# On the Asymptotic Distortion Behavior of the Distributed Karhunen-Loève Transform

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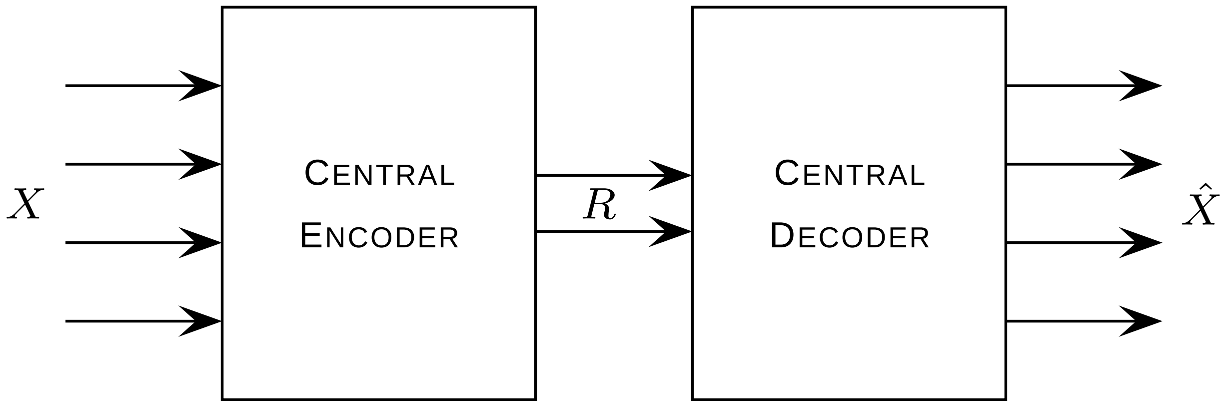
# Outline

- Centralized source coding problem (compression & approximation)
- Distributed source coding problem (compression & approximation)
- The distributed Karhunen-Loève transform
- Asymptotic normalized distortion analysis
- Example: first-order Gauss-Markov process
- Conclusions

# Centralized source coding problem (1/2)

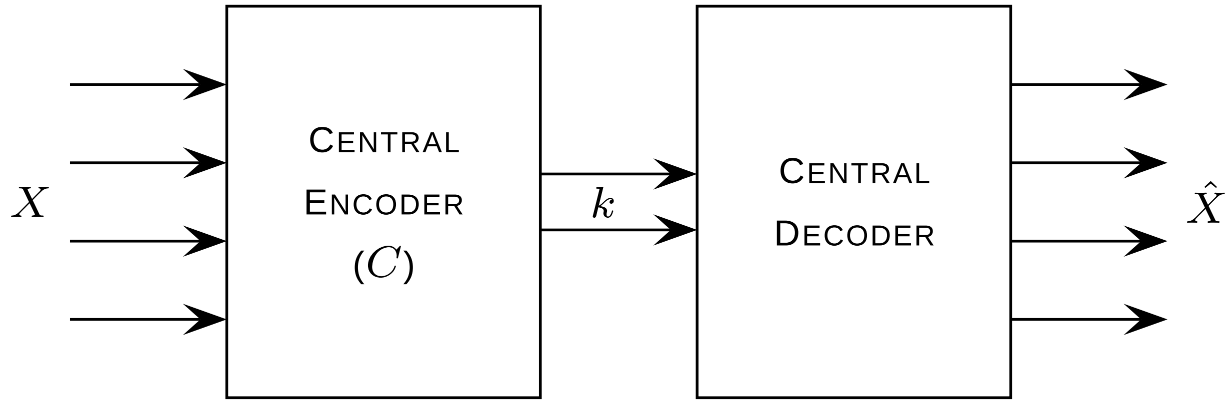
- Encoding of a jointly Gaussian random vector $X = (X_1, X_2, \ldots, X_{2n})$ with mean zero and covariance matrix $\Sigma_X$.
- Compression :




  - Central encoder provides a representation of $X$ using $R$ bits.
  - Goal : minimize the mean squared reconstruction error $\mathrm{E}[\|X - \hat{X}\|^2]$.

