A connection between an exactly solvable stochastic optimal control problem and a nonlinear reaction-diffusion equation

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Abstract. We present an exactly soluble optimal stochastic control problem involving a diffusive two-states random evolution process and connect it to a non-linear reaction-diffusion type of equation by using the technique of logarithmic transformations. The work generalizes the recently established connection between the non-linear Boltzmann like equations introduced by Ruijgrok and Wu and the optimal control of a two-states random evolution process. In the sense of this generalization, the non-linear reaction-diffusion equation is identified as the natural diffusive generalization of the Ruijgrok-Wu Boltzmann model.

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1. Introduction

In a recent address Hongler et al. connected the exactly soluble, nonlinear, discrete two-velocities Boltzmann model of Ruijgrok and Wu (the RW-model, [22]) to an optimal stochastic control problem, involving a two-states random evolution process [14]. The authors apply the technique of logarithmic transformations (LT) which, in the context of stochastic control, goes back to the works of Fleming [6] and Holland [10] in the late 1970's. The starting point of this rather general technique is the positive solution ϕ of the linear equation $A\phi = 0$ where A is a backward evolution operator of some given Markov process X. The function V, obtained via the LT: $V = -\ln(\phi)$, satisfies a nonlinear evolution equation which is the dynamic programming (DP) equation of some stochastic control problem specified by a particular cost function L. The associated optimal control u^* is closely related to the space derivative of V (see [7]).
Using these relations, Hongler et al. showed that in the particular case of a simple two-states random evolution, the total derivative of V is governed by the exactly solvable, non-linear Boltzmann like RW-model. The diffusive analogue of this connection, obtained by a central limit theorem (CLT) procedure, is the well known relation

between the heat equation and the Burgers field equation, realized via the Hopf-Cole transformation [15]. The authors indeed established the particular relations presented in table (1).

I Markov dynamics $A\phi = 0$	$\xrightarrow[V=-\ln(\phi)]{LT}$	*II* DP for V with cost functional L	$\xrightarrow{\mathcal{O}}$	*III* non-linear field equation
Rand. Evolution	$\longrightarrow$	hyperbolic eq. for V $L_{RE} \propto u \ln(u) - u + 1$	$\xrightarrow{\partial_t \pm \partial_x}$	RW-model
$\downarrow$ CLT		$\downarrow$		$\downarrow$
Diffusion	$\longrightarrow$	parabolic eq. for V $L_D \propto u^2$	$\xrightarrow{\partial_x}$	Burgers eq.

Table 1. Starting with Markovian dynamics in I, the LT leads to a stochastic control problem II, whose value function V can be related via a differential operator $\mathcal{O}$ to physically relevant non-linear field equations III. This construction leads in case of a standard Brownian motion to the Burgers equation and in case of a two-states random evolution to the RW-model.

The aim of this short note is threefold: (i), we put in section 2 the two-states dynamics and the diffusive dynamics together and construct and solve via the logarithmic transformation an exactly soluble stochastic control problem. (ii), we connect in section 3 the dynamic programming equation associated to this stochastic control problem to a non-linear reaction-diffusion type of equation which appeared in an ad hoc manner in [12]. This extends table (1) by the construction presented in table (2). (iii), we remark

I diffusive Rand. evolution	$\xrightarrow[V=-\ln(\phi)]{LT}$	*II* 4th. order eq. for V $L = L_{RE} + L_D$	$\xrightarrow{\mathcal{O}^{\pm}}$	*III* non-linear reaction-diffusion eq.

Table 2. Starting from a diffusive random evolution (I) we construct an exactly solvable optimal control problem with running costs L (II) and connect it to a reaction-diffusion type of equation (III) which is subsequently identified as the diffusive generalization of the RW model.

in section 4 that the cost functions L_{RE} and L_D associated respectively to the control of the two-states random evolution and the diffusion, are related to a large deviations principle. The running costs L_{RE} and L_D therefore tax in a natural way deviations away from the uncontrolled trajectories.

