Non-Analyticity and the van der Waals Limit

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Abstract

We study the analyticity properties of the free energy $f_\gamma(m)$ of the Kac model at points of first order phase transition, in the van der Waals limit $\gamma \searrow 0$. We show that there exists an inverse temperature β_0 and $\gamma_0 > 0$ such that for all $\beta \geq \beta_0$ and for all $\gamma \in (0, \gamma_0)$, $f_\gamma(m)$ has no analytic continuation along the path $m \searrow m^*$ (m^* denotes spontaneous magnetization). The proof consists in studying high order derivatives of the pressure $p_\gamma(h)$, which is related to the free energy $f_\gamma(m)$ by a Legendre transform.

Keywords: Non-analyticity, singularity at first order phase transition, Pirogov-Sinai Theory, Kac potentials, van der Waals limit.

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1 Introduction

The first equation of state giving precise predictions on the liquid-vapor equilibrium at low temperature was given by van der Waals [vdW]:

$$\Big(p + \frac{a}{v^2}\Big)\Big(v - b\Big) = RT\,. \tag{1.1}$$

This equation follows from the hypothesis that the molecules interact via 1) a short range hard core repulsion, due to the assumption that molecules are extended in space, 2) an attractive potential, whose range is assumed to be comparable to the size of the system. Nowadays, such an approximation is called a *mean field* approximation. As well known, there exists a critical temperature $T_c = T_c(a,b)$ such that for $T < T_c$, $\frac{\partial}{\partial v}p \geq 0$ for some values of v, which implies thermodynamic instability. On physical and geometrical grounds, the graph of the pressure was modified by Maxwell who replaced $p(v)$, on a suitably chosen interval $[v_l, v_g]$, by a flat horizontal segment (the "equal area rule"). The new function obtained, written $\mathrm{MC}\,p(v)$, describes

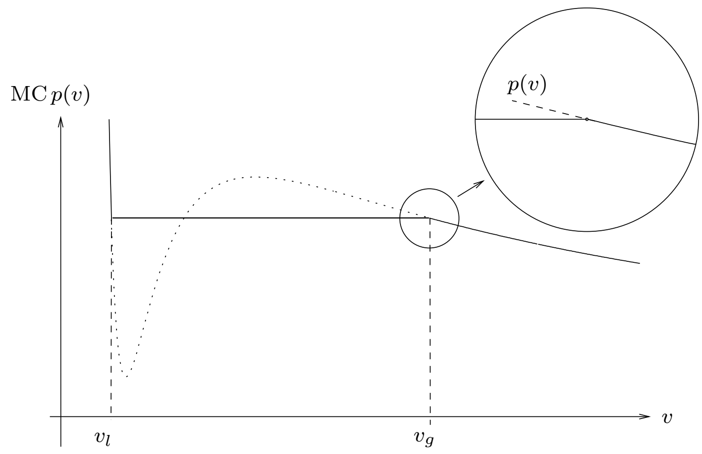

Figure 1: The equation of state modified by Maxwell and the analytic continuation at the condensation point.

precisely what is observed in the laboratory: v_l is called the evaporation point and v_g is the condensation point (see Figure 1).
A particularity of this scenario is that $\mathrm{MC}\,p$ can be continued analytically along the paths $v \nearrow v_l$ and $v \searrow v_g$: the liquid and gas branches can be joined analytically by a *single* function, which is nothing but the original isotherm p given in (1.1). The pressure obtained by analytic continuation was originally considered as the pressure of a meta-stable state (see Figure 1). For instance, the meta-stable state obtained by analytic continuation along the path $v \searrow v_g$ is called a *super-saturated vapor*.

Much later, Kac, Uhlenbeck and Hemmer [KUH] showed how the Maxwell construction could be rigorously justified for a one dimensional model, from

first principles of statistical mechanics, using a double limiting process: if the range of interaction diverges *after* the thermodynamic limit, then convexity is preserved and the free energy converges to the convex envelope of mean field theory. Later this was generalized and extended to higher dimensions by Lebowitz and Penrose [LP]. From the point of view of analyticity, these results imply, as in the theory of van der Waals, that the free energy can be continued analytically across condensation/evaporation points.
In the mean time, arguments were given, saying that when the range of interaction is finite, the free energy might have some singularities that forbid analytic continuation across the transition points. In [F] and [L], Fisher and Langer analyzed in details simple models to illustrate this phenomenon, but it was not until the seminal work of Isakov [I1] that this was shown for the Ising model.

An important issue is thus to understand how the breakdown of analyticity at a first order phase transition point relates to the range of interaction. Since Kac potentials give a way of interpolating finite range systems and mean field, it seems an interesting problem to study the dependence on the scaling parameter γ of the analyticity properties of the Kac model at low temperature. The aim of this work is to show that for the Kac-Ising ferromagnet on $\mathbf{Z}^d$ ($d \geq 2$) at low temperature, the free energy has no analytic continuation at first order phase transition points *as long as the range of interaction is finite* ($\gamma > 0$). Analytic continuation occurs only *after* the van der Waals limit ($\gamma \searrow 0$). This result answers a question raised by Joel Lebowitz at a conference devoted to Kac potentials, *Inhomogeneous Random Systems*, held in Paris, January 2001.

In Section 1.1 we remind the main properties of the free energy for mean field and Kac potentials in the case of Ising spins. In Section 1.2 we state our main results and give the strategy of the proof.

1.1 Mean Field and Kac Potentials

We consider the lattice $\mathbf{Z}^d$, $d \geq 2$, with a distance $d(x,y) = \|x-y\|$, where

$$\|x\| := \max_{i=1,\dots,d} |x_i| \,. \tag{1.2}$$

This distance will also be used for points of $\mathbb{R}^d$. The letter Λ will always denote a finite subset of $\mathbf{Z}^d$. At each site $i \in \mathbf{Z}^d$ lives a spin $\sigma_i \in \{\pm 1\}$. The configuration space is $\Omega = \{\pm 1\}^{\mathbf{Z}^d}$. For any set Λ, $\Omega_\Lambda = \{\pm 1\}^\Lambda$. Our notations are often inspired by those of Presutti [Pr].
Mean Field. In a mean field model, the interactions ignore the spatial positions of the spins, and the hamiltonian in a volume Λ containing N

sites is ($\sigma \in \Omega_\Lambda$)

$$H_\Lambda^{MF}(\sigma) := -\frac{1}{N} \sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} \sigma_i \sigma_j \,. \tag{1.3}$$

As is well known, the free energy can be easily computed. For $m \in [-1, +1]$,

$$f_{MF}(m) = -\frac{1}{2}m^2 - \frac{1}{\beta} I(m)\,, \tag{1.4}$$

where

$$I(m) := -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}\,. \tag{1.5}$$

When $\beta \leq 1$ f_{MF} is strictly convex, but when $\beta > 1$, f_{MF} has two minima at $\pm m^*(\beta)$, where $m^*(\beta)$ is the positive solution of $m = \tanh(\beta m)$. $\beta_c := 1$ is the critical temperature of mean field theory. As in van der Waals theory, f_{MF} is non convex when $\beta > \beta_c$, in contradiction with thermodynamic stability.

Kac Potentials. Kac potentials are defined as follows. Consider $J : \mathbb{R}^d \to \mathbb{R}^+$ supported by the cube $\{y \in \mathbb{R}^d : \|y\| \leq 1\} = [-1, +1]^d$ such that the overall strength equals unity, i.e.

$$\int_{\mathbb{R}^d} J(x)\mathrm{d}x = 1\,. \tag{1.6}$$

Let $\gamma \in (0, 1)$ be the scaling parameter. Define $J_\gamma : \mathbf{Z}^d \to \mathbb{R}^+$ as follows:

$$J_\gamma(x) := c_\gamma \gamma^d J(\gamma x)\,, \tag{1.7}$$

where c_γ is defined so that

$$\sum_{x\neq 0} J_\gamma(x) = 1\,. \tag{1.8}$$

It is easy to see that (1.6) implies $\lim_{\gamma\searrow 0} c_\gamma = 1$. Since $J_\gamma(x) = 0$ if $\|x\| > \gamma^{-1}$, we call $R := \gamma^{-1}$ the range of the interaction.

Convention: Unless stated explicitly, R will always denote the range of interaction, i.e. γ^{-1}. For simplicity, we will usually omit γ from the notations of the quantities that will appear in the sequel (hamiltonian, partition function).

For a finite Λ, $\sigma \in \Omega_\Lambda$, the Kac hamiltonian is defined by

$$H_\Lambda^h(\sigma) = -\sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} J_\gamma(i-j)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i\,, \tag{1.9}$$

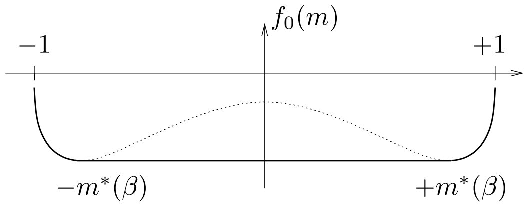

Figure 2: The free energy $f_0(m)$ when $\beta > 1$. The dotted line is the analytic continuation provided by $f_{MF}(m)$.

where $h \in \mathbb{R}$ is the magnetic field. The magnetization in Λ is

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma_i \,, \tag{1.10}$$

and takes values in a set $\chi_\Lambda \subset [-1, +1]$. The canonical partition function is defined by ($\beta > 0$ is the inverse temperature, $m \in \chi_\Lambda$):

$$Z(\Lambda, m) = \sum_{\substack{\sigma_\Lambda \in \Omega_\Lambda: \\ m_\Lambda(\sigma) = m}} \exp\big(-\beta H^0_\Lambda(\sigma_\Lambda)\big) \,. \tag{1.11}$$

The free energy density is, for $m \in [-1, +1]$,

$$f_\gamma(m) = -\lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{\beta |\Lambda|} \log Z(\Lambda, m(\Lambda)) \,, \tag{1.12}$$

where the thermodynamic limit $\Lambda \nearrow \mathbf{Z}^d$ is along a sequence of cubes, and the sequence $m(\Lambda)$ is such that $m(\Lambda) \to m$. The function f_γ exists and is convex. The Theorem of Lebowitz-Penrose [LP] gives a closed form for the free energy in the van der Waals limit $\gamma \searrow 0$. For a function $f(x)$, let $\mathrm{CE}\, f(x)$ denote its convex envelope.

Theorem 1.1. **[LP]** *For any $\beta > 0$, $m \in [-1, +1]$,*

$$f_0(m) := \lim_{\gamma \searrow 0} f_\gamma(m) = \mathrm{CE}\, f_{MF}(m) \,. \tag{1.13}$$

When $\beta > 1$, the graph of $f_0(m)$ is thus horizontal between $-m^*(\beta)$ and $+m^*(\beta)$, giving a rigorous justification of the Maxwell construction (see Figure 2).
From the point of view of analyticity, we have

Corollary 1.1. *When $\beta > 1$, f_0 is analytic everywhere except at $\pm m^*(\beta)$, and has analytic continuations along the (real) paths $m \nearrow -m^*(\beta)$, $m \searrow +m^*(\beta)$. The unique analytic continuation is given by the mean field free energy f_{MF}.*

That is: *after* the van der Waals limit, all the analyticity properties of the free energy are known explicitly. There exists no formula for f_γ when $\gamma > 0$, and it was not shown, until the papers of Cassandro and Presutti [CP] and Bovier and Zahradník [BZ1], that the system exhibits a first order phase transition *before* reaching the mean field regime: for all $\beta > 1$, the graph of $f_\gamma(m)$ already has a plateau $[-m^*(\beta,\gamma), +m^*(\beta,\gamma)]$ when γ is small enough. In this sense, one can say that mean field, together with the Maxwell construction, is a good approximation to long but finite range interactions (and vice versa). Our purpose is to show that from the point of view of analyticity, the situation is very different.

1.2 Obstruction for $\gamma > 0$; Main Results

Our results hold for Kac potentials for which Lemmas 2.1 and 2.8 hold, but we believe them to be true for any ferromagnetic potential satisfying (1.6). For the sake of simplicity, we focus on a particular potential, i.e. on the step function

$$J(x) := 2^{-d} 1_{\|x\| \leq 1}(x) . \tag{1.14}$$

In this setting, our main result for the free energy density is the following:

Theorem 1.2. *There exists β_0 and $\gamma_0 > 0$ such that for all $\beta \geq \beta_0$, $\gamma \in (0, \gamma_0)$, f_γ is analytic everywhere except at $\pm m^*(\beta,\gamma)$, but has no analytic continuation along the paths $m \nearrow -m^*(\beta,\gamma)$, $m \searrow +m^*(\beta,\gamma)$.*

This result is in favor of the original ideas of Fisher and Langer, saying that *finiteness of the range of interaction is responsible for absence of analytic continuation.* In particular it excludes the possibility of obtaining the free energy by a Maxwell construction: when $\gamma > 0$ the phases $+$ and $-$ cannot be joined analytically.

The proof of Theorem 1.2 will be done by working in the more appropriate grand canonical ensemble (in the lattice gas terminology), in which the constraint on the magnetization is replaced by a magnetic field. Let

$$Z(\Lambda) = \sum_{\sigma \in \Omega_\Lambda} \exp\big(-\beta H_\Lambda^h(\sigma)\big) . \tag{1.15}$$

Define the pressure density by

$$p_\gamma(h) := \lim_{\Lambda \nearrow \mathbf{Z}^d} p_{\gamma,\Lambda}(h) , \text{ where } p_{\gamma,\Lambda}(h) = \frac{1}{\beta|\Lambda|} \log Z(\Lambda) . \tag{1.16}$$

The free energy and pressure densities are related by a Legendre transform:

$$f_\gamma(m) = \sup_{h \in \mathbb{R}} (hm - p_\gamma(h)) . \tag{1.17}$$

See for instance [Pr] for a proof of this property. The analytic properties of f_γ at $\pm m^*(\beta,\gamma)$ will be obtained from those of p_γ at $h=0$. By the Theorem of Yang and Lee [YL], p_γ is analytic outside the imaginary axis. The main result of the paper is the following characterization of the analyticity properties of the pressure at $h=0$.

Theorem 1.3. *There exists β_0, $\gamma_0 > 0$ and a constant $C_r > 0$ such that for all $\beta \geq \beta_0$, $\gamma \in (0,\gamma_0)$, the following holds:*

1) The directional derivatives $p_\gamma^{(k),\leftarrow}(0)$ exist for all $k \in \mathbb{N}$, i.e. p_γ is C^∞ at $h=0$. Moreover, there exists a constant $C_+ > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{0 \leq \mathrm{Re}\, h \leq \epsilon} |p_\gamma^{(k),\leftarrow}(h)| \leq \bigl(C_+ \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\bigr)^k k!^{\frac{d}{d-1}} + C_r^k k! \,. \tag{1.18}$$

2) The pressure has no analytic continuation at $h=0$. More precisely, there exists $C_- > 0$ and an unbounded increasing sequence of integers $k_1, k_2, \dots$ such that for all $k \in \{k_1, k_2, \dots\}$,

$$|p_\gamma^{(k),\leftarrow}(0)| \geq \bigl(C_- \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\bigr)^k k!^{\frac{d}{d-1}} - C_r^k k! \,. \tag{1.19}$$

The lower bound (1.19) becomes irrelevant when $\gamma \searrow 0$. Moreover, we should mention that each integer k_i depends on γ and β, with $\lim_{\gamma \searrow 0} k_i = +\infty$: information about non-analyticity is lost in the van der Waals limit. Since we know from the Lebowitz-Penrose Theorem that p_γ converges, when $\gamma \searrow 0$, to a function that is is analytic at $h=0$, it is worthwhile considering the low order derivatives of p_γ. Considering the upper bound (1.18), it easy to show the

Corollary 1.2. *There exists $C = C(\beta)$ such that for small values of k, i.e. for $k \leq \gamma^{-d}$, we have the upper bound*

$$\sup_{0 \leq \mathrm{Re}\, h \leq \epsilon} |p_\gamma^{(k),\leftarrow}(h)| \leq C^k k! \,. \tag{1.20}$$

This shows that a close inspection of the derivatives of the pressure allows to detect how analyticity starts to manifest when γ approaches 0. These different behaviours are illustrated on Figure 3.

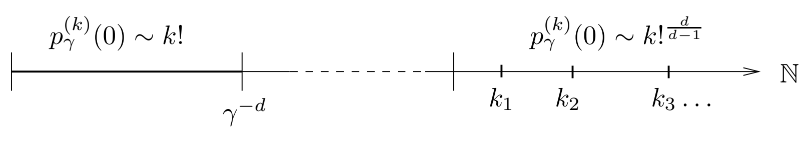

Figure 3: The derivatives of the pressure at $h=0$, when $\gamma > 0$. The first ones ($k \leq \gamma^{-d}$) behave like those of an analytic function, but non-analyticity always dominates for large k.

