# GLOBAL BIFURCATION OF HOMOCLINIC SOLUTIONS OF HAMILTONIAN SYSTEMS

S. Secchi[1] and C. A. Stuart[2]

[1]SISSA
via Beirut 2/4
I-34014 Trieste, Italy

[2]Département de Mathématiques
École Polytechnique Fédérale
1015 Lausanne, Switzerland

**Abstract.** We provide global bifurcation results for a class of nonlinear Hamiltonian systems.

1. **Introduction.** In this paper we present some results about the bifurcation of global branches of homoclinic solutions for the following class of Hamiltonian systems:

$$Jx'(t) = \nabla H(t, x(t), \lambda), \tag{1.1}$$

where $x \in H^1(\mathbb{R}, \mathbb{R}^{2N})$, $J$ is a real $2N \times 2N-$ matrix such that $J^T = J^{-1} = -J$ and the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently smooth. Moreover $\lambda$ is the bifurcation parameter and $\nabla H(t, \xi, \lambda) = D_\xi H(t, \xi, \lambda)$ for $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{2N}$ and $\lambda \in \mathbb{R}$. We suppose that $x \equiv 0$ satisfies (1.1) for all values of the real parameter $\lambda$ and we study the existence of solutions which are homoclinic to this trivial solution in the sense that

$$\lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} x(t) = 0. \tag{1.2}$$

Our approach is based on the topological degree for proper Fredholm operators of index zero, as developed by Fitzpatrick, Pejsachowicz and Rabier in [3, 4, 8]. This tool has been applied recently by Rabier and Stuart (see [10]) to get bifurcation results for some classes of quasilinear elliptic partial differential equations on $\mathbb{R}^N$ with possibly a non-variational structure.

A first step is to express the problem (1.1)(1.2) as the set of zeros of some suitable function $F \in C^1(\mathbb{R} \times X, Y)$ where $X$ and $Y$ are real Banach spaces. Then we have to find conditions on $H$ under which this $F$ is a proper Fredholm operator of index zero. The Fredholm property holds provided that the linearisation of (1.1) at the $x = 0$ tends to periodic linear systems

$$Ju'(t) = A^{\pm}_{\lambda}(t)u(t) \text{ as } t \to \pm\infty$$

which have no characteristic multipliers on the unit circle. This is proved in Theorem 5.1. A criterion for properness is obtained provided that the nonlinear system (1.1) tends to periodic (possibly autonomous) Hamiltonian systems as $t \to \pm\infty$

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which have no solutions homoclinic to zero. The precise statement of this result is given as Theorem 4.9 and conditions which can be used to check for the absence of homoclinics are established in Theorem 5.5. A general result concerning the global bifurcation of solutions of the system (1.1)(1.2) is then formulated as Theorem 5.5 and we give one example illustrating how its hypotheses can be checked.

**Notation**

- $B(X,Y)$ is the space of bounded linear operators from $X$ into $Y$.
- $GL(X,Y)$ is the space of continuous isomorphisms from $X$ into $Y$.
- deg is the ordinary Leray–Schauder degree.
- The kernel of a linear operator $L$ is denoted by $\ker L$, and its range by $\operatorname{rge} L$.
- An operator $L \in B(X,Y)$ is said to be *Fredholm of index zero* if $\operatorname{rge} L$ is closed in $Y$, $\ker L$ is finite-dimensional and $\dim\ker L = \operatorname{codim} \operatorname{rge} L$. We set $\Phi_0(X,Y) = \{L \in B(X,Y) : L \text{ is a Fredholm operator of index zero}\}$.
- $L^2 = L^2(\mathbb{R},\mathbb{R}^{2N})$ with $\|x\|_2 = (\int_{\mathbb{R}} \|x(t)\|^2\, dt)^{1/2}$ for $x \in L^2$ where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{2N}$. The scalar product on $\mathbb{R}^{2N}$ will be denoted by $\langle\cdot,\cdot\rangle$ and that on $L^2$ by $\langle\cdot,\cdot\rangle_2$. Thus
$$\langle x,y\rangle_2 = \int_{-\infty}^{\infty} \langle x(t),y(t)\rangle\, dt \text{ for } x,y \in L^2.$$
- $H^1 = H^1(\mathbb{R},\mathbb{R}^{2N})$ with $\|x\| = \{\|x\|_2^2 + \|x'\|_2^2\}^{1/2}$ for $x \in H^1$. Recall that, for all $x \in H^1$, $x$ is continuous (after modification on a set of measure zero) and $\lim_{|t|\to\infty} x(t) = 0$.
- $C_d = \{x \in C(\mathbb{R},\mathbb{R}^{2N}) : \lim_{|t|\to\infty} x(t) = 0\}$ is Banach space with the norm $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$. $H^1$ is continuously embedded in $C_d$.
- $\|M\|$ will also be used to denote the Euclidean norm of a matrix $M$.

2. **A review of the topological degree for Fredholm maps.** Consider two real Banach spaces $X$ and $Y$. The notion of topological degree for $C^1-$Fredholm operator of index zero from $X$ to $Y$ has been introduced in [3, 4, 8] in several steps. First of all, one defines the **parity** of a continuous path $\lambda \in [a,b] \mapsto A(\lambda)$ of bounded linear Fredholm operators with index zero from $X$ into $Y$. It is always possible to find a **parametrix** for this path, namely a continuous function $B\colon [a,b] \to GL(Y,X)$ such that the composition $B(\lambda)A(\lambda)\colon X \to X$ is a compact perturbation of the identity for every $\lambda \in [a,b]$. If $A(a)$ and $A(b)$ belong to $GL(X,Y)$, then the parity of the path $A$ on $[a,b]$ is by definition
$$\pi(A(\lambda) \mid \lambda \in [a,b]) = \deg(B(a)A(a)) \deg(B(b)A(b)).$$
This is a good definition in the sense that it is independent of the parametrix $B$. The following criterion can be useful for evaluating the parity of an admissible path.

**Proposition 2.1.** *Let $A\colon [a,b] \to B(X,Y)$ be a continuous path of bounded linear operators having the following properties.*

(i) $A \in C^1([a,b],B(X,Y))$.
(ii) $A(\lambda)\colon X \to Y$ *is a Fredholm operator of index zero for each* $\lambda \in [a,b]$.
(iii) *There exists* $\lambda_0 \in (a,b)$ *such that*
$$A'(\lambda_0)[\ker A(\lambda_0)] \oplus \operatorname{rge} A(\lambda_0) = Y \tag{2.3}$$
*in the sense of a topological direct sum.*

*Then there exists* $\varepsilon > 0$ *such that* $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a, b]$,

$$A(\lambda) \in GL(X, Y) \text{ for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon] \tag{2.4}$$

*and*

$$\pi(A(\lambda) \mid \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k \tag{2.5}$$

*where* $k = \dim \ker A(\lambda_0)$.

The proof of this proposition is essentially contained in [2, 3].

We remark that given a continuous path $A\colon [a, b] \to \Phi_0(X, Y)$ and any $\lambda_0 \in [a, b]$ such that $A(\lambda) \in GL(X, Y)$ for all $\lambda \neq \lambda_0$, the parity $\pi(A(\lambda) \mid \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon])$ is the same for all $\varepsilon > 0$ sufficiently small. This number is then called the parity of $A$ across $\lambda_0$.

As in the case of the Leray–Schauder degree, the parity plays a role in bifurcation theory.

**Definition 2.2.** *Let* $X$ *and* $Y$ *be real Banach spaces and consider a function* $F \in C^1(\Lambda \times X, Y)$ *where* $\Lambda$ *is an open interval. Let* $P(\lambda, x) = \lambda$ *be the projection of* $\mathbb{R} \times X$ *onto* $\mathbb{R}$. *We say that* $\Lambda$ *is an admissible interval for* $F$ *provided that*

- (i) *for all* $(\lambda, x) \in \Lambda \times X$, *the bounded linear operator* $D_x F(\lambda, x)\colon X \to Y$ *is a Fredholm operator of index zero;*
- (ii) *for any compact subset* $K \subset Y$ *and any closed bounded subset* $W$ *of* $\mathbb{R} \times X$ *such that*

$$\inf \Lambda < \inf PW \leq \sup PW < \sup \Lambda,$$

$F^{-1}(K) \cap W$ *is a compact subset of* $\mathbb{R} \times X$.

**Theorem 2.3.** *Let* $X$ *and* $Y$ *be real Banach spaces and consider a function* $F \in C^1(\Lambda \times X, Y)$ *where* $\Lambda$ *is an admissible open interval for* $F$. *Suppose that* $\lambda_0 \in \Lambda$ *and that there exists* $\varepsilon > 0$ *such that* $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset \Lambda$,

$$D_x F(\lambda, 0) \in GL(X, Y) \text{ for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$$

*and*

$$\pi(D_x F(\lambda, 0) \mid [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = -1.$$

*Let* $Z = \{(\lambda, u) \in \Lambda \times X \mid u \neq 0 \text{ and } F(\lambda, u) = 0\}$ *and let* $C$ *denote the connected component of* $Z \cup \{(\lambda_0, 0)\}$ *containing* $(\lambda_0, 0)$.

*Then* $C$ *has at least one of the following properties:*

1. $C$ *is unbounded.*
2. *The closure of* $C$ *contains a point* $(\lambda_1, 0)$ *where* $\lambda_1 \in \Lambda \setminus [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ *and* $D_x F(\lambda_1, 0) \notin GL(X, Y)$.
3. *The closure of* $PC$ *intersects the boundary of* $\Lambda$.

*Proof.* See [10]. ∎

In the rest of the present paper, we want to present some explicit conditions under which Theorem 2.3 can be applied to our problem (1.1)(1.2).

3. **The functional setting.** In our analysis of the problem, the following terminology will help us to formulate conditions on the Hamiltonian $H$ which ensure that the system (1.1)(1.2) is equivalent to an equation of the form $F(\lambda, x) = 0$ where $F \in C^1(\mathbb{R} \times H^1, L^2)$.

Consider a function $f \colon \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$. This $f$ can be identified with the application

$$(t, \xi) \in \mathbb{R} \times \mathbb{R}^M \mapsto (t, f(t, \xi)) \in \mathbb{R} \times \mathbb{R}.$$

**Definition 3.1.** *We say that $f$ is an equicontinuous $C^0_\xi$–bundle map if $f$ is continuous on $\mathbb{R} \times \mathbb{R}^M$ and the collection $\{f(t, \cdot)\}_{t \in \mathbb{R}}$ is equicontinuous at every point $\xi$ of $\mathbb{R}^M$. For $k \in \mathbb{N}$, we say that $f$ is an equicontinuous $C^k_\xi$–bundle map if all the partial derivatives $\partial^\kappa f / \partial \xi^\kappa$ exist for all $|\kappa| \leq k$ and are equicontinuous $C^0_\xi$–bundle maps.*

We shall discuss the system (1.1)(1.2) under the following hypotheses on the Hamiltonian $H(t, \xi, \lambda)$ where $t, \lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^{2N}$.

(H1) $H \in C(\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R})$ with $H(t, \cdot, \lambda) \in C^2(\mathbb{R}^{2N})$ and $D_\xi H(t, 0, \lambda) = 0$ for all $t, \lambda \in \mathbb{R}$

(H2) The partial derivatives $D_\xi H, D^2_\xi H, D_\lambda D_\xi H, D_\lambda D^2_\xi H$ and $D_\xi D_\lambda D_\xi H$ exist and are continuous on $\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}$.

(H3) For each $\lambda \in \mathbb{R}$, $D_\xi H(\cdot, \cdot, \lambda) : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a $C^1_\xi$–bundle map, and $D_\lambda D^2_\xi H : \mathbb{R} \times (\mathbb{R}^{2N} \times \mathbb{R}) \to \mathbb{R}$ is a $C^0_{(\xi, \lambda)}$–bundle map.

(H4) $D^2_\xi H(\cdot, 0, 0)$ and $D_\lambda D^2_\xi H(\cdot, 0, 0) \in L^\infty(\mathbb{R})$.

