

LOW ORDER DISCONTINUOUS GALERKIN METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. We consider DG-methods for 2nd order scalar elliptic problems using piecewise affine approximation in two or three space dimensions. We prove that both the symmetric and the non-symmetric version of the DG-method are well-posed also without penalization of the interelement solution jumps provided boundary conditions are imposed weakly. Optimal convergence is proved for sufficiently regular meshes and data. We then propose a discontinuous Galerkin method using piecewise affine functions enriched with quadratic bubbles. Using this space we prove optimal convergence in the energy norm for both a symmetric and non-symmetric DG-method without stabilization. All these proposed methods share the feature that they conserve mass locally independent of the penalty parameter.

Key words. Discontinuous Galerkin; elliptic equation; Crouzeix-Raviart approximation; interior penalty; local mass conservation.

AMS subject classifications. 65M160, 65M15

1. Introduction. The discontinuous Galerkin method (DG) for $(2n)$ th-order elliptic problems was introduced and analysed by Baker [2] with special focus on the fourth order case. In this work both the jumps of the solution and its gradient were penalized. Douglas and Dupont however announced already in [10] a forthcoming analysis for discontinuous approximations and penalty on the solution jumps. This analysis was then realized in the work of Wheeler [22], in the framework of collocation methods for discontinuous Galerkin and Arnold [1] in the framework of parabolic problems leading to the symmetric interior penalty DG-method (SIPG).

During the nineties there was a strong development of discontinuous Galerkin methods for elliptic problems. Bassi and Rebay proposed a formulation for Navier-Stokes equations [3], introducing an auxiliary variable for the diffusive fluxes, this method was then analysed by Cockburn and co workers in [8, 7] and more recently in the framework of Friedrichs systems by Ern and Guermond [12].

Babuska, Baumann and Oden proposed a non-symmetric method for elliptic problems with a less stiff penalization term [17] (NIPG). This method was modified by Oden, Prudhomme and Romkes in [19] where they proposed a DG-method for reaction-diffusion equations with penalization of the diffusive fluxes to enhance stability for the non-symmetric version without perturbing the local massconservation properties. The DG-methods for second order elliptic problems have been further analysed in the works by Girault, Rivière and Wheeler [18] and Larson and Niklasson [15]. In the last reference the authors proved that in the non-symmetric case when using high order polynomial approximation the stabilization term may be dropped since control of the solution jumps can be recovered from the anti-symmetric part of the diffusion-operator using an inf-sup argument.

For a review of discontinuous Galerkin methods for elliptic problems we refer to Arnold et al. [1] and for a review of stabilization mechanisms in discontinuous Galerkin methods we refer to Brezzi et al. [5].

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One of the advantages of the discontinuous Galerkin method is that it has enhanced local conservation compared to the continuous Galerkin method. On the other hand, as was pointed out by Larson and Niklasson in [13] this local conservation is not independent of the penalization operator. Only in the case of the non-symmetric DG method without penalization of the solution jumps or with penalization on the fluxes is it possible to obtain local mass conservation independently of the penalization parameter. In both cases however the formulation is not adjoint consistent and one may not prove L^2 -convergence using the Nitsche trick.

The question of how much the discontinuous Galerkin method needs to be stabilized was discussed in the case of mixed formulations of elliptic problems by Sherwin et al. in [20] and by Marazzina in [16]. It was found that stabilization needs to be applied on the boundary of the domain only to assure well-posedness of the discrete system.

In this paper we discuss the relation between stabilization, well posedness and optimal convergence in the case of scalar second order elliptic problems. The aim is to design a low order DG-method that

1. has optimal convergence in the H^1 - and L^2 -norm,
2. is locally massconservative independently of the penalty parameter.

We will show that for the symmetric DG-method the only thing required to obtain well-posedness of the discrete system is weakly imposed boundary conditions. No interelement penalization of the solution jumps is required. In this case of course local mass conservation is obtained independently of the penalty parameter. Optimal convergence is also obtained only imposing boundary conditions, however under strong assumptions on the computational mesh and data. Either the mesh has to satisfy as certain macro element property, or mesh and data have to be sufficiently regular. If these conditions are not satisfied the convergence of the solution jumps can be perturbed by the appearance of a checkerboard mode that vanishes too slowly in the absence of interior penalty. We exemplify the checkerboard mode numerically and show how it is quenched by penalization.

To reduce the constraints on the mesh we enrich the space with quadratic non-conforming bubble functions. These stabilizing bubbles eliminate the checkerboard mode. This allows for optimal convergence in the energy norm without stabilization in the symmetric and non-symmetric case. In the symmetric case we obtain additionally optimal convergence in the L^2 -norm. In both cases the analysis relies on a discrete inf-sup condition drawing from earlier ideas on minimal stabilization for DG-methods in [6] and [15]. The analysis is also inspired by Larson and Niklasson [14] and extends some of their results to two and three space dimensions for first order polynomial approximations.

2. The problem setting. Let Ω be a convex polygon (polyhedron in three space dimensions) in \mathbb{R}^d , $d = 2, 3$, with outer normal n . Let \mathcal{K} be a subdivision of $\Omega \subset \mathbb{R}^d$ into non-overlapping d -simplices κ . We consider the following elliptic problem with homogeneous Dirichlet boundary conditions:

Find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot \sigma \nabla u &= f, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{cases} \quad (2.1)$$

where $f \in L^2(\Omega)$ and with diffusion coefficient $\sigma(x)|_{\kappa} = \sigma_{\kappa} \in \mathbb{R}$ and $\sigma(x) > \sigma_0 > 0$. We assume that there exists a constant $\rho > 0$ such that $\sigma|_{\kappa_1} \leq \rho \sigma|_{\kappa_2}$ for any two

elements satisfying $\partial\kappa_1 \cap \partial\kappa_2 \neq \emptyset$. The fact that the boundary conditions are of homogeneous type is not a limitation of the presented methods but rather to avoid technical details.

Let \mathcal{F}_i denote the set of interior faces $((d-1)$ -manifolds) of the mesh, i.e. the set of faces that are not included in the boundary $\partial\Omega$. The set \mathcal{F}_e denotes the faces that are included in $\partial\Omega$ and define $\mathcal{F} = \mathcal{F}_i \cup \mathcal{F}_e$. Note that for a subset $\mathcal{K}_0 \subset \mathcal{K}$ we use the notation $\mathcal{F}_i(\mathcal{K}_0)$, $\mathcal{F}_e(\mathcal{K}_0)$ to indicate the interior resp. exterior (with respect to \mathcal{K}_0) faces of the set \mathcal{K}_0 .

Assume that \mathcal{K} is shape-regular, does not contain any hanging node and covers $\overline{\Omega}$ exactly. Suppose that each $\kappa \in \mathcal{K}$ is an affine image of the reference element $\hat{\kappa}$, i.e. for each element κ there exists an affine transformation $T_\kappa : \hat{\kappa} \rightarrow \kappa$. For an element $\kappa \in \mathcal{K}$, h_κ denotes its diameter. Set $h = \max_{\kappa \in \mathcal{K}} h_\kappa$ and let \tilde{h} , \tilde{m} be the functions such that $\tilde{h}|_\kappa = h_\kappa$ resp. $\tilde{m}|_\kappa = \text{meas}(\kappa)$. We will say that a family of subdivisions $\{\mathcal{K}\}_h$ is asymptotically uniform with some $\zeta > 0$ if there exists a constant $c > 0$ such that for every $F \in \mathcal{F}_i$ there holds $|\tilde{m}|_{\kappa_1} - \tilde{m}|_{\kappa_2}| \leq ch_\kappa^\zeta \tilde{m}|_{\kappa_1}$ where $F = \partial\kappa_1 \cap \partial\kappa_2$ with $\kappa_1, \kappa_2 \in \{\mathcal{K}\}_h$.

For a face $F \in \mathcal{F}$, h_F denotes its diameter and let \tilde{h}_F be the function such that $\tilde{h}_F|_F = h_F$.

For a subset $R \subset \Omega$ or $R \subset \mathcal{F}$, $(\cdot, \cdot)_R$ denotes the $L^2(R)$ -scalar product, $\|\cdot\|_R = (\cdot, \cdot)_R^{1/2}$ the corresponding norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$ -norm. The element-wise counterparts will be distinguished using the discrete partition as subscript, for example $(\cdot, \cdot)_\mathcal{K} = \sum_{\kappa \in \mathcal{K}} (\cdot, \cdot)_\kappa$. For $s \geq 1$, let $H^s(\mathcal{K})$ be the space of piecewise Sobolev H^s -functions and denote its norm by $\|\cdot\|_{s,\mathcal{K}}$.

For $v \in H^1(\mathcal{K})$, $\tau \in [H^1(\mathcal{K})]^d$ and an interior face $F = \kappa_1 \cap \kappa_2 \in \mathcal{F}_i$, where κ_1 and κ_2 are two distinct elements of \mathcal{K} with respective outer normals n_1 and n_2 , define the jump and average by

$$\begin{aligned} [v] &= (v|_{\kappa_1} n_1 + v|_{\kappa_2} n_2), & \{v\} &= \frac{1}{2} (v|_{\kappa_1} + v|_{\kappa_2}), \\ [\tau] &= (\tau|_{\kappa_1} \cdot n_1 + \tau|_{\kappa_2} \cdot n_2), & \{\tau\} &= \frac{1}{2} (\tau|_{\kappa_1} + \tau|_{\kappa_2}). \end{aligned}$$

Additionally we define on each face $F \in \mathcal{F}$ the normal n_F in an arbitrary but fixed manner.

On outer faces $F = \partial\kappa \cap \partial\Omega \in \mathcal{F}_e$, for some $\kappa \in \mathcal{K}$ with outer normal n , the jump and the average are defined as $[v] = v|_F n$ and $\{v\} = v|_F$ resp. $[\tau] = \tau|_F \cdot n$ and $\{\tau\} = \tau|_F$.

The shape-regularity implies that there exists a constant $c > 0$ independent of the mesh size h such that on any face $F \in \mathcal{F}$

$$h_F \leq \{\tilde{h}\} \leq ch_F.$$

In this paper $c > 0$ denotes a generic constant and can change at each occurrence, while an indexed constant stays fix. Any constant is independent of the mesh size h . We also introduce the average of the jump over a given face. Let $v \in H^1(\mathcal{K})$, then $\overline{[v]} \in \mathbb{R}^d$ is defined by

$$\int_F \overline{[v]} ds = \int_F [v] ds \quad \forall F \in \mathcal{F}.$$

3. Stabilization mechanisms for finite element methods. In this section we will recall some notions of the framework for stabilization mechanisms proposed in [5]. We first present the standard SIPG method and then we will introduce the

modified methods that will be discussed in this paper. To this aim we introduce the broken space

$$H_\sigma^1(\mathcal{K}) := \{v|_\kappa \in H^1(\kappa) \text{ such that } \nabla \cdot (\sigma \nabla v)|_\kappa \in L^2(\kappa), \forall \kappa \in \mathcal{K}\}.$$

A function $u \in H_\sigma^1(\mathcal{K})$ will coincide with a solution of (2.1) if and only if

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= f \text{ in each } \kappa \in \mathcal{K}, \\ [u] &= 0 \text{ on each } F \in \mathcal{F}, \\ [\sigma \nabla u] &= 0 \text{ on each } F \in \mathcal{F}_i. \end{aligned}$$

Following [5] we introduce a method in $H_\sigma^1(\mathcal{K})$ where these residuals are imposed to be zero weakly: Find $u \in H_\sigma^1(\mathcal{K})$ such that

$$(\nabla \cdot (\sigma \nabla u) - f, B_0 v)_\mathcal{K} + ([u], \mathbf{B}_1 v)_\mathcal{F} + ([\sigma \nabla u], B_2 v)_{\mathcal{F}_i} = 0, \quad \forall v \in H_\sigma^1(\mathcal{K}) \quad (3.1)$$

where the three operators B_0 , \mathbf{B}_1 and B_2 remain to be chosen and will define our method. The classical case of SIPG using penalization of the solution jumps is obtained by the choice

$$\begin{aligned} B_0 v &= v \\ \mathbf{B}_1 v &= -\{\sigma \nabla v\} + \gamma[v] \\ B_2 v &= \{v\}. \end{aligned}$$

Inserting this in (3.1) and integrating by parts leads to the well known formulation: Find $u \in H_\sigma^1(\mathcal{K})$ such that

$$(\sigma \nabla u, \nabla v)_\mathcal{K} - (\{\sigma \nabla u\}, [v])_\mathcal{F} - (\{\sigma \nabla v\}, [u])_\mathcal{F} + (\gamma[u], [v])_\mathcal{F} = (f, v)_\mathcal{K}, \quad (3.2)$$

for all $v \in H_\sigma^1(\mathcal{K})$. This formulation is symmetric and continuity is imposed by a penalization of the solution jump over element faces. The uniqueness of solutions to (3.2) is assured by Theorem 1 of [5] and the solution coincides with that of (2.1). In this paper we will investigate the possibility of choosing $\gamma = 0$ on interior faces in (3.2). We show that the resulting discrete system always is well posed but that convergence depends on the regularity of the mesh and data. To relax the constraints on the data we then propose to work in a bubble enriched space.

