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EDGE STABILIZATION FOR THE GENERALIZED STOKES PROBLEM: A CONTINUOUS INTERIOR PENALTY METHOD

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Abstract. In this note we introduce a new stabilized finite element method for the generalized Stokes equation. The method uses least square stabilization of the gradient jumps across element boundaries and can be seen as a higher order version of the Brezzi-Pitkäranta penalty stabilization [6]. The method gives better resolution on the boundary for the Stokes equation than does classical Galerkin Least Squares fomulation and has quasi optimal convergence properties for the porous media models of Darcy and Brinkman. Some numerical examples are given.

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1. Introduction

The use of equal order interpolation of the pressure and the velocities for the Stokes problem are not stable if implemented without stabilization. Over the years many stabilization methods have been proposed and stabilization is by now a well established discipline with different well explored methods like the SUPG/SD-method [13], the residual free bubbles [5] and more recent contributions like local projection methods [3, 9] for Stokes problem. The relation between the different approaches is also well understood in most cases. In this paper we present a method which stabilizes both Stokes problem and Darcy's problem by adding a least-squares term based on the jump in the gradient over element boundaries. The method has many of the advantages of the above methods, but no additional degrees of freedom are added, no hierarchical meshes are needed, the formulation remains symmetric, and the mass can be lumped for efficient time marching and treatment of stiff source terms. The price to pay is an increased number of non-zero elements in the jacobian due to the fact that the gradient jump term couple neighboring elements. This method has been successfully applied to the problem of convection–diffusion in [7] and it was noted that the stabilization parameter was independent of the diffusion parameter, hence making the method very well suited also for degenerate diffusion problems. For the Stokes problem the behavior is somewhat different and, depending on how the stabilization parameter scales with respect to the meshsize h, the analysis gives different results. Using the optimal choice yields the following a priori error estimates for the Stokes' and the Darcy's problems respectively,

- Stokes

$$\|u - u_h\|_{0,\Omega} + h\Big(\|\nabla(u - u_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega}\Big) \le Ch^2\Big(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}\Big)$$

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- Darcy

$$\|u-u_h\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}+h^{1/2}\Big(\|\nabla\cdot(u-u_h)\|_{0,\Omega}+\|\nabla(p-p_h)\|_{0,\Omega}\Big) \leq Ch^{3/2}\Big(\|u\|_{2,\Omega}+\|p\|_{2,\Omega}\Big).$$

We observe that this is optimal for the case of Stokes equation. For Darcy's equation we have optimality for the divergence of the velocities and the gradient of the pressure and suboptimality with a gap of half a power of h for the pressures and the velocities in the L^2 norm. This result for the vanishing viscosity case is very similar to the corresponding convection–diffusion result.

2. Generalized Stokes' problem

We propose to study a generalized Stokes problem, with two parameters σ and ν including the Darcy's equation as a special case. We consider the problem of solving the partial differential equation

$$\begin{aligned} \sigma u-\nu\Delta u+\nabla p &= f \quad \text{in } \Omega,\\ \nabla\cdot u &= g \quad \text{in } \Omega,\\ u\cdot n &= 0 \quad \text{on } \partial\Omega\\ \nu u\cdot t &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

where Ω is bounded polygonal domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $d=2,3$ and σ and ν are two positive parameters, that may not vanish simultaneously. This problem can be written in weak form as follows: Find $u\in V=\{v\in[H^1(\Omega)]^d:\ v|_{\partial\Omega}=0\}$ when $\nu>0$ ($u\in V=\{v\in[L^2(\Omega)]^d,\nabla\cdot v\in L^2(\Omega):\ v\cdot n|_{\partial\Omega}=0\}$ for $\nu=0$) and $p\in Q=L_2(\Omega)/\mathbb{R}$ when $\nu>0$ ($p\in H^1(\Omega)$ for $\nu=0$) such that

$$a(u,v)+b(p,v)-b(q,u)=L(v,q),\quad \forall(v,q)\in V\times Q, \tag{2}$$

where

$$a(u,v)=\int_\Omega\sum_{i=1}^d \sigma u_iv_i+\nu\nabla u_i\cdot\nabla v_i\,\mathrm{d}x,\quad b(p,v)=-\int_\Omega p\ \nabla\cdot v\,\mathrm{d}x,$$

and

$$L(v,q)=\int_\Omega f\cdot v\,\mathrm{d}x-\int_\Omega gq\,\mathrm{d}x.$$

The finite element method consists of seeking piece wise polynomial approximations u_h of u and p_h of p, where $u_h\in V^h\subset V$ and $p_h\in Q^h\subset Q$, with V^h and Q^h built from continuous functions. Consider a partitioning of Ω into a conforming triangulation T_h of affine simplicies K. We shall be concerned with the approximation

$$V^h=\{v\in[V\cap C^0(\Omega)]^d:\ v|_K\in[P^1(K)]^d\quad \forall K\in T_h\},$$

and a continuous pressure space,

$$Q^h=\{q\in Q\cap C^0(\Omega):\ q|_K\in P^1(K)\quad \forall K\in T_h\}.$$

It is well known that the combination $V^h\times Q^h$ is unstable (see, e.g., [4]).

The edge stabilization method can be formulated as follows: Find $(u_h,p_h)\in V^h\times Q^h$ such that

$$\begin{aligned} a(u_h,v)+b(p_h,v)+\tilde{j}(u_h,v) &= L(v,0) \quad \text{in } \Omega,\\ b(q,u_h)-j(p_h,q) &= L(0,q) \quad \text{in } \Omega, \end{aligned} \tag{3}$$

for all $(v,q) \in V^h \times Q^h$, where

$$j(p,q) := \sum_K \frac{1}{2} \int_{\partial K} \gamma h_K^{s+1} \, [n \cdot \nabla p_h] \, [n \cdot \nabla q] \, \mathrm{d}s \tag{4}$$

and

$$\tilde{j}(u,v) := \sum_K \frac{1}{2} \int_{\partial K} \gamma h_K^{s+1} \, [\nabla \cdot u_h] \, [\nabla \cdot v] \, \mathrm{d}s, \tag{5}$$

where $[x]$ denotes the jump of quantity x over edge ∂K when $\partial K \cap \partial\Omega = \emptyset$ else $[x] = 0$. The coefficient s takes the values $s = 2$ in the case $\nu \geq h$ and $s = 1$ in the case $\nu < h$.