# Centralized source coding problem (2/2)

- Approximation :

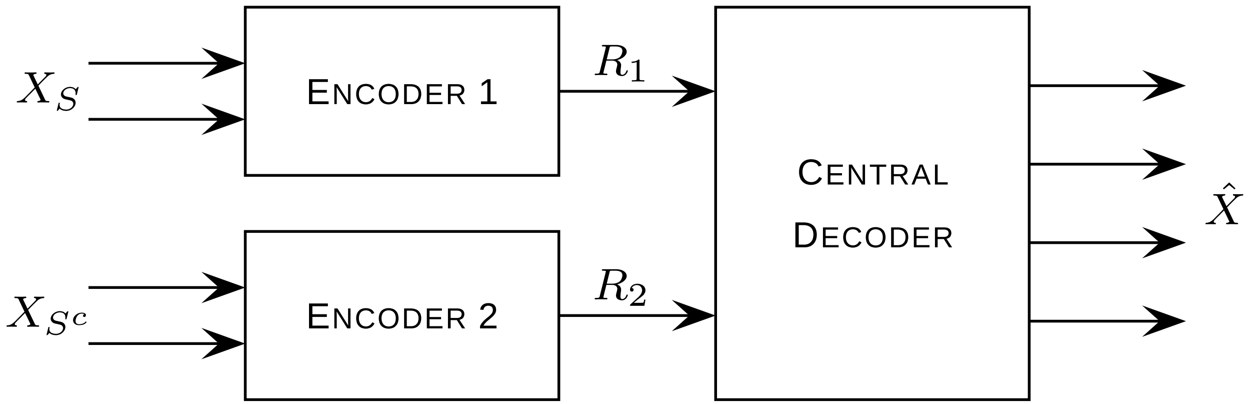



  - Central encoder applies a linear transform on $X$ and provides a representation using $k$ coefficients.
  - Goal : minimize the mean squared reconstruction error $\mathrm{E}[\|X - \hat{X}\|^2]$.
  - Optimal transform $C$ : Karhunen-Loève Transform (KLT).

- Approximation and compression perspectives nicely related : optimality of transform coding.

# Distributed source coding problem (1/2)

- Compression :



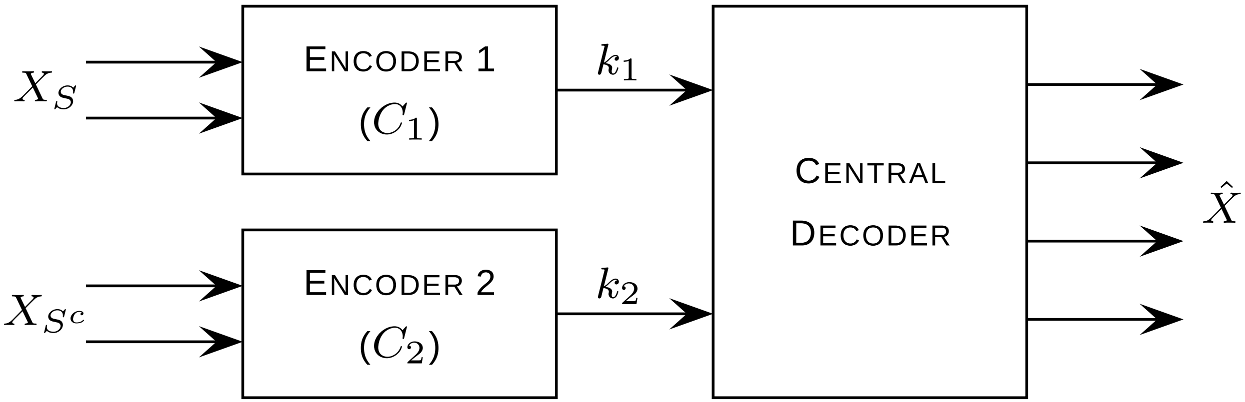

where

$$X = (X_S, X_{S^c})$$
$$X_S = (X_1, X_2, \ldots, X_n)$$
$$X_{S^c} = (X_{n+1}, X_{n+2}, \ldots, X_{2n})$$

- Encoders 1 and 2 provide a representation of $X_S$ and $X_{S^c}$ using $R_1$ and $R_2$ bits, respectively.
- Goal : minimize the mean squared reconstruction error $\mathrm{E}[\|X - \hat{X}\|^2]$.
- An instance of lossy Slepian-Wolf : long standing open problem.