For the modified methods proposed in this paper there is no coercivity and stability must be recovered using an *inf-sup* argument. This may seem unnatural for a method approximating an elliptic problem, but shows that when relaxing the stabilization the domain decomposition aspect of the DG-method is accentuated.

4. Finite Element Spaces. We will consider two low order finite element spaces in the present work, the space of piecewise affine discontinuous functions and the space of piecewise affine discontinuous functions enriched with nonconforming quadratic bubbles. We show that every function in the former space can be written as a sum of a midpoint continuous function (in the Crouzeix-Raviart space) and a “midpoint discontinuous” function in a space that will be specified later. The motivation for this decomposition is that the symmetric DG-bilinear form of (3.2) results in a block diagonal matrix for this choice of basis and hence the continuous and the discontinuous contributions may be analysed separately.

Define the piecewise affine discontinuous finite element space by

$$V_h^1 = \{v_h \in L^2(\Omega) : v_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{K}\}.$$

and the enriched space by

$$V_h^b = V_h^1 \oplus V^b,$$

with

$$V^b = \{v \in L^2(\Omega) : v(x)|_\kappa = \alpha_\kappa x \cdot x ; \text{ where } \alpha_\kappa \in \mathbb{R}\},$$

where $x = (x_1, \dots, x_d)$ denotes the physical variables. Additionally we define

$$V_{h,0}^1 = \left\{ v_h \in V_h^1 : \int_F v_h ds = 0, \forall F \in \mathcal{F}_e \right\},$$

the space of piecewise affine elements where the homogeneous Dirichlet boundary conditions are imposed on the midpoints of each exterior face.

4.1. Splitting of the finite element space $V_{h,0}^1$. The idea is to split $V_{h,0}^1$ into a midpoint continuous space, the Crouzeix-Raviart space, and a midpoint discontinuous space. Recall the definition of the Crouzeix-Raviart space

$$V^C = \left\{ v_h \in V_h^1 : \int_F [v_h] ds = 0, \forall F \in \mathcal{F}_i; \int_F v_h ds = 0, \forall F \in \mathcal{F}_e \right\}.$$

Its "midpoint discontinuous" counterpart is defined by

$$V^D = \left\{ v_h \in V_h^1 : \int_F \{v_h\} ds = 0, \forall F \in \mathcal{F} \right\}.$$

Denote by N_i the number of interior faces of the mesh \mathcal{K} . Let us denote $\{\phi_i^c\}_{i=1}^{N_i}$ the Crouzeix-Raviart basis defined such that

$$\int_{F_j} \{\phi_i^c\} ds = \delta_{i,j} \text{meas}(F_i), \quad \forall F_j \in \mathcal{F}.$$

This builds a basis for the space V^C . Now let us define a basis for the space V^D . For each face F consider the basis function ϕ_F^d defined as

$$(\phi_F^d)|_\kappa = \frac{1}{2}(\phi_F^c)|_\kappa \frac{(\nabla \phi_F^c)|_\kappa \cdot n_F}{|(\nabla \phi_F^c)|_\kappa \cdot n_F|}$$

with n_F a fixed but arbitrary normal associated to face F . It is easy to verify that $\phi_i^d \in V^D$ and that

$$\int_{F_j} [\phi_i^d] \cdot n_{F_j} ds = \delta_{i,j} \text{meas}(F_i), \quad \forall F_i \in \mathcal{F}.$$

Now we are ready to prove the following lemma.

LEMMA 4.1. *The splitting of $V_{h,0}^1$ into V^C and V^D is a direct sum, i.e. $V_{h,0}^1 = V^C \oplus V^D$. In addition any function v_h in $V_{h,0}^1$ can be written as*

$$v_h(x) = \sum_{i=1}^{N_i} c_i \phi_i^c(x) + \sum_{i=1}^{N_i} d_i \phi_i^d(x)$$

where $c_i = \frac{1}{\text{meas}(F_i)} \int_{F_i} \{v_h\} ds$ and $d_i = \frac{1}{\text{meas}(F_i)} \int_{F_i} [v_h] \cdot n_{F_i} ds$.

REMARK 4.2. Note that for $v \in H^1(\Omega)$ the function

$$i_c v(x) = \sum_{i=1}^{N_i} c_i \phi_i^c(x) \quad (4.1)$$

with $c_i = \frac{1}{\text{meas}(F_i)} \int_{F_i} v \, ds$ is the Crouzeix-Raviart interpolant and has optimal approximation properties.

Proof. First assume that $v_h \in V^C \cap V^D$ and show that $v_h \equiv 0$. Denote by x_F the midpoint associated to the face $F \in \mathcal{F}$. Take an arbitrary exterior face F and since $v_h \in V^D$

$$0 = \int_F \{v_h\} \, ds = \int_F v_h \, ds = \text{meas}(F) v_h(x_F), \quad (4.2)$$

using the midpoint integration rule which is exact for polynomials of order one. On the other hand for an interior face $F \in \mathcal{F}_i$ such that $F = \partial\kappa_1 \cap \partial\kappa_2$ and since $v_h \in V^D$, we can write

$$0 = \int_F \{v_h\} \, ds = \frac{\text{meas}(F)}{2} (v_h|_{\kappa_1}(x_F) + v_h|_{\kappa_2}(x_F)),$$

and since $v_h \in V^C$

$$0 = \int_F [v_h] \cdot n_F \, ds = \text{meas}(F) (v_h|_{\kappa_1}(x_F)n_1 + v_h|_{\kappa_2}(x_F)n_2) \cdot n_F.$$

Independent of the choice $n_F = n_1$ or $n_F = n_2$ we get

$$v_h|_{\kappa_1}(x_F) = v_h|_{\kappa_2}(x_F) = 0 \quad \forall F \in \mathcal{F}_i. \quad (4.3)$$

Considering an arbitrary element $\kappa \in \mathcal{K}$ and respecting (4.2) and (4.3), observe that v_h vanishes in the $d+1$ midpoints of the adjacent faces. Thus the function $v_h \in V_{h,0}^1$ must be identically zero on all elements.

We show that $V^C \oplus V^D$ covers $V_{h,0}^1$ entirely. Assume that $v_h \in V_{h,0}^1$ and define $u_h \in V^C \oplus V^D$ by

$$u_h(x) = \sum_{i=1}^{N_i} (c_i \phi_i^c(x) + d_i \phi_i^d(x))$$

with $c_i = \frac{1}{\text{meas}(F_i)} \int_{F_i} \{v_h\} \, ds$ and $d_i = \frac{1}{\text{meas}(F_i)} \int_{F_i} [v_h] \, ds$. Let us prove that $u_h \equiv v_h$. Integrating u_h over an exterior face $F \in \mathcal{F}_e$ yields

$$\int_F u_h \, ds = 0 = \int_F v_h \, ds$$

Using again the midpoint integration rule implies

$$u_h(x_F) = v_h(x_F).$$

On interior faces $F \in \mathcal{F}_i$ observe that

$$\begin{aligned} \int_F \{u_h\} \, ds &= c_F \int_F \{\phi_F^c(x)\} \, ds = \text{meas}(F) c_F = \int_F \{v_h\} \, ds, \\ \int_F [u_h] \cdot n_F \, ds &= d_F \int_F [\phi_F^d(x)] \cdot n_F \, ds = \text{meas}(F) d_F = \int_F [v_h] \cdot n_F \, ds, \end{aligned}$$

and thus

$$u_h|_{\kappa_1}(x_F) = v_h|_{\kappa_1}(x_F) \quad \text{and} \quad u_h|_{\kappa_2}(x_F) = v_h|_{\kappa_2}(x_F).$$

Now since $v_h, u_h \in V_{h,0}^1$ and since these two functions coincide at $d+1$ points per element they are equal. By consequence every function v_h in $V_{h,0}^1$ can be written as $v_h = v_c + v_d$ with $v_c \in V^C$ and $v_d \in V^D$. \square

LEMMA 4.3 (Asymptotic L^2 -orthogonality between V^C and V^D). *Assume that the mesh is asymptotically uniform with some $\zeta > 0$, then the spaces V^C and V^D satisfy the following weak L^2 -orthogonality property: there exists a constant $c > 0$ independent of h , such that*

$$|(v_c, v_d)_\Omega| \leq ch^\zeta \|v_c\|_\Omega \|v_d\|_\Omega + c \|\tilde{h} \nabla v_c\|_\Omega^* \|\tilde{h} \nabla v_d\|_\Omega^* \quad \forall v_c \in V^C, v_d \in V^D,$$

where the norms marked by \star appear only in the case of $d = 3$.

Proof. In the 2-dimensional case we can proceed as follows. Take $v_c \in V^C$ and $v_d \in V^D$ and develop

$$|(v_c, v_d)_\mathcal{K}| = \left| \sum_{\kappa \in \mathcal{K}} (v_c, v_d)_\kappa \right| = \frac{1}{d+1} \left| \sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \tilde{m} v_c(x_F) v_d(x_F) \right| \quad (4.4)$$

using the midpoint integration rule which integrates over the volume κ in function of the integrand evaluated in the midpoints of each face. This integration rule is exact of order 2 in the 2-dimensional case but not in the 3-dimensional one. The points x_F denote the midpoints of the face F . Since $v_d(x_F) = 0$ for exterior faces $F \in \mathcal{F}_e$ and since v_c is midpoint continuous we can rearrange the sum

$$|(v_c, v_d)_\mathcal{K}| = \frac{2}{d+1} \left| \sum_{F \in \mathcal{F}_i} v_c(x_F) \{\tilde{m} v_d\}(x_F) \right|.$$

Since $\{uv\} = \{u\}\{v\} + \frac{1}{4}[u] \cdot [v]$ and $\{v_d\}(x_F) = 0$ for all interior faces

$$|(v_c, v_d)_\mathcal{K}| \leq \frac{2}{d+1} \sum_{F \in \mathcal{F}_i} |v_c(x_F)| |[\tilde{m}](x_F) \cdot n_F| |[v_d](x_F) \cdot n_F|.$$

The regularity assumption on the mesh implies that $|[\tilde{m}] \cdot n_F| \leq ch^\zeta \{\tilde{m}\}$ with $\zeta > 0$ and since $\frac{1}{2}|[u_d](x_F) \cdot n_F| = |u_d|_{\kappa_i}|$, for $i = 1, 2$, we can rearrange the sum again

$$|(v_c, v_d)_\mathcal{K}| \leq ch^\zeta \sum_{\kappa \in \mathcal{K}} (|v_c|, |v_d|)_\kappa \leq ch^\zeta \|v_c\|_\Omega \|v_d\|_\Omega.$$

In the 3-dimensional case, since the midpoint integration rule is not anymore of order two, we introduce the local midpoint interpolation i_h^κ for each element. Then using the triangle inequality yields

$$|(v_c, v_d)_\mathcal{K}| \leq \left| \sum_{\kappa \in \mathcal{K}} \int_\kappa i_h^\kappa(v_c v_d) dx \right| + \left| \sum_{\kappa \in \mathcal{K}} \int_\kappa (v_c v_d - i_h^\kappa(v_c v_d)) dx \right|. \quad (4.5)$$

The first term of the right hand side of (4.5) can be developed as in (4.4) since now for the midpoint interpolation i_h^κ the midpoint integration rule is exact. For the second term of the right hand side of (4.5) one can show using standard interpolation results that

$$\left| \sum_{\kappa \in \mathcal{K}} \int_\kappa (v_c v_d - i_h^\kappa(v_c v_d)) dx \right| \leq c \|\tilde{h} \nabla v_c\|_\mathcal{K} \|\tilde{h} \nabla v_d\|_\mathcal{K}.$$

\square

4.2. Properties of the enriched space V_h^b . The motivation for the particular form of the enriched space is given in the following Lemma. The key idea is that the gradient of a function in V_h^b restricted to an element is in fact locally in the Raviart-Thomas space and equivalently the curl of a function in $[V_h^b]^d$ is locally in the lowest order Nedelec space. Here we will only exploit the former property. Let RT_0 denote the space of Raviart-Thomas elements of order zero.

LEMMA 4.4. *For all $w_h \in V_h^b$ there holds*

$$\nabla w_h|_\kappa \in RT_0(\kappa),$$

and for all $r_h \in RT_0(\kappa)$ there exists $w_h \in V_h^b$ such that $\nabla w_h|_\kappa = r_h$, for all $\kappa \in \mathcal{K}$.

Proof. Let $w_h \in V_h^b$, restricting w_h to an arbitrary element κ we can write

$$w_h|_\kappa(x) = \alpha x \cdot x + \beta \cdot x + \gamma$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$ are the local degrees of freedom. Then

$$\nabla w_h|_\kappa(x) = 2\alpha x + \beta.$$

To show that this function lies in the Raviart-Thomas finite element space we have to map it on the reference element using the *Piola* transformation. But let us first introduce the *affine* transformation T_κ between the reference element $\hat{\kappa}$ defined by its vertices $a_i = e_i$ for $i = 1, \dots, d$ and $a_{d+1} = \mathcal{O}$ and the physical element κ . The vectors e_i denote the unit vectors corresponding to the i -th coordinate. The affine transformation may be written as

$$T_\kappa(\hat{x}) = J_\kappa \hat{x} + t_\kappa$$

where $\hat{x} = (\hat{x}_1, \dots, \hat{x}_d)^\top \in \hat{\kappa}$ denotes the variable in the reference element. Then we denote by ψ_κ the Piola transformation between the physical element and the reference element defined by

$$\psi_\kappa(v)(\hat{x}) = |J_\kappa| J_\kappa^{-1} v(T_\kappa(\hat{x})).$$

Thus

$$\psi_\kappa(\nabla w_h|_\kappa)(\hat{x}) = |J_\kappa| J_\kappa^{-1} (2\alpha T_\kappa(\hat{x}) + \beta) = |J_\kappa| (2\alpha \hat{x} + J_\kappa^{-1}(\beta + 2\alpha t_\kappa))$$

and this function is clearly an element of the Raviart-Thomas finite element space on the reference element.