2. An optimal controlled diffusive random evolution

We start with the backward evolution equation $A\phi = 0$ corresponding to a dynamical system driven by the independent sum of the white Gaussian noise and a two-states random evolution. The Langevin equation of motion of our state variable X_t (e.g. the position of an overdamped particle moving on the line) is therefore of the form

$$dX_s = Z_s ds + \sigma dB_s, \tag{1}$$

where B_s stands for the standard Brownian motion, $\sigma > 0$ is constant, Z_s is a time-continuous two-states Markov process taking values in the set $\{\nu, -\nu\} \subset \mathbb{R}$. The symmetric jump rates between ν and $-\nu$ are supposed to be constant and, for convenience, we set them equal to 1. The pair process (X_s, Z_s) is Markov with state space $\Sigma = \mathbb{R} \times \{\pm\nu\}$. The associated backward evolution operator $\mathbf{A}$ acts on functions $\phi(t, x, z)$ in $\mathcal{C}_0^{2,1}([0, t_1] \times \mathbb{R} \times \{\pm\nu\})$ with $t_1 > 0$ fixed and is given by

$$\mathbf{A}\phi(t, x, z) = \partial_t \phi(t, x, z) + \frac{\sigma^2}{2}\partial^2_{x,x}\phi(t, x, z) + z\partial_x\phi(t, x, z) + (\phi(t, x, -z) - \phi(t, x, z)). \tag{2}$$

In this paper we restrict ourselves to functions $\phi(\cdot, \cdot, \pm z) \overset{\text{not.}}{=} \phi^{\pm z}$ solving $\mathbf{A}\phi = 0$ which are sufficiently regular (*i.e.* $\phi^z \in \mathcal{C}_0^{4,1}$, $z \in \{\pm\nu\}$). In this case we can isolate ϕ^z (resp. ϕ^{-z}) from the system (2) and get two uncoupled fourth order equations for ϕ^z and ϕ^{-z}:

$$([\partial_t + \frac{\sigma^2}{2}\partial^2_{x,x}]^2 - \nu^2\partial^2_{x,x})\phi^z = (2\partial_t + \sigma^2\partial^2_{x,x})\phi^z, \quad z \in \{\pm\nu\}. \tag{3}$$

Having introduced the Markovian dynamics via the evolution operator eq.(2), we construct now the control problem by

a) controlling the jump rates $1 \to u(s) \equiv u(s, X_s, Z_s)$
b) and by adding a drift $v(s) \equiv v(s, X_s, Z_s)$ to the Brownian motion *i.e.* we replace σdB_s by $v(s, X_s, Z_s)ds + \sigma dB_s$.

The backward evolution operator for the controlled process $\mathbf{A}^{u,v}$ is given by

$$\mathbf{A}^{u,v}(\phi^z, \phi^{-z}) = \partial_t\phi^z + \Big(z + v(\cdot, \cdot, z)\Big)\partial_x\phi^z + \frac{\sigma^2}{2}\partial^2_{x,x}\phi^z + u(\cdot, \cdot, z)(\phi^{-z} - \phi^z). \tag{4}$$

The controllers aim is to minimize the expected costs

$$J(t, x, z; (u, v)) := \mathbb{E}_{t,x,z}\Big(\int_t^{t_1} L(s, u(s), v(s))ds + \psi(t_1, X_{t_1}, Z_{t_1})\Big), \tag{5}$$

incurred during the finite interval $[t, t_1]$, $0 \le t \le t_1$, when starting the process at the initial point $(X_t, Z_t) = (x, z)$ and where ψ accounts for the final costs. The minimum, denoted $V(t, x, z)$, is taken over all admissible pairs of Markov controls $(u(s), v(s))$, $s \in [t, t_1]$ with $u(s) = u(s, \cdot, z) \in \mathcal{C}_0(\mathbb{R}; \mathbb{R}^+)$ and $v(s) = v(s, \cdot, z) \in \mathcal{C}_0(\mathbb{R}; \mathbb{R})$. For the cost function (or likewise, the Lagrangian) L, discussed in section 4, we choose

$$L(s, u, v) = L(u, v) = L_D(v) + L_{RE}(u), \tag{6}$$

where the cost function accounting for the diffusive part is given by

$$L_D(v) = \frac{\mu^2}{2}v^2 \tag{7}$$

and where the one associated to the random evolution is

$$L_{RE}(u) = \begin{cases} \lambda\Big(\frac{u}{2|z|\lambda}\ln(\frac{u}{2|z|\lambda}) - \frac{u}{2|z|\lambda} + 1\Big) & \text{if} \quad u \geq 0 \\ \infty & \text{if} \quad u < 0. \end{cases} \tag{8}$$