# Distributed source coding problem (2/2)

- Approximation :

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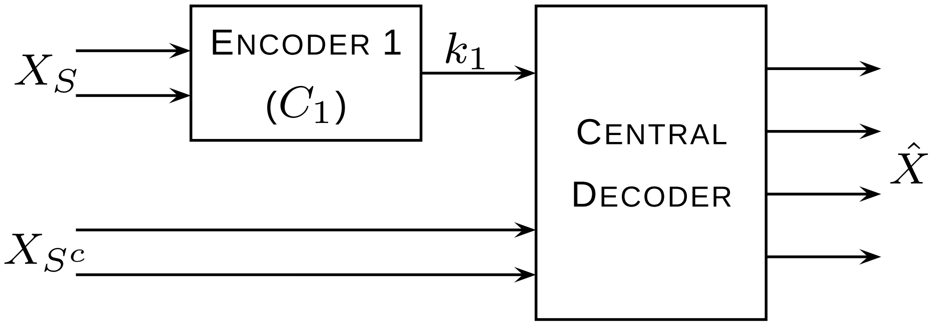
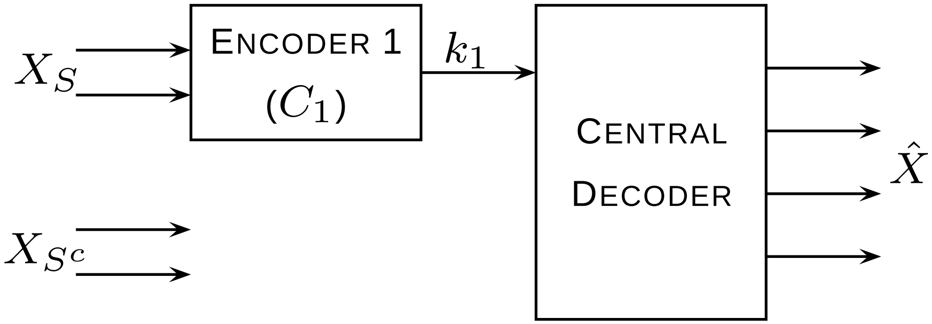
- Encoders 1 and 2 apply each a linear transform on $X_S$ and $X_{S^c}$ and provide representations using $k_1$ and $k_2$ coefficients, respectively.
- Goal : minimize the mean squared reconstruction error $\mathrm{E}[\|X - \hat{X}\|^2]$.
- What are the optimal transforms $C_1$ and $C_2$ to apply at Encoder 1 and 2 ?

# The distributed Karhunen-Loève transform (1/2)

- Problem considered under the name of distributed Karhunen-Loève Transform (dKLT) [Gastpar et al.] (see also [Zhang et al.]).
- Particular cases :
  - Conditional KLT (cKLT) : $k_1 = n$ or $k_2 = n$ (side information)




  - Partial KLT (pKLT) : $k_1 = 0$ or $k_2 = 0$ (hidden part)



# The distributed Karhunen-Loève transform (2/2)

- General solution unknown : approximation counterpart to lossy Slepian-Wolf.
- Distributed algorithm to find locally optimal solutions.
- Expected distortion behavior :
  - Lower bound : joint KLT (jKLT).
  - Upper bound : marginal KLT (mKLT).
- Questions :
  - What is the gain/loss in having side information/hidden part ?
  - How do these scenarios relate to the general distortion problem ?

# Asymptotic normalized distortion analysis (1/4)

- Finite dimensional regime :
  - Encoder $l \in \{0, 1\}$ provides the reconstruction point with a description of size $k_l$ with $k_l \leq n$.
  - The distortion can be expressed as

$$D_n(k_1, k_2) = \mathrm{E}[\|X - \hat{X}\|^2] = \mathrm{tr}\left(\Sigma_X - \Sigma_X C^T (C \Sigma_X C^T)^{-1} C \Sigma_X\right)$$

where $C$ is given by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}.$$

  - Little can be said about the distortion.
  - Difficult to compare analytically the different scenarios of interest.