On the other hand if $r_h \in RT_0$ then $\psi_\kappa \circ r_h|_\kappa$ is of the form

$$\psi_\kappa(r_h|_\kappa)(\hat{x}) = a\hat{x} + b$$

where $a \in \mathbb{R}$, and $b = (b_1, \dots, b_d)^\top \in \mathbb{R}^d$. Thus

$$r_h|_\kappa(x) = \frac{1}{|J_\kappa|} (ax + J_\kappa b - t_\kappa).$$

Defining locally

$$w_h|_\kappa(x) = \frac{1}{|J_\kappa|} \left(\frac{a}{2} x \cdot x + (J_\kappa b - t_\kappa) \cdot x \right)$$

yields that

$$\nabla w_h|_\kappa(x) = r_h|_\kappa(x).$$

□

5. Poincaré Inequalities. The analysis of the model problem (2.1) relies on the Poincaré inequality. It is therefore important to show that Poincaré inequalities hold for the different spaces used. First we give an elementary result showing that only the average solution jump needs to be controlled for a Poincaré inequality to hold in a DG-method. Then we prove Poincaré inequalities for the two spaces V^C and V^D separately, a result that is of essential importance to prove the well posedness of the symmetric unstabilized formulation. Finally we prove a stronger Poincaré inequality for the space V^D under some mesh-assumption in order to prove optimal convergence of the unstabilized formulation.

LEMMA 5.1. *There is a constant $c > 0$ independent of h such that for all $u_h \in V_h^b$ there holds*

$$c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_h]\|_{\mathcal{F}}^2 \leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2 + \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2.$$

Proof. Immediate by the approximation properties of the average jump and a discrete trace inequality:

$$\begin{aligned} \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_h]\|_{\mathcal{F}}^2 &= \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} ([u_h] - \overline{[u_h]})\|_{\mathcal{F}}^2 \\ &\leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2 + c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{\frac{1}{2}} [\nabla u_h \times n]\|_{\mathcal{F}}^2 \\ &\leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2 + c \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2. \end{aligned}$$

□

COROLLARY 5.2. *The following Poincaré inequality for broken H^1 -spaces holds, for all $u_h \in V_h^b$*

$$c \|\sigma^{\frac{1}{2}} u_h\|_{\mathcal{K}}^2 \leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2 + \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2.$$

Proof. An immediate consequence of the previous lemma and the Poincaré inequality

$$c \|\sigma^{\frac{1}{2}} u_h\|_{\mathcal{K}}^2 \leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_h]\|_{\mathcal{F}}^2 + \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2 \quad (5.1)$$

proved by Brenner [4]. □

REMARK 5.3. *We give the proofs for $u_h \in V_h^b$ but since $V_h^1 \subset V_h^b$ the above results hold also for the standard space of piecewise affine discontinuous functions.*

PROPOSITION 5.4 (Poincaré inequality for V^C). *There exists a constant $c > 0$ depending only on Ω such that, for all $h < 1$,*

$$\forall u_c \in V^C, \quad c \|u_c\|_{\mathcal{K}} \leq |u_c|_{1,\mathcal{K}}.$$

Proof. See Temam [21] or Ern-Guermond [11] □

PROPOSITION 5.5 (Poincaré inequality for V^D). *There exists a constant $c > 0$ depending only on Ω such that, for all $h < 1$,*

$$\forall u_d \in V^D, \quad c \|u_d\|_{\mathcal{K}} \leq |u_d|_{1,\mathcal{K}}.$$

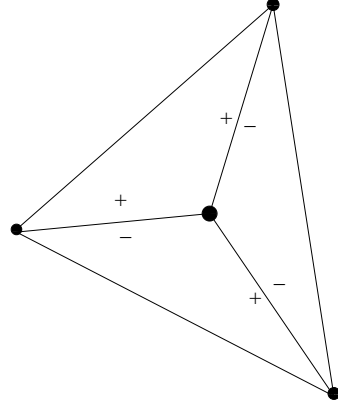


FIG. 5.1. Illustration of the macroelement argument of Lemma 5.6 the '+' and '-' signs refer to the sign of u_d at the midpoints of the faces

Proof. Let $u_d \in V^D$ be fixed. Then define the splitting of Ω into two parts \mathcal{K}_1 and \mathcal{K}_2 by

$$\begin{aligned}\mathcal{K}_1 &= \{\kappa \in \mathcal{K} : \exists x \in \kappa \text{ s.t. } u_d(x) = 0\}, \\ \mathcal{K}_2 &= \{\kappa \in \mathcal{K} : u_d(x) \neq 0, \forall x \in \kappa\}.\end{aligned}$$

Firstly prove the inequality for the region \mathcal{K}_1 . Fix an element $\kappa_1 \in \mathcal{K}_1$ and define

$$Z(\kappa_1) = \{x \in \kappa_1 : u_d(x) = 0\}.$$

Observe that $Z(\kappa_1)$ is either κ_1 itself, a line or one of the vertices since $u_d|_{\kappa_1} \in \mathbb{P}_1(\kappa_1)$. However we may write in all three cases

$$u_d(x) = \nabla u_d \cdot (x - x^*) \quad \text{with } x^* \in Z(\kappa_1).$$

Thus we conclude immediately that

$$\|u_d\|_{\mathcal{K}_1} \leq \|\tilde{h} \nabla u_d\|_{\mathcal{K}_1}. \quad (5.2)$$

Secondly split \mathcal{K}_2 in maximal subsets $\{\mathcal{K}_2^j\}_{j=1}^m$ in order that each

$$\Omega_2^j = \left(\bigcup_{\kappa \in \mathcal{K}_2^j} \kappa \right)^\circ$$

is connex. Fix a subset \mathcal{K}_2^j and observe that $|u_d|$ is midpoint continuous on interior faces of \mathcal{K}_2^j . In consequence $|u_d|$ lies in the Crouzeix-Raviart space over the domain Ω_2^j and thus let us proceed analogous to the proof of Proposition 5.4. Details of the remaining part are given in the appendix. \square

In case the mesh has a certain macro-element structure we may prove a stronger Poincaré inequality for the space V^D .

PROPOSITION 5.6 (Strong Poincaré inequality for V^D). *Let \mathcal{K} be a mesh. Assume that there exists a coarse mesh \mathcal{T} covering $\bar{\Omega}$ such that each macro element (d -simplex)*

$T \in \mathcal{T}$ contains exactly $d+1$ elements $\kappa_1, \dots, \kappa_{d+1}$ of \mathcal{K} and such that $\kappa_i \cap \kappa_j \neq \emptyset$, for all $1 \leq i, j \leq d+1$. Then the following inequality holds

$$\forall u_d \in V^D, \quad c \|u_d\|_{\mathcal{K}} \leq |\tilde{h} u_d|_{1, \mathcal{K}}.$$

Proof. Let $u_d \in V^D$ and fix an element $\kappa \in \mathcal{K}$. If there exists $x^* \in \kappa$ such that $u_d(x^*) = 0$ we conclude analogous to (5.2) that

$$\|u_d\|_{\kappa} \leq \|h_{\kappa} \nabla u_d\|_{\kappa}.$$

Otherwise there exists in the same macro element $T \in \mathcal{T}$ a neighbour element $\kappa' \in \mathcal{K}$ such that there exists $x^* \in \kappa'$ with $u_d(x^*) = 0$. Indeed assume that $u_d(x) \neq 0$ for all $x \in T$. Observe that u_d changes sign in the midpoint of each face since it lies in V^D and hence $\int_F \{u_d\} = 0$. Consider all three elements of T in 2 dimensions and an arbitrary selection of three elements in 3 dimensions. The solution changes sign over each face. However in the three elements the sign has to change four times. Hence it has to change sign within one element which leads to a contradiction. See the illustration in Figure 5.1. Thus there exists at least one element $\kappa^* \in \mathcal{K}$ of the macro element T such that there exists a point $x^* \in \kappa^*$ with $u_d(x^*) = 0$. Since $\kappa \cap \kappa^* \neq \emptyset$ we conclude that

$$\|u_d\|_{\kappa} \leq c \|h_{\kappa} \nabla u_d\|_{\kappa}.$$

□

6. Discontinuous Galerkin Methods. Define the following bilinear form

$$a_s(u_h, v_h) = (\sigma \nabla u_h, \nabla v_h)_{\mathcal{K}} - (\{\sigma \nabla u_h\}, [v_h])_{\mathcal{F}} - s(\{\sigma \nabla v_h\}, [u_h])_{\mathcal{F}}, \quad (6.1)$$

for $s \in \{-1, 1\}$ and the stabilization term

$$j(u_h, v_h) = (\omega \tilde{h}_{\mathcal{F}}^{-1} [\overline{u_h}], [\overline{v_h}])_{\mathcal{F}_i},$$

where $\omega|_F = \min(\sigma|_{\kappa_1}, \sigma|_{\kappa_2})$ for $F = \partial\kappa_1 \cap \partial\kappa_2$. Note that we have replaced the penalization of the jumps of formulation (3.2) with the penalization of the average jumps in the spirit of Lemma 5.1.

Let us define two methods to approximate the solution of (2.1):

Method 1: Find $u_h^1 \in V_{h,0}^1$ such that

$$a_s(u_h^1, v_h) + \gamma j(u_h^1, v_h) = (f, v_h)_{\mathcal{K}} \quad \forall v_h \in V_{h,0}^1, \quad (6.2)$$

for some $\gamma \in \mathbb{R}$ and $s \in \{-1, 1\}$.

REMARK 6.1. Except for the penalization of the average jump only this corresponds to the SIPG-method for $s = 1$ and the NIPG-method for $s = -1$. The well-posedness of the particular case $\gamma = 0$ is discussed in Lemma 8.6, the convergence behavior in Section §8.2.

Method 2: Find $u_h^b \in V_h^b$ such that

$$a_s(u_h^b, v_h) = (f, v_h)_{\mathcal{K}} \quad \forall v_h \in V_h^b. \quad (6.3)$$

REMARK 6.2 (Local mass conservation property). The solutions u_h^1, u_h^b of (6.2) resp. (6.3) satisfy

$$\begin{aligned} - \int_{\partial\kappa} \{\sigma \nabla u_h^1\} \cdot n_{\kappa} ds + \gamma \int_{\partial\kappa} \omega \tilde{h}_{\mathcal{F}}^{-1} [\overline{u_h^1}] \cdot n_{\kappa} ds &= \int_{\kappa} f dx, \\ - \int_{\partial\kappa} \{\sigma \nabla u_h^b\} \cdot n_{\kappa} ds &= \int_{\kappa} f dx. \end{aligned}$$

Remark that u_h^b is locally mass conservative while for u_h^1 with $\gamma \neq 0$ the local flux includes the penalty term depending on the stabilization parameter γ .

LEMMA 6.3 (Consistency of methods). *If the exact solution u of problem (2.1) satisfies $u \in H^1(\Omega)$, then the formulations defined by (6.2) and (6.3) are consistent (and adjoint consistent if the bilinear form is symmetric). Moreover the following Galerkin orthogonalities hold*

$$\begin{aligned} a_s(u - u_h^1, v_h) + \gamma j(u - u_h^1, v_h) &= 0 \quad \forall v_h \in V_{h,0}^1 \\ a_s(u - u_h^b, v_h) &= 0 \quad \forall v_h \in V_h^b \end{aligned}$$

where $u_h^1 \in V_{h,0}^1$ and $u_h^b \in V_h^b$ denote the discrete solutions of (6.2) resp. (6.3).

Proof. Since the form $a_s(\cdot, \cdot)$ coincides with the SIPG formulation for $s = 1$ resp. NIPG for $s = -1$ it is consistent. Moreover for $s = 1$ the method is adjoint consistent. We only show the primal consistency, dual consistency follows by symmetry. Since $u \in H^1(\Omega)$ we have

$$[u]|_F = 0, \quad \forall F \in \mathcal{F}, \quad \forall F \in \mathcal{F}_i$$

and by partial integration

$$a_s(u, v_h) = (\sigma \nabla u, \nabla v_h)_{\mathcal{K}} - (\sigma \nabla u, [v_h])_{\mathcal{F}} = (\nabla \cdot (\sigma \nabla u), v_h)_{\mathcal{K}} = (f, v_h)_{\mathcal{K}}$$

for all $v_h \in V_h^b$. Note that $V_{h,0}^1 \subset V_h^b$. Observe also that for u , solution of (2.1), there holds

$$j(u, v_h) = (\omega \tilde{h}_{\mathcal{F}}^{-1} [\overline{u}], [\overline{v_h}])_{\mathcal{F}_i} = 0,$$

which concludes the proof. \square

7. Technical results. For the analysis it is useful to introduce the following norms

$$\begin{aligned} |||v_h|||^2 &= \|\sigma^{\frac{1}{2}} v_h\|_{1,\mathcal{K}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [v_h]\|_{\mathcal{F}}^2, \\ |||v_h|||_d^2 &= \|\sigma^{\frac{1}{2}} \nabla v_h\|_{\mathcal{K}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [v_h]\|_{\mathcal{F}}^2, \\ |||v_h|||_c^2 &= \|\sigma^{\frac{1}{2}} \nabla v_h\|_{\mathcal{K}}^2 + \|\tilde{h}_{\mathcal{F}}^{\frac{1}{2}} \{\sigma \nabla v_h\}\|_{\mathcal{F}_i}^2, \end{aligned}$$

for all $v_h \in V_h^b$. Using the Poincaré inequality (5.1) we have

$$|||v_h||| \leq c |||v_h|||_d. \quad (7.1)$$

We have the following standard approximability results.

