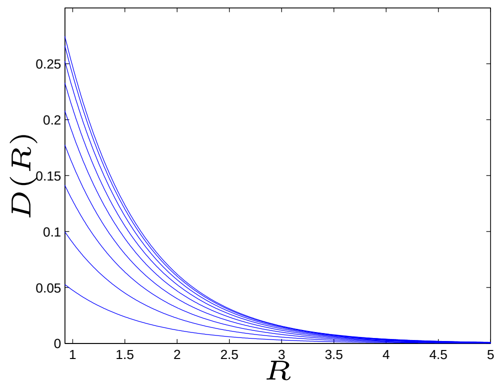
Distortion in the compression framework

- Joint KLT: central encoding ($R = R_1 + R_2$)

$$D(R) = (1 - \rho^2)2^{-2R}$$

for $R \geq \log_2(1 + \rho)$ [bits/source coefficient]

- Corresponds to $D(R)$ of a Gaussian random variable with mean 0 and variance $1 - \rho^2$



for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom)

Distortion in the compression framework

- Conditional KLT: encoding with side information ($R_1 = R, R_2 \to \infty$)

$$D(R, \infty) = \frac{1-\rho^2}{1+\rho^2} 2^{-2R}$$

  for $R \geq 0$ [bits/source coefficient]

- Corresponds to $D(R)$ of a Gaussian random variable with mean 0 and variance $(1-\rho^2)/(1+\rho^2)$

Distortion in the compression framework

- Partial KLT: encoding with hidden part ($R_1 = R, R_2 = 0$)

$$
\begin{aligned}
D(R,0) \quad &= \quad \frac{1}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right) \\
&\quad \cdot \left(\frac{(1+\rho^2)^2 + 2\rho^2 + (1+\rho^2)\sqrt{1+6\rho^2+\rho^4}}{2} 2^{-2R} + 1\right)
\end{aligned}
$$

for $R \geq \frac{1}{2}\log_2\left((1+\rho^2)^2 + 2\rho^2 + (1+\rho^2)\sqrt{1+6\rho^2+\rho^4}\right) - \frac{1}{2}$

Distortion in the compression framework

- Gain due to side information

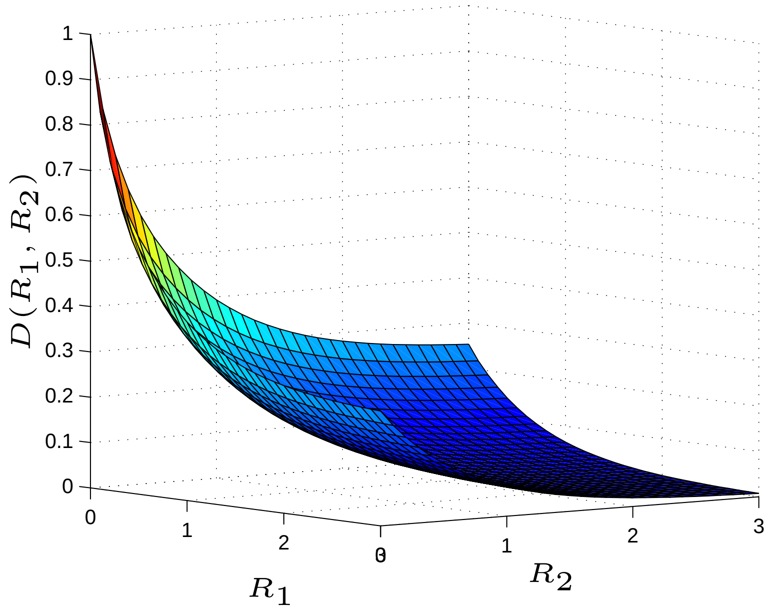
$$\frac{D(R)}{D(R,\infty)} = (1+\rho^2)^2$$

  for $R \geq \log_2(1+\rho^2)$ [bits/source coefficient]

- Loss due to hidden part: we do not display the formula here for the sake of esthetism ☺

Distortion in the compression framework

- General two-encoder setup



for $\rho = 0.6$

- Global optimum only corresponds to best-known achievable rate-distortion region

Conclusions

- Analysis of the asymptotic distortion behavior of the dKLT for stationnary sources (e.g. first-order Gauss-Markov process)
- Distortion analysis from a pure approximation point of view
- Closed-form formulas for the boundaries of the general distortion surface
- Extension to the rate-distortion framework and similar interpretation of the distortion surface boundaries

Thanks for your attention

Questions?