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☞ Asymptotic analysis

# Asymptotic normalized distortion analysis (2/4)

- Infinite dimensional regime :
  - Encoder $l \in \{0,1\}$ provides the reconstruction point with a description of size $\lfloor \alpha_l n \rfloor$ with $\alpha_l \in [0,1]$, i.e. using a fraction $\alpha_l \sim k_l/n$ of transformed coefficients.
  - The asymptotic normalized distortion is defined as

$$D(\alpha_1, \alpha_2) = \lim_{n \to \infty} \frac{D_n(k_1, k_2)}{n}$$

    if the limit exists.
  - Allows a convenient representation of the distortion surface as a function of parameters in $[0,1]$.

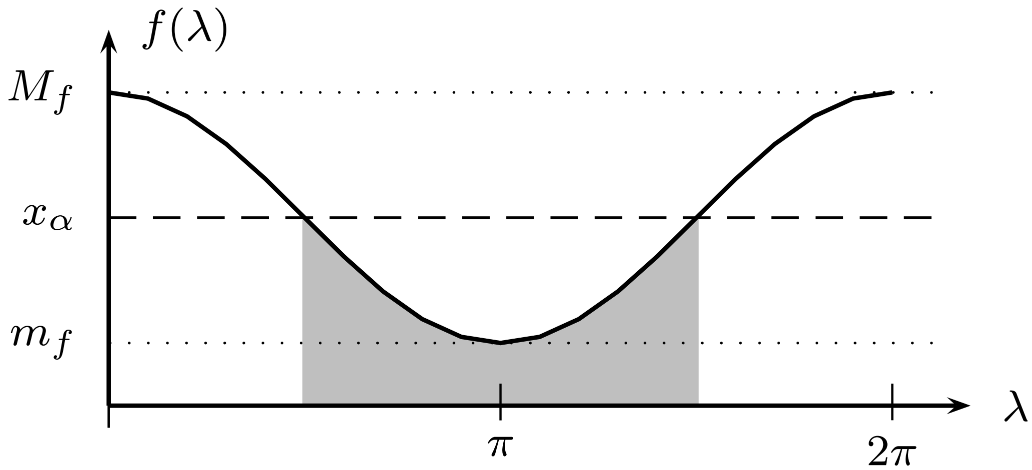
# Asymptotic normalized distortion analysis (3/4)

- Centralized scenario : $D(\alpha)$.
- We consider a stationary process with power spectral density $f$.
- Computation of the asymptotic normalized distortion based on Szegö's theorem.
- We can show the following :

$$D(\alpha) = \frac{1}{2\pi} \int_{\lambda : f(\lambda) \leq x_\alpha} f(\lambda)\, d\lambda$$

where $x_\alpha$ satisfies $F(x_\alpha) = 1 - \alpha$.

- "Water-filling" solution :



# Asymptotic normalized distortion analysis (4/4)

- Distributed scenario : $D(\alpha_1, \alpha_2)$.
- We consider in particular :
  - the cKLT ($D(\alpha_1, 1)$).
  - the pKLT ($D(\alpha_1, 0)$).
- They correspond to the KLT of a process with a modified covariance matrix.
- The computation of the asymptotic normalized distortion in these cases thus reduces to the centralized scenario with appropriate power spectral densities.

# Example : first-order Gauss-Markov process (1/8)

- First-order Gauss-Markov process :

$$X_n = \rho X_{n-1} + Z_n$$

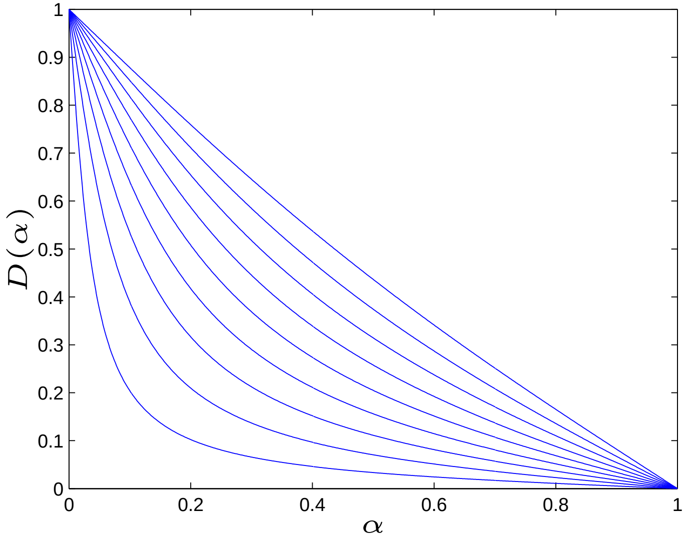
where $\rho \in [0, 1]$ and $Z_n$ are i.i.d. Gaussian random variables with mean 0 and variance $1 - \rho^2$.