To show Theorem 1.3, we first construct the phase diagram of the Kac model with a complex magnetic field, at low temperatures, γ small. Then, we adapt the technique of Isakov to obtain lower bounds on the derivatives of the pressure in a finite volume. These two essential steps deserve a few comments.

1. Phase diagrams of lattice systems can be studied in the general framework of Pirogov-Sinai Theory ([PS], [Z1]), which applies when the system under consideration has a finite number of ground states, and for which the unperturbed hamiltonian satisfies the Peierls condition. In our case, the Kac potential has two ground states which are the pure $+$ and pure $-$ configurations, but the Peierls constant (computed with respect to these two ground states) goes to zero when $\gamma \searrow 0$ since in the van der Waals limit, the interaction between two arbitrary spins vanishes. Therefore, a direct application of Pirogov-Sinai Theory would lead to a range of temperature shrinking to zero in the van der Waals limit.
   We will use a technique useful for the study of spin systems with long but finite range interactions, invented recently by Bovier and Zahradník [BZ2]. Their technique allows to study, for instance, the Kac model with a magnetic field, in a range of temperature that is *uniform in* γ. In their approach, the ground states of Pirogov-Sinai Theory are replaced by *restricted phases*, i.e. by *sets* of configurations. In the $+$-restricted phase, for example, all the points are $+$-correct, i.e. their γ^{-1}-neighbourhood contains a majority of spins $+$. When a point is in neither of the restricted phases, it is in the support of a contour Γ, and it can then be shown that the contours defined in this way satisfy the Peierls condition with a Peierls constant ρ that is *uniform* in γ: $\|\Gamma\| \geq \rho|\Gamma|$ where $\|\Gamma\|$ is the surface energy of Γ. In Section 3 we show that a polymer representation can be obtained for the restricted phases, and that their corresponding free energies behave analytically at $h = 0$. The full phase diagram is then completed in Section 4: we give precise domains in which the partition function can be exponentiated. These domains are made optimal by introducing special isoperimetric constants associated to contours (see the discussion hereafter, and (2.44)). Complications arise from the fact that polymers of the restricted phases induce interactions among contours. Besides the definition of the restricted ensembles, our analysis of the phase diagram is independent of the paper [BZ2]. In a different setting, restricted ensembles were also studied in [BS], [DS], [BKL], and [LMP].

2. To implement the mechanism used by Isakov, we consider the pressure $p^+_{\gamma,\Lambda}$ in a finite box Λ, with a pure $+$-boundary condition. By introducing an order among the contours inside Λ, the pressure can

be written as a *finite* sum:

$$p^{+}_{\gamma,\Lambda} = \frac{1}{\beta|\Lambda|}\log Z^{+}_{r}(\Lambda) + \frac{1}{\beta|\Lambda|}\sum_{\Gamma\in\mathcal{C}^{+}(\Lambda)} u^{+}_{\Lambda}(\Gamma)\,, \tag{1.21}$$

where $Z^{+}_{r}(\Lambda)$ is the restricted partition function and $\mathcal{C}^{+}(\Lambda)$ is the family of all contours of type $+$ in Λ. With the analysis of Sections 3 and 4, the derivatives of the functions $u^{+}_{\Lambda}(\Gamma)$ can be estimated using a stationary phase analysis. When Λ is sufficiently large, the contributions to $p^{+(k)}_{\gamma,\Lambda}(0)$ are the following: since it is analytic, the restricted phase contributes a factor $C_r^k k!$. Then, a class of contours called k-large gives a contribution of order $k!^{\frac{d}{d-1}}$. The rest of the contours is shown to have a negligible contribution in comparison of the k-large ones. This gives a lower bound

$$|p^{+(k)}_{\gamma,\Lambda}(0)| \geq \bigl(C_{-}\gamma^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\bigr)^{k} k!^{\frac{d}{d-1}} - C_r^k k!\,. \tag{1.22}$$

In the last step of the proof we show that $\lim_{\Lambda} p^{+,(k)}_{\gamma,\Lambda}(0) = p^{(k),\leftarrow}_{\gamma}(0)$, and so (1.22) extends to the thermodynamic limit $\Lambda \nearrow \mathbf{Z}^d$, which gives (1.19).

Before going further, we make an important remark. In [I1], Isakov proved Theorem 1.3 for the Ising model. An attempt was then made, in a second paper [I2], to extend the method to any two phase model for which the Peierls condition holds. Unfortunately, this extension could only be done under two additional assumptions which we briefly describe. Associate to each phase a discrete isoperimetric problem of the following type: let $V(\Gamma)$ denote the volume of the contour Γ (of a given type) and $\|\Gamma\|$ its surface energy. For $N\in\mathbb{N}$, consider the problem:

$$\mathscr{P}(N)\quad \begin{cases} \text{Find the best constant } C(N) \text{ such that } \frac{V(\Gamma)}{\|\Gamma\|} \leq C(N)V(\Gamma)^{\frac{1}{d}} \\ \text{for all contour } \Gamma \text{ with } V(\Gamma)\leq N\,. \end{cases}$$

The assumptions of Isakov are then that in the limit $N\to\infty$, 1) the asymptotic behaviour of the constant $C(N)$ is the *same* for the two phases, 2) there exist maximizers of arbitrary large size.
Clearly, these assumptions are satisfied by the Ising model, for which $\|\Gamma\| = |\Gamma|$ (the number of dual bonds on the dual lattice) and the maximizers are always given by cubes, i.e. $C(N)=(2d)^{-1}$ for all N. But for a model with no symmetry or with interactions that are more complicated than nearest neighbours, these assumptions can be very hard to check. The problem comes from the fact that the surface energy $\|\Gamma\|$ depends on the detailed structure of the hamiltonian. In our case, symmetry reduces the difficulty to the existence of large maximizers. We will see that the

construction of the phase diagram can be done when the isoperimetric problem is formulated as follows:

$$\mathscr{P}'(N) \begin{cases} \text{Find the best constant } K(N) \text{ such that } \frac{V(\Gamma)}{\|\Gamma\|} \leq K(N)V(\Gamma)^{\frac{1}{d}} \\ \text{for all contour } \Gamma \text{ with } V(\Gamma) \geq N\,. \end{cases}$$

By formulating the problem in this way, the existence of large maximizers is immediate, and we avoid the necessity of solving the isoperimetric problem explicitly.
It was actually shown in [FP] that the two assumptions of Isakov can be swept out, and that the result of [I2] can be extended to the whole class of two phase models treated generally in Pirogov-Sinai theory, the only necessary ingredient for non-analyticity being the Peierls condition. The general theorem of [FP] applies to the Kac model but with some restriction $\beta \geq \beta_0(\gamma)$ where $\beta_0(\gamma)$ diverges when $\gamma \searrow 0$. In the present paper we study the van der Waals limit at fixed β.

The description of the model in terms of contours and the verification of the Peierls condition for $\|\Gamma\|$ will be done in Section 2. Section 3 is entirely devoted to the study of restricted phases and to their analyticity properties, adapting the technique of [BZ2]. Section 4 is the construction of the phase diagram in the complex plane of the magnetic field. Section 5 contains the proofs of our main results, and Appendix A contains basic definitions for the cluster expansion technique.

Conventions: we will often use the norm $\|f\|_D := \sup_{z\in D}|f(z)|$. When G is a graph we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges.

Acknowledgments: We wish to thank Anton Bovier and Miloš Zahradník for many useful discussions concerning [BZ2], and Daniel Ueltschi for suggesting the method used in the proof of Corollary 3.2.

2 Contour Description

For the description of configurations in terms of contours, we use the notion of correct/incorrect point introduced by Bovier and Zahradník in [BZ2]. There are two major requirements for the way in which contours should be defined.

1. They are defined on a coarse-grained scale, and a Peierls condition must hold for the surface energy of each contour, with a Peierls constant that is *uniform* in γ. See Proposition 2.2.

2. Outside contours, a partial re-summation over configurations will lead to restricted phases. To obtain convergent expansions for these

phases, care must be taken in the definition of contours. See the parameter $\tilde{\delta}$ in (2.16).

Remark: In the study of Kac potentials, one finds in the literature another definition of contour. For instance in [CP] and [BZ1], contours are defined by comparing the local (empirical) magnetization to the mean field spontaneous magnetization $\pm m^*(\beta)$. This allows to study the system very close to the critical temperature, by using explicitly the mean field functionals. Unfortunately, this technique hasn't yet been extended to the study of the Kac model with a magnetic field. In our case, the local magnetization is always compared with ± 1 (rather than $\pm m^*(\beta)$), and we must therefore work at low temperature, not reaching the whole coexistence regime. Moreover, we need to introduce a *complex* magnetic field, which definitely rules out the possibility of using the standard techniques existing for Kac models.

2.1 Definition of Contours

We introduce some more notations. We have $d(x,\Lambda) = \inf\{d(x,y) : y \in \Lambda\}$. For $N \geq 1$, define the box $B_N(x) := \{y \in \mathbf{Z}^d : d(x,y) \leq N\}$, and $B_N^\bullet(x) := B_N(x) \backslash \{x\}$. The N-neighbourhood of Λ is

$$[\Lambda]_N := \bigcup_{x\in\Lambda} B_N(x)\,, \tag{2.1}$$

and the boundaries

$$\partial_N^+\Lambda = \{x \in \Lambda^c : d(x,\Lambda) \leq N\}\,, \tag{2.2}$$

$$\partial_N^-\Lambda = \{x \in \Lambda : d(x,\Lambda^c) \leq N\}\,. \tag{2.3}$$

A set Λ is N-connected if for all $x, y \in \Lambda$ there exists a sequence $x_1, x_2, \ldots, x_{n-1}, x_n$ with $x_1 = x$, $x_n = y$, $x_i \in \Lambda$, and $d(x_i, x_{i+1}) \leq N$. If $\sigma_\Lambda \in \Omega_\Lambda$, $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$, we define the concatenation $\sigma_\Lambda\eta_{\Lambda^c} \in \Omega$ in the usual way:

$$(\sigma_\Lambda\eta_{\Lambda^c})_i = \begin{cases} (\sigma_\Lambda)_i & \text{if } i \in \Lambda\,, \\ (\eta_{\Lambda^c})_i & \text{if } i \in \Lambda^c\,. \end{cases} \tag{2.4}$$

We often use the symbol # to denote either of the symbols + or −, or the constant configuration taking the value # at each site of $\mathbf{Z}^d$. We define

$$\phi_{ij}(\sigma_i,\sigma_j) := -\frac{1}{2}J_\gamma(i-j)(\sigma_i\sigma_j - 1)\,, \tag{2.5}$$

Let $\phi_{ij} := \phi_{ij}(+,-)$. The overall interaction strength is the upper bound on the energy of interaction of a single spin with the rest of the system, and equals

$$\sum_{j:j\neq i} \phi_{ij} = \sum_{j:j\neq i} J_\gamma(i-j) = 1\,. \tag{2.6}$$

Relevant functions for the study of nearly constant spin regions are the following (they will appear naturally later when reformulating the hamiltonian):

$$w_{ij}^{\#}(\sigma_i, \sigma_j) := \phi_{ij}(\sigma_i, \sigma_j) - \phi_{ij}(\#, \sigma_j) - \phi_{ij}(\sigma_i, \#) . \tag{2.7}$$

Notice that $w_{ij}^{\#}(\#, \sigma_j) = w_{ij}^{\#}(\sigma_i, \#) = 0$. Let $\delta \in (0, 1)$, $\sigma \in \Omega$. With regard to the step function J defined in (1.14), we define a point i to be $(\delta, +)$-correct for σ if

$$|B_R^{\bullet}(i) \cap \{j : \sigma_j = -1\}| \leq \tfrac{\delta}{2} |B_R(i)| . \tag{2.8}$$

That is, the R-neighbourhood of a $(\delta, +)$-correct point contains a majority of $+$ spins. Although we will always consider the step function, it is often easier to formulate proofs with the help of the functions $w_{ij}^{\#}$, since they will appear naturally later in the re-formulation of the hamiltonian. We thus define the notion of correct/incorrect point in the general case.

Definition 2.1. *Let $\delta \in (0, 1)$, $\sigma \in \Omega$, $i \in \mathbf{Z}^d$.*

1. *i is* $(\delta, +)$-correct for σ *if* $\sum_{j : j \neq i} |w_{ij}^{+}(-, \sigma_j)| \leq \delta$.
2. *i is* $(\delta, -)$-correct for σ *if* $\sum_{j : j \neq i} |w_{ij}^{-}(+, \sigma_j)| \leq \delta$.
3. *i is* δ-correct for σ *if it is either $(\delta, +)$- or $(\delta, -)$-correct for σ.*
4. *i is* δ-incorrect for σ *if it is not δ-correct.*

It is easy to see that this definition coincides with (2.8) when J is the step function.
The notion of correctness for a point i depends on the spins in the R-neighbourhood of i but neither on the value of σ_i, nor on the magnetic field. Notice that if $\delta = 0$ this notion of correct point essentially coincides with the one of Zahradník in [Z1]. We first show that when δ is small, regions of $(\delta, +)$- and $(\delta, -)$-correct points are distant. In particular, a point i cannot be at the same time $(\delta, +)$- and $(\delta, -)$-correct.

Lemma 2.1. *Let $\delta \in (0, 2^{-d})$, $\sigma \in \Omega$. Then*
1) If i is $(\delta, +)$-correct, the box $B_R(i)$ contains either $(\delta, +)$-correct, or δ-incorrect points (but no $(\delta, -)$-correct points).
2) If i is $(\delta, -)$-correct, the box $B_R(i)$ contains either $(\delta, -)$-correct, or δ-incorrect points (but no $(\delta, +)$-correct points).

Proof. Suppose i is $(\delta, +)$-correct for σ. Consider $j \in B_R(i)$ and compute

$$\sum_{k : k \neq j} |w_{jk}^{-}(+, \sigma_k)| = \sum_{\substack{k \in B_R^{\bullet}(j) \\ \sigma_k = +1}} 2\phi_{jk} \geq \sum_{\substack{k \in B_R^{\bullet}(j) \cap B_R^{\bullet}(i) \\ \sigma_k = +1}} 2\phi_{jk} . \tag{2.9}$$

Using the properties of the function $J(\cdot)$ [1], we can exchange j and i and write

$$\sum_{\substack{k\in B^{\bullet}_R(j)\cap B^{\bullet}_R(i)\\ \sigma_k=+1}} 2\phi_{jk} = \sum_{\substack{k\in B^{\bullet}_R(j)\cap B^{\bullet}_R(i)\\ \sigma_k=+1}} 2\phi_{ik} = \sum_{\substack{k\neq i\\ \sigma_k=+1}} 2\phi_{ik} - \sum_{\substack{k\notin B^{\bullet}_R(j)\cap B^{\bullet}_R(i)\\ \sigma_k=+1}} 2\phi_{ik} \tag{2.10}$$

Using (2.6) and $|B_R(j)\cap B_R(i)| \geq 2^{-d}|B_R(i)|$, this last sum can be bounded by

$$\sum_{\substack{k\notin B^{\bullet}_R(j)\cap B^{\bullet}_R(i)\\ \sigma_k=+1}} \phi_{ik} \leq \frac{2^d-1}{2^d}\,. \tag{2.11}$$

Then, since i is $(\delta,+)$-correct for σ,

$$\sum_{\substack{k\neq i\\ \sigma_k=+1}} 2\phi_{ik} = 2 - \sum_{\substack{k\neq i\\ \sigma_k=-1}} 2\phi_{ik} = 2 - \sum_{k:k\neq i} |w^+_{ik}(-,\sigma_k)| \geq 2-\delta\,. \tag{2.12}$$

We thus have the lower bound

$$\sum_{k:k\neq j} |w^-_{jk}(+,\sigma_k)| \geq 2-\delta-2\frac{2^d-1}{2^d} > \delta\,, \tag{2.13}$$

i.e. j cannot be $(\delta,-)$-correct for σ, which finishes the proof. $\square$

In the sequel we will always assume that $\delta\in(0,2^{-d})$ is fixed. The cleaned configuration $\overline{\sigma}\in\Omega$ is defined as follows:

$$\overline{\sigma}_i := \begin{cases} +1 & \text{if } i \text{ is } (\delta,+)\text{-correct for } \sigma\,,\\ -1 & \text{if } i \text{ is } (\delta,-)\text{-correct for } \sigma\,,\\ \sigma_i & \text{if } i \text{ is } \delta\text{-incorrect for } \sigma\,. \end{cases} \tag{2.14}$$

For any set $M\subset \mathbf{Z}^d$, we can always consider the partial cleaning $\sigma_M\overline{\sigma}_{M^c}$ which coincides with σ on M and with $\overline{\sigma}$ on M^c. In the sequel, the cleaning and partial cleaning are always done according to the *original* configuration σ, with a fixed δ. Notice that if a point i is, say, $(\delta,+)$-correct for σ, then the cleaning of σ has the only effect, in the box $B_R(i)$, of changing $-$ spins into $+$ spins (and not $+$ spins into $-$ spins). This is a consequence of Lemma 2.1. We denote by $I_\delta(\sigma)$ the set of δ-incorrect points of the configuration σ. The important property of the cleaning operation is stated in the following lemma.