The positive numbers λ and μ are two parameters of the system. For the associated dynamic programming equation $0 = \min_{(u,v)}\left[\mathbf{A}^{u,v}V + L\right]$ (see e.g. [7] Chapt. III) we get:

$$\begin{aligned} 0 &= \min_{u,v}\Big\{\Big(\partial_t + \frac{\sigma^2}{2}\partial^2_{x,x} + v(t)\partial_x\Big)V(t,x,z) + z\partial_x V(t,x,z) + u(t)[V(t,x,-z) - V(t,x,z)] \\ &\qquad + \frac{\mu^2}{2}v(t)^2 + L_{RE}(u(t))\Big\} \\ &= \Big(\partial_t + \frac{\sigma^2}{2}\partial^2_{x,x} + z\partial_x\Big)V(t,x,z) + \min_v\Big\{\Big(\frac{\mu}{\sqrt{2}}v(t) + \frac{1}{\sqrt{2}\mu}\partial_x V(t,x,z)\Big)^2\Big\} \\ &- \Big(\frac{1}{\sqrt{2}\mu}\partial_x V(t,x,z)\Big)^2 + \min_u\Big\{u(t)[V(t,x,-z) - V(t,x,z)] + L_{RE}(u(t))\Big\} \end{aligned}$$

and the minima, satisfying

$$\Big(\partial_t + \frac{\sigma^2}{2}\partial^2_{x,x} + z\partial_x\Big)V(t,x,z) = \Big(\frac{1}{\sqrt{2}\mu}\partial_x V(t,x,z)\Big)^2 + \lambda\{e^{[V(t,x,z)-V(t,x,-z)]2|z|} - 1\}, \tag{9}$$

is attained for

$$u(t,x,z) = u^*(t,x,z) \equiv 2|z|\lambda \exp\Big(2|z|[V(t,x,z) - V(t,x,-z)]\Big), \tag{10}$$

$$v(t,x,z) = v^*(t,x,z) \equiv -\frac{1}{\mu^2}\partial_x V(t,x,z). \tag{11}$$

According to the LT we define now the functions

$$h(t,x,z) = \exp(-2|z|V(t,x,z)), \quad z \in \{\pm\nu\} \tag{12}$$

which, using eq.(9), solve the system

$$\Big(\partial_t + \frac{\sigma^2}{2}\partial^2_{x,x} + z\partial_x\Big)h(t,x,z) + 2|z|\lambda(h(t,x,-z) - h(t,x,z)) = \Big(\frac{\sigma^2}{2} - \frac{1}{4\mu^2|z|}\Big)\frac{(\partial_x h(t,x,z))^2}{h(t,x,z)}.$$

Comparing this equation with eq.(2), we see that $h(\cdot,\cdot,\pm\nu)$ is in the kernel of the uncontrolled backward operator $\mathbf{A}$ exactly if we choose the parameters λ and μ according to the relations

$$\lambda = \frac{1}{2|z|} \qquad \text{and} \qquad \mu^2 = \frac{1}{2|z|\sigma^2}. \tag{13}$$

We suppose from now on eq.(13) to hold. The cost function takes than the form

$$L(u,v) = \frac{1}{2|z|}\Big(\frac{v^2}{2\sigma^2} + (u\ln(u) - u + 1)\Big) \tag{14}$$

and the optimal controlled dynamics (X^*_s, Z^*_s) read as

$$dX^*_s = \Big(Z^*_s + \sigma^2\partial_x \ln(h(t,x,z))\Big)ds + \sigma dB_s. \tag{15}$$

The controlled two-states process Z^*_s has now inhomogeneous jump rates from z to $-z$, $z \in \{\pm\nu\}$, given by the ratio $u^* = h(t,x,-z)/h(t,x,z)$. The functions $h(t,x,\pm z)$ are solutions to the linear system (3) which have to match the final conditions $h(t_1, x, \pm z) = \exp(\psi(t_1, x, \pm z))$. We note that eq.(3) already appeared in [18]. To find the fundamental solutions to eq.(3) with initial conditions $\phi(0, x, z) = \delta_x \delta_{\nu - z}$ we follow [1]. We apply the time inversion $t \to \tau = t_1 - t$ and rewrite the equation as

$$\det \begin{pmatrix} -(\mathcal{O}^- + 1) & 1 \\ 1 & -(\mathcal{O}^+ + 1) \end{pmatrix} \phi(\tau, x, z) = 0, \tag{16}$$