Under these hypotheses the system (1.1) can be expressed as $F(\lambda, x) = 0$ where

$$F(\lambda, x) = Jx' - h(\lambda, x)$$

and

$$h(\lambda, x)(t) = D_\xi H(t, x(t), \lambda) \text{ for } t \in \mathbb{R}$$

is the Nemytskii operator generated by the function $D_\xi H$. To proceed we must establish some basic properties of this Nemytskii operator. First we observe that

$$D_\xi H(t, \xi, \lambda) = \int_0^1 \frac{d}{ds} D_\xi H(t, s\xi, \lambda) ds = \int_0^1 D^2_\xi H(t, s\xi, \lambda) \xi ds,$$

and so

$$\|D_\xi H(t, \xi, \lambda)\| \leq \|\xi\| \int_0^1 \left\| D^2_\xi H(t, s\xi, \lambda) \right\| ds. \tag{3.6}$$

**Lemma 3.2.** *(i) For every $K > 0$, there is a constant $C(K) > 0$ such that*

$$\left\| D_\lambda D^2_\xi H(t, \xi, \lambda) \right\| \leq C(K) \textit{ for all } t \in \mathbb{R} \textit{ and } (\xi, \lambda) \in \mathbb{R}^{2N+1} \textit{ with } \|(\xi, \lambda)\| \leq K.$$

*(ii) For each $\lambda \in \mathbb{R}$ and $K > 0$, there exists a constant $C(\lambda, K)$ such that*

$$\left\| D^2_\xi H(t, \xi, \lambda) \right\| \leq C(\lambda, K) \textit{ for all } t \in \mathbb{R} \textit{ and } \xi \in \mathbb{R}^{2N} \textit{ with } \|\xi\| \leq K.$$

*Proof.* (i) The hypothesis (H4) means that $\sup_{t \in \mathbb{R}} \|D_\lambda D^2_\xi H(t, 0, 0)\| < \infty$, and (H3) implies that, for all $(\xi, \lambda) \in \mathbb{R}^{2N+1}$, there exists $\delta(\xi, \lambda) > 0$ such that

$$\left\| D_\lambda D^2_\xi H(t, \xi, \lambda) - D_\lambda D^2_\xi H(t, \eta, \mu) \right\| < 1 \text{ for all } t \in \mathbb{R}$$

provided that $\|(\xi, \lambda) - (\eta, \mu)\| < \delta(\xi, \lambda)$. A straight forward compactness argument now leads to the first assertion.

(ii) First we note that

$$D_\xi^2 H(t,0,\lambda) - D_\xi^2 H(t,0,0) = \int_0^1 \frac{d}{ds} D_\xi^2 H(t,0,s\lambda)ds = \lambda \int_0^1 D_\lambda D_\xi^2 H(t,0,s\lambda)ds.$$

Hence, from (H4) and part (i), we see that

$$\sup_{t\in\mathbb{R}} \left\| D_\xi^2 H(t,0,\lambda) \right\| \leq \sup_{t\in\mathbb{R}} \left\| D_\xi^2 H(t,0,0) \right\| + |\lambda|\, C(|\lambda|).$$

Furthermore, by (H3), for any $\xi \in \mathbb{R}^{2N}$, there exists $\delta(\xi,\lambda) > 0$ such that

$$\left\| D_\xi^2 H(t,\xi,\lambda) - D_\xi^2 H(t,\eta,\lambda) \right\| < 1 \text{ for all } t \in \mathbb{R}$$

provided that $\|\xi - \eta\| < \delta(\xi,\lambda)$. A compactness argument now yields the conclusion (ii).

This lemma shows that the Nemytskii operator $h(\lambda,\cdot)$ maps $H^1$ into $L^2$. Indeed, for any $x \in H^1$, we have that $\|x\|_\infty < \infty$, and so by the lemma, there exists a constant $C(\lambda, \|x\|_\infty)$ such that

$$\left\| D_\xi^2 H(t,\xi,\lambda) \right\| \leq C(\lambda, \|x\|_\infty) \text{ for all } t \in \mathbb{R} \text{ and } \|\xi\| \leq \|x\|_\infty$$

Hence, by (3.6),

$$\|D_\xi H(t,x(t),\lambda)\| \leq \|x(t)\| \int_0^1 \left\| D_\xi^2 H(t,sx(t),\lambda) \right\| ds \leq \|x(t)\|\, C(\lambda, \|x\|_\infty),$$

showing that

$$h(\lambda,x) \in L^2 \text{ with } \|h(\lambda,x)\|_2 \leq C(\lambda, \|x\|_\infty)\, \|x(t)\|_2\,. \tag{3.7}$$

From now on we can consider $h$ as a mapping from $\mathbb{R} \times H^1$ into $L^2$ and the system (1.1)(1.2) can be written as

$$\begin{aligned} &F(\lambda,x) = 0 \text{ where } F : \mathbb{R} \times H^1 \to L^2 \text{ is defined by} \\ &F(\lambda,x) = Jx' - h(\lambda,x). \end{aligned} \tag{3.8}$$

Note that if $(\lambda,x) \in \mathbb{R} \times H^1$ and $F(\lambda,x) = 0$, it follows that $x \in C^1(\mathbb{R})$ and $\lim_{|t|\to\infty} x(t) = 0$. Furthermore, it follows from the argument leading to (3.7) that $F(\lambda,\cdot)$ maps $H^1$ boundedly into $L^2$. □

We now investigate the smoothness of the function $F : \mathbb{R} \times H^1 \to L^2$. The Hamiltonian $H$ is assumed to have the properties (H1) to (H4) from now on.

**Theorem 3.3.** (1) *$F \in C^1(\mathbb{R} \times H^1, L^2)$ with*

$$D_x F(\lambda,x)u = Ju' - M(\lambda,x)u \text{ for all } \lambda \in \mathbb{R} \text{ and } x,u \in H^1$$

*where $M(\lambda,x)(t) = D_\xi^2 H(t,x(t),\lambda)$ for all $t \in \mathbb{R}$.*

(2) *$D_x F(\cdot,0) \in C^1(\mathbb{R}, B(H^1,L^2))$ and*

$$D_\lambda D_x F(\lambda,0)u = -D_\lambda D_\xi^2 H(t,0,\lambda)u \text{ for all } \lambda \in \mathbb{R} \text{ and } u \in H^1.$$

(3) *Let $W$ be any bounded subset of $H^1$. The family of functions $\{F(\cdot,u) : \mathbb{R} \to L^2\}_{u\in W}$ is equicontinuous at $\lambda$ for every $\lambda \in \mathbb{R}$.*

(4) *For all $\lambda \in \mathbb{R}$, the function $F(\lambda,\cdot) : H^1 \to L^2$ is weakly sequentially continuous.*

*Proof.* See the Appendix. □

Noting that $M(\lambda,x)(t)$ is a symmetric $2N\times 2N-$matrix, we see that the equation

$$D_xF(\lambda,x)u=0$$

is a linear Hamiltonian system. We have already established that, for all $\lambda\in\mathbb{R}$ and $x\in H^1$, $D_xF(\lambda,x):H^1\to L^2$ is a bounded linear operator. It is important to know when it is a Fredholm operator of index zero. In fact, $D_xF(\lambda,x)$ can also be considered as an unbounded self-adjoint operator acting in $L^2$ and this means that its index must be zero whenever it is Fredholm. The next result summarizes the situation. Later we shall give explicit conditions on $H$ which ensure that $D_xF(\lambda,x)\in\Phi_0(H^1,L^2)$.

**Theorem 3.4.** *For $\lambda\in\mathbb{R}$ and $x\in H^1$, set $L=D_xF(\lambda,x)$.*

(1) $L\in B(H^1,L^2)$

(2) *$L:H^1\subset L^2\to L^2$ is an unbounded self-adjoint operator in $L^2$, and so*

$$L^2=\ker L\oplus\overline{\operatorname{rge} L}$$

*where $\overline{\operatorname{rge} L}$ is the closure of the range of $L$ in $L^2$ and $\oplus$ denotes an orthogonal direct sum in $L^2$.*

(3) *The following statements are equivalent:*
   (a) $\operatorname{rge} L=\overline{\operatorname{rge} L}$
   (b) $L\in\Phi_0(H^1,L^2)$.

(4) *Let $w\in H^1$. Then $L=D_xF(\lambda,x)\in\Phi_0(H^1,L^2)\Longleftrightarrow D_xF(\lambda,w)\in\Phi_0(H^1,L^2)$.*

*Proof.* (1) For any $u\in H^1$,

$$\|Lu\|_2=\|Ju'-M(\lambda,x)u\|_2\le\|Ju'\|_2+\|M(\lambda,x)u\|_2$$

by Theorem 3.3 where $\|M(\lambda,x)(t)\|\le C(\lambda,\|x\|_\infty)$ for all $t\in\mathbb{R}$ by Lemma 3.2 Thus

$$\|Lu\|_2\le\|u'\|_2+C(\lambda,\|x\|_\infty)\,\|u\|_2$$

showing that $L\in B(H^1,L^2)$.

(2) This is well-known. See [6], for example.

(3) Clearly, (b)$\Longrightarrow$(a) and so it suffices to prove that (a)$\Longrightarrow$(b). First we observe that if $u\in\ker L$, the $u\in C^1(\mathbb{R})$ and $Ju'=M(\lambda,x)u$. But the set of all solutions of this linear system is a vector space of dimension $2N$ and hence $\dim\ker L\le 2N$. Furthermore, by part (2), $\operatorname{codim}\overline{\operatorname{rge} L}=\dim\ker L$. Thus (a)$\Longrightarrow$(b).

(4) For all $x,w,u\in H^1$,

$$\{D_xF(\lambda,x)-D_xF(\lambda,w)\}u=\{M(\lambda,w)-M(\lambda,x)\}u$$

where, for all $t\in\mathbb{R}$,

$$M(\lambda,w)(t)-M(\lambda,x)(t)=\\ \{D_\xi^2H(t,w(t),\lambda)-D_\xi^2H(t,0,\lambda)\}-\{D_\xi^2H(t,x(t),\lambda)-D_\xi^2H(t,0,\lambda)\}.$$

Given any $\varepsilon>0$, it follows from (H3) that there exists $\delta>0$ such that

$$\left\|D_\xi^2H(t,\xi,\lambda)-D_\xi^2H(t,0,\lambda)\right\|<\varepsilon$$

for all $t\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^{2N}$ such that $|\xi|<\delta$. But, since $x,w\in H^1$, there exists $R>0$ such that $|x(t)|<\delta$ and $|w(t)|<\delta$ whenever $|t|>R$. Thus,

$$\|M(\lambda,w)(t)-M(\lambda,x)(t)\|<2\varepsilon \text{ for all } t\in\mathbb{R} \text{ such that } |t|>R,$$

showing that $\lim_{|t|\to\infty}\|M(\lambda,w)(t)-M(\lambda,x)(t)\|=0$. From this it follows easily that multiplication by $M(\lambda,w)-M(\lambda,x)$ defines a compact linear operator $K$ from $H^1$ into $L^2$. Since $D_xF(\lambda,x)-D_xF(\lambda,w)=K$, this implies that $D_xF(\lambda,x)\in\Phi_0(H^1,L^2)\Longleftrightarrow D_xF(\lambda,w)\in\Phi_0(H^1,L^2)$. □

4. **Admissible intervals.** In this section we give some useful criteria for the existence of admissible intervals for $F$. For this we shall assume henceforth that, in addition to the properties (H1) to (H4), the Hamiltonian $H$ is asymptotically periodic in the following sense.

($\mathrm{H}^\infty$) For all $\lambda \in \mathbb{R}$, there exist two $C^1_\xi$−bundle maps $g^+(\cdot,\cdot,\lambda)$ and $g^-(\cdot,\cdot,\lambda)$ : $\mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that

(1) $g^+(t,0,\lambda) = g^-(t,0,\lambda) = 0$ for all $t, \lambda \in \mathbb{R}$

(2) $\lim_{t\to\infty}\{D^2_\xi H(t,\xi,\lambda) - D_\xi g^+(t,\xi,\lambda)\} = \lim_{t\to-\infty}\{D^2_\xi H(t,\xi,\lambda) - D_\xi g^-(t,\xi,\lambda)\} = 0$, uniformly for $\xi$ in bounded subsets of $\mathbb{R}^{2N}$

(3) $g^+(t+T^+,\xi,\lambda) - g^+(t,\xi,\lambda) = g^-(t+T^-,\xi,\lambda) - g^-(t,\xi,\lambda) = 0$ for some $T^+, T^- > 0$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^{2N}$.

**Remarks** (1) The periods $T^+$ and $T^-$ may depend on $\lambda$.

(2) It follows easily from this assumption that $D_\xi g^+(t,\xi,\lambda)$ and $D_\xi g^-(t,\xi,\lambda)$ are symmetric matrices for all $(t,\xi,\lambda) \in \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}$, and that

$$\lim_{t\to\infty}\{D_\xi H(t,\xi,\lambda) - g^+(t,\xi,\lambda)\} = \lim_{t\to-\infty}\{D_\xi H(t,\xi,\lambda) - g^-(t,\xi,\lambda)\} = 0, \tag{4.9}$$

uniformly for $\xi$ in bounded subsets of $\mathbb{R}^{2N}$.

