Localization in single Landau bands

T. C. Dorlas[a)]
Department of Mathematics, University College of Swansea, Singleton Park, Swansea SA2 8PP, Wales

N. Macris[b)]
Institut de Physique Théorique Ecole Polytechnique Fédérale de Lausanne, CH 1015 Lausanne, Switzerland

J. V. Pulé[c),d)]
Department of Mathematical Physics, University College, Belfield, Dublin 4, Ireland

(Received 10 April 1995; accepted for publication 12 October 1995)

We consider a single-band approximation to the random Schrödinger operator in an external magnetic field. The random potential is taken to be constant on unit squares and i.i.d. on each square with a bounded distribution. We prove that the eigenstates corresponding to energies at the edges of the Landau band are localized. This is an important ingredient in the theory of the Quantum Hall Effect. © *1996 American Institute of Physics.* [S0022-2488(96)03403-1]

I. INTRODUCTION

We consider a two-dimensional infinite system of noninteracting electrons moving in a uniform magnetic field of strength B and a random potential V. In the symmetric gauge the vector potential is given by $A(x)=[(B/2)x_2,-(B/2)x_1]$, $x=(x_1,x_2)\in\mathbb{R}^2$ and the Hamiltonian is

$$H=(-i\nabla-A(x))^2+V(x). \tag{1.1}$$

The effect of the random potential is to broaden the Landau levels into bands. When the potential is bounded and the magnetic field is strong enough these bands do not overlap. It is generally expected that the states lying near the edges of the bands are exponentially localized and the corresponding spectrum is pure point.[1,2] Near the center of the bands the situation is more controversial. One possibility is that there exist truly extended states in some finite-energy range. Instead, it could happen that the localization length remains finite for all energies, except for one value, where it diverges (e.g., like a power law).[3–5] This picture is essential for understanding the occurrence of plateaus in the conductivity as a function of the magnetic field measured in Quantum Hall experiments. In this connection, Kunz[6] has shown that the localization length must be infinite for at least one energy in each band, assuming that the states with energy at the edges of the bands are exponentially localized.

Rigorous results on random Schrödinger operators with magnetic fields are still rare. A few exact results concerning the density of states have been obtained.[7–11] In the present paper we address the problem of proving that the energies at the edges of the bands correspond to localized states. For the random potential we choose a model already considered in previous works in the absence of magnetic field.[12,13] The two-dimensional plane is decomposed into unit squares, on each of which the potential is taken to be constant. The values of the potential on the squares are i.i.d.s with a bounded probability distribution. The precise hypotheses on the probability distribu-

[a)]Electronic mail: T.C.Dorlas@swansea.ac.uk
[b)]Electronic mail: macris@eldp.epfl.ch
[c)]Electronic mail: jpule@acadamh.ucd.ie
[d)]Research Associate, School of Theoretical Physics, Dublin Institute for Advanced Studies.

0022-2488/96/37(4)/1574/22/$10.00

tion are stated in Sec. II. When the magnetic field is strong it is reasonable to consider only the projections of the Hamiltonian onto each Landau level and to neglect the cross terms. The Hamiltonian restricted to the nth level is

$$H_n = B(2n+1)P_n + P_n V P_n, \tag{1.2}$$

where P_n denotes the projection onto the level. The term $B(2n+1)P_n$ comes from the decomposition of the purely kinetic part of (1.1) and can be dropped as it modifies the energy only by a constant. Note that the resulting Hamiltonian is a random integral operator instead of a differential operator and that the kernels of P_n are known explicitly.

Our main result is that, for the Hamiltonian H_n, the states at the edges of the corresponding band are exponentially localized. For simplicity in this paper, we restrict ourselves to the case $n=0$. Our proof depends on a modification of a theorem of Von Dreifus and Klein.[14] This theorem is only stated here and the proof is given in a companion paper,[15] which deals with the easier case when the distribution of the random potential is unbounded, e.g., Gaussian. In this paper we concentrate on the case when the distribution is bounded.

We now describe the main features of our analysis. In Sec. II we prove (see Lemma 2.3) that it is enough to show that, because of the special form of the Hamiltonian, the wave function decays exponentially on the lattice. This simplification enables us to use the methods of Spencer,[16] Von Dreifus and Klein[14] for lattice models. However, the fact that the model is formulated in the continuum makes the model considerably more difficult to analyze and nontrivial modifications are required, because the relevant Green's identity (3.3) is more difficult to handle. These are described in Sec. III. This section also contains the basic step in the proof of localization, Theorem 3.1, which as in Ref. 14 reduces to two main conditions (P1) and (P2). The proof of this theorem, which does not depend on the boundedness of the potential, can be found in Ref. 15. In Sec. IV we verify the conditions (P1) and (P2). The condition (P1) is an estimate of the type first proved by Wegner[17] on the probability that an energy E lies within some small distance from the spectrum of the Hamiltonian for a finite box. This requires bounds on the integrated density of states in finite boxes. Our Hamiltonian, when restricted to a finite box, turns out to be a Hilbert–Schmidt operator. Therefore the spectrum has an accumulation point at zero that requires an adaptation of Wegner's argument. This feature is intimately related to the fact that the original Laudau levels are infinitely degenerate. Condition (P2) states that there exist a length scale L such that the Green's function for a box of size L decays exponentially fast, with a high probability depending on L. For bounded potentials the usual proofs in the absence of a magnetic field use the fact that the density of states is exponentially small near the band edge. These are the so-called Lifshitz tails. Here we verify (P2) directly using a Combes–Thomas argument[18] and the explicit form of the eigenfunctions of P_0. The main part of the paper is concerned with this problem.

While this paper was being written we received a preprint by Combes and Hislop[19] with similar results, and recently W.-M. Wang also obtained results along the same lines.[20] We wish to compare briefly these papers with the present one.

In Refs. 19 and 20, localization is proved for the Hamiltonian (1.1) in the case where the random potential V is sufficiently smooth. Mathematical techniques of percolation theory and microlocal analysis are used (also see Ref. 21). The regime studied is that of large magnetic field, that is, the magnetic length ($\approx 1/\sqrt{B}$) has to be smaller than the characteristic length over which the potential varies. In this situation the one-band problem is well approximated by the classical effective Hamiltonian $(2n+1)B+V(x)$. As a consequence the problem is mapped onto a percolation problem for the equipotential lines of $V(x)$. As far as we know this physical picture goes back to Ref. 22.

In contrast, the effective Hamiltonian used in this paper for the single band problem is $P_n V P_n$, and therefore the kinetic energy that is contained in P_n is not quenched. As a consequence our main theorem holds for arbitrary strength of the magnetic field (for the single band). In particular, localization at the band edges occurs even when the magnetic length is large with

respect to the characteristic length of variation of V, that is, when the percolation picture loses its validity. Of course, we have neglected the interband coupling, and if that is taken into account a condition on the strength of B would be necessary. However, it is not clear what the optimal condition would be. We remark that in the present study the random potential is of a different kind from that in Refs. 19 and 20 since it is discontinuous.

II. THE HAMILTONIAN

Let ω_n, $n \in \mathbb{Z}^2$ be i.i.d. random variables with distribution given by a probability measure μ with supp $\mu = X = [a,b]$, a compact interval in $\mathbb{R}$. We let $\Omega = X^{\mathbb{Z}^2}$ and $\mathbb{P} = \Pi_{n \in \mathbb{Z}^2} \mu$. For $m \in \mathbb{Z}^2$ let τ_m be the measure preserving automorphism of Ω defined by

$$(\tau_m \omega)_n = \omega_{n-m}. \tag{2.1}$$

The group $\{\tau_m : m \in \mathbb{Z}^2\}$ is ergodic for the probability measure $\mathbb{P}$.

Let $\mathscr{H} = L^2(\mathbb{R}^2)$ and let $\mathscr{H}_0$ be the eigenspace corresponding to the lowest eigenvalue (first Landau level) of the Hamiltonian H_0 defined in (1.1). Let P_0 be the orthogonal projection onto $\mathscr{H}_0$. The Hamiltonian for our model is the operator on $\mathscr{H}_0$, given by

$$H(\omega) = P_0 V(\cdot, \omega) = P_0 V(\cdot, \omega) P_0, \tag{2.2}$$

where $\omega \in \Omega$ and

$$V(x, \omega) = \sum_{n \in \mathbb{Z}^2} \mathbf{1}_{\Lambda_1(n)}(x) \omega_n, \tag{2.3}$$

$\Lambda_1(n)$ being the square of the unit side centered at n,

P_0 is an integral operator with kernel

$$P_0(x,y) = \frac{2\kappa}{\pi} \exp[-\kappa |x-y|^2 + 2i\kappa x \wedge y], \tag{2.4}$$

where $\kappa = B/4$. Since we shall be using both the Euclidean norm and the maximum norm on $\mathbb{R}^2$, we shall use the following convention:

$$|x| = (x_1^2 + x_2^2)^{1/2}, \quad \|x\| = \max(|x_1|, |x_2|),$$

and for $L > 0$ and $x \in \mathbb{R}^2$,

$$B(x,L) = \{y \in \mathbb{R}^2 : |y-x| \leqslant L\}, \quad \Lambda_L(x) = \{y \in \mathbb{R}^2 : \|y-x\| \leqslant \tfrac{1}{2} L\}.$$

Let $\{U_y : y \in \mathbb{R}^2\}$ be the family of unitary operators on $\mathscr{H}$ corresponding to the magnetic translations:

$$(U_y f)(x) = e^{2i\kappa y \wedge x} f(x+y). \tag{2.5}$$

Then for $n \in \mathbb{Z}^2$,

$$U_n H(\omega) U_n^{-1} = H(\tau_n \omega). \tag{2.6}$$

Note that $[P_0, U_y] = 0$ for all $y \in \mathbb{R}^2$, so that $U_y \mathscr{H}_0 \subset \mathscr{H}0$. Also $U_{y_1} U_{y_2} = e^{2i\kappa y_z \wedge y_1} U_{y_1 + y_2}$. The ergodicity of $\{\tau_m : m \in \mathbb{Z}^2\}$ and Eq. (2.6) together imply that the spectrum of $H(\omega)$ and its components are nonrandom (see, for example, Carmona and Lacroix, Theorem V.2.4); it is easy to prove that almost surely the spectrum of $H(\omega)$ is equal to X (cf. Ref. 23).