where the differential operators, $\mathcal{O}^\pm$ defined on $\mathcal{C}_0^{2,1}(\mathbb{R} \times \mathbb{R}^+; \mathbb{R})$, are given by

$$\mathcal{O}^\pm = \partial_\tau \mp \nu \partial_x - \frac{\sigma^2}{2} \partial^2_{x,x} \tag{17}$$

and where "det" denotes the determinant. It is then shown in [1] that a solution to eq.(16) is, as expected by the independence of the two processes Z_t and B_t, given by the space-convolution

$$\phi(\tau, x, z) = \int_{\mathbb{R}} G(y, \tau) T_z(x - y, \tau) dy \tag{18}$$

of the fundamental solutions G and T_z corresponding respectively to the heat equation operator $\partial_\tau - \frac{\sigma^2}{2} \partial^2_{x,x}$ and the two-states random evolution operator $\partial^2_{\tau,\tau} + 2\partial_\tau - \nu^2 \partial^2_{x,x}$ with initial state z. This solves the stochastic optimal control problem expressed in eq.(5).

3. A solvable nonlinear Reaction-Diffusion Model

In this section we present a coupled set of nonlinear, exactly solvable 1 dimensional reaction-diffusion equations and connect it to the above explicitly solved optimal control problem. The reaction-diffusion equations studied in [12] read as:

$$(\partial_\tau \pm \nu \partial_x) f_\pm = \frac{\sigma^2}{2} (\partial^2_{x,x} f_\pm) \pm B(f_+, f_-) \pm K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-), \quad (\tau, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{19}$$

where the quadratic Boltzmann-like collision operator B is given by

$$B(f_+, f_-) = f_+ f_- - f_+ + f_-$$

and where the operator K_σ is defined as:

$$\begin{aligned} K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-) = {} & \frac{\sigma^2}{2\nu} \Big[\frac{1}{2\nu} S^2 - S D_x + \frac{1}{4\nu} D S^2 \Big] + \\ & \frac{\sigma^4}{4\nu^2} \Big[- \frac{1}{16\nu^2} S^4 - \frac{1}{2} (S_x)^2 + \frac{1}{2\nu} S^2 S_x \Big]. \end{aligned} \tag{20}$$

Here we used the short hands $S := f_+(x, \tau) + f_-(x, \tau)$ and $D := f_+(x, \tau) - f_-(x, \tau)$, where D_x (resp. S_x) stands for the space derivative of the quantity D (resp. S). We interpret f_- (resp. f_+) as the distribution of particles moving on the line to the left (the $-$ particles) and right (the $+$ particles) both with absolute speed $\nu > 0$.

The eqs. (19) contain, for $\sigma^2 = 0$, the Boltzmann-like equations of Th. W. Ruijgrok and T.T. Wu, the RW-model, [22]:

$$(\partial_\tau \pm \nu \partial_x) \rho_\pm = \pm B(\rho_+, \rho_-), \quad \rho_\pm = \rho_\pm(x, \tau). \tag{21}$$

This two-velocities Boltzmann-like model generalizes the viscous Burgers equation:

$$\partial_\tau \rho(x,\tau) - \frac{\nu}{2}\rho(x,\tau)\partial_x\rho(x,\tau) = \frac{\nu^2}{2}\partial^2_{x,x}\rho(x,\tau), \tag{22}$$

which indeed can be obtained from the RW-model by performing a diffusive limit [13]. Note that the explicit nature of this scaling limit is very appealing to investigate numerical schemes and has recently regained some attention [17, 9]. For the fundamental role played by the Burgers equation in shock wave analysis and fluid dynamics we refer to [15] and [2]. The physical content of the RW-model in the domain of thermal fusion is given in [22], its relevance for the theory of weak solutions to conservation laws resp. for the theory of car traffic modeling is mentioned in [9] resp. [11].

The solutions to the generalized RW-model (19) are, for a large class of initial conditions, explicitly given in terms of two linear differential operators $\mathcal{O}^+$ and $\mathcal{O}^-$ defined in eq.(17). They act on the logarithm of the convolution product of the densities associated to the Gaussian measure and the two state random evolution measure [12]. We indeed have that for sufficiently regular initial conditions, the two functions

$$f_\pm(x,\tau) = \pm\mathcal{O}^\pm \ln\left(\phi(\tau,x,\pm\nu)\right) \tag{23}$$

solve the system (19) with ϕ defined in eq.(18). The connection with the stochastic control problem culminates therefore in the following

Proposition. *The dynamic programming equations (9) are equivalent to the reaction-diffusion type of equations given in (19).*

The equivalence is understood in the sense that sufficiently regular solutions to (9) (typically $\mathcal{C}^{4,1}$) are, upon a time inversion $t \to t_1 - t$, solutions to (19) and vice versa. This establishes the relation exposed in table (2).