LEMMA 7.1 (Approximability in V_h^1). *Let $u \in H^2(\Omega)$ and $\pi_h u \in V_h^1$ denote the element-wise L^2 -projection of u onto V_h^1 , then there holds*

$$|||u - \pi_h u||| \leq c h \|u\|_{2,\mathcal{K}} \quad \text{and} \quad |||u - \pi_h u|||_c \leq c h \|u\|_{2,\mathcal{K}}. \quad (7.2)$$

Proof. The claim is an immediate consequence of the following inequalities

$$\|u - \pi_h u\|_{\mathcal{K}} \leq c h^2 \|u\|_{2,\mathcal{K}}, \quad \|\nabla(u - \pi_h u)\|_{\mathcal{K}} \leq c h \|u\|_{2,\mathcal{K}}$$

and

$$\|\tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u - \pi_h u]\|_{\mathcal{F}} \leq c h \|u\|_{2,\mathcal{K}}, \quad \|\tilde{h}_{\mathcal{F}}^{\frac{1}{2}} \{\nabla(u - \pi_h u)\}\|_{\mathcal{F}} \leq c h \|u\|_{2,\mathcal{K}}$$

that are proved using trace inequalities and standard approximation results. \square

LEMMA 7.2 (Approximability in V^C). *Let $u \in H^\beta(\Omega)$, with $\beta \in \{1, 2\}$, and $i_c u \in V^C$ denote the Crouzeix-Raviart-interpolant of u onto V^C defined by (4.1), then there holds*

$$\|u - i_c u\|_{\mathcal{K}} \leq c h^\beta \|u\|_{\beta, \mathcal{K}}. \quad (7.3)$$

If $u \in H^2(\Omega)$, then

$$\| \|u - i_c u\| \| \leq c h \|u\|_{2, \mathcal{K}}, \quad (7.4)$$

$$\| \|u - i_c u\| \|_c \leq c h \|u\|_{2, \mathcal{K}}. \quad (7.5)$$

Proof. We refer to the original paper of Crouzeix and Raviart [9] for the optimal approximation properties of the interpolant. \square

8. Analysis of method 1.

8.1. Stability. In this section we will use orthogonality properties of V^C and V^D to obtain coercivity results also for the unstabilized method. These results ensures existence of the discrete solution.

LEMMA 8.1 (Orthogonality relations). *The bilinear forms $a_s(\cdot, \cdot)$ and $j(\cdot, \cdot)$ satisfy the following orthogonality relations:*

$$\begin{aligned} a_s(v_c, v_d) &= 0 & \forall v_c \in V^C, \forall v_d \in V^D, \\ a_s(v_d, v_c) &= (1-s)(\sigma \nabla v_d, \nabla v_c)_{\mathcal{K}} & \forall v_c \in V^C, \forall v_d \in V^D, \\ j(v_c, v_d) &= j(v_d, v_c) = 0 & \forall v_c \in V^C, \forall v_d \in V^D. \end{aligned}$$

REMARK 8.2. *The spaces V^C and V^D are orthogonal with respect to the symmetric bilinear form $a_1(\cdot, \cdot)$.*

Proof. Let $v_c \in V^C$ and $v_d \in V^D$. Since $\int_F [v_c] dx = 0$ for all interior faces $F \in \mathcal{F}_i$ it follows directly that

$$j(v_c, v_d) = j(v_d, v_c) = 0,$$

and that

$$a_s(v_c, v_d) = (\sigma \nabla v_c, \nabla v_d)_{\mathcal{K}} - (\{\sigma \nabla v_c\}, [v_d])_{\mathcal{F}_i}.$$

An integration by parts yields

$$a_s(v_c, v_d) = -(\nabla \cdot \sigma \nabla v_c, v_d)_{\mathcal{K}} + ([\sigma \nabla v_c], \{v_d\})_{\mathcal{F}} = 0,$$

since $\int_F \{v_d\} dx = 0$ for all faces $F \in \mathcal{F}$ and $\nabla \cdot \sigma \nabla v_c \equiv 0$. Analogously we prove that

$$a_s(v_d, v_c) = (\sigma \nabla v_d, \nabla v_c)_{\mathcal{K}} - s(\{\sigma \nabla v_c\}, [v_d])_{\mathcal{F}_i} = (1-s)(\sigma \nabla v_d, \nabla v_c)_{\mathcal{K}}.$$

\square

LEMMA 8.3. *The bilinear forms $a_s(\cdot, \cdot)$ and $j(\cdot, \cdot)$ satisfy the following relations:*

$$\begin{aligned} a_s(u_c, v_c) &= (\sigma \nabla u_c, \nabla v_c)_{\mathcal{K}} & \forall u_c, v_c \in V^C, \\ a_s(u_d, v_d) &= -s(\sigma \nabla u_d, \nabla v_d)_{\mathcal{K}} & \forall u_d, v_d \in V^D, \\ j(u_c, v_c) &= 0 & \forall u_c, v_c \in V^C. \end{aligned}$$

Proof. The proof is similar to the one of Lemma 8.1 and uses the properties of the spaces V^C and V^D . \square

LEMMA 8.4 (Splitting of method 1). *The first method defined by (6.2) is equivalent to: Find $u_c \in V^C$, $u_d \in V^D$ such that*

$$(\sigma \nabla u_c, \nabla v_c)_K + (1-s)(\sigma \nabla u_d, \nabla v_c)_K = (f, v_c)_K \quad \forall v_c \in V^C, \quad (8.1)$$

$$-s(\sigma \nabla u_d, \nabla v_d)_K + \gamma(\omega \tilde{h}_{\mathcal{F}}^{-1}[\overline{u_d}], [\overline{v_d}])_{\mathcal{F}_i} = (f, v_d)_K \quad \forall v_d \in V^D. \quad (8.2)$$

REMARK 8.5. *Observe that for $s = 1$, (8.1) is the Crouzeix-Raviart method for problem (2.1). In consequence the stability and convergence analysis is known for this problem. Since in this case u_c is totally uncoupled of the second problem (8.2), u_c always converges to the exact solution with optimal rate independently of the choice of the stabilization parameter, $\gamma = 0$ included. Moreover note that (8.2) is independent of (8.1). Hence we can solve first for the discontinuous field u_d and then for the continuous field u_c also in the case $s = -1$.*

Proof. Let $v_h \in V_{h,0}^1$. Since $V_{h,0}^1 = V^C \oplus V^D$ we can write $v_h = v_c + v_d$ with $v_c \in V^C$ and $v_d \in V^D$. Analogously we can write $u_h^1 = u_c + u_d$. Testing in (6.2) with v_c and v_d separately yields the problem: Find $u_c \in V^C$ and $u_d \in V^D$ such that

$$a_s(u_c + u_d, v_c) + \gamma j(u_c + u_d, v_c) = (f, v_c)_K \quad \forall v_c \in V^C,$$

$$a_s(u_c + u_d, v_d) + \gamma j(u_c + u_d, v_d) = (f, v_d)_K \quad \forall v_d \in V^D.$$

Applying Lemma 8.1 leads directly to: Find $u_c \in V^C$ and $u_d \in V^D$ such that

$$a_s(u_c + u_d, v_c) = (f, v_c)_K \quad \forall v_c \in V^C, \quad (8.3)$$

$$a_s(u_d, v_d) + \gamma j(u_d, v_d) = (f, v_d)_K \quad \forall v_d \in V^D. \quad (8.4)$$

Note that the equivalences between the problems (8.3) and (8.1) resp. (8.4) and (8.2) follow directly from Lemma 8.3. \square

LEMMA 8.6 (Coercivity of method 1). *The discrete problem (8.2) is well posed for all $s\gamma \leq 0$ and $s\gamma > C_{stab}$, where $C_{stab} > 0$ is a certain constant independent on h , i.e.*

$$C(\gamma) |||u_d|||_d^2 \leq |a_s(u_d, u_d) + \gamma j(u_d, u_d)|,$$

where

$$C(\gamma) = \begin{cases} c \min(1, |\gamma|) & \text{if } s\gamma < 0, \\ c(\gamma - C_{stab}) & \text{if } s\gamma > C_{stab}, \end{cases}$$

and if $\gamma = 0$ we have

$$c \|\sigma^{\frac{1}{2}} u_d\|_K^2 \leq |a_s(u_d, u_d)|,$$

on general meshes and for meshes described in Proposition 5.6 we have

$$c |||u_d|||_d^2 \leq |a_s(u_d, u_d)|. \quad (8.5)$$

REMARK 8.7. *Observe that the method is well posed also for $\gamma = 0$.*

Proof. Let us prove first the well-posedness of (8.2) for $s\gamma < 0$. Observe that

$$\begin{aligned} c \min(1, |\gamma|) \|u_d\|_d^2 &\leq \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 + |\gamma| \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_d}]\|_{\mathcal{F}_i}^2 \\ &= -s a_s(u_d, u_d) + |\gamma| j(u_d, u_d) \\ &= |a_s(u_d, u_d) + \gamma j(u_d, u_d)| \end{aligned}$$

applying Lemma 5.1 and since $\|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_d}]\|_{\mathcal{F}_e}^2 = 0$.

For $s\gamma > 0$ observe that using the inverse and trace inequalities yields

$$\|u_d\|_d^2 = \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_d]\|_{\mathcal{F}}^2 \leq c \|\tilde{h}^{-1} \sigma^{\frac{1}{2}} u_d\|_{\mathcal{K}}^2.$$

On the other hand by norm equivalence in discrete spaces there exists a constant $c_\star > 0$, independent of the mesh size h , such that

$$\|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_d}]\|_{\mathcal{F}_i}^2 \geq c_\star \|\tilde{h}^{-1} \sigma^{\frac{1}{2}} u_d\|_{\mathcal{K}}^2,$$

since $\omega|_F \geq \rho\sigma|_{\kappa_i}$, $i = 1, 2$. Thus using the inverse inequality with constant c_{ie} yields

$$\begin{aligned} |a_s(u_d, u_d) + \gamma j(u_d, u_d)| &\geq -\|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 + s\gamma \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_d}]\|_{\mathcal{F}_i}^2 \\ &\geq (s\gamma c_\star - c_{ie}) \|\tilde{h}^{-1} \sigma^{\frac{1}{2}} u_d\|_{\mathcal{K}}^2. \end{aligned}$$

Observe that coercivity holds under the assumption that $s\gamma = |\gamma| > \frac{c_{ie}}{c_\star} =: C_{stab}$.

For $\gamma = 0$ on general meshes observe that

$$c \|\sigma^{\frac{1}{2}} u_d\|_{\mathcal{K}}^2 \leq \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 = -s a_s(u_d, u_d) = |a_s(u_d, u_d)|$$

using the Poincaré inequality, Proposition 5.5.

For $\gamma = 0$ on meshes described in Proposition 5.6 we have

$$\begin{aligned} \|u_d\|_d^2 &= \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_d]\|_{\mathcal{F}}^2 \leq \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 + c \|\tilde{h}^{-1} \sigma^{\frac{1}{2}} u_d\|_{\mathcal{K}}^2 \\ &\leq c \|\sigma^{\frac{1}{2}} \nabla u_d\|_{\mathcal{K}}^2 = c |a_s(u_d, u_d)| \end{aligned}$$

using the trace inequality and the strong Poincaré inequality, Proposition 5.6. \square

8.2. Convergence. We will now address the question of optimal convergence for different values of the stabilization parameter. In the case where the stabilization parameter is set to zero the lack of continuity of the bilinear form may perturb convergence. However if the mesh has the macro element structure of Proposition 5.6 optimal convergence is recovered.