Lemma 2.2. *Let $M_1\subset M_2$, $\delta'\in(0,\delta]$. Then $I_{\delta'}(\sigma_{M_1}\overline{\sigma}_{M_1^c})\subset I_{\delta'}(\sigma_{M_2}\overline{\sigma}_{M_2^c})$.*

[1] At this point we use the particularity of the step function: ϕ_{jk} is constant on the intersection $B^{\bullet}_R(j)\cap B^{\bullet}_R(i)$.

Proof. Let i be a $(\delta', +)$-correct point of $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. Using the fact that $\sigma_{M_1}\overline{\sigma}_{M_1^c}$ and $\sigma_{M_2}\overline{\sigma}_{M_2^c}$ coincide on M_1 and M_2^c, we decompose

$$\sum_{k:k\neq i} |w^+_{ik}(-, (\sigma_{M_1}\overline{\sigma}_{M_1^c})_k)| = \sum_{\substack{k:k\neq i \\ k\in M_1\cup M_2^c}} |w^+_{ik}(-, (\sigma_{M_2}\overline{\sigma}_{M_2^c})_k)| + \sum_{\substack{k:k\neq i \\ k\in M_2\setminus M_1}} |w^+_{ik}(-, \overline{\sigma}_k)|$$

There are at most three possibilities for a point k of the last sum. 1) If k is $(\delta, +)$-correct for σ then $\overline{\sigma}_k = +1$ and so $|w^+_{ik}(-, \overline{\sigma}_k)| = 0$. 2) If k is δ-incorrect for σ then $\overline{\sigma}_k = \sigma_k = (\sigma_{M_2}\overline{\sigma}_{M_2^c})_k$. 3) If k is $(\delta, -)$-correct for σ then it is also $(\delta, -)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. By Lemma 2.1, i is not $(\delta, +)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. This is a contradiction with the fact that i is $(\delta', +)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$, so there are no such k.
We can then bound the whole sum by δ'. This shows that i is $(\delta', +)$-correct for $\sigma_{M_1}\overline{\sigma}_{M_1^c}$, and finishes the proof. $\square$

Contours are defined on a coarse-grained scale. Consider the partition of $\mathbf{Z}^d$ into disjoint cubes $C^{(l)}$ of side length $l \in \mathbb{N}$, $l > 2R$, whose centers lie on the sites of a square sub-lattice of $\mathbf{Z}^d$. We denote by $C_i^{(l)}$ the unique box of this partition containing the site $i \in \mathbf{Z}^d$. $\mathcal{C}^{(l)}$ will denote the family of all subsets of $\mathbf{Z}^d$ that are unions of boxes $C^{(l)}$. For any set $A \subset \mathbf{Z}^d$, consider the thickening (compare with (2.1))

$$\{A\}_l := \bigcup_{i\in A} C_i^{(l)} . \tag{2.15}$$

In the sequel we always consider l such that $l = \nu R$, with $\nu > 2$.
We will need to decouple contours from the rest of the system. Since interactions are of arbitrary large finite range, we follow [BZ2] and introduce a second parameter $\tilde{\delta} \in (0, \delta)$. This new parameter is crucial; its importance will be seen later, for instance in the proof of the analyticity of the restricted phases. For each $\sigma \in \Omega$ with $|I_{\tilde{\delta}}(\sigma)| < \infty$, consider the following set:

$$\mathcal{E}(\sigma) := \left\{M \in \mathcal{C}^{(l)} : M \supset [I_\delta(\sigma)]_R,\ M \supset [I_{\tilde{\delta}}(\sigma_M\overline{\sigma}_{M^c})]_R\right\} . \tag{2.16}$$

First we show that $\mathcal{E}(\sigma)$ is not empty. Consider $M_0 := \{[I_{\tilde{\delta}}(\sigma)]_R\}_l$. If $M_0 = \emptyset$ then $I_{\tilde{\delta}}(\sigma) = I_\delta(\sigma) = \emptyset$ and any subset of $\mathbf{Z}^d$ is in $\mathcal{E}(\sigma)$. So we assume $M_0 \neq \emptyset$. This gives $\mathcal{E}(\sigma) \neq \emptyset$ since $M_0 \in \mathcal{C}^{(l)}$, $M_0 \supset [I_{\tilde{\delta}}(\sigma)]_R \supset [I_\delta(\sigma)]_R$ and $M_0 \supset [I_{\tilde{\delta}}(\sigma)]_R \supset [I_{\tilde{\delta}}(\sigma_{M_0}\overline{\sigma}_{M_0^c})]_R$ by Lemma 2.2. We then show that $\mathcal{E}(\sigma)$ is stable by intersection. Suppose $A, B \in \mathcal{E}(\sigma)$. Then clearly $A \cap B \supset [I_\delta(\sigma)]_R$ and using again Lemma 2.2,

$$A \supset [I_{\tilde{\delta}}(\sigma_A\overline{\sigma}_{A^c})]_R \supset [I_{\tilde{\delta}}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R , \tag{2.17}$$

$$B \supset [I_{\tilde{\delta}}(\sigma_B\overline{\sigma}_{B^c})]_R \supset [I_{\tilde{\delta}}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R , \tag{2.18}$$

which implies $A \cap B \in \mathcal{E}(\sigma)$. The following set is thus well defined, and is the candidate for describing the contours of the configuration σ:

$$I^*(\sigma) := \bigcap_{M \in \mathcal{E}(\sigma)} M \,. \tag{2.19}$$

By construction, $I^*(\sigma)$ is the smallest element of $\mathcal{E}(\sigma)$. A first important property of $I^*(\sigma)$ is the following, which will be essential to obtain the Peierls bound on the surface energy of contours.

Lemma 2.3. *There exists, in the $2R$-neighbourhood of each box $C^{(l)} \subset I^*(\sigma)$, a point $j \in I^*(\sigma)$ which is $\tilde{\delta}$-incorrect for the configuration $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$.*

Proof. Let $C^{(l)} \subset I^*(\sigma)$. First, suppose $I_\delta(\sigma) \cap [C^{(l)}]_{2R} \neq \emptyset$. Then each $j \in I_\delta(\sigma) \cap [C^{(l)}]_{2R}$ is δ-incorrect for σ, and hence $\tilde{\delta}$-incorrect for $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$, since $\tilde{\delta} < \delta$ and σ and $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$ coincide on $B_R(j)$.
Suppose there exists a box $C^{(l)}$ such that [2] $[I_\delta(\sigma)]_R \cap [C^{(l)}]_R = \emptyset$. If $I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) \cap [C^{(l)}]_{2R} = \emptyset$, i.e. $[I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \cap [C^{(l)}]_R = \emptyset$, then we define $I' := I^* \backslash C^{(l)}$ and show that $I' \in \mathcal{E}(\sigma)$, a contradiction with the definition of I^*. First, $I' \supset [I_\delta(\sigma)]_R$. Using Lemma 2.2, $I^* \supset [I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \supset [I_{\tilde{\delta}}(\sigma_{I'}\overline{\sigma}_{I'^c})]_R$. Since we have $[I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \cap [C^{(l)}]_R = \emptyset$, this implies $I' \supset [I_{\tilde{\delta}}(\sigma_{I'}\overline{\sigma}_{I'^c})]_R$, i.e. $I' \in \mathcal{E}(\sigma)$. □

When studying restricted phases, we will need to re-sum over the set of configurations that have the same set of contours, that is to consider, for a fixed σ (we assume $I^*(\sigma) \neq \emptyset$),

$$\mathcal{A}(\sigma) := \big\{\sigma' : \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\, I^*(\sigma') = I^*(\sigma)\big\} \,. \tag{2.20}$$

It is important to have an *explicit* characterization of the set $\mathcal{A}(\sigma)$. Let $\Lambda^\#(\sigma)$ denote the set of points of $I^*(\sigma)^c$ that are $(\delta, \#)$-correct for σ. By Lemma 2.1 we have $d(\Lambda^+(\sigma), \Lambda^-(\sigma)) > l$, and we have the partition

$$\mathbf{Z}^d = I^*(\sigma) \cup \Lambda^+(\sigma) \cup \Lambda^-(\sigma) \,. \tag{2.21}$$

We now show that the set $\mathcal{A}(\sigma)$ can be characterized explicitly by

$$\mathcal{D}(\sigma) := \big\{\sigma' : \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\ \text{each } i \in [\Lambda^\#(\sigma)]_R \text{ is } (\delta, \#)\text{-correct for } \sigma'\big\} \,.$$

Proposition 2.1. *If $I^*(\sigma) \neq \emptyset$, then $\mathcal{A}(\sigma) = \mathcal{D}(\sigma)$.*

Proof. 1) Assume $\sigma' \in \mathcal{A}(\sigma)$. Then $I^* \equiv I^*(\sigma) = I^*(\sigma') \supset [I_\delta(\sigma')]_R$, so that each $i \in [I^{*c}]_R$ is δ-correct for σ'. Let A be a maximal connected component of $[I^{*c}]_R$. There exists $i \in A$ such that $i \in I^*$, since we assumed $I^* \neq \emptyset$. By Lemma 2.1, it suffices to show that i is $(\delta, +)$-correct for σ if

[2] Here we use the fact that $A \cap [B]_{2R} = \emptyset$ if and only if $[A]_R \cap [B]_R = \emptyset$.

and only if it is $(\delta,+)$-correct for σ'. Assume this is not the case, e.g. suppose i is $(\delta,+)$-correct for σ and $(\delta,-)$-correct for σ'. That is,

$$\sum_{j\neq i} |\omega_{ij}^{+}(-,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| = \sum_{j\in B_R^{\bullet}(i)\cap I^*} |w_{ij}^{+}(-,\sigma_j)| \le \tilde{\delta}\,, \tag{2.22}$$

$$\sum_{j\neq i} |\omega_{ij}^{-}(+,(\sigma'_{I^*}\overline{\sigma}'_{I^{*c}})_j)| = \sum_{j\in B_R^{\bullet}(i)\cap I^*} |w_{ij}^{-}(+,\sigma_j)| \le \tilde{\delta}\,. \tag{2.23}$$

Since $i\in I^*$ we have [3]

$$\sum_{j\in B_R^{\bullet}(i)\cap I^{*c}} |w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \le \sum_{j\in B_R^{\bullet}(i)\cap I^{*c}} |w_{ij}^{-}(+,+)| \le 2(1-2^{-d})\,.$$

Therefore we get a contradiction, since,

$$\begin{aligned} 2 &= \sum_{j\neq i} |w_{ij}^{+}(-,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| + |w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \\ &\le 2\tilde{\delta} + 2\sum_{j\in B_R^{\bullet}(i)\cap I^{*c}} |w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \le 2\tilde{\delta}+2(1-2^{-d}) < 2\,, \end{aligned} \tag{2.24}$$

where we used the fact that $\tilde{\delta}<\delta<2^{-d}$.
2) Suppose $\sigma'\in\mathcal{D}(\sigma)$. Since σ' coincides with σ on $I^*(\sigma)$ and all points of $[I^*(\sigma)^c]_R$ are δ-correct for σ', we have $I_\delta(\sigma') = I_\delta(\sigma)$. This gives $I^*(\sigma)\supset [I_\delta(\sigma)]_R = [I_\delta(\sigma')]_R$. Then, since $\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c} = {\sigma'}_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c}$, we have $I^*(\sigma)\supset [I_{\tilde{\delta}}(\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c})]_R = [I_{\tilde{\delta}}({\sigma'}_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c})]_R$. This implies $I^*(\sigma)\in\mathcal{E}(\sigma')$, i.e. $I^*(\sigma')\subset I^*(\sigma)$. Assume $I^*(\sigma)\backslash I^*(\sigma')\neq\emptyset$. Using the fact that σ and σ' coincide on $I^*(\sigma)\backslash I^*(\sigma')$, we have $\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c} = \sigma'_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c}$. This gives, like before, $I^*(\sigma')\supset [I_{\tilde{\delta}}({\sigma'}_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c})]_R = [I_{\tilde{\delta}}(\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c})]_R$. With $I^*(\sigma')\supset [I_\delta(\sigma')]_R = [I_\delta(\sigma)]_R$, this implies $I^*(\sigma')\in\mathcal{E}(\sigma)$, i.e. $I^*(\sigma')\supset I^*(\sigma)$. So $\sigma'\in\mathcal{A}(\sigma)$. □

In particular, Proposition 2.1 implies that $\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c}$ is an element of $\mathcal{A}(\sigma)$.

Definition 2.2. *The connected components of $I^*(\sigma)$ form the support of the* contours *of the configuration σ, and are written $\operatorname{supp}\Gamma_1,\ldots,\operatorname{supp}\Gamma_n$. A* contour *is thus a couple $\Gamma = (\operatorname{supp}\Gamma,\sigma_\Gamma)$, where σ_Γ is the restriction of σ to Γ.*
A family of contours $\{\Gamma_1,\ldots,\Gamma_n\}$ is admissible *if there exists a configuration σ such that $\{\Gamma_1,\ldots,\Gamma_n\}$ are the contours of σ* [4].

[3]Here we use a property of the step function, but this can be done for any Kac potential whose function J has the symmetry $J(x)=J(y)$ when $\|x\|=\|y\|$.

[4]Note that the configuration σ is not unique, unlike in the usual situation treated in Pirogov-Sinai Theory.

The fact that the contours are defined on a coarse-grained scale will be crucial when dealing with their entropy, which we must control uniformly in γ. Notice that two (distinct) contours are at distance at least l from each other. We will usually denote $\operatorname{supp}\Gamma$ also by Γ. Contours should always be considered together with their type and labels, which we are about to define. The following topological property is needed for the definition of labels.

Lemma 2.4. *Fix $R \geq 1$. Let $B \subset \mathbf{Z}^d$ be R-connected and bounded. Then $\partial_R^+ A$ and $\partial_R^- A$ are R-connected, where A is any maximal R-connected component of $B^c = \mathbf{Z}^d \backslash B$.*

Proof. Let A be any maximal R-connected component of B^c. Then A^c is R-connected. Indeed, let $x, y \in A^c$, and consider a path $x_1 = x, x_2, \dots, x_n = y$, $d(x_i, x_{i+1}) \leq R$. If $x_i \in A^c$ for all i there is nothing to show. So suppose there exists $1 \leq i_- \leq i_+ \leq n$ such that $\{x_1, \dots, x_{i_- -1}, x_{i_-}\} \subset A^c$, $x_{i_-+1} \in A$, $x_{i_+-1} \in A$, $\{x_{i_+}, x_{i_++1}, \dots, x_n\} \subset A^c$. Since A is maximal, we have $x_{i_-} \in B$, $x_{i_+} \in B$, and we can find a path from x_{i_-} to x_{i_+} entirely contained in B, i.e. in A^c.
We then show that $\partial_1^+ A$ is R-connected. Fix $\epsilon > 0$ and consider the sets

$$X = \left\{ x \in \mathbb{R}^d : d(x, A) \leq \tfrac{R}{2} + \epsilon \right\}, \tag{2.25}$$

$$Y = \left\{ y \in \mathbb{R}^d : d(y, A^c) \leq \tfrac{R}{2} + \epsilon \right\}. \tag{2.26}$$

Then X, Y are closed arc-wise connected subsets of $\mathbb{R}^d$, and $X \cup Y = \mathbb{R}^d$. By a Theorem of Kuratowski, $X \cap Y$ is arc-wise connected [5]. Let $\epsilon' > 0$ and consider $x, y \in \partial_1^+ A$, together with $\tilde{x}, \tilde{y} \in X \cap Y$ such that $d(x, \tilde{x}) < \frac{1}{2}$, $d(y, \tilde{y}) < \frac{1}{2}$. Then consider any sequence $\tilde{x}_1 = \tilde{x}, \dots, \tilde{x}_n = \tilde{y}$, $\tilde{x}_i \in X \cap Y$, $d(\tilde{x}_i, \tilde{x}_{i+1}) \leq \epsilon'$. For each i we have $d(\tilde{x}_i, A) \leq \frac{R}{2} + \epsilon$, $d(\tilde{x}_i, A^c) \leq \frac{R}{2} + \epsilon$. This implies that each box $B_{\frac{R}{2}+\epsilon}(\tilde{x}_i)$ contains at least one element $x'_i \in \partial_1^+ A$, i.e. $d(\tilde{x}_i, x'_i) \leq \frac{R}{2} + \epsilon$. We have

$$d(x'_i, x'_{i+1}) \leq d(x'_i, \tilde{x}_i) + d(\tilde{x}_i, \tilde{x}_{i+1}) + d(\tilde{x}_{i+1}, x'_{i+1}) \leq R + 2\epsilon + \epsilon'. \tag{2.27}$$

If $2\epsilon + \epsilon' < \frac{1}{2}$, this shows that $\partial_1^+ A$ is R-connected, which implies that $\partial_R^+ A$ is R-connected. The same proof holds when $\partial_R^+ A$ is replaced by $\partial_R^- A$. $\square$

Let Γ be a contour of σ, A a maximal R-connected component of $(\operatorname{supp}\Gamma)^c$. Let $i \in \partial_R^- A$. By definition, i is $(\delta, \#)$-correct for σ for some $\# \in \{\pm 1\}$. By Lemmas 2.4 and 2.1, each $i' \in \partial_R^- A$ is $(\delta, \#)$-correct for σ for the *same* value $\#$. We call $\#$ the label of the component A. There exists a unique unbounded component of Γ^c. The label of this component is called the type of the contour Γ. Let Γ be of type $+$ (resp. $-$). The union of all components of Γ^c with label $-$ (resp. $+$) is called the interior of Γ,

[5] This property of $\mathbb{R}^d$ is called *unicoherence*. See [Ku], vol. 2, Theorem 9 of Chapter 57.I, and Theorem 2 of Chapter 57.II.

and is denoted intΓ. Notice that there is only one type of interior. We define $V(\Gamma) := |\text{int}\Gamma|$. The union of the remaining components is called the exterior of Γ, and is denoted by extΓ. A contour is external if it is not contained in the interior of another contour.
Let Γ be a contour of some configuration σ. Assume Γ is of type $+$. Consider the configuration $\sigma[\Gamma]$, which coincides with σ_Γ on the support of Γ, and which equals $+1$ on extΓ, -1 on intΓ. Using Proposition 2.1, it is easy to see that $\sigma[\Gamma]$ has a single contour, which is exactly Γ. This can be generalized to a family of external contours of the same type, as in the second part of the following lemma.