Furthermore, setting

$$H^\pm(t,\xi,\lambda) = H(t,0,\lambda) + \int_0^1 \langle g^\pm(t,s\xi,\lambda),\xi\rangle\, ds,$$

we have that

$$D_\xi H^\pm(t,\xi,\lambda) = g^\pm(t,\xi,\lambda) \tag{4.10}$$

and

$$\lim_{t\to\infty}\{H(t,\xi,\lambda) - H^+(t,\xi,\lambda)\} = \lim_{t\to-\infty}\{H(t,\xi,\lambda) - H^-(t,\xi,\lambda)\} = 0 \tag{4.11}$$

uniformly for $\xi$ in bounded subsets of $\mathbb{R}^{2N}$. In particular, the differential equations

$$Jx'(t) - g^+(t,x(t),\lambda) = 0 \text{ and } Jx'(t) - g^-(t,x(t),\lambda) = 0$$

are periodic Hamiltonian systems. Let $h^\pm$ denote the Nemytskii operators induced by $g^\pm, h^\pm(\lambda,x)(t) = g^\pm(t,x(t),\lambda)$ and then define $F^\pm$ by

$$F^\pm(\lambda,x) = Jx' - h^\pm(\lambda,x). \tag{4.12}$$

**Theorem 4.1.** *Under the hypotheses (H1) to (H4) and ($H^\infty$), for every $\lambda \in \mathbb{R}$, $F^\pm(\lambda,\cdot)$ maps $H^1$ boundedly into $L^2$. Furthermore, $F^\pm(\lambda,\cdot) : H^1 \to L^2$ is weakly sequentially continuous.*

*Proof.* See the Appendix. ☐

**Lemma 4.2.** *Let $B$ be a bounded subset of $H^1$ and consider $\lambda \in \mathbb{R}$ and $\varepsilon, L > 0$. There exists $R = R(\varepsilon,B,L,\lambda) > 0$ such that*

$$\left\|F(\lambda,x) - F^+(\lambda,x)\right\|_{L^2(I^+)} < \varepsilon \text{ and } \left\|F(\lambda,x) - F^-(\lambda,x)\right\|_{L^2(I^-)} < \varepsilon$$

*for all $x \in B$ where $I^\pm$ are any intervals of length less than $L$ with $I^+ \subset [R,\infty)$ and $I^- \subset (-\infty,-R]$.*

*Proof.* Since $B$ is bounded in $H^1$, there is a constant $b > 0$ such that

$$\|x(t)\| \leq b \text{ for all } t \in \mathbb{R} \text{ and all } x \in B.$$

By (4.9), there exists $R = R(b, \varepsilon, \lambda, L) > 0$ such that

$$\left\|D_\xi H(t, \xi, \lambda) - g^+(t, \xi, \lambda)\right\| < \sqrt{\frac{\varepsilon}{L}} \text{ for all } t \geq R \text{ and } \|\xi\| \leq b,$$

and hence,

$$\left\|D_\xi H(t, x(t), \lambda) - g^+(t, x(t), \lambda)\right\| < \sqrt{\frac{\varepsilon}{L}} \text{ for all } t \geq R \text{ and } x \in B.$$

It follows that, for any interval $I^+$ of length less than $L$ with $I^+ \subset [R, \infty)$,

$$\begin{aligned}\left\|F(\lambda, x) - F^+(\lambda, x)\right\|^2_{L^2(I^+)} &= \int_{I^+} \left\|D_\xi H(t, x(t), \lambda) - g^+(t, x(t), \lambda)\right\|^2 dt \\ &\leq \varepsilon \text{ for all } x \in B.\end{aligned}$$

The other case is similar. □

We now introduce a notation for the translate of a function. Given $h \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^M$, let $\tau_h(f)$ be the function defined by

$$\tau_h(f)(t) = f(t + h) \text{ for all } t \in \mathbb{R}.$$

In particular, $\tau_h(F(\lambda, x))$ is the function

$$\tau_h(F(\lambda, x))(t) = Jx'(t + h) - D_\xi H(t + h, x(t + h), \lambda).$$

**Lemma 4.3.** *Let $B$ be a bounded subset of $H^1$ and consider $\lambda \in \mathbb{R}$ and $\varepsilon, \omega > 0$. There exists $h_0 = h_0(\varepsilon, B, \omega, \lambda) > 0$ such that*

$$\left\|\tau_h(F(\lambda, x)) - \tau_h(F^+(\lambda, x))\right\|_{L^2(-\omega,\omega)} < \varepsilon \textit{ for all } x \in B \textit{ and } h \geq h_0$$

$$\textit{and } \left\|\tau_h(F(\lambda, x)) - \tau_h(F^-(\lambda, x))\right\|_{L^2(-\omega,\omega)} < \varepsilon \textit{ for all } x \in B \textit{ and } h \leq -h_0.$$

*Proof.* Since

$$\left\|\tau_h(F(\lambda, x)) - \tau_h(F^\pm(\lambda, x))\right\|_{L^2(-\omega,\omega)} = \left\|F(\lambda, x) - F^\pm(\lambda, x)\right\|_{L^2(-\omega+h,\omega+h)},$$

the result follows immediately from Lemma 4.2. □

**Lemma 4.4.** *Let $\{h_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} |h_n| = \infty$. For any $x \in L^2$, let $\widetilde{x_n} = \tau_{h_n}(x)$. Then $\tilde{x}_n \rightharpoonup 0$ weakly in $L^2$.*

*Proof.* Since $\|\widetilde{y_n}\|_2 = \|y\|_2$ for all $n$, it is enough to show that

$$\langle \widetilde{y_n}, \varphi \rangle_2 \to 0 \text{ as } n \to \infty \text{ for all } \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^{2N})$$

where $\langle \cdot, \cdot \rangle_2$ denotes the usual scalar product on $L^2$. Suppose that $\varphi(t) = 0$ for all $|t| > L$. Then,

$$\begin{aligned}|\langle \widetilde{y_n}, \varphi \rangle_2| &\leq \int_{-L}^{L} \|\widetilde{y_n}(t)\| \, \|\varphi(t)\| \, dt \leq \sqrt{2L} \, \|\varphi\|_\infty \left(\int_{-L}^{L} \|\widetilde{y_n}(t)\|^2 \, dt\right)^{1/2} \\ &= \sqrt{2L} \, \|\varphi\|_\infty \left(\int_{-L+h_n}^{L+h_n} \|y(t)\|^2 \, dt\right)^{1/2}\end{aligned}$$

and

$$\int_{-L+h_n}^{L+h_n} \|y(t)\|^2 \, dt \to 0 \text{ as } n \to \infty$$

since $y \in L^2$ and $|h_n| \to \infty$. □

**Definition 4.5.** *We say that a sequence $\{x_n\}$ in $H^1$* ***vanishes uniformly at infinity*** *if, for all $\varepsilon > 0$, there exists $R > 0$ such that $\|x_n(t)\| \le \varepsilon$ for all $|t| \ge R$ and all $n \in \mathbb{R}$.*

Recalling that $H^1$ is continuously embedded in $C_d$, we observe that $\{x_n\} \subset H^1$ vanishes uniformly at infinity if, for all $\varepsilon > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that $\|x_n(t)\| \le \varepsilon$ for all $|t| \ge R$ and all $n \ge n_0$.

**Lemma 4.6.** *Let $\{x_n\}$ be a bounded sequence in $H^1$ and let $x \in H^1$. The following statements are equivalent.*
*(1) $\|x_n - x\|_\infty \to 0$,*
*(2) $x_n \rightharpoonup x$ weakly in $H^1$ and $\{x_n\}$ vanishes uniformly at infinity.*

*Proof.* We begin by showing that (1) implies (2). If $\{x_n\}$ does not converge weakly to $x$, there are a number $\delta > 0$, an element $\varphi \in H^1$ and a subsequence $\{x_{n_k}\}$ such that

$$\left|\langle x_{n_k} - x, \varphi\rangle_2 + \left\langle x'_{n_k} - x', \varphi'\right\rangle_2\right| \ge \delta \text{ for all } k.$$

Then, passing to a further subsequence, we can suppose that $\{x_{n_k}\}$ converges weakly in $H^1$ to some element $y$. This implies that $\{x_{n_k}\}$ converges uniformly to $y$ on any compact interval and so, by (1), $y = x$. This contradicts the choice of $\{x_{n_k}\}$, proving that $\{x_n\}$ must converge weakly to $x$ in $H^1$.
Now fix $\varepsilon > 0$. There exists $R > 0$ such that $\|x(t)\| < \varepsilon$ for all $|t| \ge R$, and there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x\|_\infty < \varepsilon$ for all $n \ge n_0$. Hence, for all $|t| \ge R$ and all $n \ge n_0$,

$$\|x_n(t)\| \le \|x_n - x\|_\infty + \|x(t)\| < 2\varepsilon,$$

showing that $\{x_n\}$ vanishes uniformly at infinity.

Now we show that (2) implies (1). For any $\varepsilon > 0$, there exists $R > 0$ such that

$$\|x_n(t)\| < \varepsilon \text{ for all } |t| \ge R \text{ and all } n \in \mathbb{N},$$

since $\{x_n\}$ vanishes uniformly at infinity. Increasing $R$ if necessary, we also have that

$$\|x(t)\| < \varepsilon \text{ for all } |t| \ge R$$

since $x \in H^1 \subset C_d$. Thus,

$$\|x_n(t) - x(t)\| < 2\varepsilon \text{ for all } |t| \ge R \text{ and all } n.$$

But the weak convergence of $\{x_n\}$ in $H^1$ implies that $x_n$ converges uniformly to $x$ on $[-R, R]$, so there exists $n_1 \in \mathbb{N}$ such that

$$\|x_n(t) - x(t)\| < \varepsilon \text{ for all } |t| \le R \text{ and all } n \ge n_1.$$

Thus we have that

$$\|x_n - x\|_\infty \le 2\varepsilon \text{ for all } n \ge n_1,$$

showing that $\|x_n - x\|_\infty \to 0$. □

**Theorem 4.7.** *Recalling the hypotheses (H1) to (H4), suppose that there is an element $(\lambda, x^0) \in \mathbb{R} \times H^1$ such that $D_x(\lambda, x^0) \in \Phi_0(H^1, L^2)$. Then the following statements are equivalent.*
*(1) The restriction of $F(\lambda, \cdot) : H^1 \to L^2$ to the closed bounded subsets of $H^1$ is proper.*

*(2) Every bounded sequence $\{x_n\}$ in $H^1$ such that $\{F(\lambda, x_n)\}$ is convergent in $L^2$ contains a subsequence converging in $C_d$.*

*Proof.* We show first that (1) implies (2). Indeed, let $\{x_n\}$ be a bounded sequence in $H^1$ such that $\{F(\lambda, x_n)\}$ converges in $L^2$. We want to prove that some subsequence of $\{x_n\}$ converges in $C_d$. We know that $\|x_n\| \leq M$ for every $n \in \mathbb{N}$ and some constant $M > 0$; moreover, there is an element $y \in L^2$ such that $\|F(\lambda, x_n) - y\|_2 \to 0$. Let $K = \{F(\lambda, x_n)\} \cup \{y\}$ and $W = \overline{B(0, 2M)} \subset H^1$. Then $K$ is compact in $L^2$ and $W$ is closed and bounded in $H^1$. From assumption (1) we know that

$$F(\lambda, \cdot)^{-1}(K) \cap W \text{ is compact.}$$

We conclude that $\{x_n\}$ has a strongly convergent subsequence in $H^1$, and *a fortiori* in $C_d$.

To show that (2) implies (1), we proceed as follows. Let $W$ be a closed and bounded subset of $H^1$, and let $K$ be a compact subset of $L^2$. We wish to prove that

$$F(\lambda, \cdot)^{-1}(K) \cap W \text{ is compact.}$$

Let $\{x_n\}$ be a sequence from $F(\lambda, \cdot)^{-1}(K) \cap W$. In particular, there exists a constant $M > 0$ such that $\|x_n\| \leq M$ for all $n$. Moreover, passing to a subsequence, we can suppose that

$$F(\lambda, x_n) \to y \in K \text{ strongly in } L^2.$$

We can also assume that $x_n \rightharpoonup x$ in $H^1$, and by (2) that $\|x_n - x\|_\infty \to 0$. By the weak sequential continuity of $F(\lambda, \cdot)$, we get $F(\lambda, x_n) \rightharpoonup F(\lambda, x)$ in $L^2$, so that $y = F(\lambda, x)$.