- Setup under consideration :
  - Encoder 1 : odd coefficients, sends a fraction $\alpha_1 \in [0, 1]$ of transformed coefficients.
  - Encoder 2 : even coefficients, sends a fraction $\alpha_2 \in [0, 1]$ of transformed coefficients.

# Example : first-order Gauss-Markov process (2/8)

- Joint KLT : central encoding ($\alpha = (\alpha_1 + \alpha_2)/2$).

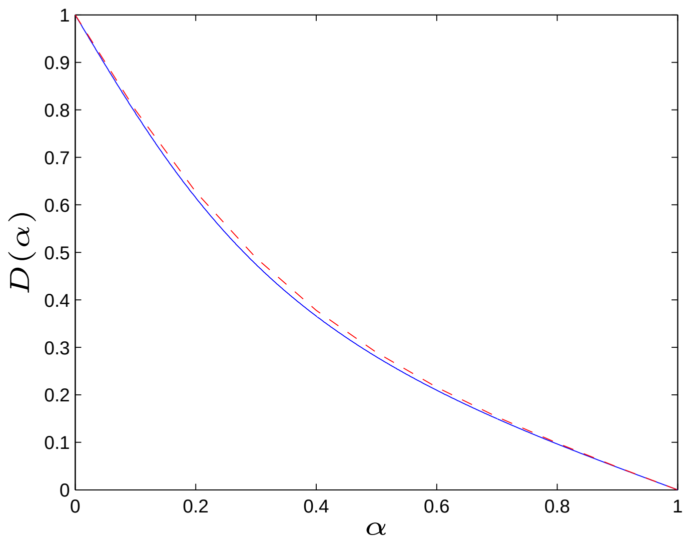
$$D(\alpha) = 1 - \frac{2}{\pi} \arctan\left( \frac{1+\rho}{1-\rho} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

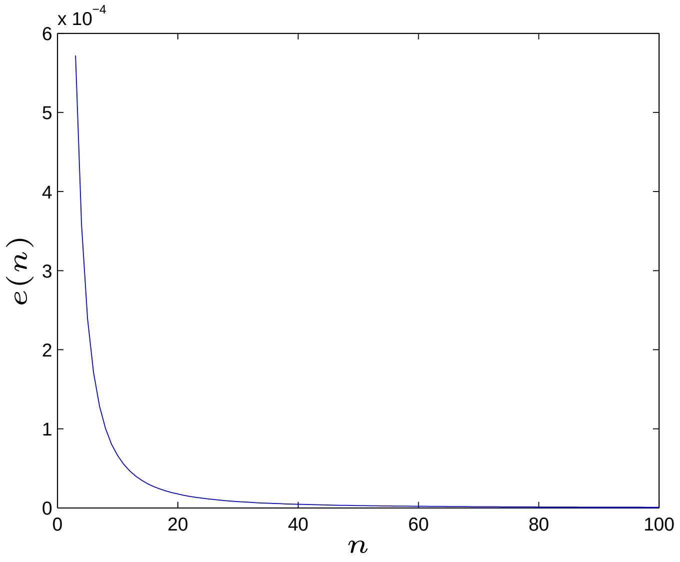



for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom).

# Example : first-order Gauss-Markov process (3/8)

- Joint KLT : finite vs. infinite dimensional regime.