Lemma 2.1: For P-almost all $\omega\in\Omega$,

$$\sigma(H(\omega))=X. \tag{2.7}$$

Proof: For $\psi\in\mathscr{H}_0$,

$$a\|\psi\|^2\leqslant\langle\psi,H(\omega)\psi\rangle\leqslant b\|\psi\|^2,$$

and therefore

$$\sigma(H(\omega))\subset X.$$

To prove the reverse inclusion, it is sufficient to prove that[24] for each $E\in X$ and for all $\delta>0$ there exist $\Omega'\subset\Omega$ with $\mathbb{P}(\Omega')>0$ and $\psi\in\mathscr{H}_0$ with $\|\psi\|=1$, such that for all $\omega\in\Omega'$, $\|(H(\omega)-E)\psi\|<\delta$. Let $E\in X$ and $\psi\in\mathscr{H}_0$ with $\|\psi\|=1$. For $R>0$, let $\psi_R=P_0\mathbf{1}_{B(0,R)}\psi$. Since $\|\psi-\psi_R\|\leqslant\|\mathbf{1}_{B(0,R)^c}\psi\|$, we can choose R large enough such that $\|\psi_R\|>1/2$. Let $\Omega'=\{\omega:|V(x,\omega)-E|<1/2\delta,\forall x\in B(0,2R)\}$ then clearly $\mathbb{P}(\Omega')>0$. Now

$$\begin{aligned}\|(H(\omega)-E)\psi_R\|^2&=\|P_0(V(\cdot,\omega)-E)P_0\mathbf{1}_{B(0,R)}\psi\|^2\\&\leqslant\|(V(\cdot,\omega)-E)P_0\mathbf{1}_{B(0,R)}\psi\|^2\\&\leqslant\int_{B(0,2R)}dx(V(x,\omega)-E)^2|\psi_R(x)|^2+\int_{B(0,2R)^c}dx(V(x,\omega)-E)^2|\psi_R(x)|^2.\end{aligned} \tag{2.8}$$

If $\omega\in\Omega'$ for the first integral in (2.8), we have

$$\int_{B(0,2R)}dx(V(x,\omega)-E)^2|\psi_R(x)|^2\leqslant\frac{1}{4}\,\delta^2\|\psi_R\|^2\leqslant\frac{1}{4}\,\delta^2. \tag{2.9}$$

We now estimate the second integral in (2.8),

$$|\psi_R(x)|^2=\left(\int_{B(0,R)}dy\;P_0(x,y)\psi(y)\right)^2\leqslant\int_{B(0,R)}dy|P_0(x,y)|^2, \tag{2.10}$$

using the Schwarz inequality and $\|\psi\|=1$. If $x\in B(0,2R)^c$ and $y\in B(0,R)$,

$$|P_0(x,y)|^2\leqslant\frac{4\kappa^2}{\pi^2}\exp[-\kappa R^2-\kappa|x-y|^2], \tag{2.11}$$

so that we have, for the second integral in (2.8),

$$\begin{aligned}\int_{B(0,2R)^c}dx(V(x,\omega)-E)^2|\psi_R(x)|^2&\leqslant\frac{4(b-a)^2\kappa^2}{\pi^2}\,e^{-\kappa R^2}\int_{B(0,R)^c}dx\int_{B(0,R)}dy\;e^{-\kappa|x-y|^2}\\&\leqslant\frac{4(b-a)^2\kappa^2R^2}{\pi}\,e^{-\kappa R^2}\int_{\mathbb{R}^2}dx\;e^{-\kappa|x|^2}<\frac{1}{4}\,\delta^2,\end{aligned} \tag{2.12}$$

if R is sufficiently large. □

The next lemma describes the generalized eigenfunctions of $H(\omega)$. It is proved in Ref. 15 (see Theorem 2.3 and Lemma 6.2) in the case where X may be unbounded.

Lemma 2.2: For almost every $E \in X$ with respect to the spectral measure of H, there exists ψ, a polynomially bounded C^∞ function on $\mathbb{R}^2$ such that $H\psi = E\psi$ and $P_0\psi = \psi$. Moreover, if $\psi \in \mathscr{H}_0$ then E is in the pure-point spectrum of H.

The object of this paper is to prove that almost surely the generalized eigenfunctions of H corresponding to points of X near its edges are localized, in the sense that they decay exponentially and therefore those points are in the pure-point spectrum. The next definition makes precise what is meant by exponential decay.

Definition: $\psi: \mathbb{R}^2 \to \mathbb{R}$ decays exponentially with a rate greater or equal to m if

$$\limsup_{x \to \infty} \frac{\ln|\psi(x)|}{|x|} \leqslant -m. \tag{2.13}$$

The main result of this paper is the following theorem, which is proved in Sec. IV.

Theorem 2.3: If the probability measure corresponding to the i.i.d. random variables ω_n is absolutely continuous with respect to the Lebesgue measure and its density ρ satisfies a Lipshitz condition of order $\sigma > 0$ and supp $\rho = [a,b]$, where $-\infty < a < b < \infty$, then there is a $\Delta > 0$ and $m > 0$, such that almost surely $[a, a+\Delta] \cup [b-\Delta, b]$ is in the pure-point spectrum of H and the corresponding eigenfunctions of H decay with a rate greater or equal to m.

The last lemma of this section provides an important simplifying feature in our proof of localization. It shows that to prove that an eigenfunction decays exponentially it is sufficient to prove that its average on unit squares decays exponentially.

Lemma 2.4: If ψ is a generalized eigenfunction of H and

$$\limsup_{\substack{n \to \infty \\ n \in \mathbb{Z}^2}} \frac{\ln\langle \mathbf{1}_{\Lambda_1(n)}, |\psi| \rangle}{|n|} \leqslant -m, \tag{2.14}$$

then ψ decays exponentially with rate greater or equal to m.

Proof: If ψ is a generalized eigenfunction of H then, by Lemma 2.2, $\psi \in C^\infty$ and is polynomially bounded, $|\psi(x)| < C(1+|x|)^t$, say. If ψ satisfies (2.14) then, given $\epsilon > 0$, we can choose R such that for $n \in \mathbb{Z}^2$ with $|n| > R - 1/\sqrt{2}$,

$$\int_{\Lambda_1(n)} |\psi(x)| dx \leqslant e^{-(m-\epsilon)|n|}. \tag{2.15}$$

Since $\psi = R_0\psi$, for all $x \in \mathbb{R}^2$,

$$\begin{aligned} |\psi(x)| &\leqslant \frac{2\kappa}{\pi} \int_{\mathbb{R}^2} e^{-\kappa|x-y|^2} |\psi(y)| dy \\ &= \frac{2\kappa}{\pi} \int_{|y| \leqslant R} e^{-\kappa|x-y|^2} |\psi(y)| dx + \frac{2\kappa}{\pi} \int_{|y|>R} e^{-\kappa|x-y|^2} |\psi(y)| dy \\ &= \frac{2\kappa}{\pi} (I_1 + I_2). \end{aligned} \tag{2.16}$$

For the first term, we have

$$I_1 \leqslant C(1+|R|)^t \pi R^2 e^{-\kappa(|x|-R)^2}. \tag{2.17}$$

We now obtain an exponential bound on the second term,

$$
\begin{aligned}
I_2 &\leqslant \sum_{\substack{n\in\mathbb{Z}^2\\ |n|>R-1/\sqrt{2}}} \int_{\Lambda_1(n)} e^{-\kappa|x-y|^2}|\psi(y)|dy \\
&\leqslant \sum_{\substack{n\in\mathbb{Z}^2|x-n|\leqslant 1\\ |n|>R-1}} \int_{\Lambda_1(n)} e^{-\kappa|x-y|^2}|\psi(y)|dy \\
&\quad + \sum_{\substack{n\in\mathbb{Z}^2|x-n|>1\\ |n|>R-1}} \int_{\Lambda_1(n)} e^{-\kappa|x-y|^2}|\psi(y)|dy = I_3+I_4. \qquad (2.18)
\end{aligned}
$$

Now

$$
I_3 \leqslant \sum_{n\in\mathbb{Z}^2|x-n|\leqslant 1} e^{-(m-\epsilon)|n|} \leqslant 4e^{-(m-\epsilon)(|x|-1)} \qquad (2.19)
$$

and

$$
\begin{aligned}
I_4 &\leqslant \sum_{n\in\mathbb{Z}^2} e^{-(m-\epsilon)|n|} e^{-\kappa(|x-n|-1)^2} \\
&\leqslant e^{-(m-\epsilon)|x|} \sum_{n\in\mathbb{Z}^2} e^{(m-\epsilon)|x-n|} e^{-\kappa(|x-n|-1)^2} \\
&\leqslant e^{-(m-\epsilon)|x|} \sum_{n\in\mathbb{Z}^2} e^{(m-\epsilon)(|n|+1)} e^{-\kappa(|n|-2)^2} \leqslant C' e^{-(m-\epsilon)|x|}. \qquad (2.20)
\end{aligned}
$$

□

III. THE METHOD

In this short section we describe our method. Our proof is based on the paper of Von Dreifus and Klein[14] (also see Refs. 25 and 16). Here we give a summary of the main differences. The details can be found in Ref. 15.