We claim that

$$\lim_{n \to +\infty} \|F(\lambda, x_n) - F(\lambda, x) - D_x F(\lambda, x)(x_n - x)\|_{L^2} = 0. \tag{4.13}$$

Indeed,

$$\begin{aligned}
F(\lambda, x_n) &- F(\lambda, x) - D_x F(\lambda, x)(x_n - x) \\
&= -D_\xi H(\cdot, x_n, \lambda) + D_\xi H(\cdot, x, \lambda) + D_\xi^2 H(\cdot, x, \lambda)(x_n - x) \\
&= D_\xi^2 H(\cdot, x, \lambda)(x_n - x) - \int_0^1 \frac{d}{d\tau} D_\xi H(\cdot, \tau x_n + (1-\tau)x, \lambda)\, d\tau \\
&= D_\xi^2 H(\cdot, x, \lambda)(x_n - x) - \int_0^1 D_\xi^2 H(\cdot, \tau x_n + (1-\tau)x, \lambda)(x_n - x)\, d\tau \\
&= \int_0^1 \{D_\xi^2 H(\cdot, x, \lambda) - D_\xi^2 H(\cdot, \tau x_n + (1-\tau)x, \lambda)\}(x_n - x)\, d\tau.
\end{aligned}$$

Since $D_\xi H(\cdot, \cdot, \lambda)$ is a $C_\xi^1$–bundle map, a standard compactness argument shows that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sup_{t \in \mathbb{R}} \|D_\xi^2 H(t, \xi, \lambda) - D_\xi^2 H(t, \eta, \lambda)\| < \varepsilon$$

for all $\xi, \eta \in \mathbb{R}^{2N}$ such that $\|\xi\|, \|\eta\| \leq \|x\|_\infty + 1$ and $\|\xi - \eta\| < \delta$. But

$$\|[\tau x_n + (1-\tau)x] - x\|_\infty = \tau \|x_n - x\|_\infty \leq \|x_n - x\|_\infty$$

for all $\tau \in [0,1]$, and so there exists $n_0 \in \mathbb{N}$ such that $\|[\tau x_n + (1-\tau)x] - x\|_\infty < \delta$ for all $n \geq n_0$ and all $\tau \in [0,1]$. Thus,

$$\sup_{t \in \mathbb{R}} \|D_\xi^2 H(t, x(t), \lambda) - D_\xi^2 H(t, \tau x_n(t) + (1-\tau)x(t), \lambda)\| < \varepsilon$$

for all $n \geq n_0$ and all $\tau \in [0,1]$. Hence,

$$\|F(\lambda, x_n) - F(\lambda, x) - D_x F(\lambda, x)(x_n - x)\|_2 \leq \varepsilon \|x_n - x\|_2 \leq \varepsilon 2M$$

for all $n \geq n_0$, since $\|\cdot\|_2 \leq \|\cdot\|$. This proves that

$$\|F(\lambda, x_n) - F(\lambda, x) - D_x F(\lambda, x)(x_n - x)\|_2 \to 0 \text{ as } n \to \infty.$$

But $\|F(\lambda, x_n) - F(\lambda, x)\|_2 = \|F(\lambda, x_n) - y\|_2 \to 0$ and so we have that

$$\|L(x_n - x)\|_2 \to 0 \text{ where } L = D_x F(\lambda, x) \in \Phi_0(H^1, L^2).$$

By Theorem 3.15 of [1], there exist $S \in B(L^2, H^1)$ and a compact linear operator $C : H^1 \to H^1$ such that $SL = I + C$. But then,

$$\begin{aligned} \|x_n - x\| = \|(SL - C)(x_n - x)\| &\leq \|SL(x_n - x)\| + \|C(x_n - x)\| \\ &\leq \|S\| \, \|L(x_n - x)\|_2 + \|C(x_n - x)\| \end{aligned}$$

where $\|C(x_n - x)\| \to 0$ by the compactness of $C$ and the weak convergence of $\{x_n\}$ in $H^1$. Thus $\|x_n - x\| \to 0$ and the compactness of $F(\lambda, \cdot)^{-1}(K) \cap W$ is established. □

**Lemma 4.8.** *Let $\{x_n\}$ be a bounded sequence in $H^1$ and consider any numbers $T^+, T^- \in (0, \infty)$. At least one of the following properties must hold.*

*(1) $\{x_n\}$ vanishes uniformly at infinity*

*(2) There is a sequence $\{l_k\} \subset \mathbb{Z}$ with $\lim_{k\to\infty} l_k = \infty$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\widetilde{x_k} = \tau_{l_k T^+}(x_{n_k}) = x_{n_k}(\cdot + l_k T^+)$ converges weakly in $H^1$ to an element $\widetilde{x} \neq 0$*

*(3) There is a sequence $\{l_k\} \subset \mathbb{Z}$ with $\lim_{k\to\infty} l_k = -\infty$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\widetilde{x_k} = \tau_{l_k T^-}(x_{n_k}) = x_{n_k}(\cdot + l_k T^-)$ converges weakly in $H^1$ to an element $\widetilde{x} \neq 0$*

*Proof.* Assume that $\{x_n\}$ does not satisfy (1). Then there exists $\varepsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $t_k \in \mathbb{R}$ with $|t_k| \geq k$, and there exists $n_k \in \mathbb{N}$ with $n_k > k$ such that $\|x_{n_k}(t_k)\| \geq \varepsilon$. By passing to a subsequence we may suppose that $\{t_k\}$ diverges either to $+\infty$ or to $-\infty$.
Suppose that $t_k \to \infty$ as $k \to \infty$. Then there exists $l_k \in \mathbb{Z}$ such that $t_k - l_k T^+ \in [0, T^+]$ and $l_k \to \infty$ as $k \to \infty$. Clearly, $\|\tilde{x}_k\| = \|x_{n_k}\|$ and hence also $\{\tilde{x}_k\}$ is bounded in $H^1$. Passing to a further subsequence which we still denote by $\{\tilde{x}_k\}$, we can suppose that $\tilde{x}_k$ converges weakly in $H^1$ to some element $\tilde{x}$. By the compactness of the embedding of $H^1([0, T^+], \mathbb{R}^{2N})$ into $C([0, T^+], \mathbb{R}^{2N})$, $\tilde{x}_k \to \tilde{x}$ uniformly on $[0, T^+]$. In particular, $\max_{t\in[0,T^+]} \|\tilde{x}(t)\| = \lim_{k\to\infty} \max_{t\in[0,T^+]} \|\tilde{x}_k(t)\|$. But

$$\max_{t\in[0,T^+]} \|\tilde{x}_k(t)\| \geq \|\tilde{x}_k(t_k - l_k T^+)\| = \|x_{n_k}(t_k)\| \geq \varepsilon \text{ for all } k,$$

so $\max_{t\in[0,T^+]} \|\tilde{x}(t)\| \geq \varepsilon$. Thus we see that (2) holds when $t_k \to \infty$. as $k \to \infty$.
A similar argument shows that (3) holds if $t_k \to -\infty$. as $k \to \infty$, completing the proof. □

**Theorem 4.9.** *Under the hypotheses (H1) to (H4) and ($H^\infty$), suppose that*
*(1) there is an element $(\lambda, x^0) \in \mathbb{R} \times H^1$ such that $D_x(\lambda, x^0) \in \Phi_0(H^1, L^2)$, and*
*(2)$\{x \in H^1 : F^+(\lambda, x) = 0\} = \{x \in H^1 : F^-(\lambda, x) = 0\} = \{0\}$.*
*Then the restriction of $F(\lambda, \cdot) : H^1 \to L^2$ to the closed bounded subsets of $H^1$ is proper.*

*Proof.* According to Theorem 4.7 and Lemma 4.6, it suffices to show that any bounded sequence $\{x_n\}$ in $H^1$ such that $\|F(\lambda, x_n) - y\|_2 \to 0$ for some element $y \in L^2$ has a weakly convergent subsequence which vanishes uniformly at infinity. By the boundedness of $\{x_n\}$, we may assume henceforth that $x_n \rightharpoonup x$ weakly in $H^1$ for some $x \in H^1$. Furthermore, $\{x_n\}$ has at least one of the properties stated in Lemma 4.8 where $T^\pm$ are chosen to be periods of $g^\pm(\cdot, \xi, \lambda)$ as in $(\mathrm{H}^\infty)(3)$.
Let us suppose that $\{x_n\}$ has the property (2) of Lemma 4.8. That is to say, $\widetilde{x_k} \rightharpoonup \widetilde{x}$ weakly in $H^1$ where $\widetilde{x_k}(t) = x_{n_k}(t + l_k T^+)$ and $\widetilde{x} \neq 0$. The invariance by translation of the Lebesgue measure implies that $\|F(\lambda, x_n) - y\|_2 = \|\tau_{l_k T^+}(F(\lambda, x_n) - y)\|_2$ so

$$\|\tau_{l_k T^+}(F(\lambda, x_{n_k})) - \widetilde{y_k}\|_2 \to 0 \text{ where } \widetilde{y_k}(t) = y(t + l_k T^+)$$

For any $\omega \in (0, \infty)$, Lemma 4.3 shows that

$$\left\|\tau_{l_k T^+}(F(\lambda, x_{n_k})) - \tau_{l_k T^+}(F^+(\lambda, x_{n_k}))\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty.$$

Hence

$$\left\|\tau_{l_k T^+}(F^+(\lambda, x_{n_k})) - \widetilde{y_k}\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty.$$

But

$$\begin{aligned}\tau_{l_k T^+}(F^+(\lambda, x_{n_k}))(t) &= J x'_{n_k}(t + l_k T^+) - g^+(t + l_k T^+, x_{n_k}(t + l_k T^+), \lambda) \\ &= J\widetilde{x'_k}(t) - g^+(t, \widetilde{x_k}(t), \lambda) = F^+(\lambda, \widetilde{x_k})(t)\end{aligned}$$

by the periodicity of $g^+$. Consequently,

$$\left\|F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k}\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty,$$

for all $\omega \in (0, \infty)$. Since the sequence $\{F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k}\}$ is bounded in $L^2$, this implies that $F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k} \rightharpoonup 0$ weakly in $L^2$. But $\widetilde{y_k} \rightharpoonup 0$ weakly in $L^2$ by Lemma 4.4, so we now have that $F^+(\lambda, \widetilde{x_k}) \rightharpoonup 0$ weakly in $L^2$. However, the weak sequential continuity of $F^+(\lambda, \cdot) : H^1 \to L^2$ implies that

$$F^+(\lambda, \widetilde{x_k}) \rightharpoonup F^+(\lambda, \widetilde{x}) \text{ weakly in } L^2,$$

so we must have $F^+(\lambda, \widetilde{x}) = 0$, contradicting the hypothesis (2) of the theorem. This shows that the sequence $\{x_n\}$ cannot have the property (2) of Lemma 4.8.
A similar argument excludes the property (3), completing the proof of the theorem. □

**Corollary 4.10.** *Suppose that (H1) to (H4) and ($\mathrm{H}^\infty$) are satisfied. An open interval $\Lambda$ is admissible for $F : \mathbb{R} \times H^1 \to L^2$ provided that, for all $\lambda \in \Lambda$,*
*(1) $D_x F(\lambda, 0) \in \Phi_0(H^1, L^2)$ and*
*(2) $\{x \in H^1 : F^+(\lambda, x) = 0\} = \{x \in H^1 : F^-(\lambda, x) = 0\} = \{0\}$*

*Proof.* From hypothesis (1) and part (4) of Theorem 3.4, it follows that $D_x F(\lambda, x) \in \Phi_0(H^1, L^2)$ for all $(\lambda, x) \in \Lambda \times H^1$.
Let $K$ be a compact subset of $L^2$ and let $W$ be a closed bounded subset of $\mathbb{R} \times H^1$ such that

$$\inf \Lambda < \inf PW \leq \sup PW < \sup \Lambda.$$

To show that $F^{-1}(K) \cap W$ is a compact subset of $\mathbb{R} \times H^1$, we consider a sequence $\{(\lambda_n, x_n)\} \subset F^{-1}(K) \cap W$. Passing to a subsequence, we can suppose that there exist $\lambda \in [\inf PW, \sup PW] \subset \Lambda$ and $y \in K$ such that

$$\lambda_n \to \lambda \text{ and } \|F(\lambda_n, x_n) - y\|_2 \to 0.$$

But, by part (3) of Theorem 3.3, the family of functions $\{F(\cdot, x_n)\}_{n\in\mathbb{N}}$ is equicontinuous at $\lambda$ since the sequence $\{x_n\}$ is bounded in $H^1$. It follows from this that $\|F(\lambda, x_n) - y\|_2 \to 0$. By Theorem 4.9 we know that $F(\lambda, \cdot) : H^1 \to L^2$ is proper on the closed bounded subsets of $H^1$ and so there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x \in H^1$ such that $\|x_{n_k} - x\| \to 0$. Thus $\{(\lambda_{n_k}, x_{n_k})\}$ converges to $(\lambda, x)$ in $\mathbb{R} \times H^1$. Furthermore $(\lambda, x) \in W$ since $W$ is closed. This proves that $F^{-1}(K) \cap W$ is a compact subset of $\mathbb{R} \times H^1$, completing the proof that $\Lambda$ is an admissible interval. □

**Theorem 4.11.** *Consider the system (1.1)(1.2) under the assumptions (H1) to (H4) and ($H^\infty$). Let $\Lambda$ be an open interval having the following properties.*

(1) *For all* $\lambda \in \Lambda, D_xF(\lambda, 0) \in \Phi_0(H^1, L^2)$.
(2) *For all* $\lambda \in \Lambda$,
$$\{x \in H^1 : F^+(\lambda, x) = 0\} = \{x \in H^1 : F^-(\lambda, x) = 0\} = \{0\}.$$
(3) *There is a point* $\lambda_0 \in \Lambda$ *such that*
   (i) $\dim\ker D_xF(\lambda_0, 0)$ *is odd,*
   (ii) *for every* $u \in \ker D_xF(\lambda_0, 0)\backslash\{0\}$ *there is an element* $v \in \ker D_xF(\lambda_0, 0)$ *such that*
$$\int_{-\infty}^{\infty} \langle D_\lambda D_\xi^2 H(t, 0, \lambda_0)u(t), v(t)\rangle\, dt \neq 0$$
   (iii) $\dim\{D_\lambda D_\xi^2 H(\cdot, 0, \lambda_0)u : u \in \ker D_xF(\lambda_0, 0)\} = \dim D_xF(\lambda_0, 0)$.

*Then a global branch of homoclinic solutions of (1.1)(1.2) bifurcates at* $\lambda_0$ *in the sense of Theorem 2.3 with* $X = H^1$ *and* $Y = L^2$.