Remark 2.1. *The change of the order of the parameter when passing from the viscous case to the non-viscous case ressembles the behavior of the SUPG method for convection–diffusion problems. A standard way of handling this for problems where the viscosity is non-uniform in the domain is to use the ν–weighted parameter $\gamma h_K^2(1 + \frac{\nu}{h_K})^{-1}$.*

Remark 2.2. *On a uniform mesh, the jump term $j(p,q)$ (with $s = 2$) can be seen as the only remaining contribution from a discretization of $h^4\Delta^2 p$ when applying the discontinuous method proposed by Baker [1] to piece wise linear approximations p_h of p. In this sense the method is related to that of Brezzi-Pitkäranta [6], where the corresponding stabilization term can be seen as an approximation of $h^2\Delta p$.*

Remark 2.3. *A stabilization method for Stokes like the one proposed here has been independently proposed by Becker & Braack [3] as an example of a possible stabilization fitting a theoretical framework quite different from ours. In [3] the main focus is on a conceptually different stabilization method (local projections), however, and no numerical results with the jump approach were presented.*

Remark 2.4. *The term penalizing the incompressibility condition is necessary only in the case where $\nu < h$. This term is needed to give a $\|h^{1/2}\nabla \cdot u\|$ contribution to the triple norm necessary to obtain optimal order estimates in the case of Darcy flow. However it should be noted that for $u \in H^2(\Omega)$ we may use the same jump operator for the pressure and the velocity, hence stabilizing the jumps of the gradient (component wise for the velocities). This will not affect the order of the a priori estimates but gives increased control of the gradients at the cost of larger constants in the estimate.*

Remark 2.5. *The stabilizing Galerkin/Least-Squares (GLS) method in different guises has been used extensively; for pioneering work in this direction, see, e.g., [6,10,13]. However, there is in GLS a decrease of accuracy close to the boundaries due to artificial pressure boundary conditions, for which a number of remedies have been proposed, cf. [2,9,11]. Following the edge stabilization method, there is less degradation of accuracy close to the boundary. See figure 1 and 2.*

3. The inf-sup condition

For Stokes equation the essential feature of a stabilized method is the satisfaction of the *inf–sup* condition. We introduce the triple norm

$$|||(u_h,p_h)|||_s^2 = \sigma\|u_h\|^2 + \nu\|\nabla u_h\|^2 + c_d\|h^{s/2}\nabla \cdot u_h\|^2 + c_g\|h^{s/2}\nabla p_h\|^2 + c_p\|p_h\|^2,$$

where c_d, c_g, and c_p are constants, depending on the material data, which will be defined in the stability analysis below (cf. Remark 3.6). We also define the bilinear form

$$A[(u,p),(v,q)] := a(u,v) + b(p,v) - b(q,u) + \tilde{j}(u,v) + j(p,q).$$

The stability of the method is obtained by the fact that the edge operator controls the projection error of $h^s\nabla p_h$. This allows us to control $\|h^{s/2}\nabla p_h\|$, which in its turn leads to satisfaction of the inf-sup condition. We will for

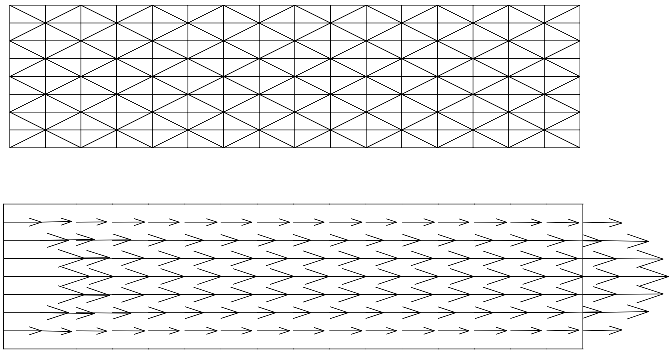

FIGURE 1. Mesh and computed velocity for Poiseuille flow

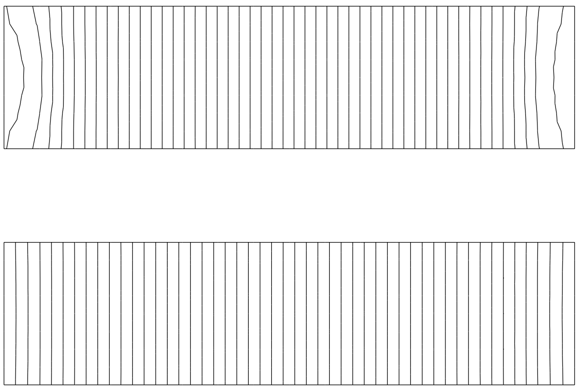

FIGURE 2. Brezzi-Pitkäranta (top) and edge stabilization (bottom) pressure isolines.

simplicity assume that h_K is uniform so that $\pi_h h^s \nabla p_h = h^s \pi_h \nabla p_h$, where π_h is an interpolation operator to be defined later, and that $h_K < 1$ for all K. By $\{\varphi_i\}$ we denote the set of finite element basis functions spanning the space V_h. Let $\mathcal{N}_i$ be the set of all triangles K^i containing node i and assume that the cardinality of $\mathcal{N}_i$ is bounded uniformly in i. Let $\mathcal{F}_K$ be the set of all test functions φ_i such that $K \in \operatorname{supp} \varphi_i$ and $\Omega_i = \bigcup_{\mathcal{N}_i} K^i$. We will consider a function $y \in [P_0(K)]^d$, and its element wise representation in the finite element basis $\tilde{y}$ defined by

$$\tilde{y}|_K = y|_K \sum_{i \in \mathcal{F}_K} \varphi_i. \tag{6}$$

It follows that $\tilde{p} = p$ everywhere except on elements adjacent to Dirichlet boundaries where the boundary nodes are not included in the finite element space. We note that, with $y := \nabla p_h$, we wish to choose as our testfunction $v = h^s \pi_h y$ to obtain after an integration by parts

$$b(p_h, v) = \|h^{s/2} y\|^2 + (y, h^s(\pi_h y - y)), \tag{7}$$

and we wish to bound the projection error using the jump term. This cannot be done exactly since $\pi_h y$ must obey the boundary conditions, unlike y. However, (7) can equally well be written $(h^s y, \tilde{y}) + (h^s y, \pi_h y - \tilde{y})$, and if we can show that $c_b \|y\|^2 \leq (y, \tilde{y})$ we have

$$c_b \|h^{s/2} y\|^2 + (y, h^s(\pi_h y - \tilde{y})) \leq (y, h^s \tilde{y}) + (y, h^s(\pi_h y - \tilde{y})),$$

and we can proceed to bound the second term on the left hand side in terms of the first together with the jumps. Thus, we need:

Lemma 3.1. *Suppose that K is an element with at least one node on a Dirichlet boundary then*

$$\|y\|_K^2 = \frac{d+1}{n_i}(y,\tilde{y}), \tag{8}$$

where n_i denotes the number of interior nodes of the element.