THEOREM 8.8. *Let $u \in H^2(\Omega)$ be the solution of (2.1) and u_h^1 the solution of (6.2) with $s\gamma < 0$, $s\gamma > C_{stab}$ or $\gamma = 0$ on the meshes defined in Proposition 5.6, then there holds*

$$\|u - u_h^1\| \leq c h \|u\|_{2, \mathcal{K}}.$$

REMARK 8.9. *On general meshes, in the particular case $\gamma = 0$, this theorem is no more valid. In this case an optimal convergence result can be shown under some restrictive regularity assumptions on f and the mesh, see Theorem 8.10.*

Proof. First note that for the bilinear form $a_s(\cdot, \cdot)$ the following continuity holds for all $w \in H^1(\Omega)$, $w_c \in V^C$ and $v_d \in V^D$ by Cauchy - Schwarz inequality

$$a_s(w - w_c, v_d) \leq \|w - w_c\|_c \|v_d\|_d. \quad (8.6)$$

1. Decompose the error in two midpoint-continuous parts and one midpoint-discontinuous part,

$$\|u - u_h^1\| \leq \|u_c - i_c u\| + \|u - i_c u\| + \|u_d\|. \quad (8.7)$$

2. Observe that by (7.1), Lemma 8.3 and Lemma 6.3:

$$\begin{aligned} \|u_c - i_c u\|^2 &\leq c \|u_c - i_c u\|_d^2 \leq c a_s(u_c - i_c u, i_c u - u_c) \\ &= c a_s(u - i_c u - u_d, i_c u - u_c) \\ &\leq c (\|u - i_c u\| + \|u_d\|) \|u_c - i_c u\|, \end{aligned}$$

since $u_c - i_c u + u - u_h^1 = u - i_c u - u_d$ and $a_s(u - u_h^1, i_c u - u_c) = 0$. Thus

$$\|u - u_h^1\| \leq c (\|u - i_c u\| + \|u_d\|). \quad (8.8)$$

3. Use Lemma 7.2 to bound the first term of (8.8),

$$\|u - i_c u\| \leq ch \|u\|_{2,\mathcal{K}}.$$

4. For the second term of (8.8) use (7.1) and the coercivity, Lemma 8.6,

$$\|u_d\|^2 \leq c \|u_d\|_d^2 \leq \frac{c}{C(\gamma)} |a_s(u_d, u_d) + \gamma j(u_d, u_d)|.$$

In the particular case $\gamma = 0$, the constant $C(0)$ denotes the constant of (8.5).

5. Use the consistency of the bilinear form, Lemma 6.3,

$$\|u_d\|^2 \leq \frac{c}{C(\gamma)} |a_s(u - u_c, u_d)|,$$

since $u_d + u - u_h^1 = u - u_c$ and $a_s(u - u_h^1, u_d) - \gamma j(u_h^1, u_d) = 0$.

6. Conclude by applying the continuity (8.6) and the approximation result (7.5).

□

Under some restrictions we can show optimal convergence also in the particular case of $\gamma = 0$ for the symmetric version on meshes without the macro element property.

THEOREM 8.10. *Let $u \in H^2(\Omega)$ be the solution of (2.1) and u_h^1 the solution of (6.2) with $\gamma = 0$ and $s = 1$. Assume that $f \in H^\beta(\Omega)$, with $\beta \in \{1, 2\}$, and that the mesh is asymptotically uniform with some $\zeta > 0$, then there holds*

$$\|\nabla(u - u_h^1)\|_{\mathcal{K}} \leq c(h\|u\|_{2,\mathcal{K}} + (h^\zeta + h^{2\star})\|f\|_{1,\Omega} + h^\beta\|f\|_{\beta,\mathcal{K}}),$$

where the term marked by \star appears only in the 3-dimensional case.

Proof. Using the triangle inequality we can split

$$\|\nabla(u - u_h^1)\|_{\mathcal{K}} \leq \|\nabla(u - u_c)\|_{\mathcal{K}} + \|\nabla u_d\|_{\mathcal{K}}.$$

Since u_c is the standard Crouzeix-Raviart solution the first term of the right hand side of the previous equation can be bounded by

$$\|\nabla(u - u_c)\|_{\mathcal{K}} \leq ch\|u\|_{2,\mathcal{K}}.$$

From (8.2) we can write

$$\|\nabla u_d\|^2 = |(f, u_d)_{\mathcal{K}}| \leq |(f - i_c f, u_d)_{\mathcal{K}}| + |(i_c f, u_d)_{\mathcal{K}}| \quad (8.9)$$

where i_c is the Crouzeix-Raviart interpolant introduced in Remark 4.2. The first term of the right hand side of (8.9) can be bounded by

$$|(f - i_c f, u_d)_K| \leq \|f - i_c f\|_K \|u_d\|_K \leq c h^\beta \|f\|_{\beta, K} \|\nabla u_d\|_K \quad (8.10)$$

by optimal approximation properties of the Crouzeix-Raviart interpolant and by the Poincaré inequality for V^D , Proposition 5.5.

For the second term of the right hand side of (8.9) we use Lemma 4.3, the Poincaré inequality for V^D , Proposition 5.5, and the stability of the Crouzeix-Raviart interpolant $\|i_c f\|_K \leq \|f\|_{1, \Omega}$ resp. $\|\nabla i_c f\|_K \leq \|f\|_{1, \Omega}$:

$$\begin{aligned} |(i_c f, u_d)_K| &\leq c (h^\zeta \|i_c f\|_K \|u_d\|_K + h^{2\star} \|\nabla i_c f\|_K \|\nabla u_d\|_K) \\ &\leq c (h^\zeta + h^2) \|f\|_{1, \Omega} \|\nabla u_d\|_K. \end{aligned} \quad (8.11)$$

The term marked by \star appears only in the 3-dimensional case. Combining (8.10) and (8.11) completes the proof. \square

REMARK 8.11. *If $\beta = 2$ and $\zeta = 2$, then optimal convergence can be shown in the triple norm $\|\cdot\|$ since in this case*

$$\|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_d]\|_{\mathcal{F}}^2 \leq c \|\tilde{h}^{-1} \nabla u_d\|_K \leq c h \|f\|_{2, K}.$$

REMARK 8.12. *Observe that on uniform meshes the convergence is only limited by the regularity of f .*

We will now show that we have optimal L^2 -convergence for the symmetric version thanks to the adjoint consistency. For the non-symmetric version the L^2 -convergence rate depends on the regularity of the mesh and the right hand side as pointed out in [14] in the one dimensional case.

THEOREM 8.13. *Let $u \in H^2(\Omega)$ with $\|u\|_{2, \Omega} \leq c \|f\|_{\Omega}$ be the solution of (2.1) and u_h^1 the solution of (6.2) with $s\gamma < 0$, $s\gamma > C_{stab}$ or $\gamma = 0$ on meshes as described in Proposition 5.6, then there holds:*

a) *If $s = 1$, then*

$$\|u - u_h^1\|_{L^2(\Omega)} \leq c h^2 \|u\|_{2, \Omega}.$$

b) *If $s = -1$, and assuming that $f \in H^\beta(\Omega)$, with $\beta \in \{1, 2\}$, and that the mesh is asymptotically uniform with some $\zeta > 0$, then*

$$\|u - u_h^1\|_{L^2(\Omega)} \leq c (h^2 \|u\|_{2, K} + (h^\zeta + h^{2\star}) \|f\|_{1, \Omega} + h^\beta \|f\|_{\beta, K}),$$

where the term marked by \star appears only in the 3-dimensional case.

Proof. Consider the dual problem: Find $\phi \in H_0^1(\Omega)$ such that

$$(\sigma \nabla \phi, \nabla z)_\Omega = (e, z)_\Omega, \quad \forall z \in H_0^1(\Omega).$$

Under the regularity assumptions on u we have $\|\phi\|_{1, \Omega} \leq c \|e\|_\Omega$ and $\|\phi\|_{2, \Omega} \leq c \|e\|_\Omega$. It follows that

$$-(\nabla \cdot \sigma \nabla \phi, z)_\Omega = (e, z)_\Omega, \quad \forall z \in L^2(\Omega).$$

Let $e = u - u_h$ then we have by the dual consistency of Lemma 6.3

$$\begin{aligned} \|e\|_\Omega^2 &= a_1(e, \phi) = a_s(e, \phi) + (s-1)(\{\sigma \nabla \phi\}, [e])_{\mathcal{F}_i} \\ &= a_s(e, \phi - i_c \phi) - (s-1)(\{\sigma \nabla \phi\}, [u_d])_{\mathcal{F}_i} \\ &\leq \|e\|_d \|\phi - i_c \phi\|_c + |s-1| \|\phi\|_{1,\mathcal{K}}^{\frac{1}{2}} \|\phi\|_{2,\mathcal{K}}^{\frac{1}{2}} \|u_d\| \\ &\leq ch \|e\|_d \|\phi\|_{2,\Omega} + |s-1| \|\phi\|_{1,\mathcal{K}}^{\frac{1}{2}} \|\phi\|_{2,\mathcal{K}}^{\frac{1}{2}} \|u_d\| \end{aligned}$$

by (8.6) and (7.5). We conclude using the a priori bound on the $H^1(\Omega)$ and $H^2(\Omega)$ norm of ϕ and Theorem 8.8,

$$\|e\|_\Omega \leq c(h \|e\| + |s-1| \|u_d\|) \leq c(h^2 \|u\|_{2,\Omega} + |s-1| \|u_d\|).$$

For the symmetric case the result follows immediately since $|s-1| = 0$.

Consider now the non-symmetric case for which $s = -1$. By (7.1) and the coercivity, Lemma 8.6, it follows that

$$C(\gamma) \|u_d\|^2 \leq C(\gamma) \|u_d\|_d^2 \leq |a_{-1}(u_d, u_d) + \gamma j(u_d, u_d)| = |(f, u_d)_\mathcal{K}|.$$

Then split f into a midpoint continuous and discontinuous part and use Lemma 4.3, (7.3) and the stability of the Crouzeix-Raviart interpolant

$$\begin{aligned} C(\gamma) \|u_d\|^2 &\leq |(f - i_c f, u_d)_\mathcal{K}| + |(i_c f, u_d)_\mathcal{K}| \\ &\leq (\|f - i_c f\|_\mathcal{K} + ch^\zeta \|i_c f\|_\mathcal{K}) \|u_d\|_\mathcal{K} + ch^{2*} \|\nabla i_c f\|_\mathcal{K} \|\nabla u_d\|_\mathcal{K} \\ &\leq (ch^\beta \|f\|_{\beta,\mathcal{K}} + c(h^\zeta + h^{2*}) \|f\|_{1,\Omega}) \|u_d\|. \end{aligned}$$

Thus the result in the non-symmetric case. \square

REMARK 8.14. For $\gamma = 0$, in the particular case of Remark 8.11 optimal convergence in the L^2 -norm can be shown also on regular meshes without the macroelement property. The details are left to the reader.

8.3. Numerical Tests. Observe that the only difference between the standard SIPG/NIPG-method and the method 1 is that in the latter case the stabilization term is composed by the facewise L^2 -projection of order 0 of the jumps. From an implementational viewpoint this can be realized by reducing the order of the quadrature formula for the numerical integration on the faces, i.e. applying the midpoint integration rules for the computation of the stabilization term.

8.3.1. Test problems. Let us briefly present the test problems used for the numerical tests.

a) *Problem with smooth solution*

We consider problem (2.1) with $\sigma = 1$ and $f = 2(2 - x^2 - y^2)$ on the square $\Omega = (-1, 1)^2$. The exact solution is given by $u(x, y) = (x^2 - 1)(y^2 - 1) \in C^\infty(\Omega)$. A sequence of unstructured meshes is considered.

b) *Problem with irregular solution*

Now choose the following L -shaped domain: $\Omega = (-1, 1) \times (-1, 0) \cup (0, 1)^2$. We consider problem (2.1) with $\sigma = 1$ and $f = 0$ and non-homogeneous boundary conditions such that the solution is

$$u(x, y) = (x^2 + y^2)^{\frac{1}{3}} \sin\left(\frac{2}{3} \arctan_*\left(\frac{x}{y}\right)\right)$$

where \arctan_* is chosen in the manner that it is a continuous function at points with $y = 0$. One can prove that $u \notin H^2(\Omega)$. Therefore the Theorems 8.8, 8.13, 9.5 and 9.6 are no longer valid. A sequence of unstructured meshes is considered.

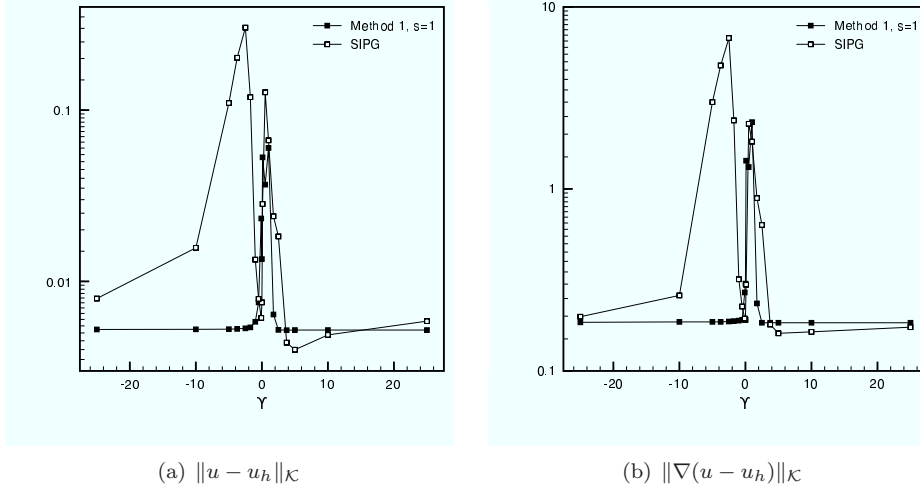


FIG. 8.1. Difference between the symmetric version of method 1 and the SIPG-method for the smooth problem measured in the L^2 -error (a) resp. H^1 -error (b) .

c) *Problem with checkerboard mode*

We consider problem (2.1) with $\sigma = 1$ and $f = -1 + 2\chi_{x>y}$ where χ denotes the characteristic function on the square $\Omega = (-1, 1)^2$. A sequence of structured meshes is considered.