4. Interpretation of the cost function L

Let us now informally discuss our choice for the cost function

$$L = \frac{1}{2|z|}\left(\frac{v^2}{2\sigma^2} + u\ln(u) - u + 1\right). \tag{24}$$

We focus in a first step on a small noise diffusion without random evolution i.e. we consider the controlled dynamics

$$dX^n_s = v(s, X^n_s)ds + \frac{1}{\sqrt{n}}\sigma dB_s, \tag{25}$$

with small noise parameter $1/\sqrt{n}$, control v and Lagrangian $L_D(v) = \frac{v^2}{2\sigma^2}$. If one thinks of v as a velocity, then L_D is just the classical action integrand of a particle of mass $1/\sigma^2$ (see also [7], Chapt. III Ex. 8.2). It reminds also the integrand of the Onsager-Machlup (OM) functional in case we interpret $v \equiv \dot{\phi}$ as the time derivative of a smooth curve ϕ because the probability for the uncontrolled X of moving close along ϕ for asymptotically large n is $\propto \exp(-\int_0^{t_1} L_D(v)dt)$. This classical large deviations result reads more precisely [8]:

$$\lim_{\delta\searrow 0}\lim_{n\to\infty}\frac{1}{n}\ln P(|X^n_t - \phi(t)| < \delta, \text{ for all } t\in[0,t_1]) = -\int_0^{t_1} L_D(\dot{\phi}(s))ds. \tag{26}$$

The quadratic OM-functional L_D associated to eq.(25) is formally obtained via the Fenchel-Legendre transform of the function H defined on $\mathbb{R} \times \mathbb{R}$ by (see e.g., [16]):

$$H(x,\xi) = \lim_{t \searrow 0} t^{-1} \ln \Big(E_x[\exp(\xi(X_t - x))] \Big) \tag{27}$$

where E_x denotes the conditional expectation with respect to the uncontrolled process with $X_0 = x$. We have $H(x,\xi) = \xi^2\sigma^2/2$ and the Fenchel-Legendre transform of $H(x,\xi)$ is:

$$\sup_{\xi \in \mathbb{R}} \Big\{ \xi u - H(x,\xi) \Big\} = \frac{v^2}{2\sigma^2} = L_D(v). \tag{28}$$

The OM function has an entropy-like nature and is related to the entropy production density of the most probable path *i.e.*, the trajectory ϕ which maximizes the asymptotic estimate in eq.(26). It is for example shown in [3] that the two diffusion processes given in (25) with $v \equiv 0$ and $v = v^* \equiv \sigma^2 \partial_x \ln(h(t,x))$ – where h is in the kernel of the evolution operator associated to the process (25)– do have the same extremal trajectories. In this sense, the optimal control ($v = v^*$) interferes as less as possible with the uncontrolled trajectories ($v = 0$). It is this spare and clever interaction with the uncontrolled trajectory which realizes the minimization of the costs. Note that this connection between the variational principle for controlling diffusions and the Onsager-Machlup principle is explicitly studied in [19] (see also [4] and [5] for associated results in large deviations theory). The case with jump diffusions has been treated in [21].
Let us now exploit this connection in the case of the two-state noise Z_s with switching rates λ. Similar to eq.(27) we define

$$H_Z(z,\xi) = \lim_{t \searrow 0} t^{-1} \ln \Big(E_z[\exp(\xi(Z_t - z))] \Big), \quad z \in \{\pm\nu\}, \quad \xi \in \mathbb{R}. \tag{29}$$

Straightforward calculation of the conditional expectation E_z, using the measure of Z_t with $Z_0 = z$ (see e.g. [20] Prop. 0.1.), yield $H_Z(z,\xi) = \lambda(e^{-2z\xi} - 1)$. The Fenchel-Legendre transform of $H_Z(z,\xi)$ for $u \in \mathbb{R}_+$ is:

$$\sup_{\xi \in \mathbb{R}} \Big\{ \xi u - H_Z(z,\xi) \Big\} = \lambda \Big(\frac{u}{2|z|\lambda} \ln \Big(\frac{u}{2|z|\lambda} \Big) - \frac{u}{2|z|\lambda} + 1 \Big) = L_{RE}(u). \tag{30}$$

Hence our cost functional $L_D + L_{RE}$ appears to be the natural candidate for the local rate function of the large deviations principle associated to the small noise stochastic dynamics defined by eq.(4).