Proof. The proof is immediate noting that

$$(y,\tilde{y}) = |y_K|^2 \int_K \sum_{i\in\mathcal{F}_K} \varphi_i \mathrm{d}x = \frac{n_i}{d+1}|y_K|^2 m(K).$$

□

We will now recall some results from [7] essential for the analysis. The stability argument is based on the fact that the projection error of the gradient is controlled by the edge stabilization term.

$$\|h^{s/2}(\tilde{y}-\pi_h y)\|^2 \leq \tilde{J}_s(y,y)$$

with

$$\tilde{J}_s(y,y) = \sum_K \int_{\partial K} \gamma h^{s+1}[y]^2 \mathrm{d}s.$$

The operator $\pi_h : \nabla Q_h \to V_h$, which denotes the lowest order Clément operator is constructed as follows.

$$\pi_h y = \sum_i y_i \varphi_i \tag{9}$$

with

$$y_i = \frac{1}{m(\Omega_i)} \sum_{\mathcal{N}_i} y|_{K^i} m(K^i). \tag{10}$$

We shall frequently use the following inequalities, which we collect in a Lemma.

Lemma 3.2. *For the Clément operator there holds*

$$\|\pi_h u\|_{s,\Omega} \leq C_c \|u\|_{s,\Omega}, \quad \forall u \in H^s(\Omega), \tag{11}$$

for $s = 0, 1$. Further,

$$\|h_K \nabla \pi_h p_h\| \leq C_i \|p_h\|, \quad \forall p_h \in Q_h. \tag{12}$$

Finally, we have the trace inequality

$$\|v\|_{0,\partial K}^2 \leq C_t\Big(h_K^{-1}\|v\|_{0,K}^2 + h_K\|v\|_{1,K}^2\Big), \quad \forall v \in H^1(K), \tag{13}$$

Proof. Inequality (11) follows from the interpolation estimate

$$\|u-\pi_h u\|_{s,\Omega} \leq c_i \|u\|_{s,\Omega}, \quad s = 0, 1,$$

cf. [8], and (12) follows from (11) and the well known inverse inequality

$$\|v\|_{1,K} \leq C h_K^{-1}\|v\|_{0,K}, \quad \forall v \in V_h. \tag{14}$$

Finally, a proof of (13) is given in [16]. □

In [7] we proved the following lemma.

Lemma 3.3. *If y is some piecewise constant function, $\tilde{y}$ is defined by (6) and π_h is the Clément interpolant on V_h, then the edge stabilization term satisfies*

$$\|h^{s/2}(\pi_h y - \tilde{y})\|^2 \leq \gamma \tilde{J}_s(y, y) \tag{15}$$

for some $\gamma \geq \gamma_0 > 0$ independent of h but not of the mesh regularity.

Finally, we shall also need the following Lemma.

Lemma 3.4. *For $v \in V_h$, the jump operator fulfills*

$$\tilde{j}(v, v)^{1/2} \leq C_s \gamma^{1/2} \|h^s \nabla \cdot v\|. \tag{16}$$

Proof. For each edge E shared by two elements K_1 and K_2, we have

$$\int_E \gamma h^{s+1} [\nabla \cdot v]^2 ds \leq \sum_{i=1}^{2} \int_E \gamma h^{s+1} (\nabla \cdot v|_{K_i})^2 ds \leq C\gamma \sum_{i=1}^{2} \|h^s \nabla \cdot v\|^2_{K_i}$$

by scaling. Summing over all edges gives the result. □

Theorem 3.5. *Suppose that either $\sigma \geq 1$ or $\nu \geq 1$ and that $s = 1$ when $\nu < h$ and $s = 2$ when $\nu > h$ then the formulation (3) satisfies the* inf–sup *condition*

$$|||(u_h, p_h)||| \leq c_0 \sup_{(v,q) \in V_h \times Q_h} \frac{A[(u_h, p_h), (v, q)]}{|||(v, q)|||}$$

Proof. First we take $(v, q) = (u_h, p_h)$ to obtain

$$A[(u_h, p_h), (u_h, p_h)] = \sigma \|u_h\|^2 + \nu \|\nabla u_h\|^2 + \tilde{j}(u_h, u_h) + j(p_h, p_h). \tag{17}$$

By taking $(v, q) = (\pi_h(h^s \nabla p_h), 0)$ we obtain the desired control of $\|h^{s/2} \nabla p_h\|$ in the following fashion. We have

$$\begin{aligned} A[(u_h, p_h), (\pi_h(h_K^s \nabla p_h), 0)] &= \sigma(u_h, \pi_h(h^s \nabla p_h)) + \nu(\nabla u_h, \nabla \pi_h(h^s \nabla p_h)) \\ &\quad - (p_h, \nabla \cdot \pi_h(h^s \nabla p_h)) + \tilde{j}(u_h, \pi_h(h^s \nabla p_h)). \end{aligned}$$

Estimating termwise, we have

$$(\sigma u_h, \pi_h h^s \nabla p_h) \geq -C_c \sigma^{1/2} h^{s/2} \|\sigma^{1/2} u_h\| \, \|h^{s/2} \nabla p_h\|, \tag{18}$$

and

$$\begin{aligned} (\nu \nabla u_h, \nabla \pi_h h^s \nabla p_h) &\geq -\|\nu^{1/2} \nabla u_h\| \, \|\nu^{1/2} \nabla (\pi_h h^s \nabla p_h)\| \\ &\geq -\|\nu^{1/2} \nabla u_h\| C_i C_c \nu^{1/2} \|h^{s-1} \nabla p_h\|. \end{aligned}$$

Thus, for $s = 2$ we find

$$(\nu \nabla u_h, \nabla \pi_h h^s \nabla p_h) \geq -\|\nu^{1/2} \nabla u_h\| C_i C_c \nu^{1/2} \|h^{s/2} \nabla p_h\|, \tag{19}$$

and for $s = 1$, i.e., the case $\nu \leq h$,

$$(\nu \nabla u_h, \nabla \pi_h h^s \nabla p_h) \geq -\|\nu^{1/2} \nabla u_h\| C_i C_c \|\nu^{1/2} \nabla p_h)| \geq -\|\nu^{1/2} \nabla u_h\| C_i C_c \|h^{s/2} \nabla p_h\|. \tag{20}$$