**Theorem 4.12.** *For some point* $t_0 \in \mathbb{R}$, $\det D_\lambda D_\xi^2 H(t_0, 0, \lambda_0) \neq 0$.

*Proof.* In view of Corollary 4.10, we only need to check that the assumption (3) ensures that the condition (2.3) is satisfied with $A(\lambda) = D_xF(\lambda, 0)$. By (ii), we have that $D_\lambda D_\xi^2 H(, 0, \lambda_0)u \notin [\ker D_xF(\lambda_0, 0)]^\perp$ for $u \in \ker D_xF(\lambda_0, 0)\backslash\{0\}$ and so $A'(\lambda_0)[\ker D_xF(\lambda_0, 0)] \cap \operatorname{rge} A(\lambda_0) = \{0\}$ by Theorem 3.4(2). Since
$$A'(\lambda_0)[\ker D_xF(\lambda_0, 0)] = \{D_\lambda D_\xi^2 H(\cdot, 0, \lambda_0)u : u \in \ker D_xF(\lambda_0, 0)\}$$
and $\operatorname{codim} \operatorname{rge} A(\lambda_0) = \dim \ker D_xF(\lambda_0, 0)$ by (1), it follows from (3)(iii) that
$$A'(\lambda_0)[\ker D_xF(\lambda_0, 0)] \oplus \ker A(\lambda_0) = L^2.$$
It now follows from (i) and Proposition 2.1 that $\pi(D_xF(\lambda_0, 0) \mid [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = -1$ for some small $\varepsilon > 0$.

**Remark** The condition (3)(iii) is satisfied whenever
$$(iii)' \text{ for some point } t_0 \in \mathbb{R}, \det D_\lambda D_\xi^2 H(t_0, 0, \lambda_0) \neq 0.$$
In fact, if$\{u_1, ..., u_k\}$ is a basis for $\ker D_xF(\lambda_0, 0)$, then for every $t \in \mathbb{R}$, the vectors $u_1(t), ..., u_k(t)$ are linearly independent in $\mathbb{R}^{2N}$ since the functions $u_1, ..., u_k$ all satisfy the linear system $Ju'(t) = D_\xi^2 H(t, 0, \lambda_0)u(t)$. It follows from $(iii)\prime$ that the vectors
$$D_\lambda D_\xi^2 H(t_0, 0, \lambda_0)u_1(t_0), \ldots, D_\lambda D_\xi^2 H(t_0, 0, \lambda_0)u_k(t_0)$$
are linearly independent in $\mathbb{R}^{2N}$ and hence that $\dim A'(\lambda_0)[\ker D_xF(\lambda_0, 0)] = k$. □

The rest of this paper is devoted to the formulation of explicit conditions on the Hamiltonian $H$ which enable us to verify the properties (1) to (3) in the above result.

5. **More explicit criteria.** The first objective is to formulate conditions on the Hamiltonian which ensure that $D_xF(\lambda,0)\in\Phi_0(H^1,L^2)$. Recalling that

$$D_xF(\lambda,0)u = Ju' - D_\xi^2 H(\cdot,0,\lambda)u \text{ for all } u\in H^1,$$

we set

$$A_\lambda(t) = D_\xi^2 H(t,0,\lambda) \text{ for } t\in\mathbb{R}.$$

Then, assuming that $(H^\infty)$ is satisfied, we set

$$A_\lambda^+(t) = D_\xi g^+(t,0,\lambda) \text{ and } A_\lambda^-(t) = D_\xi g^-(t,0,\lambda).$$

We observe that $A_\lambda(t), A_\lambda^+(t)$ and $A_\lambda^-(t)$ are all real, symmetric $2N\times 2N-$matrices and that $A_\lambda^+(t)$ and $A_\lambda^-(t)$ are periodic in $t$. Furthermore,

$$\lim_{t\to\infty}\{A_\lambda(t)-A_\lambda^+(t)\} = \lim_{t\to-\infty}\{A_\lambda(t)-A_\lambda^-(t)\} = 0.$$

**Theorem 5.1.** *Suppose that (H1) to (H4) and $(H^\infty)$ are satisfied and that the periodic, linear Hamiltonian systems*

$$Jx' - A_\lambda^+(t)x = 0 \textit{ and } Jx' - A_\lambda^-(t)x = 0$$

*have no characteristic multipliers on the unit circle.*
*Then $D_xF(\lambda,0)\in\Phi_0(H^1,L^2)$.*
*Furthermore,*

$$\ker D_xF(\lambda,0) = N(\lambda) \textit{ where}$$

$$N(\lambda) = \{u\in C^1(\mathbb{R},\mathbb{R}^{2N}) : Ju'(t) - A_\lambda(t)u(t)\equiv 0 \textit{ and } \lim_{|t|\to\infty} u(t) = 0\}.$$

**Remarks** (1) In fact, our proof shows that functions in $\ker D_xF(\lambda,0)$ decay exponentially to zero as $|t|\to\infty$.

(2) Characteristic multipliers on the unit circle correspond to characteristic exponents with real part equal to zero.

First we establish the following useful result.

**Proposition 5.2.** *Let $M(t)$ be a real symmetric $2N\times 2N-$matrix which depends continuously and periodically on $t\in\mathbb{R}$. Suppose that the linear system*

$$Jx' - M(t)x = 0$$

*has no characteristic multipliers on the unit circle. Then the linear operator $L: H^1\to L^2$, defined by*

$$Lu = Ju' - M(t)u \textit{ for all } u\in H^1,$$

*is an isomorphism.*

**Remark** In the case where $M(t)=M$ is constant, there are no characteristic multipliers on the unit circle precisely when the matrix $JM$ has no eigenvalues on the imaginary axis.

*Proof.* Suppose that $M(t+T)=M(t)$ for all $t\in\mathbb{R}$. Recall that a $2N\times 2N-$matrix $K$ is symplectic when $K^TJK=J$ and that such matrices are always invertible. By Floquet theory (see for example Theorem IV-5-11 of [5]) there exist

(a) a real symmetric $2N\times 2N$-matrix $C$ and
(b) a real symplectic $2N\times 2N$-matrix $P(t)$ for each $t\in\mathbb{R}$ such that $P_{ij}\in C^1(\mathbb{R})$ and $P(t+2T)=P(t)$ for all $t\in\mathbb{R}$, $Ju'(t)-M(t)u(t) = P(t)\{Jz'(t)-Cz(t)\}$ for all $t\in\mathbb{R}$ where $u(t)=P(t)z(t)$.

Furthermore, the characteristic multipliers of the system

$$Ju'(t) - M(t)u(t) = 0$$

are the complex numbers $\rho_1, ..., \rho_{2N}$ where $\rho_k = e^{2T\lambda_k}$ where $\lambda_1, ..., \lambda_{2N}$ are the eigenvalues of the matrix $JC$. For $z : \mathbb{R} \to \mathbb{R}^{2N}$, let $Wz(t) = P(t)z(t)$. It follows easily from the properties of $P$ that

$$Wz \in L^2 \iff z \in L^2 \text{ and also } Wz \in H^1 \iff z \in H^1, \text{ in fact,}$$
$$W : L^2 \to L^2 \text{ and } W : H^1 \to H^1 \text{ are isomorphisms.}$$

Thus we can define a bounded linear operator $S : H^1 \to L^2$ by setting

$$Sz = W^{-1}LWz \text{ where } Lu = Ju' - M(t)u \text{ for } u \in H^1.$$

Clearly $S : H^1 \to L^2$ is an isomorphism $\iff L : H^1 \to L^2$ is an isomorphism. But, setting $u = Wz$,

$$W^{-1}LWz(t) = P(t)^{-1}\{Ju' - M(t)u\} = Jz'(t) - Cz(t).$$

By Corollary 10.2 of [12], the bounded linear operator $Sz = Jz'(t) - Cz(t)$ is an isomorphism if and only if $\sigma(JC) \cap i\mathbb{R} = \emptyset$. The hypothesis that the system $Ju'(t) - M(t)u(t) = 0$ has no characteristic multipliers on the unit circle ensures that indeed $\sigma(JC) \cap i\mathbb{R} = \emptyset$ and so the proof is complete. $\square$

**Proof of Theorem 5.1** Set $L = D_xF(\lambda, 0)$ and recall that

$$Lx(t) = Jx'(t) - A_\lambda(t)x(t) \text{ for all } x \in H^1.$$

By Theorem 3.4, we know that $L \in B(H^1, L^2)$ and it is enough to show that $\operatorname{rge} L$ is a closed subspace of $L^2$. With this in mind, let $f \in L^2$ and suppose that there exists a sequence $\{f_n\} \subset \operatorname{rge} L$ such that $\|f - f_n\|_2 \to 0$. Clearly there is a sequence $\{x_n\} \subset H^1$ such that $Lx_n = f_n$.

Let $P$ denote the orthogonal projection of $L^2$ onto $\ker L$ and set $Q = I - P$. Then $Qx_n = x_n - Px_n \in H^1$ and $Qx_n \in [\ker(L)]^\perp$, the orthogonal complement of $\ker L$ in $L^2$. Setting

$$u_n = Qx_n \text{ we have that } u_n \in H^1 \cap [\ker(L)]^\perp \text{ and } Lu_n = f_n.$$

Let us prove that the sequence $\{u_n\}$ is bounded in $H^1$. For this we use $S^\pm : H^1 \to L^2$ to denote the bounded linear operators defined by

$$S^\pm x = Jx' - A_\lambda^\pm(t)x \text{ for all } x \in H^1.$$

By the Proposition 5.2 we know that $S^+ : H^1 \to L^2$ and $S^- : H^1 \to L^2$ are both isomorphisms and so there exists a constant $k$ such that

$$\left\|S^\pm x\right\|_2 \geq k\,\|x\| \text{ for all } x \in H^1. \tag{5.14}$$

Supposing that $\|u_n\| \to \infty$, we set $w_n = \frac{u_n}{\|u_n\|}$. Then $\{w_n\} \subset H^1$ with $\|w_n\| = 1$ for all $n \in \mathbb{N}$. Passing to a subsequence, we can suppose that $w_n \rightharpoonup w$ weakly in $H^1$, and hence that $Lw_n \rightharpoonup Lw$ weakly in $L^2$. Furthermore,

$$Lw_n = \frac{Lu_n}{\|u_n\|} = \frac{f_n}{\|u_n\|} \text{ and so } \|Lw_n\|_2 = \frac{\|f_n\|_2}{\|u_n\|} \to 0$$

since $\|f_n\|_2 \to \|f\|_2$ and $\|u_n\| \to \infty$. Thus $Lw = 0$. But $w_n \in H^1 \cap [\ker(L)]^\perp$ for all $n \in \mathbb{N}$, from which it follows that $w \in H^1 \cap [\ker(L)]^\perp$. Thus $w = 0$ and $w_n \rightharpoonup 0$ weakly in $H^1$. Consequently,

$$w_n \to 0 \text{ uniformly on } [-R, R] \text{ for any } R \in (0, \infty).$$

But, for all $t \in \mathbb{R}$,

$$Jw_n'(t) = A_\lambda(t)w_n(t) + \frac{f_n(t)}{\|u_n\|}$$

and so

$$\|w_n'\|_{L^2(-R,R)} \leq \sup_{t\in\mathbb{R}} \|A_\lambda(t)\| \, \|w_n\|_{L^2(-R,R)} + \frac{\|f_n\|_2}{\|u_n\|},$$

showing that $\|w_n'\|_{L^2(-R,R)} \to 0$ as $n \to \infty$, for all $R \in (0,\infty)$. In particular,

$$\|w_n\|_{H^1(-R,R)} \to 0 \text{ as } n \to \infty.$$

Now choose any $\varepsilon > 0$. By (H$^\infty$), there is a constant $r \in (0,\infty)$ such that

$$\left|A_\lambda(t) - A_\lambda^+(t)\right| \leq \varepsilon \text{ for all } t \geq r \text{ and } \left|A_\lambda(t) - A_\lambda^-(t)\right| \leq \varepsilon \text{ for all } t \leq -r.$$

There exist a constant $R > r + \frac{1}{\varepsilon}$ and a function $\varphi \in C^1(\mathbb{R})$ such that
$0 \leq \varphi(t) \leq 1$ for all $t \in \mathbb{R}$, $\varphi(t) = 0$ for $t \leq r$, $\varphi(t) = 1$ for $t \geq R$ and $|\varphi'(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$.