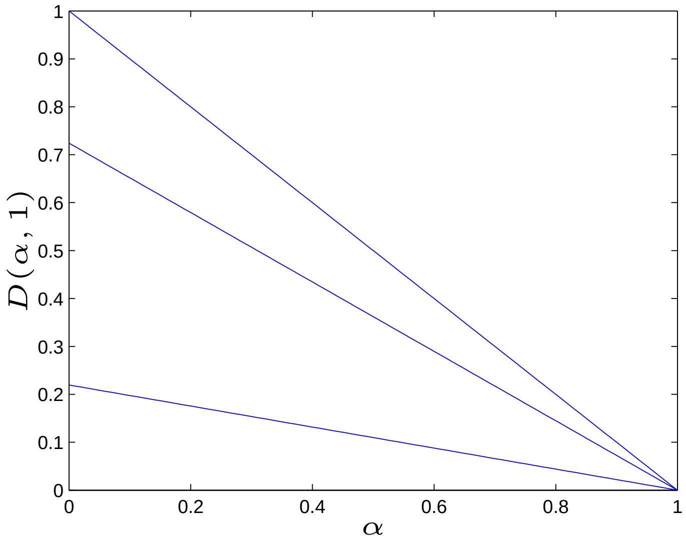




for $\rho = 0.6$ and $n = 10$.

# Example : first-order Gauss-Markov process (4/8)

- Conditional KLT : encoding with side information ($\alpha_1 = \alpha$, $\alpha_2 = 1$).

$$D(\alpha, 1) = \frac{1-\rho^2}{1+\rho^2}(1-\alpha)$$

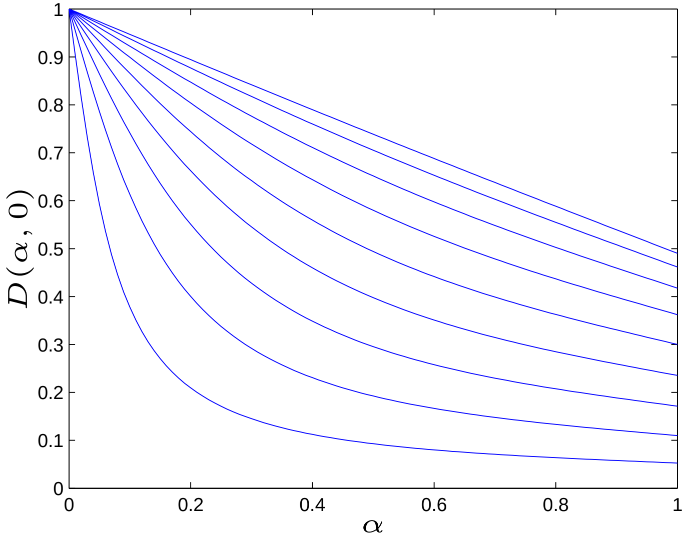



for $\rho = 0, 0.4, 0.8$ (from top to bottom).

# Example : first-order Gauss-Markov process (5/8)

- Partial KLT : encoding with hidden part ($\alpha_1 = \alpha$, $\alpha_2 = 0$).

$$D(\alpha, 0) = 1 + \frac{\alpha(1-\rho^2)}{2(1+\rho^2)} - \frac{2}{\pi} \arctan\left(\frac{1+\rho^2}{1-\rho^2} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

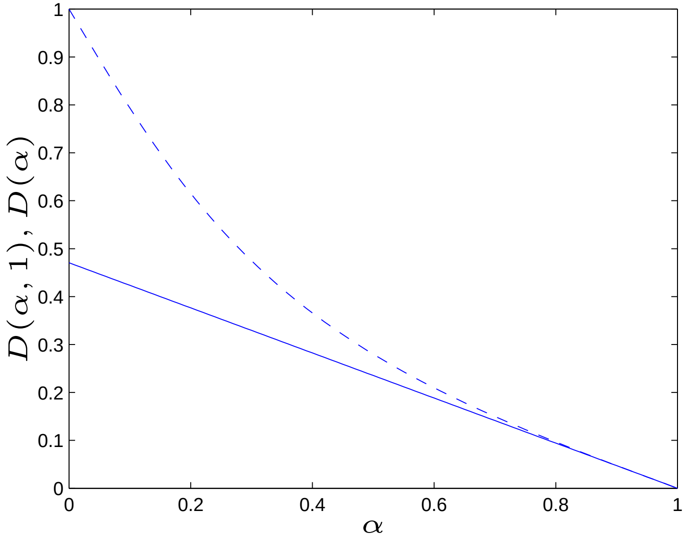



for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom).