Further,

$$\begin{aligned}(p_h, \nabla\cdot\pi_h h^s\nabla p_h) &= (h^s\nabla p_h, \pi_h\nabla p_h - \widetilde{\nabla p_h}) + (h^s\nabla p_h, \widetilde{\nabla p_h})\\ &\geq -\|h^{s/2}\nabla p_h\|\,\|h^{s/2}\pi_h\nabla p_h - \widetilde{\nabla p_h}\| + \frac{1}{d+1}\|h^{s/2}\nabla p_h\|^2\\ &\geq \frac{3}{4(d+1)}\|h^{s/2}\nabla p_h\|^2 + (d+1)\|h^{s/2}(\nabla p - \widetilde{\nabla p_h})\|^2\end{aligned} \tag{21}$$

where we used that $ab \leq a^2/4 + b^2$ for real numbers a, b. Finally, using Lemmas 3.2 and 3.4,

$$\begin{aligned}\tilde{j}(u_h, \pi_h h^s\nabla p_h) &= \sum_K \frac{1}{2}\int_{\partial K}\gamma h^{s+1}[\nabla\cdot u_h]\cdot[\nabla\cdot(\pi_h h^s\nabla p_h)]\,ds\\ &\leq \tilde{j}(u_h,u_h)^{1/2}\Big(\sum_K\frac{1}{2}\int_{\partial K}\gamma h^{s+1}[\nabla\cdot(\pi_h h^s\nabla p_h)]^2 ds\Big)^{1/2}\\ &\leq \tilde{j}(u_h,u_h)^{1/2}C_s\|\gamma^{1/2}h^s\nabla\cdot(\pi_h h^s\nabla p_h)\|\\ &\leq \tilde{j}(u_h,u_h)^{1/2}C_sC_i\|\gamma^{1/2}h^{s-1}\pi_h h^s\nabla p_h\|\\ &\leq \tilde{j}(u_h,u_h)^{1/2}C_sC_iC_c\gamma^{1/2}h^{(3s-2)/2}\|h^{s/2}\nabla p_h\|.\end{aligned} \tag{22}$$

Using (18) to (22), we deduce

$$\begin{aligned}A[(u_h,p_h),(\pi_h(h_K^s\nabla p_h),0)] \;\geq\; & c_b\|h^{s/2}\nabla p_h\|^2 + (h^{s/2}\nabla p_h, h^{s/2}(\pi_h\nabla p_h - \widetilde{\nabla p_h}))\\ & -A[(u_h,0),(u_h,0)]^{1/2}\alpha_1\|h^{s/2}\nabla p_h\|,\end{aligned} \tag{23}$$

where $\alpha_1 = \max(C_s\sigma^{1/2}h^{s/2}, C_iC_c\nu^{1/2}, C_iC_c, C_iC_sC_iC_c\gamma^{1/2}h^{(3s-2)/2})$. We conclude that by lemma 3.3 we have

$$A[(u_h,p_h),(\pi_h(h^s\nabla p_h),0)] \geq c_b\Big(1-\epsilon_1-\alpha_1^2\epsilon_1\Big)\|h^{s/2}\nabla p_h\|^2 - \frac{1}{4c_b\epsilon_1}A[(u_h,p_h),(u_h,p_h)]. \tag{24}$$

We now choose $\epsilon_1 = \frac{1}{2(1+\alpha_1^2)}$ and multiply by $c_b\epsilon_1$ to obtain

$$\frac{c_b^2\epsilon_1}{2}\|h^{s/2}\nabla p_h\|^2 - \frac{1}{4}A[(u_h,p_h),(u_h,p_h)] \leq A[(u_h,p_h),(c_b\epsilon_1\pi_h(h^s\nabla p_h),0)]. \tag{25}$$

By the surjectivity of the divergence operator (see [12]) there exists $v_p \in [H_0^1(\Omega)]^d$ such that $\nabla\cdot v_p = p_h$ and $\|v_p\|_{1,\Omega} \leq C\|p_h\|$. We now choose $(v,q) = (\pi_h v_p, 0)$ and use that $\|p_h\|^2 - (p_h, \nabla\cdot v_p) = 0$ by the properties of v_p. This gives

$$\begin{aligned}\|p_h\|^2 - (p_h,\nabla\cdot v_p) + A[(u_h,p_h),(\pi_h v_p,0)] &= \|p_h\|^2 + (p_h,\nabla\cdot(\pi_h v_p - v_p))\\ &\quad + \sigma(u_h,\pi_h v_p) + \nu(\nabla u_h,\nabla\pi_h v_p) + \tilde{j}(u_h,\pi_h v_p)\\ &\geq (1-\alpha_2^2\epsilon_2)\|p_h\|^2 + (p_h,\nabla\cdot(\pi_h v_p - v_p)) - \frac{1}{4\epsilon_2}A[(u_h,0),(u_h,0)],\end{aligned} \tag{26}$$

with $\alpha_2 = \max(\sigma^{1/2}C_cC_f, \nu^{1/2}C_cC_f, C_sC_cC_fh^{s/2}\gamma^{1/2})$, where we used the stability of the Clément interpolation operator: $\|\nabla\cdot\pi_h v_p\| \leq C\|v_p\|_{1,\Omega} \leq C_f\|p_h\|$. We have also used the following lower bound on the stabilizing term

$$\begin{aligned}\tilde{j}(u_h, \pi_h v_p) &\geq -\frac{1}{\epsilon_2}\tilde{j}(u_h, u_h) - \epsilon_2 \tilde{j}(\pi_h v_p, \pi_h v_p)\\ &\geq -\frac{1}{\epsilon_2}\tilde{j}(u_h, u_h) - \epsilon_2 C_s^2 \gamma \|h^{s/2}\nabla\cdot \pi_h v_p\|^2\\ &\geq -\frac{1}{\epsilon_2}\tilde{j}(u_h, u_h) - \epsilon_2 C_s^2 C_c^2 C_f^2 h^s \gamma \|p_h\|^2\end{aligned}$$

obtained by a scaling argument and the stability of the Clément operator. Focusing now on the second term on the right hand side we obtain by partial integration and the properties of v_p.

$$(p_h, \nabla\cdot(\pi_h v_p - v_p)) \leq \frac{c_b \epsilon_1}{4\epsilon_2}\|h^{s/2}\nabla p_h\|^2 + \frac{\epsilon_2 c_i}{c_b \epsilon_1}\|h^{(2-s)/2} p_h\|^2.$$

This leads to the following inequality for p_h

$$\begin{aligned}(1 - \alpha_2^2\epsilon_2 - \frac{\epsilon_2 c_i h^{(2-s)}}{c_b\epsilon_1})\|p_h\|^2 - \frac{c_b\epsilon_1}{4\epsilon_2}\|h^{s/2}\nabla p_h\|^2 - \frac{1}{4\epsilon_2}A[(u_h, 0), (u_h, 0)]\\ \leq A[(u_h, p_h), (\pi_h v_p, 0)].\end{aligned} \tag{27}$$

Choosing now $\epsilon_2 = \frac{1}{2}c_b\epsilon_1/(c_b\epsilon_1\alpha_2^2 + c_i h^{2-s})$ and multiplying through by ϵ_2 we have

$$\frac{\epsilon_2}{2}\|p_h\|^2 - \frac{c_b\epsilon_1}{4}\|h^{s/2}\nabla p_h\|^2 - \frac{1}{4}A[(u_h, 0), (u_h, 0)] \leq A[(u_h, p_h), (\epsilon_2\pi_h v_p, 0)]. \tag{28}$$