Consider now the function $z_n(t) = \varphi(t)w_n(t)$. Clearly $z_n \in H^1$ and

$$\begin{aligned}
S^+z_n(t) &= \varphi'(t)Jw_n(t) + \varphi(t)Jw_n'(t) - A_\lambda^+(t)z_n(t) \\
&= \varphi'(t)Jw_n(t) + \varphi(t)Lw_n(t) + \varphi(t)\{A_\lambda(t) - A_\lambda^+(t)\}w_n(t) \\
&= \varphi'(t)Jw_n(t) + \varphi(t)\frac{f_n(t)}{\|u_n\|} + \varphi(t)\{A_\lambda(t) - A_\lambda^+(t)\}w_n(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|S^+z_n\right\|_2 &\leq \varepsilon \|w_n\|_2 + \frac{\|f_n\|_2}{\|u_n\|} + \sup_{t\geq r} \left|A_\lambda(t) - A_\lambda^+(t)\right| \|w_n\|_2 \\
&\leq \varepsilon + \frac{\|f_n\|_2}{\|u_n\|} + \varepsilon
\end{aligned}$$

since $\|w_n\|_2 \leq 1$. Hence, by (5.14)

$$\|w_n\|_{H^1(R,\infty)} = \|z_n\|_{H^1(R,\infty)} \leq \|z_n\| \leq \frac{1}{k}\left(2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\right).$$

A similar argument, using $\varphi(-t)w_n(t)$ instead of $z_n$, shows that

$$\|w_n\|_{H^1(-\infty,-R)} \leq \frac{1}{k}\left(2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\right).$$

Finally, we have shown that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|w_n\|^2 &= \|w_n\|^2_{H^1(-\infty,-R)} + \|w_n\|^2_{H^1(-R,R)} + \|w_n\|^2_{H^1(R,\infty)} \\
&\leq \frac{2}{k^2}\left(2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\right)^2 + \|w_n\|^2_{H^1(-R,R)}
\end{aligned}$$

and, letting $n \to \infty$,

$$\limsup_{n\to\infty} \|w_n\|^2 \leq \frac{2}{k^2}(2\varepsilon)^2$$

since $\|w_n\|^2_{H^1(-R,R)} \to 0, \|f_n\|_2 \to \|f\|_2$ and $\|u_n\| \to \infty$. But $\|w_n\| \equiv 1$ and $\varepsilon > 0$ can be chosen so that $\frac{2}{k^2}\{2\varepsilon\}^2 < 1$. This contradiction establishes the boundedness of the sequence $\{u_n\}$ in $H^1$.

By passing to a subsequence, we can now suppose that $u_n \rightharpoonup u$ weakly in $H^1$, and consequently that $Lu_n \rightharpoonup Lu$ weakly in $L^2$. However, $Lu_n = f_n$ and $\|f_n - f\|_2 \to 0$, showing that $Lu = f$. This proves that $\operatorname{rge} L$ is a closed subspace of $L^2$ and we have shown that $D_xF(\lambda, 0) \in \Phi_0(H^1, L^2)$.

Clearly $\ker D_xF(\lambda, 0) \subset N(\lambda)$. To establish the equality we suppose that $u \in N(\lambda)$ and we must show that $u \in H^1$. In fact, we shall prove that $\|u(t)\|$ decays to zero exponentially as $|t| \to \infty$. This implies that $u \in L^2$ and then, since $A_\lambda(t)$ is bounded on $\mathbb{R}$, it follows immediately that $u' \in L^2$. Let us consider the behaviour of $u(t)$ as $t \to \infty$, the case $t \to -\infty$ being similar.

Using the notation introduced in the proof of Proposition 5.2 with $M = A_\lambda^+$, we set $u(t) = P(t)z(t)$ and find that $z \in C^1(\mathbb{R})$ and

$$P(t)\{Jz'(t) - Cz(t)\} = Ju'(t) - A_\lambda^+(t)u(t) = \{A_\lambda(t) - A_\lambda^+(t)\}u(t)$$

where $C$ is a real symmetric matrix and $JC$ has no eigenvalues with zero real part. Thus,

$$z'(t) = \{-JC + R(t)\}z(t) \text{ where } R(t) = -JP(t)^{-1}\{A_\lambda(t) - A_\lambda^+(t)\}P(t).$$

The matrix $R(t)$ depends continuously on $t$ and $\|R(t)\| \to 0$ as $t \to \infty$. According to Corollary VII-3-7 of [5], this implies that there is a number $\mu$ such that $\lim_{t\to\infty} t^{-1} \log \|z(t)\| = \mu$ and $\mu$ is the real part of an eigenvalue of the matrix $-JC$. But, since $\lim_{|t|\to\infty} u(t) = 0$, we have that $\lim_{|t|\to\infty} z(t) = 0$ and consequently $\mu \leq 0$. However, $-JC$ has no eigenvalues with zero real part and so $\mu < 0$. Thus, for any $\gamma < -\mu, \lim_{t\to\infty} e^{\gamma t} \|z(t)\| = 0$. This establishes the exponential decay of $|u(t)|$ as $t \to \infty$ and the proof is complete.

We now turn to the problem of checking the condition

$$\{x \in H^1 : F^+(\lambda, x) = 0\} = \{x \in H^1 : F^-(\lambda, x) = 0\} = \{0\}$$

in Theorem 4.11. This amounts to ensuring that certain types of Hamiltonian system have no solutions which are homoclinic to 0.

**Theorem 5.3.** *Suppose the (H1) to (H4) and ($H^\infty$) are satisfied.*

*(a) If there is a real, symmetric $2N \times 2N-$matrix $C$ such that*

$$\left\langle g^+(t, \xi, \lambda), JC\xi \right\rangle > 0 \text{ for all } \xi \in \mathbb{R}^{2N} \backslash \{0\},$$

*then $\{x \in H^1 : F^+(\lambda, x) = 0\} = \{0\}$.*

*(b) If*

$$g^+(t, \xi, \lambda) = A_\lambda^+(t)\xi \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^{2N},$$

*(that is, $g^+(t, \cdot, \lambda)$ is linear), then $\{x \in H^1 : F^+(\lambda, x) = 0\} = \{0\}$.*

*(c) If $g^+(t, \xi, \lambda)$ is independent of $t$ and $0$ is an isolated zero of $H^+(\cdot, \lambda)$ where $D_\xi H^+(\cdot, \lambda) = g^+(\cdot, \cdot, \lambda)$, then $\{x \in H^1 : F^+(\lambda, x) = 0\} = \{0\}$.*

*The same conclusions hold when $g^+$ is replaced by $g^-$.*

*Proof.* (a) Suppose that $x \in H^1$ and $F^+(\lambda, x) = 0$. Then $x \in C^1(\mathbb{R})$ and, for all $t \in \mathbb{R}$,

$$\frac{d}{dt} \langle Cx(t), x(t) \rangle = 2 \langle Cx(t), x'(t) \rangle = 2 \left\langle JCx(t), g^+(t, x(t), \lambda) \right\rangle > 0$$

whenever $x(t) \neq 0$. However $\langle Cx(t), x(t) \rangle \to 0$ as $t \to \infty$ and as $t \to -\infty$ since $x \in H^1$. Thus we must have that $\langle Cx(t), x(t) \rangle \equiv 0$ and consequently, $\langle JCx(t), g^+(t, x(t), \lambda) \rangle = 0$ for all $t \in \mathbb{R}$. This implies that $x(t) = 0$ for all $t \in \mathbb{R}$ as required.

(b) If $x \in H^1$ and $F^+(\lambda, x) = 0$, it follows that $S^+x = 0$ in the notation which was introduced in the proof of Theorem 5.1. From Floquet theory as in the proof of Proposition 5.2 with $M(t) = A_\lambda^+(t)$, it follows that

$$S^+ z = Jz'(t) - Cz(t) \text{ where } x(t) = P(t)z(t) \text{ and } z \in H^1.$$

But, as is pointed out at the beginning of Section 10 of [12], $\ker(S^+) = \{0\}$ and so $z \equiv 0$. This proves that $x = 0$.

(c) If $x \in H^1$ and $F^+(\lambda, x) = 0$, it follows that $x \in C^1(\mathbb{R})$ and $x$ satisfies the autonomous Hamiltonian system $Jx'(t) = D_\xi H(x(t), \lambda)$. Thus $H(x(t), \lambda)$ is constant, and (1.2) implies that $H(x(t), \lambda) = 0$ for all $t \in \mathbb{R}$. Since 0 is an isolated zero of $H(\cdot, \lambda)$ and $x(t) \to 0$ as $|t| \to 0$, it now follows that $x = 0$. □

Combining these results we can formulate criteria for admissible intervals which can be checked in some examples.

**Theorem 5.4.** *Suppose that (H1) to (H4) and ($H^\infty$) are satisfied. An open interval $\Lambda$ is admissible provided that, for all $\lambda \in \Lambda$, the following conditions are satisfied.*

*(1) The periodic, linear Hamiltonian systems*

$$Jx' - A_\lambda^+(t)x = 0 \text{ and } Jx' - A_\lambda^-(t)x = 0$$

*have no characteristic multipliers on the unit circle.*

*(2) The asymptotic limit $g^+$ satisfies one of the conditions (a),(b) or (c) of Theorem 5.3*

*(3) The asymptotic limit $g^-$ satisfies one of the conditions (a),(b) or (c) of Theorem 5.3.*

Finally we can reformulate Theorem 4.11 as a global bifurcation theorem concerning the system (1.1)(1.2) with hypotheses only involving properties of the Hamiltonian.

**Theorem 5.5.** *Suppose that (H1) to (H4) and ($H^\infty$) are satisfied. An open interval $\Lambda$ is admissible provided that, for all $\lambda \in \Lambda$, the following conditions are satisfied.*

*(1) The periodic, linear Hamiltonian systems*

$$Jx' - A_\lambda^+(t)x = 0 \text{ and } Jx' - A_\lambda^-(t)x = 0$$

*have no characteristic multipliers on the unit circle.*

*(2) The asymptotic limit $g^+$ satisfies one of the conditions (a),(b) or (c) of Theorem 5.3.*

*(3) The asymptotic limit $g^-$ satisfies one of the conditions (a),(b) or (c) of Theorem 5.3.*

*(4) There is a point $\lambda_0 \in \Lambda$ such that*

*(i) $k = \dim N(\lambda_0)$ is odd where $N(\lambda) = \{u \in C^2(\mathbb{R}, \mathbb{R}^{2N}) : Ju'(t) - A_\lambda(t)u(t) \equiv 0$ and $\lim_{|t|\to\infty} u(t) = 0\}$,*

*(ii) for every $u \in N(\lambda_0)\backslash\{0\}$ there exists $v \in N(\lambda_0)$ such that*

$$\int_{-\infty}^{\infty} \langle T_{\lambda_0}(t)u(t), v(t)\rangle \, dt \neq 0 \text{ where } T_\lambda(t) = D_\lambda D_\xi^2 H(t, 0, \lambda) \text{ and}$$

*(iii) $\dim\{T_{\lambda_0}(\cdot)u : u \in N(\lambda_0)\} = k$.*

*Then a global branch of homoclinic solutions of (1.1)(1.2) bifurcates at $\lambda_0$ in the sense of Theorem 2.3 with $X = H^1$ and $Y = L^2$.*

Under the hypotheses (H1) to (H4) and ($\mathrm{H}^\infty$), all solutions of the system (1.1)(1.2) decay to zero exponentially fast as $|t| \to \infty$ and so the system (1.1)(1.2) is actually equivalent to the equation $F(\lambda, x) = 0$ where $F : \mathbb{R} \times H^1 \to L^2$. The exponential decay can be established by a slight variant of the proof of the second assertion of Theorem 5.1.

**Theorem 5.6.** *Suppose that (H1) to (H4) and ($H^\infty$) are satisfied. If $x$ is a solution of (1.1)(1.2), there is a constant $\gamma > 0$ such that* $\lim_{|t|\to\infty} e^{\gamma|t|} |x(t)| = 0$.

*Proof.* First we observe that

$$D_\xi H(t,\xi,\lambda) = \int_0^1 \frac{d}{d\tau} D_\xi H(t,\tau\xi,\lambda)d\tau = M(t,\xi,\lambda)\xi$$

where the matrix $M(t,\xi,\lambda)$ is defined by

$$M(t,\xi,\lambda) = \int_0^1 D_\xi^2 H(t,\tau\xi,\lambda)d\tau.$$

Thus $x$ satisfies the linear equation

$$Jx'(t) = K(t)x(t) \text{ where } K(t) = M(t,x(t),\lambda).$$

But,

$$K(t) = \int_0^1 \{D_\xi^2 H(t,\tau x(t),\lambda) - D_\xi g^+(t,\tau x(t),\lambda)\}d\tau + \int_0^1 \{D_\xi g^+(t,\tau x(t),\lambda) - A_\lambda^+(t)\}d\tau + A_\lambda^+(t)$$

from which it is easy to see that $\|K(t) - A_\lambda^+(t)\| \to 0$ as $t \to \infty$. Now, using the Floquet change of variables and Corollary VII-3-7 of [5] as in the proof of Theorem 5.1, we see that $|x(t)| \to 0$ exponentially fast as $t \to \infty$. The behaviour as $t \to -\infty$ can be treated in the same way. □

## 6. Examples.

Consider the following Hamiltonian,

$$H(t,u,v,\lambda) = \frac{1}{2}\{v^2 + \lambda u^2 + a(t)u^2\} + \frac{A(2+\cos t)|u|^{\sigma+2}}{(\sigma+2)(1+e^{-t})} + \frac{B(2+\cos\omega t)u^2v^2}{2(1+e^t)} + r(t)Q(u,v,\lambda)$$

where $A, B, \sigma, \omega$ are constants with $A \le 0$ and $\sigma > 0$,

$$a, r \in C(\mathbb{R}) \text{ with } a \mathrel{\substack{\geq \\ \not\equiv}} 0 \text{ and } \lim_{|t|\to\infty} a(t) = \lim_{|t|\to\infty} r(t) = 0$$

and $Q \in C^3(\mathbb{R}^3)$ with

$$Q(0,0,\lambda) = \partial_i Q(0,0,\lambda) = \partial_j\partial_i Q(0,0,\lambda) = 0$$

for all $i, j = 1, 2$ and all $\lambda \in \mathbb{R}$.