5. Concluding remark

The technique of logarithmic transformations is well suited to construct exactly solvable non-linear field equations. The exactly solvable examples include the Burgers equation, the Boltzmann-like model of Rujgrok and Wu and its natural diffusive extension presented in this paper. Starting from Markovian dynamics, one gets via the logarithmic transformation a stochastic optimal control problem whose value function is directly related to the non-linear field equations. The cost structure associated to the control problem is related to the large deviations probabilities of the controlled dynamics.

Besides its physical relevance, the presented non-linear field equations are, thanks to the explicit nature of the solutions, appealing for numerical studies.

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[1] Blanchard P. and Hongler M.-O. Probabilistic solution of high order partial differential equations. *Phys. Lett. A*, 180:225, 1993.

[2] Burgers J. *A mathematical model illustrating the theory of turbulence.* Academic Press, New York, 1948.

[3] Dai Pra P. A stochastic control approach to reciprocal diffusion processes. *Appl. Math Optim*, 23:313–329, 1991.

[4] Dupius P. and Ellis R.S. *A weak Convergence Approach to the Theory of Large Deviations.* Wiley, Series in Prob. and Stat., 1997.

[5] Dupuis P. and Ellis R.S. The Large Deviation Principle for a general class of queueing systems I. *Trans. of AMS*, 347 (8):2689–2751, 1995.

[6] Fleming W.H. Exit probabilities and optimal stochastic control. *Appl. Math. Optim.*, 4:329–346, 1978.

[7] Fleming W.H. and Soner H.M. *Controlled Markov Processes and Viscosity Solutions.* Springer, 1993.

[8] Freidlin M. I. and Wentzell A.D. *Random Perturbations of Dynamical Systems. Gundleheren der mathematischen Wissenschaften 260.* Springer-Verlag, New-York, 1984.

[9] Gabetta E. and Perthame B. Scaling limits for the Ruijgrok-Wu model of the Boltzmann equation. *Mathematical Methods in the Applied Sciences*, 24:949–967, 2001.

[10] Holland C.J. A new energy characterization of the smallest eigenvalue of the Schrödinger equation. *Commun. Pure Appl. Math.*, 30:201–230, 1950.

[11] Hongler M.-O. and Filliger R. Mesoscopic derivation of a fundamental diagram of one-lane traffic. *Physics Letters A*, 301:597–602, 2002.

[12] Hongler M.-O. and Lima R. A solvable nonlinear reaction-diffusion model. *Physics Letters A*, 198:100–104, 1995.

[13] Hongler M.-O. and Streit L. A probabilistic Connection between the Burgers and a Discrete Boltzmann equation. *Europhysics Letters*, 12(3):193–197, 1990.

[14] Hongler M.O., Soner H.M. and L. Streit. Stochastic Control for a class of Random Evolution Models. *Applied mathematics and Optimization*, 49:113–121, 2004.

[15] Hopf E. The part. diff. equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3:201–230, 1950.

[16] Ito H. Derivation of the Onsager-Machlup function by a minimisation of the Kullback-Leibler entropy. *J. Phys. A: Math. Gen.*, 14:385–388, 1981.

[17] Jin S.,Pareschi L.,Toscani G. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM J. Numerical Analysis*, 35(6):2405–2439, 1998.

[18] Masoliver J., Lindenberg K. and Weiss G.H. A continuous-time generalization of the persistent random walk. *Physica A*, 157(2):891–898, 1989.

[19] Pavon M. Stochastic control and nonequilibrium thermodynamical systems. *Applied Mathematics and Optimization*, 19(1):187–202, 1989.

[20] Pinsky M.A. *Lectures on random evolution.* World Scientific, Singapore, 1991.

[21] Privault N. and Zambrini J.C. Markovian bridges and reversible diffusion processes with jumps. *Ann. I. H. Poincar*, PR 40:599–633, 2004.

[22] Ruijgrok Th.W. and Wu T.T. A completely solvable model of the nonlinear Boltzmann equation. *Physica*, 113 A:401–416, 1982.