8.3.2. Robustness with respect to the stabilization parameter. Let us consider the problem with smooth solution. We compare the robustness of the symmetric version of method 1 with the SIPG-method respectively the non-symmetric version of method 1 with the NIPG-method.

Figure 8.1 illustrates the behavior of the error in the L^2 - and H^1 -norm with varying stabilization parameter for the symmetric version of method 1 on a fixed mesh. In Figure 8.2 we present the same quantities for the non-symmetric case. Finally in Figure 8.3 we give comparisons of the condition number of the corresponding matrices. We define the condition number of a square matrix (not necessarily symmetric positive definite) as the ratio of the largest singular value of the matrix to the smallest one. Since the continuous and the discontinuous part of the approximation decouples for our formulation, we may also consider negative values of the penalization parameter.

8.3.3. Convergence. The convergence rates of method 1 with stabilization parameter $\gamma = 0$ are compared to those of the standard SIPG- and NIPG-method once for the problem a) with regular solution and once for the problem b) with irregular solution.

Figure 8.4 shows the convergence rates of the smooth problem measured in the L^2 - and H^1 -norm. The symmetric versions have slightly better convergence rates in the L^2 -norm which can be justified by Theorem 8.13.

Figure 8.5 shows the accuracy of the methods for the problem with irregular solution. The SIPG-method has a much smaller L^2 -error than the other methods, the convergence rates however seem to be the same.

The test problem c) is chosen so as to give rise a checkerboard mode in the

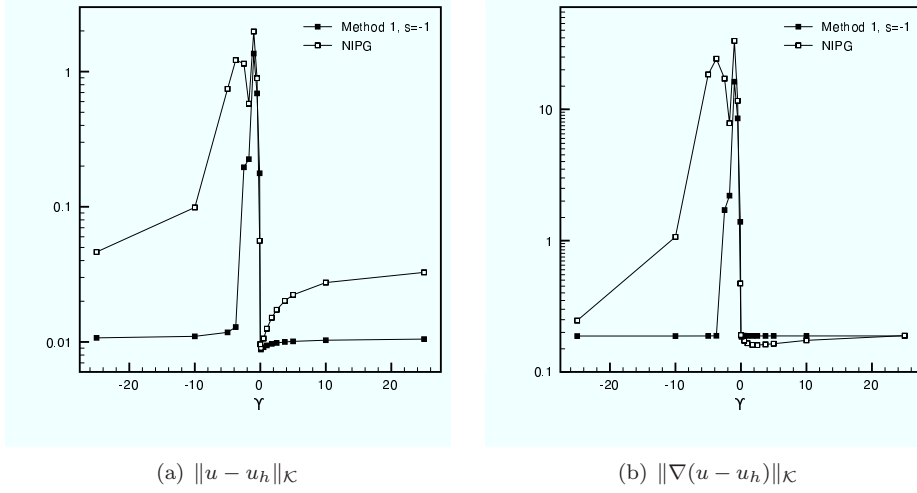


FIG. 8.2. Difference between the non-symmetric method 1 and the NIPG-method for the smooth problem measured in the L^2 -error (a) resp. H^1 -error (b) .

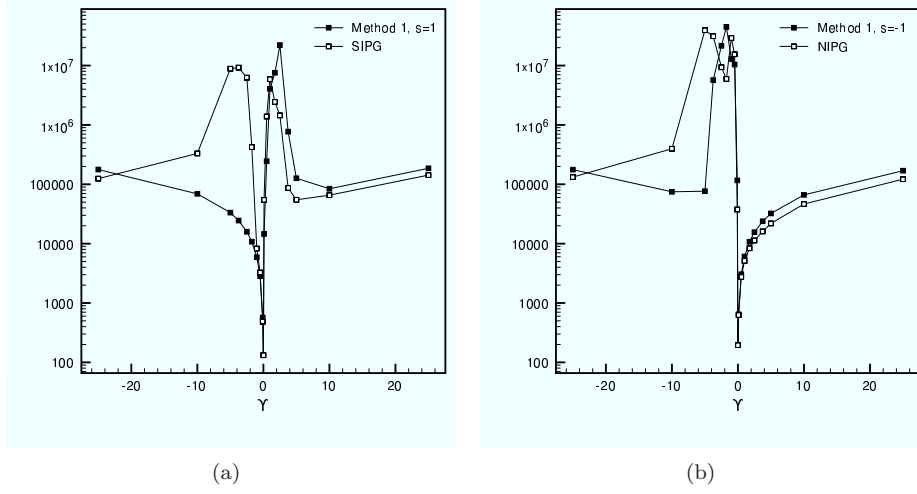


FIG. 8.3. Difference of condition number of matrices corresponding to the symmetric version (a) and non-symmetric version (b) of method 1 and the SIPG- (a) resp. NIPG-method (b) for the smooth problem.

discontinuous field. Here we give a convergence result for the symmetric version of method 1. The convergence of the jump term for different values of the stabilization parameter is given in Figure 8.6a and the convergence of the jump contribution in the broken H^1 semi-norm is given in Figure 8.6b. Clearly the broken H^1 -norm of u_d converges for the case without stabilization although the error does not converge in the norm $|||\cdot|||$ including the jumps. This lack of convergence of the interelement jumps is caused by the checkerboard mode in the field u_d . In Figure 8.7 we give plots of the u_d field for various values of the penalization parameter γ . This clearly illustrates how the penalization localizes the perturbation caused by the discontinuous data, and

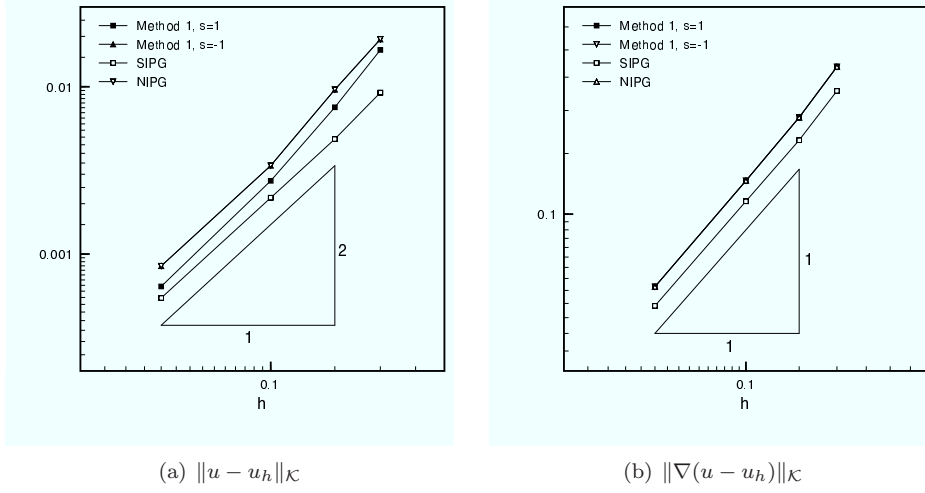


FIG. 8.4. L^2 -error (a) and H^1 -error (b) for h -refinement for the problem with smooth solution using stabilization parameters $\gamma = 0$ for method 1, $\gamma = 10$ for the SIPG-method and $\gamma = 1$ for the NIPG-method.

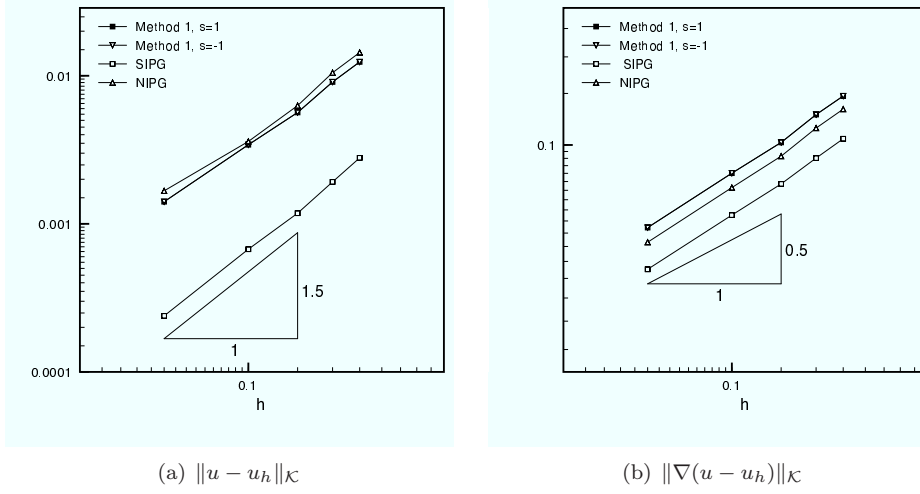


FIG. 8.5. L^2 -error (a) and H^1 -error (b) for h -refinement for the problem with irregular solution using stabilization parameters $\gamma = 0$ for method 1, $\gamma = 10$ for the SIPG-method and $\gamma = 1$ for the NIPG-method.

hence enhances convergence for $\gamma \neq 0$.

9. Analysis of method 2. In the previous section we saw that for the unstabilized symmetric DG method the appearance of a checkerboard mode for rough data destroyed convergence of the solution jumps. Optimal convergence is recovered if the mesh has a certain macro element structure. For method 2 this structure is replaced by a bubble enrichment of the space. The motivation for the DG-method using the enriched space is to obtain local mass conservation independent of the stabilization parameter for a low order DG-method while keeping optimal convergence properties

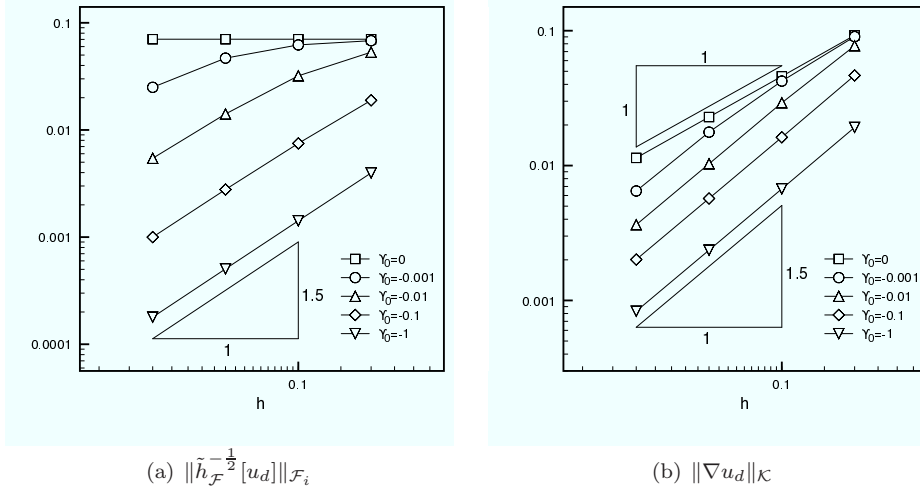


FIG. 8.6. Different norms of u_d for h -refinement and for different stabilization parameters γ .

in the general case.

9.1. Projection. In order to prove stability of the method we first need to define the following projection.

LEMMA 9.1 (Interpolant). *Let $u_h \in V_h^b$ be a fixed function. Then there exists a unique $w_h \in V_h^b$ such that*

$$\int_{\kappa} w_h \, dx = 0 \quad \forall \kappa \in \mathcal{K} \quad (9.1)$$

$$\{\sigma \nabla w_h\}_F \cdot n_F = \omega h_F^{-1} \overline{[u_h]} \cdot n_F \quad \forall F \in \mathcal{F} \quad (9.2)$$

$$\int_F \{w_h\} = 0 \quad \forall F \in \mathcal{F}_i. \quad (9.3)$$

In addition the following a priori estimate holds

$$\|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}} \leq c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}, \quad (9.4)$$

where $\omega = \min(\sigma|_{\kappa_1}, \sigma|_{\kappa_2})$ for $F = \partial\kappa_1 \cap \partial\kappa_2$.