The main tool in Refs. 14, 25, and 16 are the local Hamiltonians, the Hamiltonian restricted to bounded regions by Dirichlet boundary conditions, and the corresponding Green's functions. For $\Lambda\subset\mathbb{R}^2$, here we define the local Hamiltonian H_Λ on $L^2(\Lambda)$ by

$$
H_\Lambda = P_\Lambda V_\Lambda P_\Lambda^*, \qquad (3.1)
$$

where $P_\Lambda = \mathbf{1}_\Lambda P_0$ and $V_\Lambda = V\mathbf{1}_\Lambda$. V is also truncated to ensure that for disjoint regions the corresponding local Hamiltonians are stochastically independent. We note that for bounded Λ, H_Λ is a Hilbert–Schmidt operator and its spectrum $\sigma(H_\Lambda)$ has an accumulation point at the origin.

For $\lambda\notin\sigma(H_\Lambda)$ let

$$
G_\Lambda(E) = (H_\Lambda - E)^{-1}. \qquad (3.2)
$$

If ψ is an eigenfunction of H with eigenvalue $E\notin\sigma(H_\Lambda)$, then using the resolvent identity, we have for $x\in\Lambda$ [cf. Eq. (3.12) in Ref. 15]

$$
\psi(x) = -(G_\Lambda(E)(P_\Lambda V P_{\Lambda^c}^* + P_\Lambda V_{\Lambda^c} P_\Lambda^*)\psi)(x). \qquad (3.3)
$$

Most of the complexity in adapting the proofs of Ref. 14 to this model comes from the fact that H is not a local operator. (3.3) contains terms that couple points in Λ to points outside. However,

because of the form of the kernel of P_0, the coupling is bounded by a Gaussian. The Green's function $G_\Lambda(E)$ does not have a kernel in this case. We therefore have to modify the definition of regularity.

Definition: Let $m>0$, $0<\beta<1$, $E\in\mathbb{R}$ and $\frac{1}{2}<s<1$. A square $\Lambda_L(x)$ is (ω,m,β,E,s) regular if
(RA), $d(E,\sigma(H_{\Lambda_L}(x)))>\frac{1}{2}e^{-L^\beta}$;
(RB), for all $\phi\in L_2(\Lambda_L(x))$,

$$\langle \mathbf{1}_{\Lambda_1(x)},|G_{\Lambda_L(x)}(E)\mathbf{1}_{\tilde{\Lambda}_L(x)}\phi|\rangle<e^{-mL}\|\mathbf{1}_{\tilde{\Lambda}_L(x)}\phi\|,$$

where $\tilde{\Lambda}_L(x)=\Lambda_L(x)\backslash\Lambda_{\tilde{L}(x)}$ and $\tilde{L}=L-L^s$.

In order to state the theorem that is used in proving localization, we need to define the following two conditions: Let $E_0\in\mathbb{R}\backslash\{0\}$ and fix $\beta\in(0,1)$, $s\in(\frac{1}{2},1)$ and $p>2$. We shall say that L satisfies condition (P1) if the following occurs.

(P1) There exists $q>4p+12$ and $0<\eta<\frac{1}{2}|E_0|$, such that for all $L_1\geqslant L$ and all $E\in(E_0-\eta,E_0+\eta)$,

$$\mathbb{P}\{d(E,\sigma(H_{\Lambda_{L_1}(0)}))<e^{-L_1^\beta}\}<L_1^{-q};$$

and we shall say that L satisfies condition (P2) if the following occurs.

(P2) There exists $\gamma\in(0,1)$ and $m>L^{\gamma-1}$, such that

$$\mathbb{P}\{\Lambda_L(0)\ \text{is}\ (\omega,m,\beta,E_0,s)\ \text{regular}\}\geqslant 1-L^{-p}.$$

The following theorem is Theorem 4.1 in Ref. 15.

Theorem 3.1: There exists $L_0(\beta,s,p,q)$ such that if there is an $L\geqslant L_0$ that satisfies both conditions (P1) and (P2) then there is a $\Delta(L,\beta,s,\eta)>0$ so that almost surely, for $E_0\neq 0$, $\sigma(H)\cap(E_0-\Delta,E_0+\Delta)$ is in the pure-point spectrum and the corresponding eigenfunctions decay with mass greater or equal to m.

The proof of this theorem can be split up in two parts: one in which condition (P2) is iterated to pairs of larger and larger blocks and one in which the iterated condition is shown to imply exponential decay. Because of Lemma 2.4 it is sufficient to iterate on squares centered on points of $\mathbb{Z}^2$. This is very important in adapting the method of Ref. 14, which is for lattice Hamiltonians, to our model, which is for a continuous system, because it allows us to add probabilities.

Another difference between condition (P2) and the corresponding condition in Ref. 14 is the dependence of m on L. In most cases one checks (P2) by proving that the density of states decays very fast near the edges of the spectrum (Lifshitz tails). In this paper we check (P2) directly and this requires that we weaken (P2) to allow m to depend on L.

IV. PROOF OF LOCALIZATION

In this final section we shall show that the conditions of Theorem 3.1 are satisfied, thus establishing that the eigenfunctions corresponding to points near the edges of X are localized (Theorem 2.3).

From now on we shall assume that the probability measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and has a density ρ that satisfies a Lipshitz condition of order σ.

There exist $\sigma>0$ and $K>0$ such that

$$|\rho(x)-\rho(y)|\leqslant K|x-y|^\sigma, \tag{4.1}$$

for all $x,y\in[a,b]$. This implies that that ρ is bounded, and therefore

$$\mu[c,d]<K'(d-c). \tag{4.2}$$

In the next ten lemmas (Lemmas 4.1–4.10) we shall assume that $0<a<b$, but we emphasize that this is not necessary for the final result. Let

$$N_\Lambda^>(V,E)=\#\{j:\lambda_\Lambda^{(j)}(V)\geqslant E\}, \tag{4.3}$$

where

$$\lambda_\Lambda^{(1)}(V)\geqslant\lambda_\Lambda^{(2)}(V)\geqslant\lambda_\Lambda^{(3)}(V)\geqslant\cdots\geqslant 0$$

are the eigenvalues of H_Λ. Note that since this operator is Hilbert–Schmidt, $N_\Lambda^>(V,E)$ is finite for $E>0$. We have the following simple scaling law for $N_\Lambda^>(V,E)$: If $t>0$,

$$N_\Lambda^>(tV,tE)=N_\Lambda^>(V,E). \tag{4.4}$$

Throughout this section we shall use a simplified notation. We let $H_L = H_{\Lambda_L(0)}$, $G_L = G_{\Lambda_L(0)}$, $V_L = V_{\Lambda_L(0)}$, $P_L = P_{\Lambda_L(0)}$, $\Lambda_L=\Lambda_L(0)$, $\tilde{\Lambda}_L=\tilde{\Lambda}_L(0)$, $\Lambda_1=\Lambda_1(0)$, and $\mathscr{H}_L=L^2(\Lambda_L(0))$. The following lemma will be required for condition (P1) and for part (RA) of the regularity condition in (P2). The proof is a modification of Wegner.[17]

Lemma 4.1: There exists a constant $C>0$ such that for $E>a>0$ and $0<\epsilon<\frac{1}{2}E$,

$$\mathbb{P}(d(E,\sigma(H_L))<\epsilon)\leqslant CL^4\epsilon^{\min(1,\sigma)}.$$

Proof: We first note that $V_\Lambda^{1/2}P_\Lambda^*$ is Hilbert–Schmidt since it has a square integrable kernel and therefore H_Λ is trace class. Also,

$$\text{trace } H_\Lambda=\int_\Lambda dx\int_\Lambda dy|P_0(x,y)|^2V(y). \tag{4.5}$$

Now, since $N_\Lambda^>(V,E)$ is the number of eigenvalues greater than E, it is smaller than the sum of $\lambda_\Lambda^{(i)}(V)/a$:

$$\begin{aligned} N_\Lambda^>(V,E)\leqslant a^{-1}\text{ trace } H_\Lambda &= a^{-1}\int_\Lambda dx\int_\Lambda dy|P_0(x,y)|^2V(y)\\ &\leqslant a^{-1}b\int_{\mathbb{R}^2}dx\int_\Lambda dy|P_0(x,y)|^2\\ &\leqslant 2(\pi a)^{-1}b\kappa|\Lambda|. \end{aligned} \tag{4.6}$$

By (4.4),

$$\mathbb{E}(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon))=\mathbb{E}\left(N_\Lambda^>\left(\frac{EV}{E-\epsilon},E\right)-N_\Lambda^>\left(\frac{EV}{E+\epsilon},E\right)\right). \tag{4.7}$$

Writing (4.7) explicitly, we get

$$\mathbb{E}(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon))=\prod_{n\in\Gamma}\left(1-\frac{\epsilon}{E}\right)\int_{Ea/(E-\epsilon)}^{Eb/(E-\epsilon)}d\omega_n\ \rho\left(\left(1-\frac{\epsilon}{E}\right)\omega_n\right)N_\Lambda^>(V,E)$$
$$-\prod_{n\in\Gamma}\left(1+\frac{\epsilon}{E}\right)\int_{Ea/(E+\epsilon)}^{Eb/(E+\epsilon)}d\omega_n\ \rho\left(\left(1+\frac{\epsilon}{E}\right)\omega_n\right)N_\Lambda^>(V,E), \tag{4.8}$$

where $\Gamma=\{n\in\mathbb{Z}^2:\Lambda_1(n)\cap\Lambda_L\neq 0\}$ If we order Γ in some way we can then write (4.8) as

$$\mathbb{E}(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon))=\sum_j\left(\prod_{i<j}\left(1-\frac{\epsilon}{E}\right)\int_{Ea/(E-\epsilon)}^{Eb/(E-\epsilon)}d\omega_{n_i}\ \rho\left(\left(1-\frac{\epsilon}{E}\right)\omega_{n_i}\right)\right.$$
$$\times\left(\prod_{i>j}\left(1+\frac{\epsilon}{E}\right)\int_{Ea/(E+\epsilon)}^{Eb/(E+\epsilon)}d\omega_{n_i}\ \rho\left(\left(1+\frac{\epsilon}{E}\right)\omega_{n_i}\right)\right)\int d\omega_{n_j}$$
$$\times\left(\left(1-\frac{\epsilon}{E}\right)\mathbf{1}_{E/(E-\epsilon)[a,b]}(\omega_{n_j})\rho\left(\left(1-\frac{\epsilon}{E}\right)\omega_{n_j}\right)-\left(1+\frac{\epsilon}{E}\right)\right.$$
$$\left.\times\mathbf{1}_{E/(E+\epsilon)[a,b]}(\omega_{n_j})\rho\left(\left(1+\frac{\epsilon}{E}\right)\omega_{n_j}\right)\right)N_\Lambda^>(V,E). \tag{4.9}$$