# Example : first-order Gauss-Markov process (6/8)

- Gain due to side information :

$$D(\alpha) - D(\alpha, 1) = 1 - \frac{(1-\rho^2)}{(1+\rho^2)}(1-\alpha) - \frac{2}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right)$$

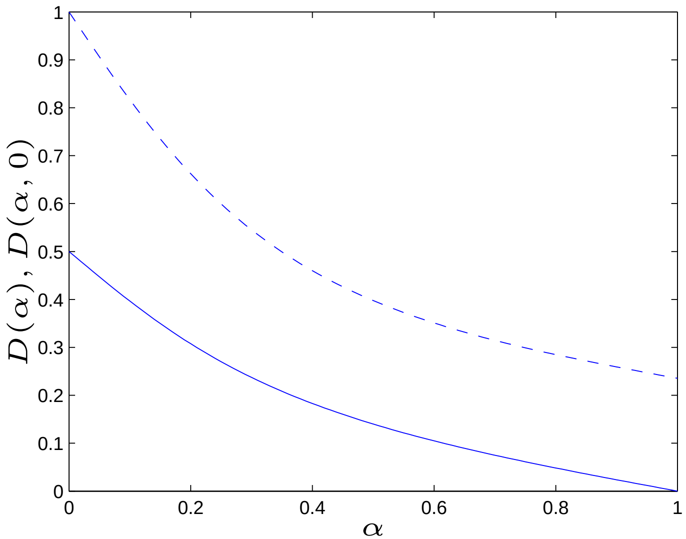



for $\rho = 0.6$.

# Example : first-order Gauss-Markov process (7/8)

- Loss due to hidden part :

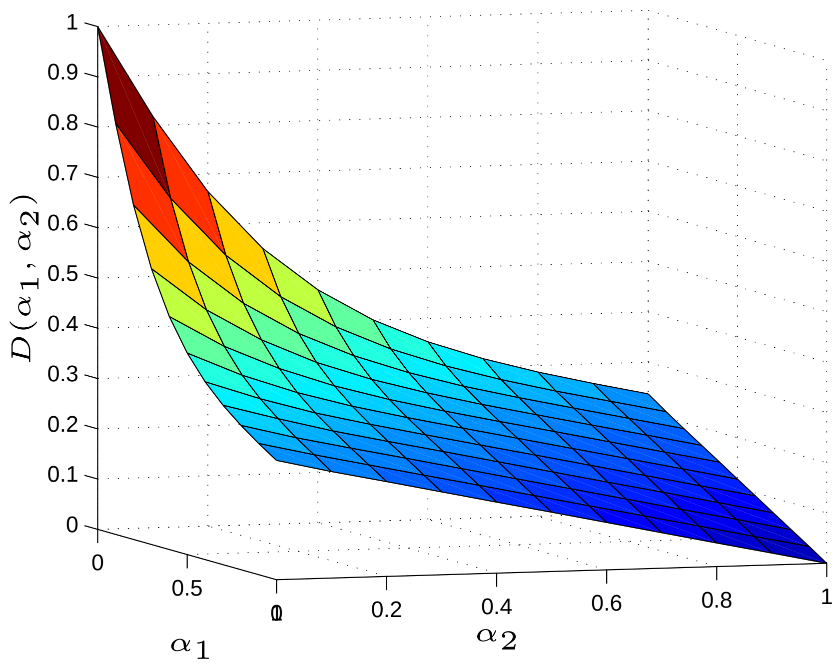
$$D(\alpha,0) - D(\alpha) = \frac{1}{2} + \frac{\alpha(1-\rho^2)}{2(1+\rho^2)} - \frac{1}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right)$$




for $\rho = 0.6$.

# Example : first-order Gauss-Markov process (8/8)

- General two-encoder setup :




for $\rho = 0.6$.

# Conclusions

- Analysis of the asymptotic distortion behavior of the dKLT for stationary sources (e.g. first-order Gauss-Markov process).
- Distortion analysis from a pure approximation point of view.
- Good approximation of the finite dimensional regime.
- Closed-form formulas for the boundaries of the general distortion surface.