Lemma 2.5. *External contours have the following properties:*
1) External contours of an admissible family have the same type.
2) Let $\{\Gamma_1, \ldots, \Gamma_n\}$ be a family of external contours, all of the same type. Then $\{\Gamma_1, \ldots, \Gamma_n\}$ is admissible if and only if $d(\Gamma_i, \Gamma_j) > l$ for all $i \neq j$.

Proof. The first statement follows easily from Lemma 2.4. For the second, we can assume that the contours are of type $+$. If $\{\Gamma_1, \ldots, \Gamma_n\}$ is admissible, then by construction the Γ_i are at distance at least l. Then, assume $d(\Gamma_i, \Gamma_j) > l$ for all $i \neq j$. Consider the configuration $\sigma[\Gamma_1, \ldots, \Gamma_n]$, which coincides with σ_{Γ_i} on the support of Γ_i, which equals $+1$ on $\bigcap_i \text{ext}\Gamma_i$ and -1 on $\bigcup_i \text{int}\Gamma_i$. Then the contours of $\sigma[\Gamma_1, \ldots, \Gamma_n]$ are given by $\{\Gamma_1, \ldots, \Gamma_n\}$. □

2.2 Re-formulation of the Hamiltonian

Consider a finite volume $\Lambda \in \mathcal{C}^{(l)}$ with the pure $+$-boundary condition $+_{\Lambda^c} \in \Omega_{\Lambda^c}$. Let $\sigma_\Lambda \in \Omega_\Lambda$. We set $\sigma := \sigma_\Lambda +_{\Lambda^c}$. The hamiltonian with boundary condition $+_{\Lambda^c}$ is defined by

$$H_\Lambda(\sigma) = H_\Lambda(\sigma_\Lambda +_{\Lambda^c}) = \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\neq j}} \phi_{ij}(\sigma_i, \sigma_j) + \sum_{i\in\Lambda} u(\sigma_i)\,, \tag{2.28}$$

where $u(\sigma_i) = -h\sigma_i$, $h \in \mathbb{R}$. Since we work in a finite volume, we will from now on identify $I^*(\sigma)$ with $I^*(\sigma) \cap \Lambda$ and $\Lambda^\pm(\sigma)$ with $\Lambda^\pm(\sigma) \cap \Lambda$. The following lemma shows how the hamiltonian can be written in such a way that spins in correct regions interact via the functions $w_{ij}^{\#}$ and are subject to an effective external field $U^{\#}$.

Lemma 2.6. *Define the potential $U^{\#}(\sigma_i) := u(\sigma_i) + \sum_{j:j\neq i} \phi_{ij}(\sigma_i, \#)$. Suppose σ_Λ is such that $I^*(\sigma) \cap \partial_R^- \Lambda = \emptyset$. Then*

$$H_\Lambda(\sigma) = H_{I^*}(\sigma_{I^*} \overline{\sigma}_{I^{*c}}) + \sum_{\#} \Big(\sum_{\substack{\{i,j\}\cap\Lambda^{\#}\neq\emptyset \\ i\neq j}} w_{ij}^{\#}(\sigma_i, \sigma_j) + \sum_{i\in\Lambda^{\#}} U^{\#}(\sigma_i) \Big)\,. \tag{2.29}$$

Proof. The proof is a simple rearrangement of the terms. Consider a pair $\{i,j\}$ appearing in $H_\Lambda(\sigma)$. Since $d(\Lambda^+,\Lambda^-)>R$ we have a certain number of cases to consider: 1) $\{i,j\}\subset\Lambda^+$. In this case, write

$$\phi_{ij}(\sigma_i,\sigma_j)=w_{ij}^+(\sigma_i,\sigma_j)+\phi_{ij}(\sigma_i,+)+\phi_{ij}(+,\sigma_j)\,. \tag{2.30}$$

The second term contributes to $U^+(\sigma_i)$, the third to $U^+(\sigma_j)$. 2) $i\in\Lambda^+$, $j\in I^*$. In this case the third term contributes to $H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})$. 3) $i\in\Lambda^+$, $j\in\Lambda^c$; in this case, $\phi_{ij}(+,\sigma_j)=0$. The other cases are similar. Notice that the case $i\in\Lambda^-$, $j\in\Lambda^c$ never occurs since points of $\partial_R^-\Lambda$ can only be $(\delta,+)$-correct. $\square$

2.3 Peierls Condition and Isoperimetric Constants

We take a closer look at the term H_{I^*}. Remember that contours are maximal R-connected components of I^*. For each contour Γ, $\sigma[\Gamma]$ and $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$ coincide on $[I^*]_R$. Since $d(\Gamma,\Gamma')>l$, we can decompose

$$H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})=\sum_\Gamma H_\Gamma(\sigma[\Gamma]) \tag{2.31}$$

$$=\sum_\Gamma\Big(\|\Gamma\|+\sum_{i\in\Gamma}u(\sigma[\Gamma]_i)\Big)\,, \tag{2.32}$$

where the sum is over contours of the configuration σ (contained in Λ), and where the surface energy is defined as

$$\|\Gamma\|:=\sum_{\substack{\{i,j\}\cap\Gamma\neq\emptyset\\ i\neq j}}\phi_{ij}(\sigma[\Gamma]_i,\sigma[\Gamma]_j)\,. \tag{2.33}$$

The central result of this section is the following.

Proposition 2.2. *The surface energy satisfies the* Peierls condition*, i.e. there exists $\rho=\rho(\tilde{\delta},\nu)>0$ such that for all contour Γ,*

$$\|\Gamma\|\geq\rho|\Gamma|\,. \tag{2.34}$$

The constant ρ is independent of γ and is called the Peierls constant*.*

Remark: $|\Gamma|$ denotes the total number of lattice sites contained in the support of Γ; in the litterature, it often denotes the number of blocks $C^{(l)}$ contained in Γ. In the latter case, the Peierls condition becomes $\|\Gamma\|\geq\rho'\gamma^{-d}|\Gamma|$ (with a different constant ρ'), and $\beta\gamma^{-d}$ is interpreted as an effective temperature for the system on the coarse-grained scale γ^{-1}.

We will need two lemmas. The first is purely geometric.

Lemma 2.7. *For any finite set $A\subset\mathbf{Z}^d$ and for all $R_0\in\mathbb{N}$, there exists $A_0\subset A$, called an R_0-*approximant *of A, such that $A\subset[A_0]_{R_0}$ and $d(x,y)>R_0$ for all $x,y\in A_0$, $x\neq y$.*

The second lemma is a property of the Kac potential. In [BZ2], this property was called "continuity" for obvious reasons.

Lemma 2.8. *Let $\sigma \in \Omega$, $i \in \mathbf{Z}^d$, $\# \in \{\pm\}$. Define*

$$V_\sigma(i;\#) := \sum_{j:j\neq i} \phi_{ij}(\#,\sigma_j)\,. \tag{2.35}$$

Then there exists $c_2 > 0$ such that for all x, y, $d(x,y) \le R$,

$$|V_\sigma(x;\#) - V_\sigma(y;\#)| \le c_2 \frac{d(x,y)}{R}\,. \tag{2.36}$$

Proof. The difference $V_\sigma(x;\#) - V_\sigma(y;\#)$ can be expressed as follows:

$$\sum_{\substack{j\in B_R(x)\\ j\notin B_R(y)}} \phi_{xj}(\#,\sigma_j) + \sum_{j\in B_R(x)\cap B_R(y)} \big(\phi_{xj}(\#,\sigma_j) - \phi_{yj}(\#,\sigma_j)\big) - \sum_{\substack{j\in B_R(y)\\ j\notin B_R(x)}} \phi_{yj}(\#,\sigma_j)$$

The first and last sum can be estimated as follows:

$$\sum_{\substack{j\in B_R(x)\\ j\notin B_R(y)}} \phi_{xj}(\#,\sigma_j) \le \big(|B_R(x)| - |B_R(x)\cap B_R(y)|\big) \sup \phi_{ij} \tag{2.37}$$

$$\le dc_\gamma\big(\sup_t J(t)\big)\Big(\frac{2R+1}{R}\Big)^{d-1}\frac{d(x,y)}{R}\,. \tag{2.38}$$

Since we are considering the step function, $\sup_t J(t) = 2^{-d}$. The middle sum vanishes [6], which finishes the proof. $\square$

Proof of Proposition 2.2: By Lemma 2.3 there exists in the $2R$-neighbourhood of each $C^{(l)} \subset \Gamma$ a point $j \in \Gamma$ that is $\tilde\delta$-incorrect for $\sigma[\Gamma]$. Let A be the set of all such points. We have $\Gamma \subset [A]_{l+2R}$. Let A_0 be any $4R$-approximant of A. We have $A \subset [A_0]_{4R}$, i.e. $\Gamma \subset [A_0]_{l+6R}$. Each $j \in A_0$ is $\tilde\delta$-incorrect for $\sigma[\Gamma]$ i.e. satisfies

$$\sum_{k:k\neq j} |w^{\pm}_{jk}(\mp,\sigma[\Gamma]_k)| > \tilde\delta\,. \tag{2.40}$$

[6] Here we use for the second time the fact that we are considering the step function (1.14). Nevertheless, if J is an arbitrary K-Lipshitz function:

$$\sum_{j\in B_R(x)\cap B_R(y)} \big|\phi_{xj}(\#,\sigma_j) - \phi_{yj}(\#,\sigma_j)\big| \le Kc_\gamma\gamma^d \sum_{j\in B_R(x)\cap B_R(y)} d(\gamma x,\gamma y)$$

$$\le Kc_\gamma\gamma^d |B_R(x)| \frac{d(x,y)}{R}\,. \tag{2.39}$$

Since $|w^{\pm}_{jk}(\mp,\sigma[\Gamma]_k)| = 2\phi_{jk}(\pm,\sigma[\Gamma]_k)$,

$$V_{\sigma[\Gamma]}(j;\pm) = \sum_{k:k\neq j} \phi_{jk}(\pm,\sigma[\Gamma]_k) > \frac{\tilde{\delta}}{2}\,. \tag{2.41}$$

We bound the surface energy from below as follows:

$$\begin{aligned}
\|\Gamma\| &\geq \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap\Gamma}\sum_{l:l\neq k}\phi_{kl}(\sigma[\Gamma]_k,\sigma[\Gamma]_l)\\
&= \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap\Gamma} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) \geq \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap C_j^{(l)}} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)\\
&\geq \frac{1}{2}\sum_{j\in A_0}\sum_{\substack{k\in B_R(j)\cap C_j^{(l)}\\ d(k,j)\leq\frac{\tilde{\delta}}{4c_2}R}} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)
\end{aligned}$$

where c_2 was defined in Lemma 2.8. Moreover we have, using (2.36), for each k of the sum,

$$V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) = V_{\sigma[\Gamma]}(j;\sigma[\Gamma]_k) + \big(V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) - V_{\sigma[\Gamma]}(j;\sigma[\Gamma]_k)\big) \tag{2.42}$$

$$\geq \frac{\tilde{\delta}}{2} - c_2\frac{d(k,j)}{R} \geq \frac{\tilde{\delta}}{2} - c_2\frac{\tilde{\delta}}{4c_2} = \frac{\tilde{\delta}}{4}\,. \tag{2.43}$$

We have used the fundamental fact that the correctness of a point j does not depend on the value taken by the spin σ_j. This gives the lower bound

$$\|\Gamma\| \geq \frac{1}{2}|A_0|\frac{1}{2^d}|B_{\frac{\tilde{\delta}}{4c_2}R}(0)|\frac{\tilde{\delta}}{4} \geq \frac{\tilde{\delta}}{2^{d+3}}|B_{\frac{\tilde{\delta}}{4c_2}R}(0)||B_{l+6R}(0)|^{-1}|\Gamma| \geq \rho|\Gamma|\,.$$

□

Since the Peierls constant is uniform in γ, we will be able to study the van der Waals limit at fixed β. Proposition 2.2 allows to define, for $N = 1,2,\ldots$, the following numbers called isoperimetric constants:

$$K(N) := \inf\Big\{\kappa > 0 : V(\Gamma)^{\frac{d-1}{d}} \leq \kappa\|\Gamma\|,\ \text{ for all } \Gamma,\ V(\Gamma)\geq N\Big\}\,. \tag{2.44}$$

These constants will play a crucial role in the construction of the phase diagram and in the study of non-analyticity. Some of their properties are given in the following lemma.