Thus

$$\begin{aligned}
&\mathbb{E}(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon))\\
&\leqslant 2(\pi a)^{-1}b\kappa|\Lambda|^2\int d\omega\left|\left(1-\frac{\epsilon}{E}\right)\mathbf{1}_{E/(E-\epsilon)[a,b]}(\omega)\rho\left(\left(1-\frac{\epsilon}{E}\right)\omega\right)\right.\\
&\quad\left.-\left(1+\frac{\epsilon}{E}\right)\mathbf{1}_{E/(E+\epsilon)[a,b]}(\omega)\rho\left(\left(1+\frac{\epsilon}{E}\right)\omega\right)\right|\\
&\leqslant 2(\pi a)^{-1}b\kappa|\Lambda|^2\left(\frac{2\epsilon}{E}+\int_{aE/(E-\epsilon)}^{bE/(E+\epsilon)}\left|\rho\left(\left(1-\frac{\epsilon}{E}\right)\omega\right)-\rho\left(\left(1+\frac{\epsilon}{E}\right)\omega\right)\right|d\omega\right.\\
&\quad\left.+\frac{E}{E+\epsilon}\,\mu\left(a,a\,\frac{E+\epsilon}{E-\epsilon}\right)+\frac{E}{E-\epsilon}\,\mu\left(b\,\frac{E-\epsilon}{E+\epsilon},b\right)\right)\\
&\leqslant C|\Lambda|^2e^{\min(1,\sigma)},
\end{aligned} \tag{4.10}$$

where we have used (4.1), (4.2), and $a\leqslant E\leqslant b$. Note that the constant C is independent of E. Now

$$\mathbb{P}(d(E,\sigma(H_L))<\epsilon)\leqslant\sum_i\ \mathbb{P}(\lambda_i\in(E-\epsilon,E+\epsilon))\leqslant\mathbb{E}(N_{\Lambda_L}^>(V,E-\epsilon)-N_{\Lambda_L}^>(V,E+\epsilon))$$
$$\leqslant CL^4e^{\min(1,\sigma)}. \tag{4.11}$$

□

We shall see later that it is sufficient to prove (P1) and part (RA) of (P2). The remaining lemmas will be used to prove part (RB) of (P2).

We first use a Combes–Thomas[18]-type argument to obtain an upper bound for $\langle\mathbf{1}_{\Lambda_1},|G_L\mathbf{1}_{\tilde{\Lambda}_L}\phi|\rangle$.

Lemma 4.2: There exists $C>0$ and $L_0>0$ such that if $0<\epsilon<1$ and $L>L_0$; then

$$\mathbb{P}\left(\langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L}\phi|\rangle < \frac{C}{\epsilon}\, e^{-C\epsilon L}\|\mathbf{1}_{\tilde{L}_L}\phi\|, \forall\, \phi \in \mathscr{H}_L\right) \geqslant \mathbb{P}(d(E,\sigma(H_L)) \geqslant \epsilon). \tag{4.12}$$

Proof: Let U be the operator on $\mathscr{H}_L$ defined by $(Uf)(x) = e^{x_0\cdot x} f(x)$, where $x_0 \in \mathbb{R}^2$ with $|x_0|<1$ and let

$$Q = UH_L U^{-1} - H_L\,. \tag{4.13}$$

Then Q has a kernel $Q(x,y)$, where

$$Q(x,y) = (e^{x_0\cdot(x-y)} - 1) H_L(x,y), \tag{4.14}$$

$H_L(x,y)$ being the kernel of H_L. Therefore

$$|(Q\phi)(x)| \leqslant \frac{2\kappa b}{\pi} \int |e^{x_0\cdot(x-y)} - 1| e^{-(\kappa/2)|x-y|^2} |\phi(y)| dy. \tag{4.15}$$

Since

$$\begin{aligned} |e^{x_0\cdot(x-y)} - 1| e^{-(\kappa/4)|x-y|^2} &\leqslant |x_0\cdot(x-y)| e^{|x_0\cdot(x-y)|} e^{-(\kappa/4)|x-y|^2} \\ &\leqslant |x_0||x-y| e^{|x_0||x-y|} e^{-(\kappa/4)|x-y|^2} \\ &\leqslant |x_0| e^{2|x-y|} e^{-(\kappa/4)|x-y|^2} \leqslant e^{1/2\kappa^{1/2}} |x_0|, \end{aligned} \tag{4.16}$$

we have

$$|(Q\phi)(x)| \leqslant (T|\phi|)(x)|x_0|$$

where T is the operator with kernel $T(x,y) = (2\kappa b/\pi) e^{1/2\kappa^{1/2}} e^{-(\kappa/4)|x-y|^2}$. Thus

$$\|Q\phi\| \leqslant \||T|\phi|\| \;\; |x_0| \leqslant \|T\| \;\; |x_0| \;\; \|\phi\|, \tag{4.17}$$

and therefore $\|Q\| \leqslant K|x_0|$.

Let E satisfy $d(E,\sigma(H_L)) \geqslant \epsilon$ and choose x_0 such that $|x_0| < \epsilon/(2K)$, so that $\|Q\| \leqslant \frac{1}{2}\epsilon$. Then, by (4.13),

$$\|UG_L(E)U^{-1}\| = \|(H_L + Q - E)^{-1}\| < \frac{2}{\epsilon}. \tag{4.18}$$

Now we split up $\tilde{\Lambda}_L$ into four parts:

$$\tilde{\Lambda}_L = \bigcup_{i=1}^{4} \tilde{\Lambda}_L^{(i)},$$

where $\tilde{\Lambda}_L^{(i)} = \{x : x \in \tilde{\Lambda}, e_i \cdot x \geqslant |x|/\sqrt{2}\}$ and $e_1 = (1,0)$, $e_2 = (-1,0)$, $e_3 = (0,1)$, and $e_4 = (0,-1)$. We have

$$\langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L}\phi|\rangle \leqslant \sum_{i=1}^{4} \langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L^{(i)}}\phi|\rangle. \tag{4.19}$$

Now

$$\begin{aligned}\langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \phi| \rangle &= \langle \mathbf{1}_{\Lambda_1}, U^{-1} | U G_L U^{-1} U \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \phi| \rangle \\ &\leqslant \| U^{-1} \mathbf{1}_{\Lambda_1} \| \| U G_L U^{-1} \| \| U \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \| \| \mathbf{1}_{\tilde{\Lambda}_L} \phi \| \\ &\leqslant \frac{2}{\epsilon} \| U^{-1} \mathbf{1}_{\Lambda_1} \| \| U \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \| \| \mathbf{1}_{\Lambda_L} \phi \|. \end{aligned} \tag{4.20}$$

Clearly,

$$\| U^{-1} \mathbf{1}_{\Lambda_1} \| \leqslant e^{\epsilon/2\sqrt{2}K} < K', \tag{4.21}$$

and by choosing $x_0 = (-\epsilon/2\sqrt{2}K, 0)$ we get

$$\| U \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \psi \|^2 = \int_{\tilde{\Lambda}_L^{(1)}} e^{2x_0 \cdot x} |\psi(x)|^2 \, dx \leqslant e^{-(\epsilon/4K)(L - L^s)} \| \psi \|^2, \tag{4.22}$$

from which it follows that

$$\| U \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \| \leqslant e^{-(\epsilon/8K)(L - L^s)} < e^{-(\epsilon/9K)L}, \tag{4.23}$$

for L sufficiently large. Thus, using (4.20)–(4.23) we get

$$\langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L^{(1)}} \phi| \rangle < \frac{2K'}{\epsilon} e^{-(\epsilon/9K)L} \| \mathbf{1}_{\tilde{\Lambda}_L} \phi \|, \tag{4.24}$$

and similarly for $i = 2,3,4$. Therefore

$$\langle \mathbf{1}_{\Lambda_1}, \ |G_L \mathbf{1}_{\tilde{\Lambda}_L} \phi| \rangle < \frac{8K'}{\epsilon} e^{-(\epsilon/9K)L} \| \mathbf{1}_{\tilde{\Lambda}_L} \phi \|. \tag{4.25}$$

□

The proof of part (RB) of (P2) is now reduced to estimating $\mathbb{P}(d(E, \sigma_L(H_L)) < \epsilon)$. However, the estimate (4.1) is not good enough and we have to obtain a better one.