It is easily seen that $H : \mathbb{R}\times\mathbb{R}^2\times\mathbb{R} \to \mathbb{R}$ satisfies the conditions (H1) to (H4) and that the system $Jx'(t) = D_{(u,v)}H(t,x(t),\lambda)$ is

$$-v'(t) = \lambda u(t) + a(t)u(t) + \frac{A(2+\cos t)|u(t)|^\sigma u(t)}{1+e^{-t}} + \frac{B(2+\cos\omega t)u(t)v(t)^2}{1+e^t} + r(t)\partial_u Q(u(t),v(t),\lambda) \tag{6.15}$$

$$u'(t) = v(t) + \frac{B(2+\cos\omega t)u(t)^2 v(t)}{1+e^t} + r(t)\partial_v Q(u(t),v(t),\lambda) \tag{6.16}$$

The condition ($\mathrm{H}^\infty$) is also satisfied with

$$
\begin{aligned}
H^+(t,u,v,\lambda) &= \frac{1}{2}\{v^2+\lambda u^2\} + \frac{A(2+\cos t)\,|u|^{\sigma+2}}{\sigma+2},\\
H^-(t,u,v,\lambda) &= \frac{1}{2}\{v^2+\lambda u^2\} + \frac{B(2+\cos \omega t)u^2v^2}{2},\\
g^+(t,u,v,\lambda) &= (\lambda u + A(2+\cos t)\,|u|^{\sigma}\,u, v)\\
g^-(t,u,v,\lambda) &= (\lambda u + B(2+\cos\omega t)uv^2, v + B(2+\cos\omega t)u^2 v).
\end{aligned}
$$

Note that

$$
A_\lambda(t) = D^2_{(u,v)}H(t,0,0,\lambda) = \begin{bmatrix} \lambda + a(t) & 0 \\ 0 & 1 \end{bmatrix} \text{ and}
$$

$$
A^\pm_\lambda(t) = D_{(u,v)}g^\pm(t,0,0,\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}.
$$

Thus $A^\pm_\lambda(t)$ is independent of $t$ and the spectrum of the matrix $J\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ is

$$
\{\pm i\sqrt{\lambda}\} \text{ for } \lambda > 0, \qquad \{0\} \text{ for } \lambda = 0 \text{ and } \{\pm\sqrt{|\lambda|}\} \text{ for } \lambda < 0.
$$

It follows from Theorem 5.1 that $D_xF(\lambda,0) \in \Phi_0(H^1,L^2)$ for all $\lambda < 0$.

Setting $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we find that $JC(u,v)^T = (-u,v)^T$ and hence that

$$
\left\langle g^+(t,u,v,\lambda), JC(u,v)^T \right\rangle = -\lambda u(t)^2 - A(2+\cos t)\,|u|^{\sigma+2} + v^2 > 0
$$

for all $(u,v) \in \mathbb{R}^2\backslash\{(0,0)\}$ provided that $\lambda < 0$. Similarly,

$$
\left\langle g^-(t,u,v,\lambda), JC(u,v)^T \right\rangle = -\lambda u(t)^2 + v^2 > 0
$$

for all $(u,v) \in \mathbb{R}^2\backslash\{(0,0)\}$ provided that $\lambda < 0$.

By Theorem 5.4 we now have that $(-\infty,0)$ is an admissible interval for the system (6.15)(6.16). In particular, $D_xF(\lambda,0) : H^1 \to L^2$ is an isomorphism for $\lambda < 0$, except at the values of $\lambda$ for which $\ker D_xF(\lambda,0) \neq \{0\}$. But, $x \in \ker D_xF(\lambda,0)\backslash\{0\}$ means that $x$ is a homoclinic solution of the linear system

$$
\begin{aligned}
-x_2'(t) &= \{\lambda + a(t)\}x_1(t)\\
x_1'(t) &= x_2(t)
\end{aligned}
$$

which is equivalent to the second order equation

$$
x_1''(t) = -\{\lambda + a(t)\}x_1(t).
$$

Under our hypotheses on the coefficient $a$, there is always at least one value of $\lambda$ in the interval $(-\infty,0)$ for which this equation has a homoclinic solution. Setting

$$
\lambda_0 = \inf\{\textstyle\int_{\mathbb{R}} \varphi'(t)^2 - a(t)\varphi(t)^2 dt : \varphi \in H^1(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \varphi(t)^2 dt = 1\},
$$

it is well-known (see [7], theorem 11.5) that $\lambda_0 \in (-\infty,0)$ and that there exists an element $\varphi_0 \in H^1(\mathbb{R})$ such that

$$
\varphi_0(t) > 0 \text{ for all } t \in \mathbb{R} \text{ and } \lambda_0 \int_{\mathbb{R}} \varphi(t)_0^2 dt = \int_{\mathbb{R}-} \varphi_0'(t)^2 - a(t)\varphi_0(t)^2 dt.
$$

Furthermore, $\varphi_0 \in H^2(\mathbb{R}) \cap C^2(\mathbb{R})$ and satisfies the equation

$$
x_1''(t) = -\{\lambda_0 + a(t)\}x_1(t).
$$

Setting $x_0 = (\varphi_0, \varphi_0')$, we find that $\ker D_xF(\lambda_0,0) = \mathrm{span}\{x_0\}$.

Finally we observe that

$$D_\lambda D^2_{(u,v)} H(t,0,0,\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and so, in the notation of Theorem 5.5,

$$T_{\lambda_0}(t)x_0(t) = (\varphi_0, 0).$$

Thus all the hypotheses of Theorem 5.5 are satisfied by the system (6.15)(6.16).

7. **Appendix. Proof of Theorem 3.3** (1) It is sufficient to prove that $D_x F$ and $D_\lambda F$ exist and are continuous on $\mathbb{R} \times H^1$.
For $D_x F(\lambda, x)$, we consider $\lambda \in \mathbb{R}$ and $x, y \in H^1$ with $(\lambda, x)$ fixed. Then

$$\begin{aligned} & F(\lambda, x+y) - F(\lambda, x) - \{Jy' - M(\lambda, x)y\} \\ &= -D_\xi H(\cdot, x+y, \lambda) + D_\xi H(\cdot, x, \lambda) + D^2_\xi H(\cdot, x, \lambda)y \\ &= \int_0^1 D^2_\xi H(\cdot, x, \lambda)y - \frac{d}{ds} D_\xi H(\cdot, x+sy, \lambda)ds \\ &= \int_0^1 \{D^2_\xi H(\cdot, x, \lambda) - D^2_\xi H(\cdot, x+sy, \lambda)\} y ds. \end{aligned}$$

Hence, for all $t \in \mathbb{R}$,

$$\begin{aligned} & \|F(\lambda, x+y)(t) - F(\lambda, x)(t) - \{Jy' - M(\lambda, x)y\}(t)\| \\ & \leq \|y(t)\| \sup_{t \in \mathbb{R}} \int_0^1 \left\| D^2_\xi H(t, x(t), \lambda) - D^2_\xi H(t, x(t) + sy(t), \lambda) \right\| ds \end{aligned}$$

and so

$$\begin{aligned} & \|F(\lambda, x+y) - F(\lambda, x) - \{Jy' - M(\lambda, x)y\}\|_2 \\ & \leq \|y\|_2 \sup_{t \in \mathbb{R}} \int_0^1 \left\| D^2_\xi H(t, x(t), \lambda) - D^2_\xi H(t, x(t) + sy(t), \lambda) \right\| ds. \end{aligned}$$

Recalling that $H^1$ is continuously embedded in $C_d$, we observe that there is a compact subset $K$ of $\mathbb{R}^{2N}$ such that $x(t)$ and $y(t) \in K$ for all $t \in \mathbb{R}$ and all $y \in H^1$ such that $\|y\| \leq 1$. Since $D^2_\xi H(\cdot, \cdot, \lambda)$ is a $C^0_\xi-$bundle map by (H3), it follows that

$$\sup_{t \in \mathbb{R}} \int_0^1 \left\| D^2_\xi H(t, x(t), \lambda) - D^2_\xi H(t, x(t) + sy(t), \lambda) \right\| ds \to 0 \text{ as } \|y\| \to 0.$$

This proves that $D_x F(\lambda, x)y$ exists and is equal to $Jy' - M(\lambda, x)y$. For the continuity of $D_x F$, we consider $(\lambda, x), (\mu, z) \in \mathbb{R} \times H^1$ and $y \in H^1$ with $(\lambda, x)$ fixed. Then

$$\begin{aligned} \{D_x F(\lambda, x) - D_x F(\mu, z)\}y &= \{M(\mu, z) - M(\lambda, x)\}y \\ &= \{D^2_\xi H(\cdot, z, \mu) - D^2_\xi H(\cdot, x, \lambda)\}y \end{aligned}$$

and hence

$$\|\{D_x F(\lambda, x) - D_x F(\mu, z)\}y\|_2 \leq \|y\|_2 \sup_{t \in \mathbb{R}} \left\| D^2_\xi H(t, z(t), \mu) - D^2_\xi H(t, x(t), \lambda) \right\|.$$

Thus $\|D_x F(\lambda, x) - D_x F(\mu, z)\|$ in $B(H^1, L^2)$ is bounded above by

$$\sup_{t \in \mathbb{R}} \left\| D^2_\xi H(t, z(t), \mu) - D^2_\xi H(t, x(t), \lambda) \right\|.$$

But

$$\begin{aligned}
&D_\xi^2 H(t,z(t),\mu) - D_\xi^2 H(t,x(t),\lambda)\\
&= \int_0^1 \frac{d}{ds} D_\xi^2 H(t,z(t),s\mu+(1-s)\lambda)ds + D_\xi^2 H(t,z(t),\lambda) - D_\xi^2 H(t,x(t),\lambda)\\
&= \int_0^1 D_\lambda D_\xi^2 H(t,z(t),s\mu+(1-s)\lambda)ds(\mu-\lambda) + D_\xi^2 H(t,z(t),\lambda) - D_\xi^2 H(t,x(t),\lambda)
\end{aligned}$$

There is a constant $K$ such that $\|(z(t),s\mu+(1-s)\lambda)\| \leq K$ for all $(\mu,z) \in \mathbb{R}\times H^1$ with $\|(\mu,z)-(\lambda,x)\| \leq 1$, and hence by part (i) of Lemma 3.2, there is a constant $C(K)$ such that

$$\left\|D_\lambda D_\xi^2 H(t,z(t),s\mu+(1-s)\lambda)\right\| \leq C(K) \text{ for all } t\in\mathbb{R} \text{ and all } s\in[0,1].$$

Thus

$$\sup_{t\in\mathbb{R}} \int_0^1 \left\|D_\lambda D_\xi^2 H(t,z(t),s\mu+(1-s)\lambda)\right\| ds \leq C(K)$$

and so

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left\|D_\xi^2 H(t,z(t),\mu) - D_\xi^2 H(t,x(t),\lambda)\right\|\\
&\leq C(K)\left|\mu-\lambda\right| + \sup_{t\in\mathbb{R}} \left\|D_\xi^2 H(t,z(t),\lambda) - D_\xi^2 H(t,x(t),\lambda)\right\|.
\end{aligned}$$

As above,

$$\sup_{t\in\mathbb{R}} \left\|D_\xi^2 H(t,z(t),\lambda) - D_\xi^2 H(t,x(t),\lambda)\right\| \to 0 \text{ as } \|z-x\| \to 0$$

since $D_\xi^2 H(\cdot,\cdot,\lambda)$ is a $C_\xi^0-$bundle map. Thus

$$\sup_{t\in\mathbb{R}} \left\|D_\xi^2 H(t,z(t),\mu) - D_\xi^2 H(t,x(t),\lambda)\right\| \to 0$$

as $(\mu,z)\to(\lambda,x)$ in $\mathbb{R}\times H^1$, establishing the continuity of $D_x F : \mathbb{R}\times H^1 \to B(H^1,L^2)$ at $(\lambda,x)$.