Lemma 2.9. *The sequence $K(N)$ is decreasing and there exists positive constants c_-, c_+ such that*

$$c_-\gamma \leq \inf_N K(N) \leq \sup_N K(N) \leq c_+\gamma\,. \tag{2.45}$$

As a consequence, the following limit exists

$$K(\infty) := \lim_{N\to\infty} K(N)\,. \tag{2.46}$$

Moreover, there exists for all $\epsilon > 0$ *a sequence* $(\Gamma_N)_{N\geq 1}$, $\lim_{N\to\infty} V(\Gamma_N) = +\infty$, *such that for* N *large enough,*

$$(1-\epsilon)K(\infty)\|\Gamma_N\| \leq V(\Gamma_N)^{\frac{d-1}{d}} \leq (1+\epsilon)K(\infty)\|\Gamma_N\|\,. \tag{2.47}$$

Proof. $K(N)$ is decreasing by definition. For the upper bound, use the Peierls condition and Lemma 2.10 hereafter: for all Γ,

$$\frac{V(\Gamma)^{\frac{d-1}{d}}}{\|\Gamma\|} \leq \frac{V(\Gamma)^{\frac{d-1}{d}}}{\rho|\Gamma|} \leq \frac{1}{\rho l} = \frac{1}{\rho\nu}\gamma \equiv c_+\gamma\,. \tag{2.48}$$

For the lower bound, we explicitly construct a large contour of cubic shape. Fix N and take $M \in \mathbb{N}$ so that $\Lambda_M = [-M;+M]^d \cap \mathbf{Z}^d$, $\Lambda_M \in \mathcal{C}^{(l)}$, $|\Lambda_M| \geq 2N$. Consider the configuration σ defined by $\sigma_i = -1$ if $i \in \Lambda_M$, $\sigma_i = +1$ if $i \in \Lambda_M^c$. Clearly, $I^*(\sigma)$ contains a single contour Γ_M (of type $+$). Using (2.6), $\|\Gamma_M\| \leq |\Gamma_M| \leq 2l|\partial_1^+\Lambda_M| = 2\nu R|\partial_1^+\Lambda_M|$. Taking M large enough guarantees $|\Lambda_M| \geq V(\Gamma_M) \geq \frac{1}{2}|\Lambda_M|$. This gives, since $|\partial_1^+\Lambda_M| = 2d|\Lambda_M|^{\frac{d-1}{d}}$,

$$\frac{V(\Gamma_M)}{\|\Gamma_M\|} \geq \frac{1}{2}\frac{1}{2\nu R}\frac{|\Lambda_M|}{|\partial_1^+\Lambda_M|} \geq \frac{\gamma}{8d\nu}V(\Gamma_M)^{\frac{1}{d}} \equiv c_-\gamma V(\Gamma_M)^{\frac{1}{d}}\,. \tag{2.49}$$

The existence of the sequence $(\Gamma_N)_{N\geq 1}$ follows from the definition of $K(N)$ and from the existence of the limit $K(\infty)$. $\square$

Lemma 2.10. *Let* $B \in \mathcal{C}^{(l)}$, *and let* A *be the union of all finite maximal* R*-connected components of* B^c. *Then*

$$|B| \geq |\partial_l^+ A| \geq l|A|^{\frac{d-1}{d}}\,. \tag{2.50}$$

Proof. Consider the edge boundary $\delta^+ A := \{e = \langle i,j\rangle : i \in A, j \in A^c\}$, where $\langle i,j\rangle$ means that i,j are nearest neighbours. Decompose $\delta^+ A = E_1 \cup \cdots \cup E_d$, where E_α is the set of edges of $\delta^+ A$ that are parallel to the coordinate axis α. Suppose $e = \langle i,j\rangle$, $i \in A$, $j \in A^c$. Since A is maximal, $C_j^{(l)} \subset B$. Moreover,

$$T_e := \big\{j, j+(j-i), j+2(j-i), \dots, j+(\tfrac{l}{2}-1)(j-i)\big\} \subset B\,. \tag{2.51}$$

For all $e, e' \in E_\alpha$, $T_e \cap T_{e'} = \emptyset$. So for all α,

$$|\partial_l^+ A| \geq \Big|\bigcup_{e\in E_\alpha} T_e\Big| = \sum_{e\in T_\alpha} |T_e| = \frac{l}{2}|E_\alpha|\,. \tag{2.52}$$

Considering the inequality $|\delta^+ A| \leq d\max_\alpha |E_\alpha|$ and the standard isoperimetric inequality $|\delta^+ A| \geq 2d|A|^{\frac{d-1}{d}}$ finishes the proof. $\square$

3 Restricted Phases

Restricted phases intervene when a set of contours $\{\Gamma\}$ is fixed (with a configuration σ_Γ on each of them) and when we re-sum over all the configurations that have this same set of contours. The set of configurations having the same set of contours was completely characterized in Proposition 2.1. We are thus naturally led to consider systems living in a volume Λ with a boundary condition η_{Λ^c}, with the constraint that each point $i \in [\Lambda]_R$ must be δ-correct. Our aim is to obtain a polymer representation for the partition function of such systems, and to show that the associated pressure behaves analytically at $h = 0$. As will be seen, the presence of the constraint will allow to treat the system in a way very similar to a high temperature expansion. The study of restricted phases we present was invented by Bovier and Zahradník in [BZ2]. At a few places our development differs slightly from theirs, so we expose all the details.
A source of complication will be that the definition of polymers, as well as their weights, will depend on the boundary conditions specified outside Λ. Typically, the Λ we want to consider is the volume between a given set of contours and the boundary of a box. That is, the boundary condition is specified partly by the spins on the contours and partly by the boundary condition outside the box. To have an idea of the objects we are going to construct, see Figures 4 and 5.

We will only treat the case $+$, the case $-$ being similar by symmetry. Fix $0 < \tilde{\delta} < \delta < 2^{-d}$. Consider any finite set $\Lambda \in \mathcal{C}^{(l)}$. First of all, we must consider boundary conditions of the following type:

Definition 3.1. *A boundary condition $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$ is $+$-*admissible *if each $i \in [\Lambda]_R$ is $(\tilde{\delta}, +)$-correct for the configuration $+_\Lambda \eta_{\Lambda^c}$.*

More intuitively, a $+$-admissible boundary condition means that when looked from any point i inside of Λ, there is a majority of spins $+1$ on the boundary. In our case (i.e. with the step function), this can be formulated as: for each $i \in [\Lambda]_R$,

$$|B_R^\bullet(i) \cap B| \leq \tfrac{\tilde{\delta}}{2} |B_R(i)|\,, \tag{3.1}$$

where the set B is defined by

$$B = B(\eta_{\Lambda^c}) := \{i \in \Lambda^c : (\eta_{\Lambda^c})_i = -1\}\,. \tag{3.2}$$

In this sense, these boundary conditions are "good"; there is hope in being able to control the $+$-phase in the volume Λ. Notice that the boundary condition specified by a contour on its interior is always admissible. This is the reason why the parameter $\tilde{\delta}$ was introduced in their definition.
We define the function that allows to realize the constraint obtained after

Proposition (2.1): consider a $+$-admissible boundary condition $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$. Let $i \in [\Lambda]_R$, $\sigma_\Lambda \in \Omega_\Lambda$, and define

$$1_i(\sigma_\Lambda) := \begin{cases} 1 & \text{if } i \text{ is } (\delta, +)\text{-correct for } \sigma_\Lambda \eta_{\Lambda^c}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Then define

$$1(\sigma_\Lambda) := \prod_{i \in [\Lambda]_R} 1_i(\sigma_\Lambda)\,. \tag{3.4}$$

Notice that $1(+_\Lambda) = 1$ since η_{Λ^c} is $+$-admissible. The hamiltonian we use for the restricted system is the one obtained after the re-formulation of Lemma 2.6 in a region of $+$-correct points. Set $\sigma := \sigma_\Lambda \eta_{\Lambda^c}$. The restricted partition function with boundary condition η_{Λ^c} is

$$Z_r{}^+(\Lambda; \eta_{\Lambda^c}) := \sum_{\sigma_\Lambda \in \Omega_\Lambda} 1(\sigma_\Lambda) \exp\Big(-\beta \sum_{\substack{\{i,j\} \cap \Lambda \neq \emptyset \\ i \neq j}} w^+_{ij}(\sigma_i, \sigma_j) - \beta \sum_{i \in \Lambda} U^+(\sigma_i) \Big)\,.$$

We will show that $Z_r{}^+$ can be put in the form $Z_r{}^+ = e^{\beta h |\Lambda|} \mathcal{Z}_r$, where $\mathcal{Z}_r$ is the partition function of a polymer model, having a normally convergent cluster expansion in the domain

$$H_+ = \left\{ h \in \mathbb{C} : \operatorname{Re} h > -\tfrac{1}{8} \right\}\,. \tag{3.5}$$

The reason for $\log Z_r{}^+$ to behave analytically at $h = 0$ is that the presence of contours is suppressed by $1(\sigma_\Lambda)$, and that on each spin $\sigma_i = -1$ acts an effective magnetic field

$$U^+(-1) = h + \sum_{j : j \neq i} \phi_{ij} = 1 + h\,, \tag{3.6}$$

which is close to 1 when h is in a neighbourhood of $h = 0$.

3.1 Representation with Polymers

The influence of a boundary condition can always be interpreted as a magnetic field acting on sites near the boundary. We thus rearrange the terms of the hamiltonian as follows:

$$\sum_{\substack{\{i,j\} \subset \Lambda \\ i \neq j}} w^+_{ij}(\sigma_i, \sigma_j) + \sum_{i \in \Lambda} \Big(U^+(\sigma_i) + \sum_{j \in \Lambda^c} w^+_{ij}(\sigma_i, (\eta_{\Lambda^c})_j) \Big)\,. \tag{3.7}$$

By defining an new effective non-homogeneous magnetic field

$$\mu^+_i(\sigma_i) := U^+(\sigma_i) + h + \sum_{j \in \Lambda^c} w^+_{ij}(\sigma_i, (\eta_{\Lambda^c})_j)\,, \tag{3.8}$$

we can extract a volume term from ${Z_r}^+$ and get ${Z_r}^+ = e^{\beta h|\Lambda|}\mathcal{Z}_r$, where

$$\mathcal{Z}_r := \sum_{\sigma_\Lambda \in \Omega_\Lambda} 1(\sigma_\Lambda) \exp\Big(-\beta \sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} w^+_{ij}(\sigma_i,\sigma_j) - \beta \sum_{i\in\Lambda} \mu^+_i(\sigma_i)\Big)\,. \tag{3.9}$$

Notice that the field $\mu^+_i(\sigma_i)$ becomes independent of η_{Λ^c} when $d(i,\Lambda^c) > R$. Since $w^+_{ij}(\sigma_i,\sigma_j) = 0$ if $\sigma_i = +1$ or $\sigma_j = +1$ and $\mu^+_i(+1) = 0$, we need only consider points i with $\sigma_i = -1$, which will be identified with the vertices of a graph. Each vertex of this graph will then get a factor $e^{-\beta\mu^+_i(-1)}$. When $h \in H_+$,

$$\mathrm{Re}\,\mu^+_i(-1) = 1 + 2\mathrm{Re}\,h + \sum_{j\in\Lambda^c} w^+_{ij}(-,(\eta_{\Lambda^c})_j) \geq 1 - 2\tfrac{1}{8} - \tilde{\delta} > \tfrac{1}{2}\,. \tag{3.10}$$

We used the fact that $\tilde{\delta} < 2^{-d}$.
The formulation of $\mathcal{Z}_r$ in terms of polymers will be a three step procedure. We first express $\mathcal{Z}_r$ as a sum over graphs, satisfying a certain constraint inherited from $1(\sigma_\Lambda)$. Then, we associate to each graph a spanning tree and re-sum over all graphs having the same spanning tree. We will see that the weights of the trees obtained have good decreasing properties. Finally, the constraint is expanded, yielding sets on which the constraint is *violated*. These sets are linked with trees. After a second partial re-summation, this yields a sum over polymers, which are nothing but particular graphs with vertices living on $\mathbf{Z}^d$ and whose edges are of length at most R.

A sum over graphs. Let $\mathcal{G}_\Lambda$ be the family of simple non-oriented graphs $G = (V,E)$ where $V \subset \Lambda$, each edge $e = \{i,j\} \in E$ has $d(i,j) \leq R$. For $e = \{i,j\}$, set $w^+_e := w^+_{ij}(-,-)$. Notice that $\omega^+_e = -2\phi_{ij} \leq 0$. Define also $\mu^+_i := \mu^+_i(-1)$. Expanding the product over edges leads to the following expression

$$\mathcal{Z}_r = \sum_{G\in\mathcal{G}_\Lambda} 1(V(G)) \prod_{e\in E(G)} (e^{-\beta w^+_e} - 1) \prod_{i\in V(G)} e^{-\beta\mu^+_i}\,, \tag{3.11}$$

where $1(V) := 1(\sigma_\Lambda(V))$, and $\sigma_\Lambda(V) \in \Omega_\Lambda$ is defined by $\sigma_\Lambda(V)_i = -1$ if $i \in V$, $+1$ otherwise. With this formulation in terms of graphs, the constraint $1(V(G)) = 1$ is satisfied if and only if

$$\sum_{\substack{e=\{i,j\} \\ j\in V(G)\cup B}} |w^+_e| \leq \delta\,, \quad \forall\, i \in [\Lambda]_R\,. \tag{3.12}$$

Moreover, the fact that the boundary condition η_{Λ^c} is $+$-admissible reduces to

$$\sum_{\substack{e=\{i,j\} \\ j\in B}} |w^+_e| \leq \tilde{\delta}\,, \quad \forall\, i \in [\Lambda]_R. \tag{3.13}$$

A sum over trees. Suppose we are given an algorithm that assigns to each connected graph G_0 a deterministic spanning tree $T(G_0)$, in a translation invariant way (that is if G_0' is obtained from G_0 by translation then $T(G_0')$ is obtained from $T(G_0)$ by the same translation). To be precise, we consider the Penrose algorithm considered in Chapter 3 of [Pf] [7]. We apply the Penrose algorithm to each component of each graph G appearing in the partition function (3.11). Let $\mathcal{T}_\Lambda \subset \mathcal{G}_\Lambda$ denote the set of all forests. We have

$$\mathcal{Z}_r = \sum_{T \in \mathcal{T}_\Lambda} 1(V(T)) \prod_{\mathtt{t} \in T} \omega^+(\mathtt{t})\,, \tag{3.14}$$

where the product is over trees of T, and the weight of each tree is defined by

$$\omega^+(\mathtt{t}) := \sum_{\substack{G \in \mathcal{G}_\Lambda: \\ T(G) = \mathtt{t}}} \prod_{e \in E(G)} (e^{-\beta w_e^+} - 1) \prod_{i \in V(G)} e^{-\beta \mu_i^+}\,. \tag{3.15}$$

Isolated sites $\{i\} \subset \Lambda$ are also considered as trees. In this case, $\omega^+(\{i\}) = e^{-\beta\mu_i^+}$. The following lemma shows how the re-formulation in terms of trees allows to take advantage of the constraint.

Lemma 3.1. *Let $T \in \mathcal{T}_\Lambda$ be a forest such that $1(V(T)) = 1$. Then for each tree $\mathtt{t} \in T$,*

$$\|\omega^+(\mathtt{t})\|_{H_+} \le \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\frac{1}{4}\beta}\,. \tag{3.16}$$

Proof. For each $\mathtt{t} \in T$, let $E^*(\mathtt{t})$ denote the set of edges of the maximal connected graph of $\{G \in \mathcal{G}_\Lambda : T(G) = \mathtt{t}\}$ (see [Pf]). We can express the weight as follows:

$$\begin{aligned}
\omega^+(\mathtt{t}) &= \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\beta\mu_i^+} \sum_{\substack{G \in \mathcal{G}_\Lambda: \\ T(G) = \mathtt{t}}} \prod_{e \in E(G)\setminus E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \\
&= \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\beta\mu_i^+} \prod_{e \in E^*(\mathtt{t})\setminus E(\mathtt{t})} e^{-\beta w_e^+}\,.
\end{aligned}$$

Since $1(V(T)) = 1$, the constraint (3.12) is satisfied, and the last product can be bounded by:

$$\prod_{e \in E^*(\mathtt{t})\setminus E(\mathtt{t})} e^{\beta |w_e^+|} \le \prod_{i \in V(\mathtt{t})} \prod_{\substack{e = \{i,j\} \\ j \in V(\mathtt{t})}} e^{\beta |w_e^+|} \tag{3.17}$$

$$= \prod_{i \in V(\mathtt{t})} \exp \beta \sum_{\substack{e = \{i,j\} \\ j \in V(\mathtt{t})}} |w_e^+| \le \prod_{i \in V(\mathtt{t})} e^{\beta\delta}\,. \tag{3.18}$$

[7] The Penrose algorithm requires the choice of an origin among the vertices of the graph. We choose this origin as the smallest vertex of the graph with respect to the lexicographical order.

This gives the result, since $\operatorname{Re}\mu_i^+ \geq \frac{1}{2}$ by (3.10), and $\delta \leq 2^{-d} \leq \frac{1}{4}$. $\square$

Notice that to obtain (3.18), we only needed that the bound

$$\sum_{\substack{e=\{i,j\}\\ j\in V(\mathtt{t})}} |w_e^+| \leq \delta\,, \quad \forall\, i \in V(\mathtt{t}) \tag{3.19}$$

be satisfied. This is weaker than (3.12) and clearly $1(V(T)) = 1$ only if (3.19) is satisfied for all $\mathtt{t} \in T$. In the sequel we can thus assume that the trees we consider always satisfy (3.19), independently of each other. So the bound (3.16) can always be used. A direct consequence of the last lemma is the following result which shows that trees and their weights satisfy the main condition ensuring convergence of cluster expansions.