Thanks for your attention.

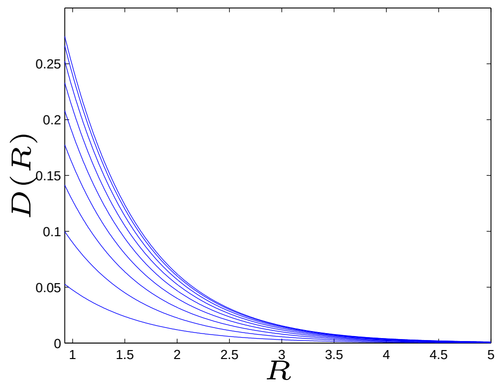
# Example : first-order Gauss-Markov process

- Joint KLT : central encoding ($R = R_1 + R_2$).

$$D(R) = (1 - \rho^2)2^{-2R}$$

for $R \geq \log_2(1 + \rho)$ [bits/source coefficient].

- Corresponds to $D(R)$ of a Gaussian random variable with mean 0 and variance $1 - \rho^2$.




for $\rho = 0.1, 0.2, \ldots, 0.9$ (from top to bottom).

# Example : first-order Gauss-Markov process

- Conditional KLT : encoding with side information ($R_1 = R, R_2 \to \infty$).

$$D(R,\infty) = \frac{1-\rho^2}{1+\rho^2} 2^{-2R}$$

for $R \geq 0$ [bits/source coefficient].

- Corresponds to $D(R)$ of a Gaussian random variable with mean 0 and variance $(1-\rho^2)/(1+\rho^2)$.

# Example : first-order Gauss-Markov process

- Partial KLT : encoding with hidden part ($R_1 = R, R_2 = 0$).

$$D(R,0) = \frac{1}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right) \cdot \frac{(1+\rho^2)^2 + 2\rho^2 + (1+\rho^2)\sqrt{1+6\rho^2+\rho^4}}{2} 2^{-2R} + 1$$

for $R \geq \frac{1}{2}\log_2\left((1+\rho^2)^2 + 2\rho^2 + (1+\rho^2)\sqrt{1+6\rho^2+\rho^4}\right) - \frac{1}{2}$.

# Example : first-order Gauss-Markov process

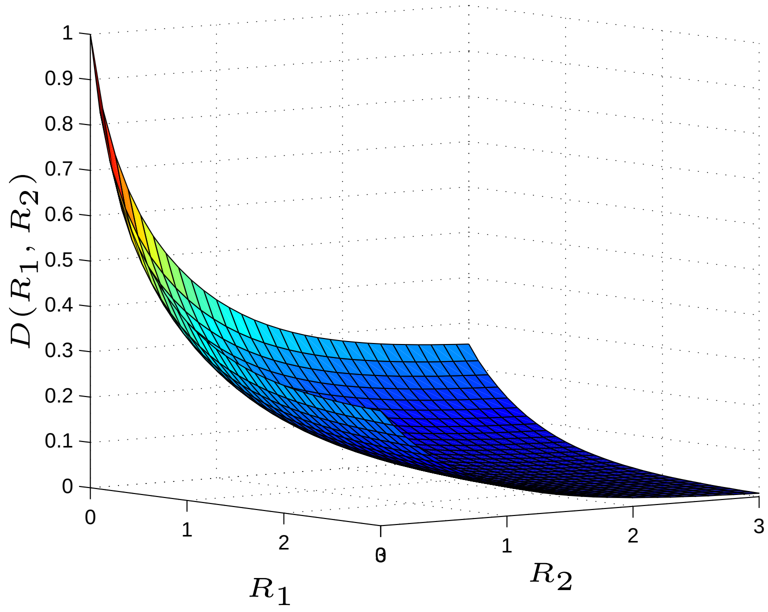
- Gain due to side information :

$$\frac{D(R)}{D(R,\infty)} = (1+\rho^2)^2$$

for $R \geq \log_2(1+\rho^2)$ [bits/source coefficient].

# Example : first-order Gauss-Markov process

- General two-encoder setup :




for $\rho = 0.6$.

- Global optimum only corresponds to best-known achievable rate-distortion region.