For the differentiability with respect to $\lambda$, we consider $\lambda,\tau\in\mathbb{R}$ and $x\in H^1$. Then

$$\begin{aligned}
&F(\lambda+\tau,x) - F(\lambda,x) + \tau D_\lambda D_\xi H(\cdot,x,\lambda)\\
&= -D_\xi H(\cdot,x,\lambda+\tau) + D_\xi H(\cdot,x,\lambda) + \tau D_\lambda D_\xi H(\cdot,x,\lambda)\\
&= \int_0^1 \{\tau D_\lambda D_\xi H(\cdot,x,\lambda) - \frac{d}{ds} D_\xi H(\cdot,x,\lambda+s\tau)\}ds\\
&= \tau \int_0^1 \{D_\lambda D_\xi H(\cdot,x,\lambda) - D_\lambda D_\xi H(\cdot,x,\lambda+s\tau)\}ds.
\end{aligned}$$

By (H1), $D_\lambda D_\xi H(\cdot,0,\lambda) \equiv 0$ and so

$$\begin{aligned}
&D_\lambda D_\xi H(\cdot,x,\lambda) - D_\lambda D_\xi H(\cdot,x,\lambda+s\tau)\\
&= \int_0^1 \frac{d}{d\sigma}\{D_\lambda D_\xi H(\cdot,\sigma x,\lambda) - D_\lambda D_\xi H(\cdot,\sigma x,\lambda+s\tau)\}d\sigma\\
&= \int_0^1 \{D_\lambda D_\xi^2 H(\cdot,\sigma x,\lambda) - D_\lambda D_\xi^2 H(\cdot,\sigma x,\lambda+s\tau)\}x d\sigma.
\end{aligned}$$

Thus

$$\left\| \frac{F(\lambda+\tau,x)-F(\lambda,x)}{\tau} + D_\lambda D_\xi H(\cdot,x,\lambda) \right\|_2$$
$$\leq \|x\|_2 \sup_{t\in\mathbb{R}} \int_0^1 \int_0^1 \left\| D_\lambda D_\xi^2 H(t,\sigma x(t),\lambda) - D_\lambda D_\xi^2 H(t,\sigma x(t),\lambda+s\tau) \right\| d\sigma\, ds.$$

Recalling that $D_\lambda D_\xi^2 H$ is a $C^0_{(\xi,\lambda)}$−bundle map, we easily deduce from this that $D_\lambda F(\lambda,x)$ exists and is equal to $-D_\lambda D_\xi H(\cdot,x,\lambda)$ for all $(\lambda,x)\in\mathbb{R}\times H^1$. For the continuity of $D_\lambda F$, we consider $(\lambda,x),(\mu,z)\in\mathbb{R}\times H^1$ with $(\lambda,x)$ fixed. Then, using (H1),

$$\begin{aligned} D_\lambda F(\lambda,x) - D_\lambda F(\mu,z) &= D_\lambda D_\xi H(\cdot,z,\mu) - D_\lambda D_\xi H(\cdot,x,\lambda) \\ &= \int_0^1 \frac{d}{ds}\{D_\lambda D_\xi H(\cdot,sz,\mu) - D_\lambda D_\xi H(\cdot,sx,\lambda)\}ds \\ &= \int_0^1 \{D_\lambda D_\xi^2 H(\cdot,sz,\mu)z - D_\lambda D_\xi^2 H(\cdot,sx,\lambda)x\}ds \end{aligned}$$

Hence

$$\begin{aligned} D_\lambda F(\lambda,x) - D_\lambda F(\mu,z) &= \\ &= \int_0^1 \{D_\lambda D_\xi^2 H(\cdot,sz,\mu)z - D_\lambda D_\xi^2 H(\cdot,sx,\lambda)z\}ds \\ &\quad + \int_0^1 \{D_\lambda D_\xi^2 H(\cdot,sx,\lambda)z - D_\lambda D_\xi^2 H(\cdot,sx,\lambda)x\}ds \end{aligned}$$

and so

$$\begin{aligned} &\|D_\lambda F(\lambda,x) - D_\lambda F(\mu,z)\|_2 \\ &\quad\leq \|z\|_2 \sup_{t\in\mathbb{R}} \int_0^1 \left\| D_\lambda D_\xi^2 H(t,sz(t),\mu) - D_\lambda D_\xi^2 H(t,sx(t),\lambda) \right\| ds \\ &\qquad + \|z-x\|_2 \sup_{t\in\mathbb{R}} \int_0^1 \left\| D_\lambda D_\xi^2 H(t,sx(t),\lambda) \right\| ds \end{aligned}$$

From (H3), it follows that

$$\sup_{t\in\mathbb{R}} \int_0^1 \left\| D_\lambda D_\xi^2 H(t,sz(t),\mu) - D_\lambda D_\xi^2 H(t,sx(t),\lambda) \right\| ds \to 0$$

as $(z,\mu)\to(x,\lambda)$ in $\mathbb{R}\times H^1$, and it follows from Lemma 3.2(i) that there is a constant $C$ such that

$$\sup_{t\in\mathbb{R}} \int_0^1 \left\| D_\lambda D_\xi^2 H(\cdot,sx,\lambda) \right\| ds \leq C.$$

Therefore

$$\|D_\lambda F(\lambda,x) - D_\lambda F(\mu,z)\|_2 \to 0 \text{ as } (z,\mu)\to(x,\lambda) \text{ in } \mathbb{R}\times H^1,$$

proving that $D_\lambda F:\mathbb{R}\times H^1\to L^2$ is continuous at $(\lambda,x)$.

(2) To prove that $D_x F(\lambda,0)$ is differentiable with respect to $\lambda$, we consider $\lambda,\tau\in\mathbb{R}$ and $y\in H^1$. Then

$$\begin{aligned} \{D_x F(\lambda+\tau,0) - D_x F(\lambda,0)\}y &= -\{D_\xi^2 H(\cdot,0,\lambda+\tau) - D_\xi^2 H(\cdot,0,\lambda)\}y \\ &= -\tau \int_0^1 D_\lambda D_\xi^2 H(\cdot,0,\lambda+s\tau)y\,ds \end{aligned}$$

and hence

$$\left\{\frac{D_xF(\lambda+\tau,0)-D_xF(\lambda,0)}{\tau}+D_\lambda D_\xi^2H(\cdot,0,\lambda)\right\}y$$
$$=\int_0^1\{D_\lambda D_\xi^2H(\cdot,0,\lambda)-D_\lambda D_\xi^2H(\cdot,0,\lambda+s\tau)\}yds.$$

Thus

$$\left\|\left\{\frac{D_xF(\lambda+\tau,0)-D_xF(\lambda,0)}{\tau}+D_\lambda D_\xi^2H(\cdot,0,\lambda)\right\}y\right\|_2$$
$$\leq\|y\|_2\sup_{t\in\mathbb{R}}\int_0^1\left\|D_\lambda D_\xi^2H(\cdot,0,\lambda)-D_\lambda D_\xi^2H(\cdot,0,\lambda+s\tau)\right\|ds$$

and, since $H^1$ is continuously embedded in $L^2$, this shows that

$$\left\|\frac{D_xF(\lambda+\tau,0)-D_xF(\lambda,0)}{\tau}+D_\lambda D_\xi^2H(\cdot,0,\lambda)[\cdot]\right\|_{B(H^1,L^2)}$$
$$\leq\sup_{t\in\mathbb{R}}\int_0^1\left\|D_\lambda D_\xi^2H(t,0,\lambda)-D_\lambda D_\xi^2H(t,0,\lambda+s\tau)\right\|ds$$

Using (H3), it follows that $D_\lambda D_xF(\lambda,0)$ exists and is equal to multiplication by $-D_\lambda D_\xi^2H(\cdot,0,\lambda)$. The continuity of $D_\lambda D_xF(\cdot,0):\mathbb{R}\to B(H^1,L^2)$ again follows from (H3). Indeed,

$$\|\{D_\lambda D_xF(\lambda,0)-D_\lambda D_xF(\mu,0)\}y\|_2=\left\|\{D_\lambda D_\xi^2H(\cdot,0,\mu)-D_\lambda D_\xi^2H(\cdot,0,\lambda)\}y\right\|_2$$
$$\leq\|y\|_2\sup_{t\in\mathbb{R}}\left\|D_\lambda D_\xi^2H(t,0,\mu)-D_\lambda D_\xi^2H(t,0,\lambda)\right\|$$

so that

$$\|D_\lambda D_xF(\lambda,0)-D_\lambda D_xF(\mu,0)\|_{B(H^1,L^2)}\leq\sup_{t\in\mathbb{R}}\left\|D_\lambda D_\xi^2H(t,0,\mu)-D_\lambda D_\xi^2H(t,0,\lambda)\right\|.$$

(3) Consider $u\in W$ and $\lambda,\mu\in\mathbb{R}$ with $\lambda$ fixed. Then

$$\|F(\lambda,u)-F(\mu,u)\|_2=\|D_\xi H(\cdot,u,\lambda)-D_\xi H(\cdot,u,\mu)\|_2$$

and

$$D_\xi H(\cdot,u,\lambda)-D_\xi H(\cdot,u,\mu)=(\lambda-\mu)\int_0^1 D_\lambda D_\xi H(\cdot,u,s\lambda+(1-s)\mu)ds$$
$$=(\lambda-\mu)\int_0^1\int_0^1 D_\lambda D_\xi^2H(\cdot,\sigma u,s\lambda+(1-s)\mu)udsd\sigma.$$

since $D_\lambda D_\xi H(\cdot,0,s\lambda+(1-s)\mu)\equiv0$ by (H1). Let $L=\sup\{\|x\|_\infty : x\in W\}$. Since $W$ is bounded in $H^1$ which is continuously embedded in $C_d$, it follows that $L<\infty$. Set $K=\sqrt{L+|\lambda|+1}$. By Lemma 3.2(i) there is a constant $C(K)$ such that

$$\left\|D_\lambda D_\xi^2H(t,\xi,\gamma)\right\|\leq C(K)\text{ for all }t\in\mathbb{R}\text{ and }\|(\xi,\gamma)\|\leq K.$$

It follows that, for all $u\in W$ and $\mu\in[\lambda-1,\lambda+1]$,

$$\|F(\lambda,u)-F(\mu,u)\|_2\leq|\lambda-\mu|\,\|u\|_2\,C(K),$$

proving that the family $\{F(\cdot,u)\}_{u\in W}$ is equicontinuous at $\lambda$.

(4) We fix $(\lambda,x)\in\mathbb{R}\times H^1$ and consider a sequence $\{x_n\}$ which converges weakly to $x$ in $H^1$. In particular, $\{x_n\}$ is bounded in $H^1$, and recalling the remarks following

(3.8), we have that $\{F(\lambda, x_n)\}$ is a bounded sequence in $L^2$. Thus it is enough to prove that

$$\langle F(\lambda, x_n) - F(\lambda, x_n), \varphi\rangle_2 \to 0 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^{2N}) = C_0^\infty$$

where $\langle\cdot,\cdot\rangle_2$ denotes the usual scalar product in $L^2$.
Given $\varphi \in C_0^\infty$, let $R > 0$ be such that $\varphi(t) = 0$ for all $t \notin [-R, R]$. Furthermore, since $H^1$ is continuously embedded in $C_d$, there is a constant $K$ such that

$$\|x_n\|_\infty \le K \text{ for all } n \in \mathbb{N} \text{ and also } \|x\|_\infty \le K.$$

Since $D_\xi H(\cdot,\cdot,\lambda)$ is uniformly continuous on $[-R, R] \times B(0, 2K) \subset \mathbb{R} \times \mathbb{R}^{2N}$, it follows that

$$\langle D_\xi H(\cdot, x_n, \lambda) - D_\xi H(\cdot, x, \lambda), \varphi\rangle \to 0 \text{ as } n \to \infty$$

since $x_n \to x$ uniformly on $[-R, R]$. Furthermore,

$$\langle Jx_n' - Jx', \varphi\rangle_2 = \langle x_n - x, J\varphi'\rangle_2 \to 0$$

again by the uniform convergence of $x_n$ to $x$ on $[-R, R]$. This completes the proof.

**Proof of Theorem 4.1** By ($\mathrm{H}^\infty$) we have that $D_\xi g^+(\cdot,\cdot,\lambda)$ is a $C_\xi^0$−bundle map and

$$g^+(t, \xi, \lambda) = \int_0^1 D_\xi g^+(t, s\xi, \lambda)\xi ds.$$

As in the proof of Lemma 3.2(ii), for any $K > 0$, there exists a constant $C(\lambda, K)$ such that

$$\left\|D_\xi g^+(t, \xi, \lambda)\right\| \le C(\lambda, K)$$

for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^{2N}$ such that $\|\xi\| \le K$. Thus,

$$\left\|g^+(t, \xi, \lambda)\right\| \le C(\lambda, K)\,\|\xi\|$$

for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^{2N}$ such that $\|\xi\| \le K$. If $W$ is a bounded subset of $H^1$, there is a constant $K$ such that $\|x\|_\infty \le K$ for all $x \in W$, and consequently,

$$\left\|g^+(t, x(t), \lambda)\right\| \le C(\lambda, K)\,\|x(t)\|$$

for all $t \in \mathbb{R}$ and $x \in W$. Thus $g^+(\cdot, x, \lambda) \in L^2$ and

$$\left\|g^+(\cdot, x, \lambda)\right\|_2 \le C(\lambda, K)\,\|x\|_2 \text{ for all } x \in W.$$

This proves that $h^+(\lambda, \cdot)$ maps $H^1$ into $L^2$ and is bounded.

The weakly sequential continuity of $F(\lambda, \cdot) : H^1 \to L^2$ follows from this and the continuity of $g^+(\cdot,\cdot,\lambda) : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ exactly as in the proof of Theorem 3.3(4).

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*E-mail address*: `secchi@sissa.it`
*E-mail address*: `charles.stuart@epfl.ch`