Corollary 3.1. *Let* $0 < c \leq \frac{1}{8}\beta$, $\epsilon > 0$. *There exists* $\gamma_0 > 0$ *and* $\beta_1 = \beta_1(\epsilon)$ *such that for all* $\gamma \in (0,\gamma_0)$, $\beta \geq \beta_1$, *the following bound holds:*

$$\sum_{\mathtt{t}:V(\mathtt{t})\ni 0} \|\omega^+(\mathtt{t})\|_{H_+} e^{c|V(\mathtt{t})|} \leq \epsilon\,. \tag{3.20}$$

Proof. Using Lemma 3.1,

$$\|\omega^+(\mathtt{t})\|_{H_+} e^{c|V(\mathtt{t})|} \leq \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{8}\beta}\,. \tag{3.21}$$

When $\mathtt{t}$ is a single isolated point (the origin), then we have a factor $e^{-\frac{1}{8}\beta}$. When $V(\mathtt{t}) \ni 0$, $E(\mathtt{t}) \neq \emptyset$, we define the generation of $\mathtt{t}$, $\mathrm{gen}(\mathtt{t})$, as the number of edges of the longest self avoiding path in $\mathtt{t}$ starting at the origin. The sum in (3.20) is bounded by

$$\begin{aligned}
e^{-\frac{1}{8}\beta} &+ \sum_{g\geq 1} \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{8}\beta} \\
&\leq e^{-\frac{1}{8}\beta} + \sum_{g\geq 1} e^{-\frac{1}{16}\beta g} \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{16}\beta} \\
&\leq e^{-\frac{1}{8}\beta} + \sum_{g\geq 1} e^{-\frac{1}{16}\beta g} \alpha_g\,,
\end{aligned}$$

where we defined ($V_l(\mathtt{t})$ is the set of leaves of the tree $\mathtt{t}$):

$$\alpha_g := \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})\setminus V_l(\mathtt{t})} e^{-\frac{1}{16}\beta} \prod_{i\in V_l(\mathtt{t})} e^{-\frac{1}{32}\beta}\,. \tag{3.22}$$

We are going to show that $\alpha_{g+1} \leq \alpha_g$ for all $g \geq 1$. Before going further, we define

$$\gamma_0 := \sup\Big\{\gamma > 0 : 2c_\gamma \gamma^d \sup_s J(s) \leq \tfrac{1}{64}\Big\}\,. \tag{3.23}$$

Since $e^{-\beta w_e^+} - 1 \leq \beta |w_e^+| e^{\beta |w_e^+|}$ and $|w_e^+| = 2\phi_{ij}$ we can bound, when $\gamma \leq \gamma_0$,

$$\sum_{e \ni 0} \big(e^{-\beta w_e^+} - 1\big) e^{-\frac{1}{32}\beta} \leq \beta e^{-\frac{1}{64}\beta} \sum_{e \ni 0} |\omega_e^+| \leq 2\beta e^{-\frac{1}{64}\beta} \equiv \beta\zeta(\beta)\,. \tag{3.24}$$

Clearly, a tree $\mathtt{t}$ of generation $g+1$ can be obtained from a sub-tree $\mathtt{t}' \subset \mathtt{t}$ of generation g by attaching edges to leaves of $\mathtt{t}'$. Let x be a leaf of $\mathtt{t}'$. The sum over all possible edges (if any) attached at x is bounded by

$$1 + \sum_{k \geq 1} \frac{1}{k!} \sum_{e_1 \ni x} \cdots \sum_{e_k \ni x} \prod_{i=1}^{k} \big(e^{-\beta w_{e_i}^+} - 1\big) e^{-\frac{1}{32}\beta} \leq 1 + \sum_{k \geq 1} \frac{1}{k!} (\beta\zeta(\beta))^k = e^{\beta\zeta(\beta)}\,.$$

Assuming β is large enough so that $\zeta(\beta) \leq \frac{1}{32}$, the weight of the leaf x changes into $e^{-\frac{1}{16}\beta} e^{\beta\zeta(\beta)} \leq e^{-\frac{1}{32}\beta}$, which is exactly what appears in α_g. This shows that $\alpha_{g+1} \leq \alpha_g$. We then have $\alpha_{g+1} \leq \alpha_g \leq \cdots \leq \alpha_1$. Like we just did, it is easy to see that $\alpha_1 \leq e^{-\frac{1}{32}\beta}$. This proves the result. $\square$

A sum over polymers. After the partial re-summation over the graphs having the same spanning tree, the constraint $1(V(T))$ in (3.14) still depends on the relative positions of the trees. This "multi-body interaction" can be worked out by expanding

$$1(V(T)) = \prod_{i \in [\Lambda]_R} 1_i(V(T)) = \prod_{i \in [\Lambda]_R} \big(1 + 1_i^c(V(T))\big) = \sum_{M \subset [\Lambda]_R} \prod_{i \in M} 1_i^c(V(T))\,,$$

where $1_i^c(V(T)) := 1_i(V(T)) - 1$. This yields

$$\mathcal{Z}_r = \sum_{T \in \mathcal{T}_\Lambda} \sum_{M \subset [\Lambda]_R} \Big(\prod_{i \in M} 1_i^c(V(T))\Big)\Big(\prod_{\mathtt{t} \in T} \omega^+(\mathtt{t})\Big)\,. \tag{3.25}$$

Consider a pair (T, M) in (3.25). Let $i \in M$. The function $1_i^c(V(T))$ is non-zero only when i is not $(\delta, +)$-correct; it depends on the presence of trees of T in the R-neighbourhood of i and possibly on the points of $B(\eta_{\Lambda^c})$ if $B_R(i) \cap \Lambda^c \neq \emptyset$. To make these dependencies only local, we are going to link the R-neighbourhood of points of M with the trees of T.
Consider the graph $N = N(M)$ defined as follows: the vertices of N are given by

$$V(N) := \bigcup_{i \in M} B_R(i)\,. \tag{3.26}$$

Then, N has an edge between x and y if and only if $\langle x, y \rangle$ is a pair of nearest neighbours of the same box $B_R(i)$ for some $i \in M$. The graph N decomposes naturally into connected components (in the sense of graph theory) $N_1, N_2, \ldots, N_K$. Some of these components can intersect Λ^c.

We then link trees $\mathtt{t}_i \in T$ with components $N_j \in N$. To this end, we define an abstract graph $\hat{G}$: to each tree $\mathtt{t}_i \in T$, associate an abstract vertex w_i and to each component N_j an abstract vertex z_j. The edges of $\hat{G}$ are defined as follows: $\hat{G}$ has only edges between vertices w_i and z_j, and this occurs if and only if $V(\mathtt{t}_i) \cap V(N_j) \neq \emptyset$. Consider a connected component of $\hat{G}$, whose vertices $\{w_{i_1}, \dots, w_{i_l}, z_{j_1}, \dots, z_{j_l}\}$ correspond to a set $P'_l = \{\mathtt{t}_{i_1}, \dots, \mathtt{t}_{i_l}, N_{j_1}, \dots, N_{j_l}\}$. We change P'_l into a set P_l, using the following decimation procedure: if $P'_l = \{\mathtt{t}_{i_1}\}$ is a single tree then $P_l := P'_l$. Otherwise,

1) delete from P'_l all trees $\mathtt{t}_{i_k}$ that have no edges,

2) for all tree $\mathtt{t}_{i_k}$ containing at least one edge, delete all edges $e \in E(\mathtt{t}_{i_k})$ whose both end-points lie in the same component N_{j_m}.

The resulting set is of the form $P_l = \{\mathtt{t}_{s_1}, \dots, \mathtt{t}_{s_l}, N_{j_1}, \dots, N_{j_l}\}$, where each tree $\mathtt{t}_{s_i}$ is a sub-tree of one of the trees $\{\mathtt{t}_{i_1}, \dots, \mathtt{t}_{i_l}\}$. P_l is called a polymer. The decimation procedure $P'_l \Rightarrow P_l$ is depicted on Figure 4.

The body of P_l is $\mathcal{B}(P_l) := V(N_{j_1}) \cup \dots \cup V(N_{j_l})$. The legs of P_l, $\mathcal{L}(P_l)$,

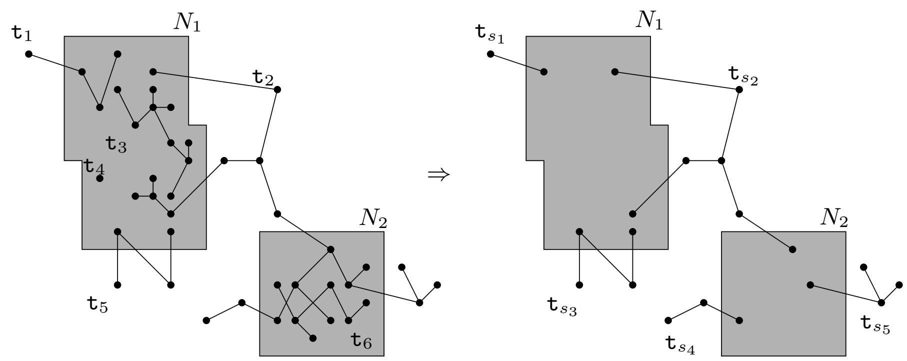

Figure 4: The decimation procedure $P'_l \Rightarrow P_l$. The hatched polygons represent the body $\mathcal{B}(P_l)$ and the legs are the trees $\{\mathtt{t}_{s_1}, \mathtt{t}_{s_2}, \mathtt{t}_{s_3}, \mathtt{t}_{s_4}, \mathtt{t}_{s_5}\}$. Each $\mathtt{t}_{s_j}$ is a sub-tree of some $\mathtt{t}_i$.

are the trees $\{\mathtt{t}_{s_1}, \dots, \mathtt{t}_{s_l}\}$.

A polymer can have no body (in which case it is a tree of $\mathcal{T}_\Lambda$), or no legs (in which case it is a single component N_{j_1}). We define the support $V(P)$ as the total set of sites:

$$V(P) := \bigcup_{\mathtt{t} \in \mathcal{L}(P)} V(\mathtt{t}) \cup \bigcup_i V(N_i)\,. \tag{3.27}$$

Often we denote $V(P)$ also by P. Two polymers are compatible if and only if $V(P_1) \cap V(P_2) = \emptyset$, denoted $P_1 \sim P_2$. We have thus associated to each pair (T, M) a family of pairwise compatible polymers $\{P\} := \varphi(T, M)$. The set of all possible polymers constructed in this way is denoted $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$.

The representation of $\mathcal{Z}_r$ in terms of polymers is then

$$\mathcal{Z}_r = \sum_{\substack{\{P\}\subset\mathcal{P}_\Lambda^+(\eta_{\Lambda^c})\\ \text{compat.}}} \prod_{P\in\{P\}} \omega^+(P)\,, \tag{3.28}$$

where the weight is defined by

$$\omega^+(P) := \sum_{\substack{(T,M):\\ \varphi(T,M)=P}} \Big(\prod_{i\in M} 1_i^c(V(T))\Big)\Big(\prod_{\mathtt{t}\in T}\omega^+(\mathtt{t})\Big)\,. \tag{3.29}$$

We should have in mind that $\omega^+(P)$ depends on the position of P inside the volume Λ, via the boundary condition η_{Λ^c}: more precisely if $\mathcal{B}(P)\cap\Lambda^c\neq\emptyset$ or if there exists a leg $\mathtt{t}\in\mathcal{L}(P)$ such that $d(\mathtt{t},\Lambda^c)\leq R$. Therefore, we define the family $\mathcal{P}^+$ of free polymers of type $+$ whose weights depends only on the intrinsic structure of P, and not on the boundary condition. The family $\mathcal{P}^+$ is translation invariant, as well as the weight of each of its polymers. To any finite family $\mathcal{P}$, we associate the partition function

$$\mathcal{Z}_r(\mathcal{P}) := \sum_{\substack{\{P\}\subset\mathcal{P}\\ \text{compat.}}} \prod_{P\in\{P\}} \omega^+(P)\,, \tag{3.30}$$

where the product equals 1 when $\{P\}=\emptyset$. For instance, we have obtained

$$Z_r{}^+(\Lambda;\eta_{\Lambda^c}) = e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))\,. \tag{3.31}$$

Everything we have done until now can be done for a $-$-admissible boundary condition τ_{Λ^c}, yielding a family of polymers $\mathcal{P}_\Lambda^-(\tau_{\Lambda^c})$, with weights $\omega^-(P)$. In this case, sites get a factor $e^{-\beta\mu_i^-}$. In particular, if we consider the spin-flipped boundary condition $-\eta_{\Lambda^c}$ defined by $(-\eta_{\Lambda^c})_i := -(\eta_{\Lambda^c})_i$, which is $-$-admissible, we have when h is purely imaginary [8],

$$\overline{\mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))} = \mathcal{Z}_r(\mathcal{P}_\Lambda^-(-\eta_{\Lambda^c}))\,. \tag{3.32}$$

3.2 Analyticity of the Restricted Phases

Define the restricted pressures by

$$p_{r,\gamma}^{\pm} := \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{\beta|\Lambda|}\log Z_r{}^{\pm}(\Lambda;\pm_{\Lambda^c})\ , \tag{3.33}$$

where the thermodynamic limit is taken along a sequence of cubes. A result of the present section is that the restricted pressures, unlike the total pressure p_γ, behave analytically at $h=0$.

We study the weight $\omega^+(P)$ ($\omega^-(P)$ is similar by symmetry). The point is

[8] Here, $\overline{z}$ denotes the complex conjugate of z.

that we linked trees with the R-neighbourhood of points of the set M, and we must now see that this thickening does not destroy, from the point of view of entropy, the uniformity we have been able to obtain with respect to the scaling parameter γ. Moreover, the body of polymers can intersect Λ^c. At this point we will see that $\delta - \tilde{\delta} > 0$ is crucial.

Lemma 3.2. *There exists β_2 and $\tau_0 > 0$ such that for all $\beta \geq \beta_2$ and for all $\gamma \in (0, \gamma_0)$, the following holds: each polymer $P \in \mathcal{P}_\Lambda^+(\eta_{\Lambda^c})$ satisfies*

$$\|\omega^+(P)\|_{H_+} \leq e^{-\tau_0 \beta |\mathcal{B}(P)|} \prod_{e \in \mathcal{L}(P)} (e^{-\beta w_e^+} - 1) \prod_{i \in \mathcal{L}(P)} e^{-\frac{1}{12}\beta} . \tag{3.34}$$

Proof. Remember that the bound (3.16) holds for each tree under consideration. If $\mathcal{B}(P) = \emptyset$, then P is a tree and the result follows from Lemma 3.1. Otherwise, $\|\omega^+(P)\|_{H_+}$ is bounded by

$$\sum_{\substack{(T,M): \\ \varphi(T,M)=P}} \Big(\prod_{i \in M} |1_i^c(V(T))| \Big) \prod_{\mathtt{t} \in T} \Big(\prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\frac{1}{4}\beta} \Big) .$$

Consider a pair (T, M) such that $\varphi(T, M) = P$. Let $i_0 \in M$, and assume $1_{i_0}^c(V(T)) \neq 0$. This implies, with regard to (3.12),

$$\sum_{\substack{e=\{i_0,j\} \\ j \in V(T) \cup B}} |w_e^+| > \delta . \tag{3.35}$$

But, according to (3.13), we have

$$\sum_{\substack{e=\{i_0,j\} \\ j \in B}} |w_e^+| \leq \tilde{\delta} . \tag{3.36}$$

This implies the crucial lower bound

$$\sum_{\substack{e=\{i_0,j\} \\ j \in V(T)}} |w_e^+| \geq \delta - \tilde{\delta} > 0 . \tag{3.37}$$

Since $|w_e^+| = 2\phi_{ij} \leq 2c_\gamma \gamma^d \sup_s J(s)$, we can find a constant c_3 such that

$$|V(T) \cap B_R^\bullet(i_0)| > (\delta - \tilde{\delta}) c_3 |B_R(i_0)| . \tag{3.38}$$

In this sense, the forests that contribute to $\omega^+(P)$ accumulate in the neighbourhood of each point $i_0 \in M$. See Figure 5. Let M_0 be any $2R$-approximant of M. Then we have $|\mathcal{B}(P)| \leq |M_0||B_{3R}(0)|$ and so

$$|V(T) \cap \mathcal{B}(P)| \geq \sum_{i_0 \in M_0} |V(T) \cap B_R(i_0)| \geq (\delta - \tilde{\delta}) c_4 |\mathcal{B}(P)| , \tag{3.39}$$

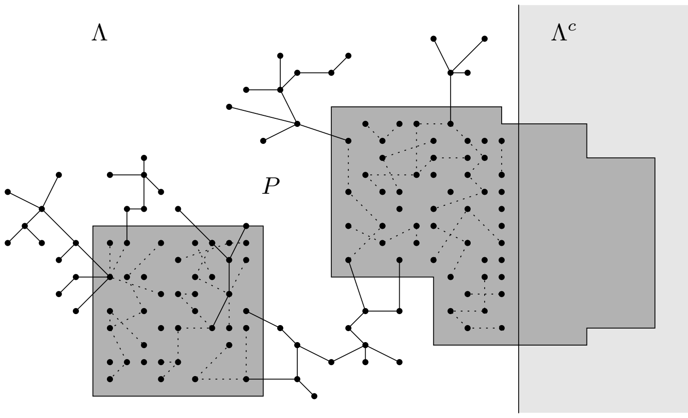

Figure 5: The re-summation of Lemma 3.2. We emphasized the fact that the forest T must have many points in $\mathcal{B}(P) \cap \Lambda$, as was shown in (3.39).

where c_4 is a constant. Now, each $i \in V(T)$ gets a factor $e^{-\frac{1}{4}\beta} = e^{-3\frac{1}{12}\beta}$. One factor $e^{-\frac{1}{12}\beta}$ contributes to extract a term decreasing exponentially fast with the size of $\mathcal{B}(P)$, using (3.39):

$$e^{-\frac{1}{12}(\delta-\tilde{\delta})c_4\beta|\mathcal{B}(P)|}\,. \tag{3.40}$$

A second factor $e^{-\frac{1}{12}\beta}$ contributes to the weight of the legs. Extracting this contribution gives

$$\prod_{e\in\mathcal{L}(P)}(e^{-\beta w_e^+}-1)\prod_{i\in\mathcal{L}(P)}e^{-\frac{1}{12}\beta}\,, \tag{3.41}$$

The last factor $e^{-\frac{1}{12}\beta}$ is used to re-sum over all the possible configurations of T inside the body $\mathcal{B}(P)$ (see Figure 5), that is over all forests T', $V(T') \subset \mathcal{B}(P)$, where each tree $\mathtt{t}' \in T'$ gets a weight bounded by

$$\omega_0(\mathtt{t}') := \prod_{e\in E(\mathtt{t}')}(e^{-\beta w_e^+}-1)\prod_{i\in V(\mathtt{t}')}e^{-\frac{1}{12}\beta}\,. \tag{3.42}$$

The remaining sum is thus bounded by

$$\sum_{T':V(T')\subset\mathcal{B}(P)}\prod_{\mathtt{t}'\in T'}\omega_0(\mathtt{t}') \equiv \Theta_0(\mathcal{B}(P))\,. \tag{3.43}$$

This partition function can be studied with a convergent cluster expansion. Proceeding as we did in Corollary 3.1, we can take β sufficiently large so that the weight $\omega_0(\mathtt{t}')$ satisfies (3.20). We can then guarantee that

$$|\log\Theta_0(\mathcal{B}(P))| \leq |\mathcal{B}(P)|\,. \tag{3.44}$$

The sum over all possible sets M such that $N(M)$ has a set of vertices given by $\mathcal{B}(P)$ is bounded by $2^{|\mathcal{B}(P)|}$. Altogether these bounds give

$$e^{-\frac{1}{12}(\delta-\tilde{\delta})c_4\beta|\mathcal{B}(P)|}2^{|\mathcal{B}(P)|}e^{|\mathcal{B}(P)|} \equiv e^{-\tau_0\beta|\mathcal{B}(P)|}\,,$$

which finishes the proof. □

We now give the consequence of this lemma, namely that polymers satisfy the main criterion needed for having a convergent cluster expansion.