Proof. Let us first observe that the number of conditions on the projection is equal to the number of unknowns. The dimension of the finite element space V_h^b is $(d+2)N_{el}$ where N_{el} denotes the number of elements in the mesh. On the other hand condition (9.1) enforces N_{el} constraints whereas conditions (9.2) and (9.3) demand $N_F + N_{F_i}$ constraints where N_F and N_{F_i} denote the number of the number of faces resp. the number of interior faces of the mesh. Observing that $N_F + N_{F_i} = (d+1)N_{el}$ implies directly a square linear system to determine the projection. Let us now establish the following a priori estimate

$$|||w_h||| \leq c |||u_h|||. \quad (9.5)$$

Since v_h has zero mean over each element it satisfies the following strong Poincaré inequality

$$\|\sigma^{\frac{1}{2}} w_h\|_{\mathcal{K}} \leq c \|\tilde{h} \sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}. \quad (9.6)$$

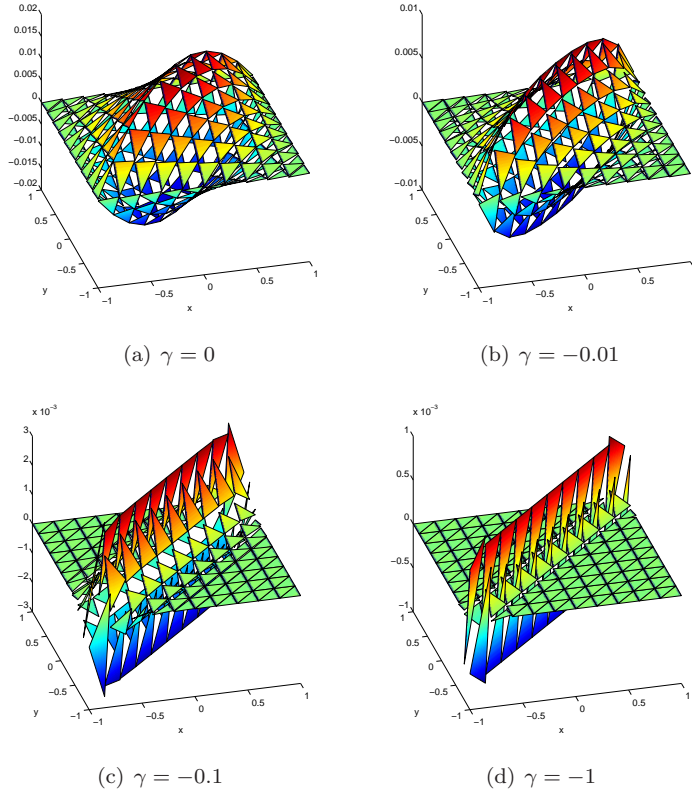


FIG. 8.7. u_d computed using different values of the stabilization parameter for the problem generating a checkerboard mode.

Hence, using a trace inequality yields

$$\|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [w_h]\|_{\mathcal{F}}^2 \leq c \|\sigma^{\frac{1}{2}} \tilde{h}^{-1} w_h\|_{\mathcal{K}}^2 \leq c \|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}^2. \quad (9.7)$$

Moreover, integrating by parts and using the properties of w_h , it follows that

$$\begin{aligned} \|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}^2 &= - \underbrace{(\nabla \cdot \sigma \nabla w_h, w_h)_{\mathcal{K}}}_{=0} + \underbrace{(\{\sigma \nabla w_h\}, [w_h])_{\mathcal{F}}}_{=0} + \underbrace{([\sigma \nabla w_h], \{w_h\})_{\mathcal{F}_i}}_{=0} \\ &= (\omega \tilde{h}_{\mathcal{F}}^{-1} \overline{[u_h]}, [w_h])_{\mathcal{F}} \leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}} \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [w_h]\|_{\mathcal{F}} \end{aligned}$$

since $\{\sigma \nabla w_h\}$ and $[\sigma \nabla w_h]$ are constant along each face, see Lemma 4.4. Applying further the strong Poincaré inequality (9.6) proves (9.4)

$$\|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}} \leq c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}.$$

Additionally applying (7.1) and (9.7) implies that

$$|||w_h||| \leq c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}} \leq |||u_h|||.$$

Since we are faced to a square linear system, existence and uniqueness of a solution of the linear system are equivalent. Let us denote by $Aw = F$ the square linear system and assume that there is a vector w_1 and w_2 such that $Aw_i = F$, $i = 1, 2$. Further let us denote the difference between them by $e = w_1 - w_2$ and therefore $Ae = 0$. The a priori estimate (9.5) implies that $e = 0$ and thus uniqueness of the solution. \square

COROLLARY 9.2. *Let $u_h \in V_h^b$ be a fixed function. Then there exists a unique $y_h \in V_h^b$ such that*

$$\begin{aligned} \int_{\kappa} y_h \, dx &= 0 \quad \forall \kappa \in \mathcal{K} \\ \{\sigma \nabla y_h\}|_F \cdot n_F &= -s \omega h_F^{-1} \overline{[u_h]} \cdot n_F \quad \forall F \in \mathcal{F} \\ \int_F \{y_h\} &= [\sigma \nabla u_h] \quad \forall F \in \mathcal{F}_i. \end{aligned}$$

Proof. Since the matrix associated to the above defined projection w_h has zero kernel, y_h exists and is unique. \square

9.2. Stability. Although we do not explicitly penalize the solution jumps, control of the solution jumps in the triple norm is recovered by an inf-sup argument which is shown in this section.

LEMMA 9.3. *There is a constant $c > 0$ independent of h such that for all $u_h \in V_h^b$ there holds*

$$|(\{\sigma \nabla u_h\}, [u_h])_{\mathcal{F}}| \leq c \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}} \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}.$$

Proof. By a decomposition of the average operator we have that

$$|(\{\sigma \nabla u_h\}, [u_h])_{\mathcal{F}}| \leq \sum_{\kappa \in \mathcal{K}} \frac{1}{2} |(\sigma \nabla u_h, [u_h])_{\partial \kappa}|,$$

however since by definition $\nabla u_h \cdot n_F \in \mathbb{R}$ we have

$$(\nabla u_h, [u_h])_F = (\nabla u_h, \overline{[u_h]})_F, \quad \forall F \in \mathcal{F}.$$

A Cauchy-Schwarz inequality and a trace inequality completes the proof noting that $\sigma|_{\kappa_1} \leq \rho \sigma|_{\kappa_2}$. \square

THEOREM 9.4. *(Discrete inf-sup condition) There exists a constant $c > 0$ independent of h such that for all $u_h \in V_h^b$ there holds*

$$c |||u_h||| \leq \sup_{v_h \in V_h^b} \frac{a_s(u_h, v_h)}{|||v_h|||},$$

for $s \in \{-1, 1\}$.

Proof. Let us prove this theorem in four steps:

Step 1: First we take $v_h = u_h$ in a standard fashion

$$a_s(u_h, u_h) = \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2 - (1+s)(\{\sigma \nabla u_h\}, [u_h])_{\mathcal{F}}.$$

Note that by Lemma 9.3 followed by an arithmetic-geometric inequality there exists a constant $c_u > 0$ independent on the mesh size h such that

$$a_s(u_h, u_h) \geq \frac{1}{2} \|\sigma^{\frac{1}{2}} \nabla u_h\|_{\mathcal{K}}^2 - c_u (1+s)^2 \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathcal{F}}^2. \quad (9.8)$$

Step 2: Second by Lemma 9.1 there exists $w_h \in V_h^b$ such that

1. $\int_{\kappa} w_h dx = 0$ for all $\kappa \in \mathcal{K}$,
2. $\{\sigma \nabla w_h\} \cdot n_F = \omega h_F^{-1} [\overline{u_h}] \cdot n_F$ on each face $F \in \mathcal{F}$,
3. $\int_F \{w_h\} = 0$ on each face $F \in \mathcal{F}_i$.

An immediate consequence is that

$$\begin{aligned} a_s(u_h, w_h) &= -(\nabla \cdot \sigma \nabla u_h, w_h)_{\mathcal{K}} + ([\sigma \nabla u_h], \{w_h\})_{\mathcal{F}_i} - s(\{\sigma \nabla w_h\}, [u_h])_{\mathcal{F}} \\ &= -s \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_h}]\|_{\mathcal{F}}^2. \end{aligned} \quad (9.9)$$

Step 3: Combining the results (9.8) and (9.9) we may take

$$v_h = u_h - s\left(\frac{1}{2} + c_u(1+s)^2\right)w_h$$

to obtain

$$a_s(u_h, v_h) \geq \frac{1}{2} \|u_h\|^2$$

using Lemma 5.1 and Corollary 5.2.

Step 4: To conclude it remains to show that there exists $c > 0$ independent of h such that

$$\|v_h\| \leq c \|u_h\|.$$

This follows by straightforward estimation

$$\|v_h\| = \|u_h + cw_h\| \leq \|u_h\| + c \|w_h\|.$$

Consider the second term of the right hand side after applying the strong Poincaré inequality (9.6)

$$\|w_h\|^2 \leq c(1+h) \|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [w_h]\|_{\mathcal{F}}^2 = I_1 + I_2.$$

It follows by (9.4) that

$$I_1 \leq c \|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}^2 \leq c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_h}]\|_{\mathcal{F}}^2,$$

and by the trace inequality, the strong Poincaré inequality and by (9.4) that

$$I_2 \leq c \|\sigma^{\frac{1}{2}} \tilde{h}^{-1} w_h\|_{\mathcal{K}}^2 \leq c \|\sigma^{\frac{1}{2}} \nabla w_h\|_{\mathcal{K}}^2 \leq c \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_h}]\|_{\mathcal{F}}^2.$$

Finally use that

$$\|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [\overline{u_h}]\|_{\mathcal{F}}^2 \leq \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} [u_h]\|_{\mathcal{F}}^2,$$

which completes the proof. \square

9.3. Convergence. Using the previously derived inf-sup condition optimal convergence is proved in a standard fashion.

THEOREM 9.5. *Let $u \in H^2(\Omega)$ be the solution of (2.1) and u_h^b the solution of (6.3) then there holds*

$$\|u - u_h^b\| \leq ch \|u\|_{2,\mathcal{K}}.$$

Proof. First note that for the bilinear form $a_s(\cdot, \cdot)$ the following continuity holds for all $v \in H^1(\mathcal{K})$ and $v_h \in V_h^b$ by Cauchy-Schwarz and trace inequality

$$a_s(v, v_h) \leq \|v\|_c \|v_h\|. \quad (9.10)$$

1. Decompose the error in a continuous and a discrete part.

$$|||u - u_h^b||| \leq |||u - \pi_h u||| + |||\pi_h u - u_h^b|||.$$

Recall that π_h denotes the element-wise L^2 -projection onto V_h^1 and observe that the convergence of the continuous part follows by (7.2).

2. Use the inf-sup condition on the discrete part and the consistency of the bilinear form, Lemma 6.3

$$c |||\pi_h u - u_h^b||| \leq \sup_{v_h \in V_h^b} \frac{a_s(\pi_h u - u_h^b, v_h)}{|||v_h|||} \leq \sup_{v_h \in V_h^b} \frac{a_s(\pi_h u - u, v_h)}{|||v_h|||}.$$

3. Conclude by applying the continuity (9.10) and the approximation result (7.2).

□

We will now show that we have optimal L^2 -convergence for the symmetric version thanks to the adjoint consistency.

THEOREM 9.6. *Let $u \in H^2(\Omega)$ with $\|u\|_{2,\Omega} \leq c\|f\|_\Omega$ be the solution of (2.1) and u_h^b the solution of (6.3) with $s = 1$, then there holds*

$$\|u - u_h^b\|_{L^2(\Omega)} \leq c h^2 \|u\|_{2,\Omega}.$$

Proof. Consider the dual problem: Find $\phi \in H_0^1(\Omega)$ such that

$$(\sigma \nabla \phi, \nabla z)_\Omega = (e, z)_\Omega, \quad \forall z \in H_0^1(\Omega).$$

Under the regularity assumptions on u we have $\|\phi\|_{2,\Omega} \leq c\|e\|_\Omega$. It follows that

$$-(\nabla \cdot \sigma \nabla \phi, z)_\Omega = (e, z)_\Omega, \quad \forall z \in L^2(\Omega).$$

Let $e = u - u_h^b$ then we have by the dual consistency of Lemma 6.3

$$\|e\|_\Omega^2 = a_1(e, \phi) = a_1(e, \phi - \pi_h \phi) \leq \|e\| \|\phi - \pi_h \phi\| \leq c h \|e\| \|\phi\|_{2,\Omega}$$

and we conclude using the a priori bound on the $H^2(\Omega)$ norm of ϕ and Theorem 9.5,

$$\|e\|_\Omega \leq c h \|e\| \leq c h^2 \|u\|_{2,\Omega}.$$

□

Additionally this method has some interesting properties as pointed out in the following remarks.

REMARK 9.7. *Let $u_h^b \in V_h^b$ be the solution of (6.3). If the right hand side f is piecewise constant, then there holds*

$$||[\sigma \nabla u_h^b]||_{\mathcal{F}_i}^2 + \|\omega^{\frac{1}{2}} \tilde{h}_{\mathcal{F}}^{-\frac{1}{2}} \overline{[u_h^b]}\|_{\mathcal{F}}^2 = 0.$$

Indeed choosing the function y_h defined in Corollary 9.2 in (6.3) and applying an integration by parts leads to the result.

9.4. Numerical Tests. We consider the same three test problems as in Section §8.3.1. In the case of method 2 the local mass conservation property is satisfied independently of the stabilization parameter.

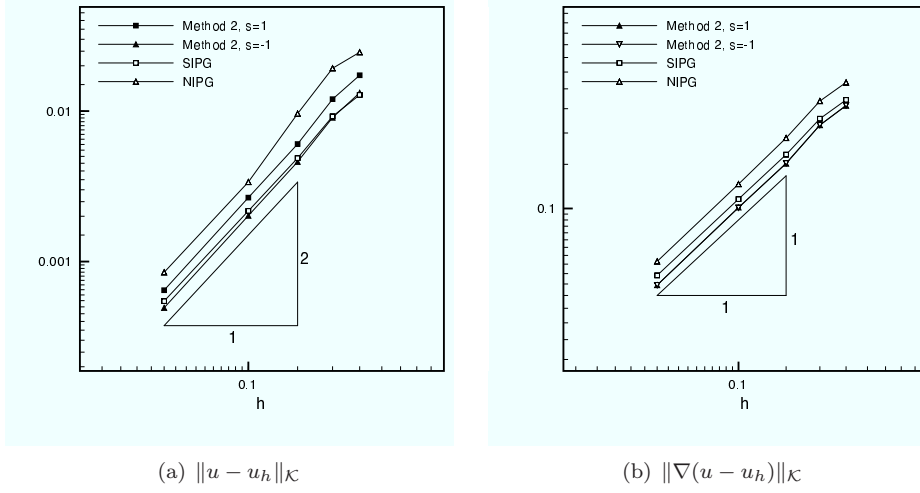


FIG. 9.1. L^2 -error (a) and H^1 -error (b) for h -refinement for the problem with smooth solution. For the SIPG- and NIPG-method a stabilization parameter of $\gamma = 10$ resp. $\gamma = 1$ is chosen.