We shall make use of the explicit form of the following basis functions for $\mathscr{H}_0$. For $m \in \mathbb{N}$ and $x \in \mathbb{R}^2$ let

$$u_m(x) = \frac{(2\kappa)^{(1/2)(m+1)}}{(\pi m!)^{1/2}} (x_1 - i x_2)^m e^{-\kappa |x|^2}. \tag{4.26}$$

Then $\{u_m : m \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{H}_0$. Note that since U_y commutes with P_0, $\{U_y u_m : m \in \mathbb{N}\}$ is also an orthonormal basis for $\mathscr{H}_0$. We also have that if $m \neq n$, then

$$\int_{B(0,r)} \overline{u_m}(x) u_n(x) \, dx = 0, \tag{4.27}$$

so that if

$$\phi = \sum_{m=0}^{\infty} c_m u_m,$$

then

$$\int_{B(0,r)} |\phi(x)|^2 \, dx = \sum_{m=0}^{\infty} |c_m|^2 \int_{B(0,r)} |u_m(x)|^2 \, dx. \tag{4.28}$$

Lemma 4.3: If $0 \leqslant r \leqslant d$ and $m \in \mathbb{N}$, then

$$\int_{B(0,d)\setminus B(0,r)} |u_m(x)|^2 \, dx \geqslant (e^{-2\kappa r^2} - e^{-2\kappa d^2}) \int_{B(0,r)} |u_m(x)|^2 \, dx. \tag{4.29}$$

Proof: A straightforward calculation gives

$$\int_{B(0,d)\setminus B(0,r)} |u_m(x)|^2 \, dx = \sum_{k=0}^{m} \left\{ \frac{(2\kappa r^2)^k}{k!} e^{-2\kappa r^2} - \frac{(2\kappa d^2)^k}{k!} e^{-2\kappa d^2} \right\} \tag{4.30}$$

and

$$\int_{B(0,r)} |u_m(x)|^2 \, dx = 1 - \sum_{k=0}^{m} \frac{(2\kappa r^2)^k}{k!} e^{-2\kappa r^2}. \tag{4.31}$$

For fixed $t \geqslant 0$, we define $F(s)$ for $s \geqslant t$ by

$$F(s) = \sum_{k=0}^{m} \frac{s^k}{k!} e^{-s} - \left(1 - \sum_{k=0}^{m} \frac{t^k}{k!} e^{-t} \right) e^{-s}. \tag{4.32}$$

Then the statement of the lemma is equivalent to the following: If $s \geqslant t \geqslant 0$, then $F(s) \leqslant F(t)$. Now

$$\begin{aligned} F'(s) &= e^{-s} \left(-\frac{s^m}{m!} + \sum_{k=m+1}^{\infty} \frac{t^k}{k!} e^{-t} \right) \\ &= \frac{e^{-s}}{m!} \left(-s^m + t^m e^{-t} \sum_{k=1}^{\infty} \frac{t^k m!}{(k+m)!} \right) \\ &\leqslant \frac{e^{-s}}{m!} \left(-s^m + t^m e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} \right) \leqslant \frac{e^{-s}}{m!} (-s^m + t^m) \leqslant 0. \end{aligned} \tag{4.33}$$

□

In the remaining lemmas we shall prove that the part (RB) of (P2) is satisfied. From Lemma 4.2 with $\epsilon = L^{\delta-1}$ and $m = 2L^{(1/2\delta-1)}$, we get, for L sufficiently large

$$\mathbb{P}(\langle \mathbf{1}_{\Lambda_1}, |G_L \mathbf{1}_{\tilde{\Lambda}_L} \phi| \rangle < e^{-mL} \|\mathbf{1}_{\tilde{\Lambda}_L} \phi\|, \quad \forall \phi \in \mathscr{H}_L) \geqslant \mathbb{P}(d(E, \sigma(H_L)) \geqslant L^{\delta-1}).$$

Now, if $E > b - \epsilon$, the inequality $H_L \leqslant (b - 2\epsilon)\mathbf{1}$ implies that $d(E, \sigma(H_L)) > \epsilon$. Therefore, it is enough to prove that for L sufficiently large with probability greater than $1 - 1/2L^p$,

$$H_L \leqslant \left(b - \frac{2}{L^{1-\delta}} \right) \mathbf{1}. \tag{4.34}$$

We shall proceed in the following way.

Let $0 < \delta < 1/4$ and put $\epsilon_L = 4L^{-1/2+\delta}$. For each configuration, $\omega \in \Omega$ and for $A \subset \mathbb{R}^2$ let

$$A^+=\{x\in A: V_L(x,\omega)>b-\epsilon_L\},$$

$$A^-=\{x\in A: V_L(x,\omega)\leqslant b-\epsilon_L\}.$$

Let t be a fixed number, such that $\kappa t>768$. We shall say that a configuration $\omega\in\Omega$ satisfies the condition (C1) if the following holds

(C1) There is a set of regions $\{B_i\}$ with $\#\{B_i\}\leqslant L^2$, such that
(i) $\Lambda_L^+\subset\cup_i B_i$,
(ii) diam $B_i\leqslant 2t\ \ln L$, and
(iii) $d(B_i,B_j)\geqslant\sqrt{t\ \ln L}$.

Let τ be a fixed number such that $16\kappa\tau^4<1$. We shall say that a configuration satisfies (C2) if the following occurs.

(C2) For each $k\in\Lambda_L\cap\mathbb{Z}^2$, we can find a ball D_k, center k, and radius ρ_k, where $(\tau^2/2)(\ln L)^{1/2}\leqslant\rho_k\leqslant\tau^2(\ln L)^{1/2}$, with a surrounding annulus $\widetilde{D}_k$ of width $\tau(\ln L)^{1/4}$, such that $\widetilde{D}_k\cap\Lambda_L^+=\emptyset$.

We shall first prove (Lemmas 4.4–4.8) that for configurations that satisfy (C1) and (C2) simultaneously, (4.34) holds. Then in Lemmas 4.9 and 4.10 we show that such configurations occur with probability greater than $1-1/2L^p$.

For a configuration that satisfies (C1), we let $B_i^{(j)}=\{x\in\mathbb{R}^2: d(x,B_i)<(j/8)\sqrt{t\ \ln L}\}$ for $j=1,2,3,4$. If $\phi\in\mathscr{H}$ we write ϕ_i for the restriction $\phi|B_i^{(2)}$.

In the following lemma we shall prove that on subsets of $B_i^{(1)}$, $P_0\phi$ can be approximated by $P_0\phi_i$.

Lemma 4.4: There exists L_0 such that if $L>L_0$ then, for all configurations that satisfy (C1), for all i, for all $\phi\in\mathscr{H}$ with $\|\phi\|\leqslant 1$, and for all $A\subset B_i^{(1)}$,

$$\left|\int_A|(P_0\phi)(x)|^2\ dx-\int_A|(P_0\phi_i)(x)|^2\ dx\right|<\frac{1}{L^4}. \tag{4.35}$$

Proof: Let $x\in B_i^{(1)}$; then

$$\begin{aligned}|(P_0\phi)(x)-(P_0\phi_i)(x)|&=\left|\int_{\mathbb{R}^2\setminus B_i^{(2)}}P_0(x,x')\phi(x')dx'\right|\\&\leqslant\left(\int_{\mathbb{R}^2\setminus B_i^{(2)}}|P_0(x,x')|^2\ dx'\right)^{1/2}\\&=\frac{2\kappa}{\pi}\left(\int_{\mathbb{R}^2\setminus B_i^{(2)}}e^{-2\kappa|x-x'|^2}\ dx'\right)^{1/2}\\&\leqslant\frac{2\kappa}{\pi}\left(e^{-(\kappa t/64)\ln L}\int_{\mathbb{R}^2}e^{-\kappa|x-x'|^2}\ dx'\right)^{1/2}\\&=2\sqrt{\frac{\kappa}{\pi}}\frac{1}{L^{\kappa t/128}}\leqslant 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{L^6}.\end{aligned} \tag{4.36}$$

Thus

$$\begin{aligned}\big||(P_0\phi)(x)|^2-|(P_0\phi_i)(x)|^2\big|&=(|(P_0\phi)(x)|+|(P_0\phi_i)(x)|)\big||(P_0\phi)(x)|-|(P_0\phi_i)(x)|\big|\\&\leqslant 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{L^6}\ (|(P_0\phi)(x)|+|(P_0\phi_i)(x)|).\end{aligned} \tag{4.37}$$

Now for L large enough,

$$\int_A |P_0\phi(x)|dx \leqslant |A|^{1/2}\|P_0\phi\| \leqslant |A|^{1/2}\|\phi\| \leqslant |A|^{1/2} < L, \tag{4.38}$$

and similarly,

$$\int_A |(P_0\phi_i)(x)| \leqslant |A|^{1/2}\|\phi_i\| \leqslant |A|^{1/2}\|\phi\| < L. \tag{4.39}$$

Therefore

$$\int_A \left|(P_0\phi)(x)|^2 - |(P_0\phi_i)(x)|^2\right| dx < 4\sqrt{\frac{\kappa}{\pi}}\,\frac{1}{L^5} < \frac{1}{L^4}, \tag{4.40}$$

for L sufficiently large. □

Lemma 4.5: There exists L_0 such that if $L > L_0$ then, for all configurations that satisfy (C1), for all i, for all $\phi \in \mathscr{H}$ with $\|\phi\| \leqslant 1$, and for all A and C subsets of $\mathbb{R}^2$ such that $B_i^{(3)} \subset C \subset A$,

$$\left| \int_A |(P_0\phi_i)(x)|^2\, dx - \int_C |(P_0\phi_i)(x)|^2\, dx \right| < \frac{1}{L^4}. \tag{4.41}$$