It remains to control $\|h^{s/2}\nabla\cdot u_h\|$. We choose $v = 0$, $q = \pi_h h^s \nabla\cdot u_h$ to obtain

$$\begin{aligned}\|h^{s/2}\nabla\cdot u_h\|^2 + (h^{s/2}\nabla\cdot u_h, h^{s/2}(\pi_h\nabla\cdot u_h - \nabla\cdot u_h)) + j(p_h, \pi_h h^s\nabla\cdot u_h)\\ = A[(u_h, p_h), (0, \pi_h h^s \nabla\cdot u_h)].\end{aligned}$$

Arguing as before we find that

$$\begin{aligned}\frac{3}{4}\|h^{s/2}\nabla\cdot u_h\|^2 - \|h^{s/2}(\pi_h\nabla\cdot u_h - \nabla\cdot u_h)\|^2 - \frac{1}{\epsilon_3}j(p_h, p_h) - \epsilon_3 j(\pi_h h^s\nabla\cdot u_h, \pi_h h^s \nabla\cdot u_h)\\ \leq A[(u_h, p_h), (0, \pi_h\nabla\cdot u_h)]\end{aligned}$$

Using now Lemma 3.4 followed by (12) we find

$$j(\pi_h h^s\nabla\cdot u_h, \pi_h h^s\nabla\cdot u_h) \leq C_s^2 C_i^2 \gamma \|h^{\frac{3s-2}{2}}\nabla\cdot u_h\|^2,$$

and we have (since $h < 1$)

$$\left(\frac{3}{4} - \epsilon_3 C_s^2 C_i^2\gamma\right)\|h^{s/2}\nabla\cdot u_h\|^2 - \tilde{j}(u_h, u_h) - \frac{1}{\epsilon_3}j(p_h, p_h) \leq A[(u_h, p_h), (0, \pi_h h^s\nabla\cdot u_h)].$$

We fix $\epsilon_3 = \frac{1}{4C_s^2C_i^2\gamma}$ and then multiply both sides with $\epsilon_4 = (4\max(1, \frac{1}{\epsilon_3}))^{-1}$ resulting in

$$\frac{\epsilon_4}{2}\|h^{s/2}\nabla\cdot u_h\|^2 - \frac{1}{4}A[(u_h, p_h), (u_h, p_h)] \leq A[(u_h, p_h), (0, \epsilon_4\pi_h h^s\nabla\cdot u_h)]. \tag{29}$$

Summing equations (17), (25), (28) and (29) yields

$$\frac{1}{4}A[(u_h,p_h),(u_h,p_h)] + \frac{\epsilon_4}{2}\|h^{s/2}\nabla\cdot u_h\|^2 + \frac{c_b\epsilon_1}{2}\|h^{s/2}\nabla p_h\|^2 + \frac{\epsilon_2}{2}\|p_h\|^2 \\ \leq A[(u_h,p_h),(u_h + c_b\epsilon_1\pi_h(h^s\nabla p_h) + \epsilon_2\pi_h v_p, p_h + \epsilon_4\pi_h h^s\nabla\cdot u_h)].$$

Setting now, $c_d = 2\epsilon_4$, $c_g = 2c_b\epsilon_1$ and $c_p = 2\epsilon_2$ we may write

$$\frac{1}{4}|||(u_h,p_h)|||^2 \leq A[(u_h,p_h),(u_h + c_b\epsilon_1\pi_h(h^s\nabla p_h) + \epsilon_2\pi_h v_p, p_h + \epsilon_4\pi_h h^s\nabla\cdot u_h)].$$

The thesis follows by noting that there exists some constant c such that $|||(v,q)||| \leq c|||(u_h,p_h)|||$. By similar arguments as above there follows

$$|||(c_b\epsilon_1\pi_h(h^s\nabla p_h),0)|||^2 \leq C\|h^{s/2}\nabla p_h\|^2$$

$$|||(\epsilon_2\pi_h v_p,0)|||^2 \leq C\|p_h\|^2$$

and

$$|||(0,\epsilon_4\pi_h h^s\nabla\cdot u_h)|||^2 \leq C\|h^{s/2}\nabla\cdot u_h\|^2$$

where the constants C depend on material data but not on h. The constant c is in fact of order unity under the condition $\sigma \geq 1$ or $\nu \geq 1$. □

Remark 3.6. *The essential dependencies of the constants c_g, c_d and c_p are*

$$c_g, c_p \approx O\left(\frac{1}{\max(\sigma,\nu)}\right)$$

$$c_d \approx O(1)$$

4. A priori error estimates

A priori estimates are obtained in the standard fashion using

- stability
- consistency
- approximation.

The first point was handled in the previous section and we will now take care of the other two, before proving our error estimates.

By definition of our method, we have the consistency condition.

Lemma 4.1. *For $(u,p) \in [H^2(\Omega)]^{d+1}$ there holds*

$$A[(u-u_h,p-p_h),(v,q)] = 0,$$

for all $(v,q) \in V^h \times Q^h$.

In addition we have the following approximation property.

Lemma 4.2. *Let $(u,p) \in [H^2(\Omega)]^3 \times H^2(\Omega)$. Then we have*

$$|||(u-\pi_h u, p-\pi_h p)||| \leq Ch\Big((h^{s/2}(c_d+\gamma^{1/2}) + \sigma^{1/2}h + \nu^{1/2})\|u\|_{2,\Omega} + h^{s/2}\max(c_g^{1/2}, c_p^{1/2}, \gamma^{1/2})\|p\|_{2,\Omega}\Big).$$

Proof. Using standard interpolation we obtain for the velocities

$$\|u - \pi_h u\|_{0,\Omega} \leq Ch^2 \|u\|_{2,\Omega}$$

$$\|\nabla(u - \pi_h u)\|_{0,\Omega} \leq Ch\|u\|_{2,\Omega},$$
$$\|h^{s/2}\nabla \cdot (u - \pi_h u)\|_{0,\Omega)} \leq Ch^{1+\frac{s}{2}}\|u\|_{2,\Omega},$$

and equivalently for the pressure

$$\|p - \pi_h p\|_{0,\Omega} \leq Ch^2\|p\|_{2,\Omega},$$
$$\|h^{s/2}\nabla(p - \pi_h p)\|_{0,\Omega} \leq Ch^{1+\frac{s}{2}}\|p\|_{2,\Omega}.$$

Further, we have, using (13),

$$\begin{aligned}\|n\cdot\nabla(p-\pi_h p)\|^2_{0,\partial K} &\leq C\Big(h_K^{-1}\|\nabla(p-\pi_h p)\|^2_{0,K}\\ &\quad +\|\nabla(p-\pi_h p)\|_{0,K}\|\nabla(p-\pi_h p)\|_{1,K}\Big)\\ &\leq Ch_K\|p\|^2_{2,K},\end{aligned}$$

and it follows by summation that $j(p - \pi_h p, p - \pi_h p)^{1/2} \leq Ch^{1+\frac{s}{2}}\|p\|_{2,\Omega}$. In the same fashion clearly $\tilde{j}(u - \pi_h u, u - \pi_h u)^{1/2} \leq Ch^{1+\frac{s}{2}}\|u\|_{2,\Omega}$.