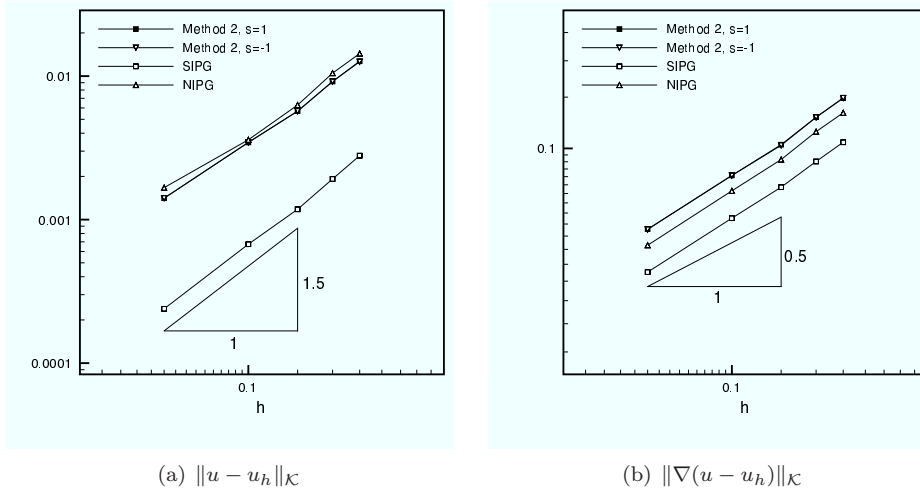


FIG. 9.2. L^2 -error (a) and H^1 -error (b) for h -refinement for the problem with smooth solution. For the SIPG- and NIPG-method a stabilization parameter of $\gamma = 10$ resp. $\gamma = 1$ is chosen.

9.4.1. Convergence. In this section we give the convergence rates of method 2 for test problems a) and b). We get optimal convergence of the error in H^1 - and L^2 -norms, and similar convergence curves as for SIPG and NIPG for the case of smooth exact solution (Figure 9.1). When the solution presents a singularity we once again observe a larger constant for method 2 than SIPG, in particular in the L^2 -norm.

10. Conclusion. In this paper we discussed low order discontinuous Galerkin methods for second order scalar elliptic problems in two and three space dimensions. The main results are given in the following points:

- (i) Midpoint imposition of Dirichlet boundary conditions is sufficient to assure

discrete well-posedness for the symmetric DG-formulation using piecewise affine approximation (no interior stabilization is needed).

(ii) The linear system resulting from method 1 can be transformed into a system with reduced couplings between the continuous and discontinuous spaces. In the symmetric case ($s = 1$) the matrix is block diagonal and can be solved in parallel. In the non-symmetric case ($s = -1$) the resulting matrix is upper triangular.

(iii) The symmetric version of method 1 without stabilization has optimal convergence in the broken H^1 -norm and in the L^2 -norm provided the meshes and data are sufficiently regular or satisfy the macro element property of Lemma 5.6.

(iv) The non-symmetric version of method 1 with stabilization has optimal convergence in the L^2 -norm provided the meshes and data are sufficiently regular.

(v) For irregular data a checkerboard mode destroys convergence for the unstabilized DG-method when using piecewise affine approximations.

(vi) Enriching the space with non-conforming quadratic bubbles allows for a DG-method where the symmetric and non-symmetric versions are stable without stabilization, yielding optimal convergence in the broken H^1 -norm and are locally mass conservative independently of the penalty parameter.

(vii) The symmetric DG-method on the enriched space has additionally optimal convergence in the L^2 -norm.

The aim of this work was to construct a symmetric (i.e. adjoint consistent) DG-method that enjoys optimal convergence and local mass conservation independently of the penalty parameter. This goal has been realized in the framework of low order approximation in the symmetric version of method 2 (see equation (6.3)) or, in the symmetric version of method 1 without penalization on interior faces when the mesh has the macro element property of Lemma 5.6.

Appendix. Here we detail the proof of the Poincaré inequality for the part \mathcal{K}_2 of the mesh in Proposition 5.5.

There exists a function $p \in H^1(\Omega_2^j)$ such that $\nabla \cdot p = |u_d|$ and $\|p\|_{1,\mathcal{K}_2^j} \leq c \|u_d\|_{0,\Omega_2^j}$. It follows that

$$\begin{aligned} \|u_d\|_{0,\Omega_2^j}^2 &= (|u_d|, |u_d|)_{0,\Omega_2^j} = -(\nabla |u_d|, p)_{0,\Omega_2^j} + ([|u|], p)_{\mathcal{F}_i(\mathcal{K}_2^j)} + (|u| \cdot n_e, p)_{\mathcal{F}_e(\mathcal{K}_2^j)} \\ &= I + II + III \end{aligned} \quad (\text{A.1})$$

where n_e denotes the exterior normal with respect to \mathcal{K}_2^j . Let us develop each term separately. Using a Cauchy-Schwarz inequality yields directly

$$I = -(\nabla |u_d|, p)_{0,\Omega_2^j} \leq \|\nabla |u_d|\|_{\mathcal{K}_2^j} \|p\|_{\mathcal{K}_2^j},$$

since $u_d \neq 0, \forall x \in \Omega_2^j$. Using the property that $|u_d|$ lies in the Crouzeix-Raviart space over \mathcal{K}_2^j , i.e. the midpoint continuity, yields

$$\begin{aligned} II &= ([|u|], p - \bar{p})_{\mathcal{F}_i(\mathcal{K}_2^j)} = ([|u| - \bar{u}], p - \bar{p})_{\mathcal{F}_i(\mathcal{K}_2^j)} \leq \sum_{F \in \mathcal{F}_i(\mathcal{K}_2^j)} \| [u - \bar{u}] \|_F \|p - \bar{p}\|_F \\ &\leq ch |u|_{1,\mathcal{K}_2^j} |p|_{1,\mathcal{K}_2^j}, \end{aligned}$$

where

$$\bar{p} = \frac{1}{\text{meas}(F)} \int_F p \, ds \quad \text{and} \quad \bar{q} = \frac{1}{\text{meas}(F)} \int_F q \, ds, \quad \forall F \in \mathcal{F}_i(\mathcal{K}_2^j).$$

For the third term of (A.1) take an arbitrary face $F \in \mathcal{F}_e(\mathcal{K}_2^j)$ of the sum. Observe that the boundary of $\partial\Omega \subset \partial\Omega_1$ where

$$\Omega_1 = \left(\bigcup_{\kappa \in \mathcal{K}_1} \kappa \right)^\circ.$$

As consequence none of the faces of Ω_2^j are contained in the boundary of Ω and thus Ω_2^j shares the boundary with Ω_1 . Therefore write $F = \partial\kappa_1(F) \cap \partial\kappa_2(F)$ with $\kappa_1(F) \in \mathcal{K}_1$ and $\kappa_2(F) \in \mathcal{K}_2^j$. Let us denote $\kappa_1(F)$, $\kappa_2(F)$ by κ_1 resp. κ_2 in the case that the face F is fixed. Use again

$$Z(\kappa_1) = \{x \in \kappa_1 : u_d(x) = 0\},$$

and we can write again

$$u_d(x)|_{\kappa_1} = \nabla u_d|_{\kappa_1} \cdot (x - x^*) \quad \text{with } x^* \in Z(\kappa_1), \quad \forall x \in \kappa_1.$$

Let x_F be the midpoint of the face F . By property of the space V^D we have that

$$0 = 2\{u_d\}(x_F) = u_d|_{\kappa_1}(x_F) + u_d|_{\kappa_2}(x_F) = \nabla u_d|_{\kappa_1} \cdot (x_F - x^*) + u_d|_{\kappa_2}(x_F).$$

Thus

$$u_d|_{\kappa_2}(x) = u_d|_{\kappa_2}(x_F) + \nabla u_d|_{\kappa_2} \cdot (x - x_F) = -\nabla u_d|_{\kappa_1} \cdot (x_F - x^*) + \nabla u_d|_{\kappa_2} \cdot (x - x_F)$$

for all $x \in \kappa_2$. Then

$$\begin{aligned} |(|u| \cdot n_e, p)_F| &\leq |(h_{\kappa_1} |\nabla u_d|_{\kappa_1}, |p|)_F + (h_{\kappa_2} |\nabla u_d|_{\kappa_2}, |p|)_F| \\ &\leq \|p\|_F (\|h_{\kappa_1} \nabla u_d|_{\kappa_1}\|_F + \|h_{\kappa_2} \nabla u_d|_{\kappa_2}\|_F) \\ &\leq c_T \left(h_{\kappa_2}^{-\frac{1}{2}} \|p\|_{0,\kappa_2} + h_{\kappa_2}^{\frac{1}{2}} |p|_{1,\kappa_2} \right) \sum_{i=1}^2 \|h_{\kappa_i}^{\frac{1}{2}} \nabla u_d\|_{\kappa_i} \end{aligned}$$

so that if $h_{\kappa_1} \leq \rho h_{\kappa_2}$ and $h_{\kappa_2} \leq 1$, then

$$|(|u| \cdot n_e, p)_F| \leq c (\|p\|_{0,\kappa_2} + h_{\kappa_2} |p|_{1,\kappa_2}) \sum_{i=1}^2 \|\nabla u_d\|_{\kappa_i} \leq c \|p\|_{1,\kappa_2} \sum_{i=1}^2 \|\nabla u_d\|_{\kappa_i}.$$

Define the boundary layer between \mathcal{K}_1 and the \mathcal{K}_2^j by

$$\begin{aligned} \mathcal{K}^{\Gamma,j} &= \mathcal{K}_1^{\Gamma,j} \cap \mathcal{K}_2^{\Gamma,j}, \\ \mathcal{K}_1^{\Gamma,j} &= \{\kappa_1 \in \mathcal{K}_1 : \exists \kappa_2 \in \mathcal{K}_2^j \text{ s.t. } \partial\kappa_1 \cap \partial\kappa_2 \neq \emptyset\}, \\ \mathcal{K}_2^{\Gamma,j} &= \{\kappa_2 \in \mathcal{K}_2^j : \exists \kappa_1 \in \mathcal{K}_1 \text{ s.t. } \partial\kappa_1 \cap \partial\kappa_2 \neq \emptyset\}. \end{aligned}$$

Then

$$\begin{aligned} |III| &\leq \sum_{F \in \mathcal{F}_e(\mathcal{K}_2^j)} |(|u| \cdot n_e, p)_F| \leq c \sum_{F \in \mathcal{F}_e(\mathcal{K}_2^j)} \|p\|_{1,\kappa_2(F)} (\|\nabla u_d\|_{\kappa_1(F)} + \|\nabla u_d\|_{\kappa_2(F)}) \\ &\leq c \|p\|_{1,\mathcal{K}_2^j} \|\nabla u_d\|_{\mathcal{K}^{\Gamma,j}} \leq c \|u_d\|_{\mathcal{K}_2^j} \|\nabla u_d\|_{\mathcal{K}^{\Gamma,j}}. \end{aligned}$$

On the other hand using similar arguments as before

$$\begin{aligned} I + II &\leq c \|\nabla u_d\|_{\mathcal{K}_2^j} \left(\|p\|_{\mathcal{K}_2^j} + h |p|_{1, \mathcal{K}_2^j} \right) \leq c \|\nabla u_d\|_{\mathcal{K}_2^j} \|p\|_{1, \mathcal{K}_2^j} \\ &\leq c \|\nabla u_d\|_{\mathcal{K}_2^j} \|u_d\|_{\mathcal{K}_2^j}. \end{aligned}$$

Thus

$$\|u_d\|_{\Omega_2^j} \leq c \left(\|\nabla u_d\|_{\mathcal{K}_2^j} + \|\nabla u_d\|_{\mathcal{K}_1^{\Gamma, j}} \right)$$

and

$$\|u_d\|_{\Omega_2}^2 \leq c \left(\|\nabla u_d\|_{\mathcal{K}_2}^2 + \sum_{j=1}^m \|\nabla u_d\|_{\mathcal{K}_1^{\Gamma, j}}^2 \right).$$

Observe that

$$\sum_{j=1}^m \|\nabla u_d\|_{\mathcal{K}_1^{\Gamma, j}}^2 \leq (d+1) \|\nabla u_d\|_{\mathcal{K}_1}^2$$

since in the worst case an element $\kappa_1 \in \mathcal{K}_1$ is at most in $(d+1)$ different $\mathcal{K}_1^{\Gamma, j}$ and thus

$$\|u_d\|_{\Omega_2} \leq c \|\nabla u_d\|_{\mathcal{K}}.$$

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