Proof: It is sufficient to prove the lemma for $A = \mathbb{R}^2$ and $C = B_i^{(3)}$,

$$\begin{aligned} |(P_0\phi_i)(x)| &\leqslant \frac{2\kappa}{\pi} \int_{B_i^{(2)}} e^{-\kappa|x-x'|^2} |\phi_i(x')| dx' \\ &\leqslant \frac{2\kappa}{\pi} \left(\int_{B_i^{(2)}} e^{-2\kappa|x-x'|^2}\, dx' \right)^{1/2} \\ &\leqslant \frac{2\kappa}{\pi} \left(\int_{B_i^{(2)}} e^{-2\kappa\{d(x,B_i^{(2)})\}^2}\, dx' \right)^{1/2} \\ &= \frac{2\kappa}{\pi} |B_i^{(2)}|^{1/2} e^{-\kappa\{d(x,B_i^{(2)})\}^2} \leqslant L e^{-\kappa\{d(x,B_i^{(2)})\}^2}, \end{aligned} \tag{4.42}$$

for L large enough. If $x \in \mathbb{R}^2 \backslash B_i^{(3)}$, $d(x, B_i^{(2)}) > \frac{1}{8}\sqrt{t \ln L}$. Also, we can find a ball B of radius L such that $B_i^{(3)} \subset B$. Let $\widetilde{B}$ be a ball of radius $2L$ concentric with B. Now

$$\begin{aligned} \int_{\mathbb{R}^2 \backslash B_i^{(3)}} |(P_0\phi_i)(x)|^2\, dx &\leqslant \int_{\mathbb{R}^2 \backslash \widetilde{B}} |(P_0\phi_i)(x)|^2\, dx + \int_{\widetilde{B} \backslash B_i^{(3)}} |(P_0\phi_i)(x)|^2\, dx \\ &\leqslant 2\pi L^2 \int_{2L}^{\infty} e^{-2\kappa(r-L)^2} r\, dr + 4\pi L^4 e^{-(\kappa/32)\ln L} \\ &= \frac{\pi L^2}{2\kappa}\, e^{-2\kappa L^2} + \frac{4\pi}{L^{\kappa t/32 - 4}} < \frac{1}{L^4}, \end{aligned} \tag{4.43}$$

for L sufficiently large. □

In the next two lemmas we obtain an upper bound for the integral of $|(P_0\phi)(x)|^2$ over that part of B_i where $V_L(x,\omega) > b - \epsilon_L$ as a fraction of the integral over $B_i^{(4)}$.

If a configuration satisfies both (C1) and (C2), for each i we let K_i be the smallest subset of $\Lambda_L \cap \mathbb{Z}^2$, such that

$$B_i^{(3)} \subset \bigcup_{k \in K_i} D_k \subset B_i^{(4)}.$$

Then $\#K_i \leqslant C \ln L$. Note that the D_k's are not disjoint.

Lemma 4.6: There exists L_0 such that if $L > L_0$ then, for all configurations that satisfy both (C1) and (C2), for all i and for all $\phi \in \mathscr{H}$ with $\|\phi\| \leqslant 1$,

$$\int_{B_i^+} |(P_0\phi_i)(x)|^2 \, dx \leqslant \left(1 - \frac{1}{L^{1/4}}\right) \int_{B_i^{(4)}} |(P_0\phi_i)(x)|^2 \, dx + \frac{1}{L^4}. \tag{4.44}$$

Proof: Let

$$P_0\phi_i = \sum_{m=0}^{\infty} c_m U_{-k} u_m, \tag{4.45}$$

where $k \in K_i$. Since for each $k \in K_i$, $B_i^{(4)-} \supset \widetilde{D}_k$,

$$\begin{aligned}\int_{B_i^{(4)-}} |(P_0\phi_i)(x)|^2 \, dx &> \int_{\widetilde{D}_k} |(P_0\phi_i)(x)|^2 \, dx \\ &= \sum_{m=0}^{\infty} |c_m|^2 \int_{\widetilde{D}_k} |u_m(x-k)|^2 \, dx \\ &\geqslant \{e^{-2\kappa\rho_k^2} - e^{-2\kappa(\rho_k + \sqrt{\rho_k})^2}\} \sum_{m=0}^{\infty} |c_m|^2 \int_{D_k} |u_m(x-k)|^2 \, dx, \end{aligned} \tag{4.46}$$

by (4.28) and Lemma 4.3. Thus

$$\int_{B_i^{(4)-}} |(P_0\phi_i)(x)|^2 \, dx \geqslant \frac{1}{L^{2\kappa r^4}} (1 - e^{-\rho_L}) \int_{D_k} |(P_0\phi_i)(x)|^2 \, dx, \tag{4.47}$$

where $\rho_L = \kappa\tau^2 (\ln L)^{1/2}$. Summing over K_i and dividing by $\#K_i$, we get

$$\int_{B_i^{(4)-}} |(P_0\phi_i)(x)|^2 \, dx \geqslant \frac{1}{L^{2\kappa\tau^4}} (1 - e^{-\rho_L}) \frac{1}{\#K_i} \int_{B_i^{(3)}} |(P_0\phi_i)(x)|^2 \, dx. \tag{4.48}$$

By Lemma 4.5 and using $\#K_i < C \ln L$ we have for L large enough,

$$\begin{aligned}\int_{B_i^{(4)-}}|(P_0\phi_i)(x)|^2\,dx &\geqslant \frac{1}{L^{2\kappa\tau^4}}\left(\frac{1-e^{-\rho_L}}{C\,\ln L}\right)\left(\int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx-\frac{1}{L^4}\right)\\ &\geqslant \frac{1}{L^{2\kappa\tau^4}}\frac{1}{2C\,\ln L}\int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx-\frac{1}{L^4}\\ &\geqslant \frac{1}{L^{1/8}}\frac{1}{2C\,\ln L}\int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx-\frac{1}{L^4}\\ &\geqslant \frac{1}{L^{1/4}}\int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx-\frac{1}{L^4}.\end{aligned}\tag{4.49}$$

Now

$$\begin{aligned}\int_{B_i^{+}}|(P_0\phi_i)(x)|^2\,dx &= \int_{B_i^{(4)+}}|(P_0\phi_i)(x)|^2\,dx\\ &= \int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx-\int_{B_i^{(4)-}}|(P_0\phi_i)(x)|^2\,dx\\ &\leqslant \left(1-\frac{1}{L^{1/4}}\right)\int_{B_i^{(4)}}|(P_0\phi_i)(x)|^2\,dx+\frac{1}{L^4}.\end{aligned}\tag{4.50}$$

□

Lemma 4.7: There exists L_0 such that if $L>L_0$ then, for all configurations that satisfy both (C1) and (C2), for all i and for all $\phi\in\mathscr{H}$ with $\|\phi\|\leqslant 1$,

$$\int_{B_i^{+}}|(P_0\phi)(x)|^2\,dx\leqslant\left(1-\frac{1}{L^{1/2}}\right)\int_{B_i^{(4)}}|(P_0\phi)(x)|^2\,dx+\frac{4}{L^4}.\tag{4.51}$$

Proof: Let $\psi=P_0\phi$ so that $P_0\psi=\psi$ and $\|\psi\|\leqslant\|\phi\|\leqslant 1$,

$$\begin{aligned}|(P_0\psi_i)(x)| &\leqslant \int_{B_i^{(2)}\setminus B_i}|P_0(x,x')||\psi(x')|dx'+\int_{B_i}|P_0(x,x')||\psi(x')|dx'\\ &\leqslant\left(\int_{B_i^{(2)}\setminus B_i}|P_0(x,x')|^2\,dx'\right)^{1/2}\left(\int_{B_i^{(2)}\setminus B_i}|\psi(x')|^2\,dx'\right)^{1/2}\\ &\quad+\left(\int_{B_i}|P_0(x,x')|^2\,dx'\right)^{1/2}\left(\int_{B_i}|\psi(x')|^2\,dx'\right)^{1/2}\\ &\leqslant\left(\frac{2\kappa}{\pi}\right)^{1/2}\left(\int_{B_i^{(2)}\setminus B_i}|\psi(x')|^2\,dx'\right)^{1/2}+\frac{2\kappa}{\pi}\,|B_i|^{1/2}e^{-\kappa\{d(x,B_i)\}^2}.\end{aligned}\tag{4.52}$$

Therefore, if L is large enough,

$$|(P_0\psi_i)(x)|^2\leqslant\frac{2\kappa}{\pi}\int_{B_i^{(2)}\setminus B_i}|\psi(x')|^2\,dx'+Le^{-\kappa\{d(x,B_i)\}^2}.\tag{4.53}$$

Hence

$$\int_{B_i^{(4)}\setminus B_i^{(1)}} |(P_0\psi_i)(x)|^2\, dx \leqslant \frac{2\kappa}{\pi}\, |B_i^{(4)}\setminus B_i^{(1)}| \int_{B_i^{(2)}\setminus B_i} |\psi(x')|^2\, dx' + L\int_{B_i^{(4)}\setminus B_i^{(1)}} e^{-\kappa\{d(x,B_i)\}^2}\, dx. \tag{4.54}$$

The last term is less than

$$\frac{L^2}{L^{(\kappa t)/64}} < \frac{1}{L^4}.$$

Using this bound and $B_i^{(2)}\setminus B_i \subset B_i^{(4)-}$,

$$\int_{B_i^{(4)}\setminus B_i^{(1)}} |(P_0\psi_i)(x)|^2\, dx \leqslant \frac{2\kappa}{\pi}\, B_i^{(4)}\setminus B_i^{(1)}| \int_{B_i^{(4)-}} |\psi(x')|^2\, dx' + \frac{1}{L^4}. \tag{4.55}$$