□

Theorem 4.3. *If $u \in [H^2(\Omega)]^d$ and $p \in H^2(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$|||(u-u_h, p-p_h)||| \leq Ch\Big(\max(c_d + \gamma^{1/2}, c_g^{-1/2})H_s + \sigma^{1/2}h + \nu^{1/2})\|u\|_{2,\Omega} + \max(c_d^{-1/2}, c_g^{1/2}, c_p^{1/2}, \gamma^{1/2})H_s\|p\|_{2,\Omega}\Big). \tag{30}$$

with $H_s = \max(h^{\frac{s}{2}}, h^{\frac{2-s}{2}})$

Proof. First of all we note that $|||(u - u_h, p - p_h)||| \leq |||(u - \pi_h u, p - \pi_h p)||| + |||(\pi_h u - u_h, \pi_h p - p_h)|||$. By Theorem 3.5 we have

$$|||(\pi_h u - u_h, \pi_h p - p_h)||| \leq c_0 \sup_{(v,q)\in V_h\times Q_h} \frac{A[(\pi_h u - u_h, \pi_h p - p_h), (v, q)]}{|||(v, q)|||}$$

and, by Lemma 4.1,

$$|||(\pi_h u - u_h, \pi_h p - p_h)||| \leq \sup_{(v,q)\in V_h\times Q_h} \frac{A[(\pi_h u - u, \pi_h p - p), (v, q)]}{|||(v, q)|||} \tag{31}$$

Writing out the terms in $A[(\pi_h u - u, \pi_h p - p), (v, q)]$ we obtain

$$\begin{aligned}A[(\pi_h u - u, \pi_h p - p), (v, q)] &= a(\pi_h u - u, v) + b(\pi_h p - p, v)\\ &\quad -b(q, \pi_h u - u) + j(\pi_h p - p, q) + \tilde{j}(\pi_h u - u, v)\\ &= i + ii + iii + iv + v.\end{aligned}$$

We bound the five terms as follows

$$i \leq |||(u - \pi_h u, 0)||| \cdot |||(v, 0)|||$$
$$ii \leq \|h^{-s/2}(\pi_h p - p)\|\|h^{s/2}\nabla\cdot v\| \leq Ch^{\frac{4-s}{2}}c_d^{-1/2}\|p\|_{2,\Omega}|||(v, 0)|||$$

and using integration by parts

$$iii = (h^{s/2}\nabla q, h^{-s/2}(u - \pi_h u)) \leq Ch^{\frac{4-s}{2}}c_g^{-1/2}\|u\|_{2,\Omega}|||(0, q)|||$$

$$iv \leq |||(0, p - \pi_h p)||| \cdot |||(0, q)|||,$$

and

$$v \leq |||(u - \pi_h u, 0)||| \cdot |||(v, 0)|||.$$

The Theorem follows by Lemma 4.2. □

Remark 4.4. *Observe that the stabilizing terms $j(p_h, p_h)$ and $\tilde{j}(u_h, u_h)$ may be included in the triple norm. This yields the following convergences of the jump terms*

$$\Big(j(p_h, p_h) + j(u_h, u_h)\Big)^{1/2} \leq Ch\Big((H_s + \sigma^{1/2} h + \nu^{1/2})\|u\|_{2,\Omega} + H_s\|p\|_{2,\Omega}\Big)$$

Let us comment briefly on the dependence on the constants in the above estimate. The important point to notice is that there is no factor ν^{-1} or σ^{-1} in the estimate. This is what allows us to treat all viscous regimes. The main dependence are on $\max(\sigma^{1/2}, \nu^{1/2})$. It is worthwhile to notice that when the viscosity becomes small the optimal choice is $s = 1$ giving $O(h^{3/2})$ convergence of the error in the L^2-norm for both the velocities and the pressure and $O(h)$ convergence of the pressure in the H^1-norm and of the velocities in the H_{div} norm.

4.1. **The Stokes problem**

For the Stokes system it is unnatural to assume that p belongs to $H^2(\Omega)$. Thus, below we prove that some classical finite element results for the Stokes problem hold also for our method, namely,

- convergence in the triple norm, assuming only $p \in H^1(\Omega)$
- optimal convergence in the L^2-norm for the velocities.

Below we always take $s = 2$ and we omit the jump term $\tilde{j}(u_h, v)$ stabilizing the incompressibility condition.

Corollary 4.5. *If the solution to the continuous problem has the regularity $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$|||(u - u_h, p - p_h)||| \leq Ch\Big(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}\Big). \tag{32}$$

Proof. The proof is very similar to the proof of Theorem 4.3, but has to be modified to account for the loss of Galerkin orthogonality. Equation (31) now becomes

$$|||(\pi_h u - u_h, \pi_h p - p_h)||| \leq \sup_{(v,q)\in V_h \times Q_h} \frac{A_c[(\pi_h u - u, \pi_h p - p), (v, q)] + j(\pi_h p, q)}{|||(v, q)|||} \tag{33}$$

with the consistent part $A_c[(\pi_h u - u, \pi_h p - p), (v, q)] = a(\pi_h u - u, v) + b(\pi_h p - p, v) - b(q, \pi_h u - u)$. We have that

$$b(\pi_h p - p, v) \leq \|\pi_h p - p\| \, \|v\|_{1,\Omega} \leq \|\pi_h p - p\| \, |||(v, 0)||| \leq h\|p\|_{1,\Omega} \, |||(v, 0)|||,$$

and

$$b(q, \pi_h u - u) \leq |||(0, q)||| \, \|\nabla \cdot (\pi_h u - u)\| \leq |||(0, q)||| \, h\|u\|_{2,\Omega}.$$

We end the proof by noting that

$$j(\pi_h p, q) \leq C\, j(\pi_h p, \pi_h p)^{1/2} |||(0, q)||| \leq C \sum_K \int_{\partial K} \gamma h^3 [\nabla \pi_h p]^2 \mathrm{d}s |||(0, q)|||$$

and that, by a scaling argument and the stability of the Clément interpolant

$$\sum_K \int_{\partial K} h^3 [\nabla \pi_h p]^2 \mathrm{d}s \leq C\|h^2 \nabla \pi_h p\|^2 \leq C\|h^2 \nabla p\|^2. \tag{34}$$

□

Corollary 4.6. *If $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$j(p_h, p_h) \leq c|||(\pi_h u - u_h, \pi_h p - p_h)|||^2 + Ch^2 \|p\|^2_{1,\Omega}.$$