Corollary 3.2. *Let $0 < c \le \min(\frac{\tau_0}{2}, \frac{1}{24})\beta$, $\epsilon > 0$. There exists $\beta_3 = \beta_3(\epsilon)$, such that for all $\beta \ge \beta_3$ and for all $\gamma \in (0,\gamma_0)$, the following holds:*

$$\sum_{P:V(P)\ni 0} \|\omega^+(P)\|_{H_+} e^{c|V(P)|} \le \epsilon\,. \tag{3.45}$$

Proof. Lemma 3.2 allows to bound

$$\|\omega^+(P)\|_{H_+} \le \Big(\prod_{N\in P}\omega_0(N)\Big)\Big(\prod_{\mathtt{t}\in\mathcal{L}(P)}\omega_0(\mathtt{t})\Big) \equiv \omega_0(P)\,, \tag{3.46}$$

where the weight of each component of the body N is $\omega_0(N) := e^{-\tau_0\beta|V(N)|}$ and the weight of each leg $\mathtt{t}$ was defined in (3.42). Fix $\epsilon > 0$ small. It is easy to show that when β is large enough,

$$\sum_{N:V(N)\ni 0} \omega_0(N)e^{(c+\epsilon)|V(N)|} \le \tfrac{1}{2}\epsilon\,, \tag{3.47}$$

and, proceeding like in Corollary 3.1,

$$\sum_{\mathtt{t}:V(\mathtt{t})\ni 0} \omega_0(\mathtt{t})e^{(c+\epsilon)|V(\mathtt{t})|} \le \tfrac{1}{2}\epsilon\,. \tag{3.48}$$

Let $n(P)$ denote the number of objects (components N and trees $\mathtt{t}$) contained in P. That is, if $P = \{\mathtt{t}_1,\dots,\mathtt{t}_L, N_1,\dots,N_K\}$, then $n(P) = L+K$. We will show by induction on $N = 1,2,\dots$ that

$$\lambda_N := \sum_{\substack{P:V(P)\ni 0\\ n(P)\le N}} \omega_0(P)e^{c|V(P)|} \le \epsilon\,, \tag{3.49}$$

which will finish the proof. If $N = 1$ then P can be either a single component N or a tree $\mathtt{t}$. The bound then follows from (3.47) and (3.48). Suppose β is large and that the bound holds for N. If P satisfies $V(P)\ni 0$, $n(P) \le N+1$, we choose an object of P that contains the origin (which can be a tree $\mathtt{t}_0$ or a component N_0), and decompose P as follows: either $P = \{N_0\}\cup\{P_1,\dots,P_k\}$ with $V(N_0)\ni 0$, $V(P_i)\cap V(N_0) \ne \emptyset$, $n(P_i)\le N$, $P_i \sim P_j$ for $i \ne j$, or $P = \{\mathtt{t}_0\}\cup\{P_1,\dots,P_k\}$ with $V(\mathtt{t}_0)\ni 0$, and

$V(P_i)\cap V(\mathtt{t}_0)\neq\emptyset$, $n(P_i)\leq N$, $P_i\sim P_j$ for $i\neq j$. In the first case, we have, using the induction hypothesis and (3.47),

$$\sum_{N_0:V(N_0)\ni 0}\omega_0(N_0)e^{c|V(N_0)|}\sum_{k\geq 0}\frac{1}{k!}\Bigl(\sum_{\substack{P:V(P)\cap V(N_0)\neq\emptyset\\ n(P)\leq N}}\omega_0(P)e^{c|V(P)|}\Bigr)^k \tag{3.50}$$

$$\leq \sum_{N_0:V(N_0)\ni 0}\omega_0(N_0)e^{c|V(N_0)|}\sum_{k\geq 0}\frac{1}{k!}\bigl(|V(N_0)|\lambda_N\bigr)^k \tag{3.51}$$

$$\leq \sum_{N_0:V(N_0)\ni 0}\omega_0(N_0)e^{c|V(N_0)|}e^{\epsilon|V(N_0)|}\leq \tfrac{1}{2}\epsilon\,. \tag{3.52}$$

In the second case the same computation yields, using (3.48),

$$\sum_{\mathtt{t}_0:V(\mathtt{t}_0)\ni 0}\omega_0(\mathtt{t}_0)e^{c|V(\mathtt{t}_0)|}\sum_{k\geq 0}\frac{1}{k!}\Bigl(\sum_{\substack{P:V(P)\cap V(\mathtt{t}_0)\neq\emptyset\\ n(P)\leq N}}\omega_0(P)e^{c|V(P)|}\Bigr)^k$$

$$\leq \sum_{\mathtt{t}_0:V(\mathtt{t}_0)\ni 0}\omega_0(\mathtt{t}_0)e^{c|V(\mathtt{t}_0)|}e^{\epsilon|V(\mathtt{t}_0)|}\leq \tfrac{1}{2}\epsilon\,. \tag{3.53}$$

This shows that $\lambda_{N+1}\leq\epsilon$ and finishes the proof. □

We now state the main result concerning restricted phases and their analyticity properties, again only for the case $\#=+$. We refer to Appendix A for notations. Here polymers play the role of animals. Clusters of polymers associated to $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ are denoted $\hat{P}\in\hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})$. By Lemma A.1, (3.45) implies

$$\sup_{x\in\Lambda}\sum_{\hat{P}\ni x}\|\omega^+(\hat{P})\|_{H_+}\leq\sup_{x\in\Lambda}\sum_{\hat{P}\ni x}|\omega_0(\hat{P})|\leq\eta(\epsilon)\,, \tag{3.54}$$

where the weights $\omega^+(\hat{P})$ and $\omega_0(\hat{P})$ are defined like in (A.3). Since ϵ can be made arbitrarily small by taking β large enough, we will replace $\eta(\epsilon)$ by a function $\epsilon_r(\beta)$, where the subscript r is to indicate that this function concerns the restricted phase. We define $\tilde{H}_+ := \{\mathrm{Re}\,h > -\frac{1}{16}\}\subset H_+$.

Theorem 3.1. *Let β be large enough, $\gamma\in(0,\gamma_0)$. Let $\Lambda\in\mathcal{C}^{(l)}$ and η_{Λ^c} be a $+$-admissible boundary condition. Then $\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))$ has a cluster expansion that converges normally in H_+, given by*

$$\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))=\sum_{\hat{P}\in\hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})}\omega^+(\hat{P})\,. \tag{3.55}$$

The maps $h\mapsto\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))$, $h\mapsto p^+_{r,\gamma}(h)$ are analytic in H_+. Moreover *there exists a function* $\epsilon_r(\beta)$, $\lim_{\beta\nearrow\infty}\epsilon_r(\beta)=0$, *such that*

$$\big\|\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\big\|_{H_+}\leq\epsilon_r(\beta)|\Lambda|\,,\qquad\sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})\\ \hat{P}\ni 0}}\|\omega^+(\hat{P})\|_{H_+}\leq\epsilon_r(\beta)\,,$$

$$\big\|\frac{\mathrm{d}}{\mathrm{d}h}\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\big\|_{\tilde{H}_+}\leq\epsilon_r(\beta)|\Lambda|\,. \tag{3.56}$$

The proof of the theorem follows easily from Lemma A.1. Analyticity follows from the fact that the convergence is normal on H_+. The bound on the first derivative is obtained by using the Cauchy formula: any disc of radius $\frac{1}{16}$ centered at $z\in\tilde{H}_+$ is contained in H_+. This also implies the existence of a constant $C_r>0$ such that for all integer $k\geq 2$,

$$\frac{1}{|\Lambda|}\left|\frac{\mathrm{d}^k}{\mathrm{d}h^k}\log {Z_r}^+(\Lambda;\eta_{\Lambda^c})\right|_{h=0}\leq C_r^k k!\,,\qquad|p^{+(k)}_{r,\gamma}(0)|\leq C_r^k k!\,. \tag{3.57}$$

4 The Phase Diagram

Throughout this section and until the end of the paper we assume $\gamma\in(0,\gamma_0)$ is fixed, where γ_0 was given in (3.23). To start with, consider the partition function

$$Z^+(\Lambda):=\sum_{\sigma_\Lambda\in\Omega^+_\Lambda}e^{-\beta H_\Lambda(\sigma_\Lambda+_{\Lambda^c})}\,, \tag{4.1}$$

where

$$\Omega^+_\Lambda:=\{\sigma_\Lambda\in\Omega_\Lambda:d(I^*(\sigma_\Lambda+_{\Lambda^c}),\Lambda^c)>l\}\,. \tag{4.2}$$

For each $\sigma_\Lambda\in\Omega^+_\Lambda$, the decomposition of $I^*(\sigma_\Lambda+_{\Lambda^c})$ into connected components yields an admissible family $\{\Gamma\}$, such that $\Gamma\subset\Lambda$ and $d(\Gamma,\Lambda^c)>l$ for each $\Gamma\in\{\Gamma\}$. Then, Λ is decomposed into $\Lambda=\{\Gamma\}\cup\Lambda^+\cup\Lambda^-$, where $\Lambda^\#$ are the points of $\Lambda\backslash\{\Gamma\}$ that are $(\delta,\#)$-correct for the configuration $\sigma_\Lambda+_{\Lambda^c}$.
In (4.1), we re-sum over the configurations σ_{Λ^+} (resp. σ_{Λ^+}) on Λ^+ (resp. Λ^-) that yield the same set of contours $\{\Gamma\}$. In Proposition 2.1 we characterized explicitly the constraints satisfied by the configurations $\sigma_{\Lambda^\pm}$: each point $i\in[\Lambda^+]_R$ must be $(\delta,+)$-correct for the configuration $\sigma_{\Lambda^+}+_{\Lambda^c}\sigma_{\{\Gamma\}}$, where $\sigma_{\{\Gamma\}}$ is the configuration specified by the contours on the union of their supports. Similarly, each point $i\in[\Lambda^-]_R$ must be $(\delta,-)$-correct for the configuration $\sigma_{\Lambda^-}\sigma_{\{\Gamma\}}$. Using the re-formulation of the hamiltonian given in Lemma 2.6 we get:

$$Z^+(\Lambda)=\sum_{\{\Gamma\}\subset\Lambda}\Big(\prod_{\Gamma\in\{\Gamma\}}\rho(\Gamma)\Big){Z_r}^+(\Lambda^+;+_{\Lambda^c}\sigma_{\{\Gamma\}}){Z_r}^-(\Lambda^-;\sigma_{\{\Gamma\}})\,, \tag{4.3}$$

where the sum is over admissible families of contours, and

$$\rho(\Gamma) := e^{-\beta H_\Gamma(\sigma[\Gamma])} \,. \tag{4.4}$$

Notice that when $\{\Gamma\} = \emptyset$, then $\Lambda \equiv \Lambda^+$ and the summand of (4.3) equals ${Z_r}^+(\Lambda; +_{\Lambda^c})$. Since they are subject to boundary conditions that depend on the family of contours $\{\Gamma\}$, the restricted phases induce an interaction among the contours. Nevertheless, the boundary conditions imposed by the contours and $+_{\Lambda^c}$ on Λ^+ and Λ^- are admissible (in the sense of Definition 3.1). This implies that the results of Section 3 can be used for the restricted partition functions appearing in (4.3).

Since we need to represent the partition function with objects whose compatibility is purely geometrical, we need to proceed by induction, and consider systems living in the interior of external contours. Therefore, we must study functions similar to (4.3), with an arbitrary +-admissible boundary condition η_{Λ^c}. We thus define

$$\Theta^+(\Lambda; \eta_{\Lambda^c}) := \sum_{\{\Gamma\} \subset \Lambda} \Big(\prod_{\Gamma \in \{\Gamma\}} \rho(\Gamma) \Big) {Z_r}^+(\Lambda^+; \eta_{\Lambda^c}\sigma_{\{\Gamma\}}) {Z_r}^-(\Lambda^-; \sigma_{\{\Gamma\}}) \,. \tag{4.5}$$

Contours always lie at least at distance l from Λ^c. The external contours of $\{\Gamma\}$ can be subject to particular constraints (as will appear, for example, in Section 5), but we omit it in the notation. Notice that for the empty family $\{\Gamma\} = \emptyset$, the summand corresponds to a pure restricted phase ${Z_r}^+(\Lambda; \eta_{\Lambda^c})$.

The aim, in the study of $\Theta^+(\Lambda; \eta_{\Lambda^c})$, is to extract from (4.5) a global contribution of the restricted phase. In the Ising model, the same operation amounts to extract the trivial term $e^{\beta h |\Lambda|}$. Here we extract ${Z_r}^+(\Lambda, \eta_{\Lambda^c}) = e^{\beta h|\Lambda|} \mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))$, and our aim is to reach the representation (4.17). The deviations from the restricted phase will be described by *chains*, i.e. contours linked by clusters of polymers (polymers describe the restricted phase). In Section 4.1, we expose this linking procedure. In Section 4.2 we show how to handle the entropy of chains, preserving the uniformity in the scaling parameter γ. In Section 4.3 we study the weights of chains and their dependence on the magnetic field near $\mathrm{Re}\, h = 0$, i.e. at coexistence. In Section 4.4 we study pure phases, i.e. $\{\mathrm{Re}\, h > 0\}$ and $\{\mathrm{Re}\, h < 0\}$.