Now

$$\begin{aligned}
\left| \int_{B_i^{(4)}} |\psi(x)|^2\, dx - \int_{B_i^{(4)}} |(P_0\psi_i)(x)|^2\, dx \right| &\leqslant \left| \int_{B_i^{(1)}} |\psi(x)|^2\, dx - \int_{B_i^{(1)}} |(P_0\psi_i)(x)|^2\, dx \right| \\
&\quad + \int_{B_i^{(4)}\setminus B_i^{(1)}} |\psi(x)|^2\, dx + \int_{B_i^{(4)}\setminus B_i^{(1)}} |(P_0\psi_i)(x)|^2\, dx \\
&\leqslant \frac{1}{L^4} + \int_{B_i^{(4)-}} |\psi(x)|^2\, dx + \int_{B_i^{(4)}\setminus B_i^{(1)}} |(P_0\psi_i)(x)|^2\, dx \\
&\qquad \text{using Lemma 4.4 and } B_i^{(4)}\setminus B_i^{(1)} \subset B_i^{(4)-}, \\
&\leqslant \left(1 + \frac{2\kappa}{\pi}\, |B_i^{(4)}\setminus B_i^{(1)}|\right) \int_{B_i^{(4)-}} |\psi(x)|^2\, dx + \frac{2}{L^4}.
\end{aligned} \tag{4.56}$$

Thus, writing $A = (2\kappa/\pi)|B_i^{(4)}\setminus B_i^{(1)}|$,

$$\begin{aligned}
\int_{B_i^{(4)}} |(P_0\psi_i)(x)|^2\, dx &\leqslant (1+A)\int_{B_i^{(4)-}} |\psi(x)|^2\, dx + \int_{B_i^{(4)}} |\psi(x)|^2\, dx + \frac{2}{L^4} \\
&= (2+A)\int_{B_i^{(4)}} |\psi(x)|^2\, dx - (1+A)\int_{B_i^{+}} |\psi(x)|^2\, dx + \frac{2}{L^4}.
\end{aligned} \tag{4.57}$$

Hence by Lemma 4.6,

$$\begin{aligned}
\int_{B_i^{+}} |(P_0\psi_i)(x)|^2\, dx &\leqslant \left(1 - \frac{1}{L^{1/4}}\right)(2+A)\int_{B_i^{(4)}} |\psi(x)|^2\, dx \\
&\quad - \left(1 - \frac{1}{L^{1/4}}\right)(1+A)\int_{B_i^{+}} |\psi(x)|^2\, dx + \frac{3}{L^4}.
\end{aligned} \tag{4.58}$$

Therefore, by using Lemma 4.4 and rearranging the inequality, we get

$$\left(1 + (1+A)\left(1 - \frac{1}{L^{1/4}}\right)\right)\int_{B_i^{+}} |\psi(x)|^2\, dx \leqslant \left(1 - \frac{1}{L^{1/4}}\right)(2+A)\int_{B_i^{(4)}} |\psi(x)|^2\, dx + \frac{4}{L^4}. \tag{4.59}$$

Thus

$$\int_{B_i^+}|\psi(x)|^2\,dx \leqslant \frac{(1-1/L^{1/4})(2+A)}{(1-1/L^{1/4})(2+A)+1/L^{1/4}}\int_{B_i^{(4)}}|\psi(x)|^2\,dx+\frac{4}{L^4}$$
$$\leqslant\left(1-\frac{1}{L^{1/2}}\right)\int_{B_i^{(4)}}|\psi(x)|^2\,dx+\frac{4}{L^4}, \tag{4.60}$$

for L sufficiently large. □

Lemma 4.8: There exists $L_1>0$ such that for all $L>L_1$ and for all configurations that satisfy both (C1) and (C2),

$$H_L \leqslant \left(b-\frac{2}{L^{1-\delta}}\right)\mathbf{1}. \tag{4.61}$$

Proof: Let $\phi\in\mathscr{H}_L$ with $\|\phi\|$. Then

$$\begin{aligned}
\langle\phi,H_L\phi\rangle &= \langle\phi,P_LV_LP_L^*\phi\rangle\\
&= \langle P_L^*\phi,V_LP_L^*\phi\rangle\\
&= \langle P_0\phi,V_LP_0\phi\rangle\\
&= \int_{\Lambda_L}V(x)|(P_0\phi)(x)|^2\,dx\\
&\leqslant (b-\epsilon_L)\int_{\Lambda_L^-}|(P_0\phi)(x)|^2\,dx+b\int_{\Lambda_L^+}|(P_0\phi)(x)|^2\,dx\\
&\leqslant (b-\epsilon_L)\int_{\Lambda_L\setminus(\cup B_i^{(4)})}|(P_0\phi)(x)|^2\,dx+(b-\epsilon_L)\sum_i\int_{B_i^{(4)-}}|(P_0\phi)(x)|^2\,dx\\
&\quad +b\sum_i\int_{B_i^{(4)+}}|(P_0\phi)(x)|^2\,dx\\
&\leqslant (b-\epsilon_L)\int_{\Lambda_L\setminus(\cup B_i^{(4)})}|(P_0\phi)(x)|^2\,dx+\sum_i\left\{(b-\epsilon_L)\int_{B_i^{(4)}}|(P_0\phi)(x)|^2\,dx\right.\\
&\quad \left.+\epsilon_L\int_{B_i^+}|(P_0\phi)(x)|^2\,dx\right\}\\
&\leqslant (b-\epsilon_L)\int_{\Lambda_L\setminus(\cup B_i^{(4)})}|(P_0\phi)(x)|^2\,dx+\sum_i\left((b-\epsilon_L)+\epsilon_L\left(1-\frac{1}{L^{1/2}}\right)\right)\\
&\quad \times\int_{B_i^{(4)}}|(P_0\phi)(x)|^2\,dx+\frac{4\epsilon_L}{L^4}L^2,
\end{aligned}$$

by the previous lemma. Thus

$$\begin{aligned}\langle \phi, H_L \phi\rangle &\leqslant (b-\epsilon_L L^{-1/2}) \int_{\Lambda_L \cup (\cup_i B_i^{(4)})} |(P_0\phi)(x)|^2 dx + \frac{4\epsilon_L}{L^2}\\ &\leqslant (b-\epsilon_L L^{-1/2}) \|P_0\phi\|^2 + \frac{4\epsilon_L}{L^2}\\ &\leqslant (b-\epsilon_L L^{-1/2}) \|\phi\|^2 + \frac{4\epsilon_L}{L^2}\\ &\leqslant \left(b-\frac{\epsilon_L}{2L^{1/2}}\right) = \left(b-\frac{2}{L^{1-\delta}}\right).\end{aligned}$$

□

Now we come to the main probabilistic estimate. The next two lemmas will be used in establishing that the configurations that satisfy (C1) and (C2) simultaneously occur with probability greater than $1-1/2L^p$.

Lemma 4.9: Let $\alpha>0$ and $p>0$. Then for all L sufficiently large, if $\alpha \ln L < a_L^2 < L^{1/4}$,

$$\mathbb{P}\left(\exists x \in \Lambda_L \cap \mathbb{Z}^2 : \#(B(x,a_L)\cap(\mathbb{Z}^2)^+) > \frac{1}{4}\, d_L^{1/2}\right) < \frac{1}{4L^p}.$$

Proof:

$$\begin{aligned}\mathbb{P}\left(\exists x \in \Lambda_L \cap \mathbb{Z}^2 : \#(B(x,a_L)\cap(\mathbb{Z}^2)^+) > \frac{1}{4}\, a_L^{1/2}\right) &\leqslant \sum_{x\in\Lambda_L\cap\mathbb{Z}^2} \mathbb{P}\left(\#(B(x,a_L)\cap(\mathbb{Z}^2)^+) > \frac{1}{4}\, a_L^{1/2}\right)\\ &\leqslant L^2 \sum_{n=n_0}^{N} \binom{N}{n} \epsilon_L^n, \qquad (4.62)\end{aligned}$$

where $n_0 = [a_L^{1/2}/4]$ and $N = \#(B(x,a_L)\cap\mathbb{Z}^2) \leqslant \gamma a_L^2$. Now $\binom{N}{n} \leqslant N^n/n! \leqslant (Ne/n)^n$, so that

$$\sum_{n=n_0}^{N} \binom{N}{n} \epsilon_L^n < \frac{1}{1-(N_{e\epsilon_L}/n_0)} \left(\frac{N_{e\epsilon_L}}{n_0}\right)^{n_0}. \qquad (4.63)$$

The result now follows from

$$\frac{N_{e\epsilon_L}}{n_0} \leqslant \frac{4\gamma a_L^2 e L^{-1/2+\delta}}{[a_L^{1/2}/4]} \leqslant 16\gamma e L^{-1/4+\delta}. \qquad (4.64)$$

□

Lemma 4.10: Let $\alpha>0$ and $p>0$. Then for all L sufficiently large, if $\alpha \ln L < a_L^2 < L^{1/4}$, the probability that for every $x \in \Lambda_L \cap \mathbb{Z}^2$, there exists $r_x \in (a_L/2, a_L - \sqrt{a_L})$, such that

$$(B(x, r_x + \sqrt{a_L}) \backslash B(x, r_x)) \cap (\mathbb{R}^2)^+ = \emptyset,$$

is greater than $1-1/4L^p$.

Proof: Suppose there exists $x \in \Lambda_L \cap \mathbb{Z}^2$ such that for all $r \in (a_L/2, a_L - \sqrt{a_L})$,

$$(B(x, r + \sqrt{a_L}) \backslash B(x, r)) \cap (\mathbb{R}^2)^+ \neq \emptyset.$$

Then each of the the concentric annuli,

$$B(x,a_L)\backslash B(x,a_L-\sqrt{a_L}),\quad B(x,a_L-\sqrt{a_L})\backslash B(x,a_L-2\sqrt{a_L}),\ldots,$$

$$B(x,a_L-([\sqrt{a_L}/2]-1)\sqrt{a_L})\backslash B(x,a_L-[\sqrt{a_L}/2]\sqrt{a_L}),$$

contains a point of $(\mathbb{R}^2)^+$, so that at least every other annulus contains a point of $(\mathbb{Z}^2)^+$. Therefore

$$\#(B(x,a_L)\cap\mathbb{Z}^2_+)>[\sqrt{a_L}/4].$$

By Lemma 4.9 this has a probability less than $1/4L^p$. □

We are going to apply this lemma in two instances: one to decouple regions of size $O(\ln L)$ and another for regions of size $O((\ln L)^{1/2})$ to get the over-spill of the wave function.