Proof. The proof is immediate noting that

$$j(p_h, p_h) = j(p_h - \pi_h p + \pi_h p, p_h - \pi_h p + \pi_h p) \leq j(p_h - \pi_h p, p_h - \pi_h p) + j(\pi_h p, \pi_h p)$$

where we now apply a scaling argument and an inverse inequality in the first term to obtain

$$j(p_h - \pi_h p, p_h - \pi_h p) \leq c|||(\pi_h u - u_h, \pi_h p - p_h)|||^2$$

and we conclude using equation (34). □

We now proceed to prove an L^2–error estimates for the velocities in the case of the Stokes equations. We introduce the following dual problem, find $(\varphi, r) \in V \times Q$ such that

$$a(v, \varphi) + b(q, \varphi) - b(r, v) = (\eta, v)_\Omega, \ \forall (v, q) \in V \times Q, \tag{35}$$

and assume that the solution enjoys the additional regularity

$$\|\varphi\|^2_{2,\Omega} + \|r\|^2_{1,\Omega} \leq C\|\eta\|^2, \tag{36}$$

valid if the boundary is sufficiently smooth, cf. [12]. We now prove the L_2–error estimate in the case when $\nu > h$.

Theorem 4.7. *If $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ is the solution to the Stokes problem and (u_h, p_h) the solution to (3), then we have*

$$\|u - u_h\| \leq Ch^2(\|u\|_{2,\Omega} + \|p\|_{1,\Omega})$$

Proof. Choosing $\eta = v = u - u_h$, $q = 0$ in (35) gives

$$\|\eta\|^2 = a(\eta, \varphi) - b(r, \eta)$$

and by Galerkin orthongonality, setting $\zeta = p - p_h$,

$$\begin{aligned}\|\eta\|^2 = a(\eta, \varphi - \pi_h \varphi) + b(r - \pi_h r, \eta) - b(\zeta, \varphi - \pi_h \varphi) + j(p_h, \pi_h r)\\ = i + ii + iii + iv.\end{aligned}$$

The terms are bounded in the following fashion.

$$i \leq |||(\eta, 0)||| \, |||(\varphi - \pi_h \varphi, 0)||| \leq Ch^2 \|\varphi\|_{2,\Omega},$$

$$ii \leq |||(\eta, 0)||| \, \|r - \pi_h r\| \leq Ch^2 \|r\|_{1,\Omega},$$
$$iii \leq |||(0, \zeta)||| \, \|\nabla \cdot (\varphi - \pi_h \varphi)\| \leq Ch^2 \|\varphi\|_{2,\Omega},$$

and finally we bound the residual part using corollary 4.6 and equation (34)

$$j(p_h, \pi_h r) \leq j(p_h, p_h)^{1/2} j(\pi_h r, \pi_h r)^{1/2} \leq Ch^2 \|r\|_{1,\Omega}.$$

We conclude using the regularity assumption on the dual problem (36). □

5. Numerical examples

In this section we will show the performance of our method on two academic examples with known solution. Since we are dealing with the generalized Stokes' problem we consider the classical Stokes' equations on the one hand with $\nu = 1$ and $\sigma = 0$ and, on the other hand, the Darcy's equations, with $\nu = 0$ and $\sigma = 1$.

5.1. **Stokes' problem**

We consider the unit square with exact flow solution (from [15]) given by $u = (20\,x\,y^3,\ 5\,x^4 - 5\,y^4)$ and $p = 60\,x^2 y - 20\,y^3 + C$. Imposing zero mean pressure ($C = -5$), we obtain the convergence shown in Figure 3; second order for the velocity and the pressure in L_2–norm.

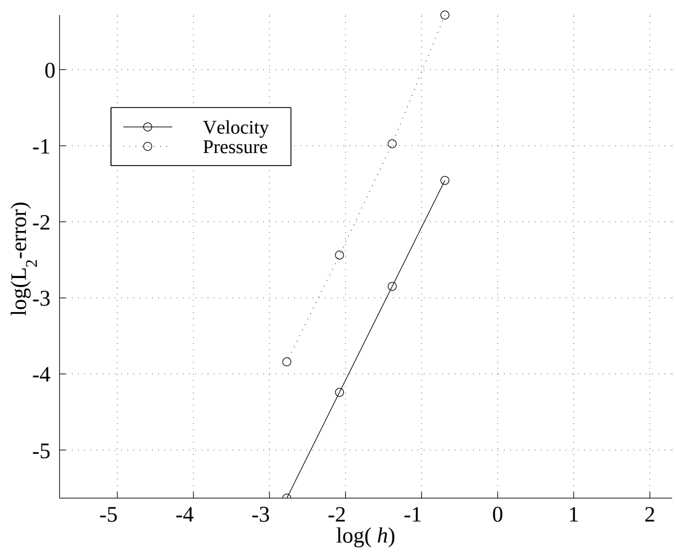

Figure 3. L_2-norm convergence of the velocity and of the pressure for Stokes.

Pressure isolines and velocity vectors on the final mesh in the sequence used to obtain the convergence plot are shown in Figures 4.

5.2. **Darcy's problem**

The second numerical example, taken from [14], is a study of convergence rates for Darcy flow. The domain under consideration is the unit square with a given exact pressure solution $p = \sin 2\pi\, x \sin 2\pi\, y$. The exact velocity field is then computed from Darcy's law to give boundary conditions and a source term for the divergence. In order to create a unique pressure field we also impose zero mean pressure.

In Figure 5, we show the approximate velocities and pressures on the final mesh in a sequence. In Figure 6, we show the convergence of the method in the L_2-norm, which yields second order accuracy for the velocities and the pressure.

Numerical experimentation indicates that even on this simple example another choice than $s = 1$ will give poorer convergence properties. In particular if the stabilization of the incompressibility condition is left out the convergence of the error in the velocities is of order $h^{3/2}$. If, on the other hand, this term becomes too dominant ($s = 0$) the convergence of the error in the pressure is of order $h^{3/2}$.