Proposition 4.11: There exists $L_1>0$ such that all $L>L_1$ satisfy (P1) and (P2) for each $E\in(b-L^{-\delta},b]$.

Proof: Putting $\epsilon = e^{-L^\beta}$ in Lemma 4.1 we get for $E>a>0$,

$$\mathbb{P}(d(E,\sigma(H_L))<e^{-L^\beta})\leqslant CL^4e^{-L^\beta\min(1,\sigma)}. \tag{4.65}$$

It follows from (4.65) that L satisfies (P1) if sufficiently large. (4.65) also shows that there is $L_1>0$ such that if $L>L_1$,

$$\mathbb{P}\left(d(E,\sigma(H_L))>\frac{1}{2}\,e^{-L^\beta}\right)\geqslant 1-\frac{1}{2L^p}.$$

It is then sufficient to prove that if $m=2L^{\gamma-1}$, where $\gamma=1/2\delta$; then

$$\mathbb{P}(\langle\mathbf{1}_{\Lambda_1}|G_L(E)\mathbf{1}_{\tilde{\Lambda}_L}\phi|\rangle<e^{-mL}\|\mathbf{1}_{\tilde{\Lambda}_L}\phi\|,\quad\forall\phi\in\mathscr{H}_L)\geqslant 1-\frac{1}{2L^p}.$$

From Lemma 4.2 with $\epsilon=L^{\delta-1}$, we get for L sufficiently large,

$$\mathbb{P}(\langle\mathbf{1}_{\Lambda_1},|G_L\mathbf{1}_{\tilde{\Lambda}_L}\phi|\rangle<e^{-mL}\|\mathbf{1}_{\tilde{\Lambda}_L}\phi\|,\quad\forall\phi\in\mathscr{H}_L)\geqslant\mathbb{P}(d(E,\sigma(H_L))\geqslant L^{\delta-1}).$$

If $E>b-\epsilon$, then $H_L\leqslant(b-2\epsilon)\mathbf{1}$ implies that $d(E,\sigma(H_L))>\epsilon$. Therefore it is enough to prove that for L sufficiently large with probability greater than $1-1/2L^p$,

$$H_L\leqslant\left(b-\frac{2}{L^{1-\delta}}\right)\mathbf{1}.$$

Let $a_L=t\ln L$ and let $\{x_i:\ i=1,\ldots,N\}$ be the points of $\Lambda_L\cap\mathbb{Z}^2$ so that $N\leqslant L^2$. By Lemma 4.10, with probability greater than $1-1/4L^p$ for each i, we can find $r_i\in(a_L/2,a_L-\sqrt{a_L})$ such that

$$(B(x_i,r_i+\sqrt{a_L})\backslash B(x_i,r_i))\cap(\mathbb{R}^2)^+=\emptyset.$$

Let

$$A_i=B(x_i,r_i)$$

and

$$\tilde{A}_i=B(x_i,r_i+\sqrt{a_L}).$$

Let $B_1=A_1$ and for $1<i\leqslant N$ let

$$B_i = A_i \setminus \cup_{j<i} \widetilde{A}_j;$$

if $B_i = \emptyset$ it is ignored. Then for all $1 < i \leqslant N$,

$$\text{diam } B_i \leqslant \text{diam } A_i = 2r_i \leqslant 2a_L,$$

and if $i > j$, since $B_j \subset A_j$ and $B_i \subset \widetilde{A}_j^c$,

$$d(B_i, B_j) > d(\widetilde{A}_j^c, A_j) = \sqrt{a_L}.$$

If $x \in \Lambda_L^+$, let i_x be the smallest i such that $x \in A_i$. Then $x \notin \cup_{j<i_x} \widetilde{A}_j$, therefore $x \in B_{i_x}$. Thus $\Lambda_L^+ \subset \cup_i B_i$. So we have proved that with probability $>1-1/4L^p$ condition (C1) is satisfied. By applying Lemma 4.10 again, this time with $a_L = \tau^2(\ln L)^{1/2}$, we see that with probability $>1-1/4L^p$, condition (C2) is satisfied. Thus, with probability $>1-1/2L^p$ both conditions (C1) and (C2) are satisfied and Lemma 4.8 gives the required result. □

We are now ready to prove the main theorem of this paper, Theorem 2.3.

Proof of Theorem 2.3: We remove the condition $a>0$ and let

$$\widetilde{\omega}_n = \omega_n + 1 - a,$$

so that the probability measure corresponding to the random variable $\widetilde{\omega}_n$ has support equal to $[1,c]$, where $c = b - a + 1$. Let

$$\widetilde{V}(x,\omega) = \sum_{n \in \mathbb{Z}^2} \mathbf{1}_{\Lambda_1(n)} \widetilde{\omega}_n,$$

and

$$\widetilde{H} = P_0 \widetilde{V} P_0.$$

Let $\delta \in (0,1/4)$ and let $L > \max(L_0, L_1)$, where L_0 is as in Theorem 3.1 and L_1 is as in Proposition 4.11. Then there an $m>0$, and for each $E \in [c - L^{-\delta}, c]$ there is a $\Delta_E > 0$ and $\Omega_E \subset \Omega$ with $\mathbb{P}(\Omega_E) = 1$, such that for $\omega \in \Omega_E$, $[1,c] \cap [E - \Delta_E, E + \Delta_E]$ is in the pure-point spectrum of $\widetilde{H}$ and the corresponding eigenfunctions decay with rate greater or equal to m. Let

$$\Omega' = \bigcap_{E \in [c - L^{-\delta}, c] \cap \mathbf{Q}} \Omega_E;$$

then $\mathbb{P}(\Omega') = 1$ and for $\omega \in \Omega'$, $[c - L^{-\delta}, c]$ is in the pure-point spectrum of $\widetilde{H}$ and the corresponding eigenfunctions decay with mass greater or equal to m. Now the eigenfunctions of H with eigenvalues in $[c - L^{-\delta}, c]$ are eigenfunctions of $\widetilde{H}$ with eigenvalues in $[b - L^{-\delta}, b]$. Thus, it follows that almost surely $[b - L^{-\delta}, b]$ is in the pure-point spectrum of H and the corresponding eigenfunctions of H decay with mass greater or equal to m. Similarly, one can prove the same result for $[a, a + L^{-\delta}]$. □

ACKNOWLEDGMENTS

This work was partially supported by the European Commission under the Human Capital and Mobility Scheme (EU Contract No. CHRX-CT93-0411).

Two of us (T.C.D. and J.V.P.) would like to thank the Institut de Physique Théorique of the Ecole Polytechnique Fédérale de Lausanne for their hospitality and financial support. They also acknowledge financial support from the British Council/Forbairt (Ireland) and J.V.P. acknowledges support from the European Commission. N.M. would like to thank University College Dublin for

inviting him to Dublin where this work was initiated, and also the Swiss National Foundation for Science for support. We are grateful to H. Kunz and A. Jori for some very useful discussions.

[1] S. Girvin and R. Prange, *The Quantum Hall Effect* (Springer-Verlag, New York, 1987).
[2] A. MacDonald, *Quantum Hall Effect: a perspective* (Kluwer Academic, New York, 1989).
[3] B. Halperin, Phys. Rev. B **25**, 2185 (1982).
[4] D. J. Thouless, J. Phys. C **14**, 3475 (1981).
[5] H. Aoki and T. Ando, Phys. Rev. Lett. **54**, 831 (1985).
[6] H. Kunz, Commun. Math. Phys. **112**, 121 (1987).
[7] F. Wegner, Z. Phys. B **51**, 279 (1983).
[8] E. Brezin, D. Gross, and C. Itzykson, Nucl. Phys. B **235**, 24 (1984).
[9] A. Klein and J. Perez, Nucl. Phys. B **25**, 199 (1985).
[10] N. Macris and J. V. Pulé, Lett. Math. Phys. **24**, 307 (1992).
[11] W.-M. Wang, ''Asymptotic expansion for the density of states of the random Schrödinger operator with a magnetic field,'' preprint, 1994, to appear in Commun. Math. Phys.
[12] H. Holden and F. Martinelli, Commun. Math. Phys **93**, 197 (1984).
[13] S. Kotani and B. Simon, Commun. Math. Phys. **112**, 103 (1987).
[14] H. von Dreifus and A. Klein, Commun. Math. Phys. **124**, 285 (1989).
[15] T. C. Dorlas, N. Macris, and J. V. Pulé, Helv. Phys. Acta **68**, 329 (1995).
[16] T. Spencer, J. Stat. Phys. **51**, 1009 (1988).
[17] F. Wegner, Z. Phys. B **44**, 9 (1981).
[18] J. M. Combes and L. Thomas, Commun. Math. Phys. **34**, 251 (1973).
[19] J. M. Combes and P. D. Hislop, ''Landau Hamiltonians with random potentials: localization and the density of states,'' Erwin Schrödinger Institute, preprint 155, 1994, to appear in Commun. Math. Phys.
[20] W.-M. Wang, ''Microlocalization, percolation and Anderson localization for the magnetic Schrodinger operator with a random potential,'' preprint Universite de Paris-Sud, 1995.
[21] J. M. Combes and P. D. Hislop, J. Funct. Anal. **124**, 149–180 (1994).
[22] S. Trugman, Phys. Rev. B **27**, 7539 (1983).
[23] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrodinger Operators* (Birkauser, Boston, 1990).
[24] H. Kunz and B. Souillard, Commun. Math. Phys. **78**, 201 (1980).
[25] F. Martinelli and E. Scoppola, Riv. Nuovo Cimento **10**, 1 (1987).