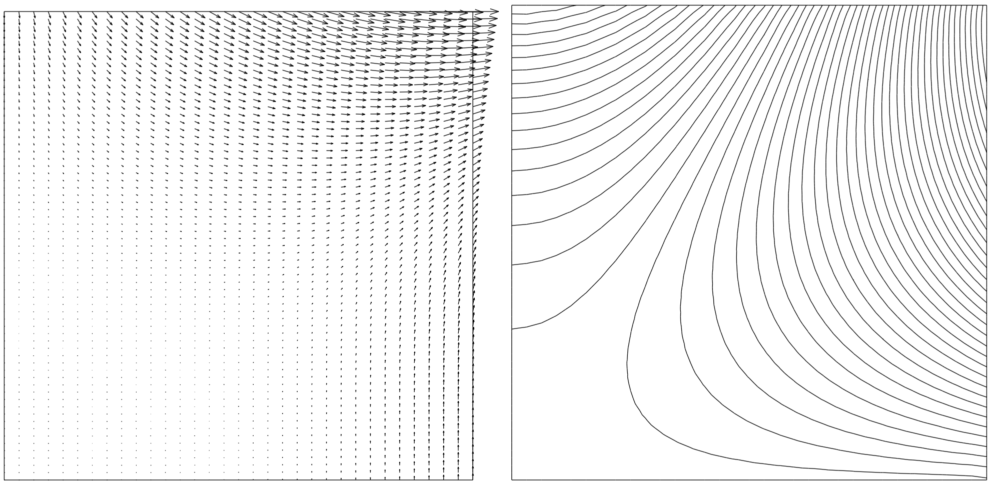

Figure 4. Approximate velocity field and pressure on the final mesh in a sequence

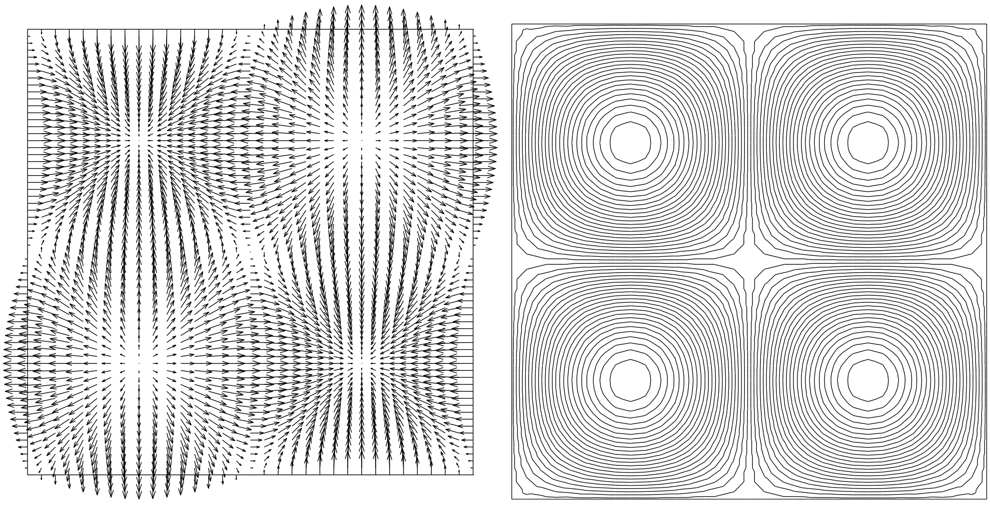

Figure 5. Approximate velocity field and pressure on the final mesh in a sequence.

6. Concluding remarks

We have suggested the use of derivative jump stabilization for P1P1–approximations of the generalized Stokes problem. We show optimal convergence for the pure Stokes case and near optimal (with a loss of one half power of h) in the case of the pure Darcy problem.

Our method has some decisive benefits: mass lumping is possible (unlike in the case of SUPG–type schemes) which is useful for extensions involving time stepping and stiff source terms. No additional unknowns are added, no special structure on the mesh is assumed. Finally, numerical evidence shows that no boundary layers appear in the pressures.

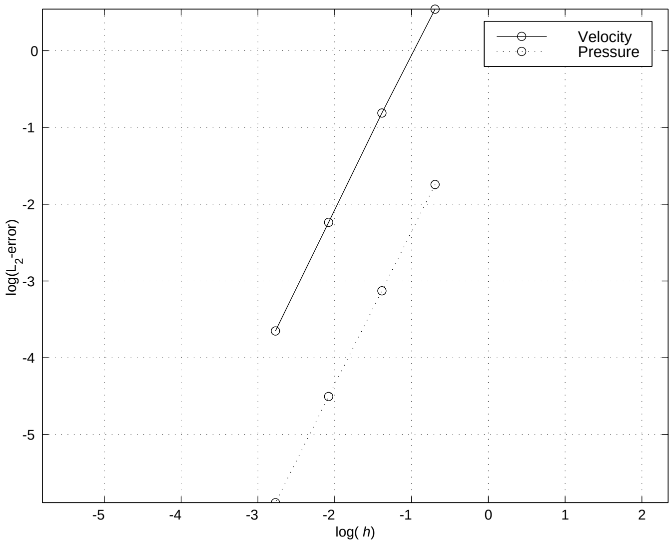

FIGURE 6. L_2-norm convergence of the velocity and of the pressure for Darcy.

REFERENCES

[1] G. A. Baker, Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.* **31** (1977) 45–59.

[2] H. Blank, M. Rudgyard, and A. Wathen, Stabilised finite element methods for steady incompressible flow. *Comput. Methods Appl. Mech. Eng.* **174** (1999) 91–105.

[3] R. Becker and M. Braack, A finite element pressure gradient stabilization for the Stokes equation based on local projection. *Calcolo* **38** (2001) 173–199.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer, Berlin, (1991).

[5] F. Brezzi, T.J.R. Hughes, L.D. Marini, A. Russo, and E.A. Süli, A priori error analysis of residual-free bubbles for advection-diffusion problems. *SIAM J. Numer. Anal.* **36** (1999) 1933–1948.

[6] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations, in *Efficient Solution of Elliptic Systems*, W. Hackbusch Ed., Vieweg (1984).

[7] E. Burman and P. Hansbo, Edge stabilization for Galerkin approximations of convection–diffusion problems, *Comput. Methods Appl. Mech Engrg.* (to appear).

[8] P. Clément, *Approximation by finite element functions using local regularization.* RAIRO Anal. Numer. **9 R-2** (1975) 77–84.

[9] R. Codina and J. Blasco, Analysis of a pressure-stabilized finite element approximation of the stationary Navier-Stokes equations. *Numer. Math.* **8** (2000) 59–81.

[10] J. Douglas and J. P. Wang, An absolutely stabilized finite-element method for the Stokes problem. *Math. Comp.* **52** (1989) 495–508.

[11] J. J. Droux and T. J. R. Hughes, A boundary integral modification of the Galerkin least-squares formulation for the Stokes problem. *Comput. Methods Appl. Mech. Eng.* **113** (1994) 173–182.

[12] V. Giarult and P.-A. Raviart, *Finite element methods for Navier–Stokes equations: Theory and algorithms*, Springer-Verlag, Berlin (1986).

[13] T. J. R. Hughes, L. P. Franca, and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accomodating equal-order interpolations. *Comput. Methods Appl. Mech. Eng.* **59** (1986) 85–99.

[14] A. Masud and T.J.R. Hughes A stabilized mixed finite element method for Darcy flow, *Comput. Methods Appl. Mech. Eng.* **191** (2002) 4341–4370.

[15] S. Norburn and D. Silvester, Stabilised vs stable mixed methods for incompressible flow. *Comput. Methods Appl. Mech. Eng.* **166** (1998) 131–141.

[16] V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer-Verlag, Berlin (1984).