4.1 The Linking Procedure

We first express $\Theta^+(\Lambda; \eta_{\Lambda^c})$ as a sum over external contours. By Lemma 2.5, each external contour is of type +. Let $\{\Gamma\}$ be a family of external contours. Then, Λ is decomposed into

$$\Lambda = \mathrm{ext}_\Lambda\{\Gamma\} \cup \{\Gamma\} \cup \bigcup_{\Gamma \in \{\Gamma\}} \mathrm{int}\Gamma \,,$$

where $\mathrm{ext}_\Lambda\{\Gamma\} := \Lambda \cap \bigcap_{\Gamma\in\{\Gamma\}} \mathrm{ext}\Gamma$. For each family of admissible external contours $\{\Gamma\}$, we re-sum over the configurations whose external contours are given exactly by $\{\Gamma\}$. This induces, for all Γ, a partition function $\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)$, which can be expressed as in (4.5). On $\mathrm{ext}_\Lambda\{\Gamma\}$, we get a restricted partition function $Z_r{}^+(\mathrm{ext}_\Lambda\{\Gamma\}; \eta_{\Lambda^c}\sigma_{\{\Gamma\}})$. We thus have

$$\Theta^+(\Lambda;\eta_{\Lambda^c}) = \\ Z_r{}^+(\Lambda;\eta_{\Lambda^c}) + \sum_{\substack{\{\Gamma\}\subset\Lambda \\ \text{ext.}}} Z_r{}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}}) \prod_\Gamma \rho(\Gamma)\Theta^-(\mathrm{int}\Gamma;\sigma_\Gamma)\,, \tag{4.6}$$

where the sum is over non-empty families of external contours. Consider the configuration $-\sigma_\Gamma$ obtained by spin-flipping σ_Γ, i.e. $(-\sigma_\Gamma)_i := -(\sigma_\Gamma)_i$ for all $i \in \Gamma$. We introduce the functions $Z_r{}^+(\mathrm{int}\Gamma; -\sigma_\Gamma)$ and $\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)$ and consider, for a while, the ratio

$$\frac{Z_r{}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}})\prod_\Gamma Z_r{}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{Z_r{}^+(\Lambda;\eta_{\Lambda^c})}\,. \tag{4.7}$$

Using the polymer representation of Section 3, we consider the family of polymers $\mathcal{P}^+_{\mathrm{ext}} := \mathcal{P}^+_{\mathrm{ext}_\Lambda\{\Gamma\}}(\eta_{\Lambda^c}\sigma_{\{\Gamma\}})$ associated to $Z_r{}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}})$, the families $\mathcal{P}^+_{\mathrm{int}\Gamma} := \mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma)$ associated to each of the $Z_r{}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)$, as well as the family $\mathcal{P}^+_\Lambda := \mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ associated to $Z_r{}^+(\Lambda;\eta_{\Lambda^c})$. Since the expansions of these functions are absolutely convergent, we can rearrange the terms. The volume contributions from $\mathrm{ext}_\Lambda\{\Gamma\}$ and $\bigcup_\Gamma \mathrm{int}\Gamma$ cancel, and we get

$$\frac{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{ext}})\prod_\Gamma \mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^+_\Lambda)} = \exp\Big(\sum_{\hat P} \pm\omega^+(\hat P) + \sum_\Gamma E^+_\Gamma\Big)\,,$$

where we used the abbreviation

$$\sum_{\hat P} \pm\omega^+(\hat P) \equiv \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+_{\mathrm{ext}} \\ d(\hat P,\{\Gamma\})\le R}} \omega^+(\hat P) - \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+_\Lambda \\ d(\hat P,\{\Gamma\})\le R \\ \hat P\cap \mathrm{ext}_\Lambda\{\Gamma\}\neq\emptyset}} \omega^+(\hat P)\,. \tag{4.8}$$

The sign $\pm$ in front of $\omega^+(\hat P)$ is chosen in function of the sum to which $\hat P$ belongs. Define $\lambda^+(\hat P) := e^{\pm\omega^+(\hat P)} - 1$ and expand

$$e^{\sum_{\hat P}\pm\omega^+(\hat P)} = \prod_{\hat P}(1+\lambda^+(\hat P)) = \sum_{\{\hat P_1,\dots,\hat P_n\}} \prod_{i=1}^n \lambda^+(\hat P_i)\,. \tag{4.9}$$

The function E^+_Γ depends only on the structure of Γ, and is given by

$$E^+_\Gamma = \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+_{\mathrm{int}\Gamma} \\ d(\hat P,\Gamma)\le R}} \omega^+(\hat P) - \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+ \\ \hat P\cap\mathrm{ext}\Gamma=\emptyset \\ d(\hat P,\Gamma)\le R}} \omega^+(\hat P)\,, \tag{4.10}$$

where $\hat{\mathcal{P}}^+$ denotes the family of clusters associated to free polymers of type $+$. Notice that E^+_Γ is analytic in H_+. Since $|[\Gamma]_R| \le 3^d|\Gamma|$ we have, if β is large enough (see Theorem 3.1)

$$\|E^+_\Gamma\|_{H_+} \le \frac{1}{3}|\Gamma|\,, \quad \|\frac{\mathrm{d}}{\mathrm{d}h}E^+_\Gamma\|_{\tilde{H}_+} \le \frac{1}{3}|\Gamma|\,. \tag{4.11}$$

If we define the weight (we denote $+\sigma_\Gamma \equiv \sigma_\Gamma$)

$$\omega^+(\Gamma) := \rho_1(\Gamma)\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\,, \tag{4.12}$$

with $\rho_1(\Gamma) := \rho(\Gamma)e^{-\beta h|\Gamma|}e^{E^+_\Gamma}$, we have

$$\frac{\Theta^+(\Lambda;\eta_{\Lambda^c})}{{Z_r}^+(\Lambda;\eta_{\Lambda^c})} = 1 + \sum_{\substack{\{\Gamma\}\subset\Lambda\\ \text{ext.}}}\sum_{\{\hat{P}_1,\dots,\hat{P}_n\}}\Big(\prod_{i=1}^n \lambda^+(\hat{P}_i)\Big)\Big(\prod_\Gamma \omega^+(\Gamma)\frac{\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{{Z_r}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}\Big)\,.$$

We can then repeat the same procedure of summing inside external contours of $\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)$, etc. This procedure continues until we reach contours whose interior can't contain any contour. At the end,

$$\frac{\Theta^+(\Lambda;\eta_{\Lambda^c})}{{Z_r}^+(\Lambda;\eta_{\Lambda^c})} = 1 + \sum_{\{\Gamma\}\subset\Lambda}\sum_{\{\hat{P}\}}\Big(\prod_{\hat{P}} \lambda^+(\hat{P})\Big)\Big(\prod_\Gamma \omega^+(\Gamma)\Big)\,, \tag{4.13}$$

where the sum over $\{\Gamma\} \subset \Lambda$ contains contours of type $+$, and each cluster $\hat{P}$ lies at distance at most R from one or several contours of $\{\Gamma\}$. For this reason, the weight of some polymers can depend on the configuration σ_Γ of the contours Γ that lie in their neighbourhood (or on η_{Λ^c}).
We get rid of these dependencies by linking polymers to contours. Like we did in Section 3 (when linking trees with components of the graph N), we associate to each pair $(\{\Gamma\},\{\hat{P}\})$ an abstract graph $\hat{G}$ as follows: each contour $\Gamma_j \in \{\Gamma\}$ is represented by an abstract vertex z_j, each cluster $\hat{P}_k \in \{\hat{P}\}$ is represented by an abstract vertex w_k. This defines $V(\hat{G})$. Then, we put an edge between z_j and w_k if and only if $d(\Gamma_j, \hat{P}_k) \le R$. We also put an edge between w_{k_1} and w_{k_2} if and only if $V(\hat{P}_{k_1}) \cap V(\hat{P}_{k_2}) \ne \emptyset$. Each connected component of $\hat{G}$, with vertices, say, $\{z_{j_1},\dots,z_{j_l},w_{k_1},\dots,w_{k_l}\}$, represents a subset of $\{\Gamma\}\cup\{\hat{P}\}$ given by $X = \{\Gamma_{j_1},\dots,\Gamma_{j_l},\hat{P}_{k_1},\dots,\hat{P}_{k_l}\}$. X is called a chain of contours, or simply a chain. We denote by $\{X\}$ the family of chains associated to the pair $(\{\Gamma\},\{\hat{P}\})$. The chains of $\{X\}$ are of type $+$, and pairwise compatible by definition. The support of X, also written X, denotes the union $\bigcup_{\Gamma\in X}\Gamma \cup \bigcup_{\hat{P}\in X}\hat{P}$. Notice that if two chains X, X' are not compatible, then $b(X)\cap b(X') \ne \emptyset$, where

$$b(X) := \bigcup_{\Gamma\in X}[\Gamma]_l \cup \bigcup_{\hat{P}\in X}\hat{P}\,. \tag{4.14}$$

The weight of a chain is defined by

$$\omega^+(X) := \Big(\prod_{\hat{P}\in X} \lambda^+(\hat{P})\Big)\Big(\prod_{\Gamma\in X} \omega^+(\Gamma)\Big)\,, \tag{4.15}$$

and depends only on the intrinsic structure of the chain X (except, maybe, if $d(X,\Lambda^c)\leq R$). The final representation of the partition function is thus

$$\Theta^+(\Lambda;\eta_{\Lambda^c}) = {Z_r}^+(\Lambda;\eta_{\Lambda^c})\sum_{\{X\}}\prod_{X\in\{X\}} \omega^+(X) \tag{4.16}$$

$$\equiv {Z_r}^+(\Lambda;\eta_{\Lambda^c})\Xi^+(\Lambda;\eta_{\Lambda^c})\,. \tag{4.17}$$

In (4.16), the product is defined to be equal to 1 when $\{X\}=\emptyset$. This last expression nicely expresses the fact that chains of contours describe deviations from a restricted phase. For the restricted phase, there corresponds a family $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ associated to ${Z_r}^+(\Lambda;\eta_{\Lambda^c})$. Similarly, there corresponds a family of chains $\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})$ associated to $\Xi^+(\Lambda;\eta_{\Lambda^c})$. The partition function can be written in terms of these families as

$$\Theta^+(\Lambda;\eta_{\Lambda^c}) = e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\Xi(\mathcal{X}^+_\Lambda(\eta_{\Lambda^c}))\,. \tag{4.18}$$

By definition, $\Xi(\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})) := 1$ when $\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})=\emptyset$. Everything that was done until now can be applied also to the case where η_{Λ^c} is $-$-admissible, yielding chains of type $-$.

4.2 The Entropy of Chains

Before starting the analysis of the weights, we show how a priori bounds on the weights $\lambda^+(\hat{P})$ and $\omega^+(\Gamma)$ allow to handle the summation of weights of chains. In this section we assume that $|\lambda^+(\hat{P})|\leq\lambda_0(\hat{P})$, $|\omega^+(\Gamma)|\leq\rho_0(\Gamma)$, i.e.

$$|\omega^+(X)| \leq \Big(\prod_{\hat{P}\in X}\lambda_0(\hat{P})\Big)\Big(\prod_{\Gamma\in X}\rho_0(\Gamma)\Big) \equiv \omega_0(X)\,. \tag{4.19}$$

Convention: Now and in the sequel we will always use a subscript “0” in the weight of an object to specify that it depends only on the geometric structure of the object (as we did in (3.46), Section 3.2). That is, such weights will always be translation invariant. When a weight is defined for an object, we use the same letter for the weight of the clusters of such objects (see Appendix A).

The proof of the following lemma is essentially the same as the one of Corollary 3.2. We use the notations $|\hat{P}| := |\bigcup_{P\in\hat{P}}V(P)|$, $|X| := \sum_{\Gamma\in X}|\Gamma| + \sum_{\hat{P}\in X}|\hat{P}|$.

Lemma 4.1. *Let $c > 0$, $\epsilon > 0$, and assume the weights $\lambda_0(\hat{P})$, $\rho_0(\Gamma)$ satisfy the bounds*

$$\sum_{\hat{P}\ni 0} \lambda_0(\hat{P}) e^{(c+\epsilon(2^d+1))|\hat{P}|} \le \frac{\epsilon}{2}, \qquad \sum_{\Gamma:[\Gamma]_l\ni 0} \rho_0(\Gamma) e^{(c+\epsilon)|[\Gamma]_l|} \le \frac{\epsilon}{2}. \tag{4.20}$$

Then the weight $\omega_0(X)$ satisfies the condition (A.4) *of Lemma A.1. Namely,*

$$\sum_{X:b(X)\ni 0} \omega_0(X) e^{c|b(X)|} \le \epsilon. \tag{4.21}$$

Proof. For a chain $X = \{\Gamma_1, \dots, \Gamma_L, \hat{P}_1, \dots, \hat{P}_M\}$, let $n(X) := L + M$ denote the number of objects composing X (a cluster $\hat{P}_i$ is considered as a single object). We show by induction on $N = 1, 2, \dots$ that

$$\xi_N := \sum_{\substack{X:b(X)\ni 0\\ n(X)\le N}} \omega_0(X) e^{c|b(X)|} \le \epsilon. \tag{4.22}$$

If $n(X) = 1$ then X contains a single object, i.e. a contour. Then $\xi_1 \le \epsilon$ follows from (4.20). So suppose (4.22) holds for N, and consider ξ_{N+1}; this sum can be bounded by a sum in which each chain X is decomposed into $[\Gamma_0]_l \ni 0$, $X \ni \Gamma_0$, or into $\hat{P}_0 \ni 0$, $X \ni \hat{P}_0$. This means:
1) in the first case, X decomposes into $X = \{\Gamma_0\} \cup \{X_1, \dots, X_K\}$ [9] with $[\Gamma_0]_l \ni 0$, $d(X_i, \Gamma_0) \le R$, $n(X_i) \le N$ for all $i = 1, \dots, K$, $X_i \cap X_j = \emptyset$ for all $i \ne j$. The contribution to ξ_{N+1} is thus bounded by

$$\begin{aligned}
&\sum_{\Gamma_0:[\Gamma_0]_l\ni 0} \rho_0(\Gamma_0) e^{c|[\Gamma_0]_l|} \sum_{K\ge 0} \frac{1}{K!} \prod_{i=1}^{K} \sum_{\substack{X_i:d(X_i,\Gamma_0)\le R\\ n(X_i)\le N}} \omega_0(X_i) e^{c|b(X_i)|} \qquad (4.23)\\
&\le \sum_{\Gamma_0:[\Gamma_0]_l\ni 0} \rho_0(\Gamma_0) e^{c|[\Gamma_0]_l|} \sum_{K\ge 0} \frac{1}{K!} \big(|[\Gamma_0]_R| \xi_N\big)^K\\
&\le \sum_{\Gamma_0:[\Gamma_0]_l\ni 0} \rho_0(\Gamma_0) e^{(c+\epsilon)|[\Gamma_0]_l|} \le \frac{\epsilon}{2},
\end{aligned}$$

where we used the induction hypothesis $\xi_N \le \epsilon$.
2) in the second case, $X = \{\hat{P}_0\} \cup \{X_1, \dots, X_K\}$ with $\hat{P}_0 \ni 0$, $d(X_i, \hat{P}_0) \le R$, $n(X_i) \le N$ for all $i = 1, \dots, K$, $X_i \cap X_j = \emptyset$ for all $i \ne j$. A chain X_i of this decomposition can be of two types: i) there exists a cluster $\hat{P} \in X_i$ such that $\hat{P} \cap \hat{P}_0 \ne \emptyset$. Then the contribution from these chains is at most

$$|\hat{P}_0| \sum_{\substack{X_i:b(X_i)\ni 0\\ n(X_i)\le N}} \omega_0(X_i) e^{c|b(X_i)|} = |\hat{P}_0| \xi_N \le |\hat{P}_0| \epsilon. \tag{4.24}$$

[9] The chains X_i are obtained as follows: consider the abstract connected graph $\hat{G}$ associated to the chain X. Then, remove all the edges of $\hat{G}$ that are adjacent to the vertex z_0 representing Γ_0 and z_0 itself, and consider the decomposition of the remaining graph into connected components. These components are exactly the representatives of $X_1, \dots, X_K$.

ii) there exists $\Gamma \in X_i$, $\Gamma \cap \{[\hat{P}_0]_R\}_l \neq \emptyset$, where the thickening $\{\cdot\}_l$ was defined in (2.15). Notice that the set $\{[\hat{P}_0]_R\}_l \in \mathcal{C}^{(l)}$ contains at most $2^d|\hat{P}_0|$ cubes $C^{(l)}$. Since contours are composed of cubes $C^{(l)}$, the contribution from these chains can be bounded by

$$2^d|\hat{P}_0|\xi_N \leq 2^d\epsilon|\hat{P}_0|\,. \tag{4.25}$$

We can then proceed like in (4.23), and get a contribution to ξ_{N+1} bounded by

$$\sum_{\hat{P}_0 \ni 0} \lambda_0(\hat{P}_0)e^{c|\hat{P}_0|}e^{\epsilon(2^d+1)|\hat{P}_0|} \leq \frac{\epsilon}{2}\,. \tag{4.26}$$

Altogether, this shows that $\xi_{N+1} \leq \epsilon$. □

4.3 Domains of Analyticity

In this section we consider the dependence of the weights $\omega^+(X)$ on the magnetic field $h \in \mathbb{C}$, in a neighbourhood of $\{\mathrm{Re}\, h = 0\}$. For obvious reasons, the domain in which $\omega^+(X)$ can be shown to be analytic depends on the contour $\Gamma \in X$ that has the largest interior. Everything we say in this section holds for chains of both types, but for the sake of simplicity, the statements will be given only for chains of type $+$.
The domains of analyticity depend on the isoperimetric constants $K(N)$ defined in (2.44). Consider the reals

$$R(N) := \frac{\theta}{2K(N)N^{\frac{1}{d}}}\,, \tag{4.27}$$

where $\theta \in (0,1)$ will play an important role later in the study of the derivatives. We know from Lemma 2.9 that $R(N)N^{\frac{1}{d}}$ is increasing and that

$$\lim_{N\to\infty} R(N)N^{\frac{1}{d}} = \frac{\theta}{2K(\infty)}\,. \tag{4.28}$$

Since we want the domains of analyticity to be decreasing with the size of the contours, we define

$$R^*(N) := \min\{R(N') : 1 \leq N' \leq N\}\,. \tag{4.29}$$

The sequences $R^*(N)$ and $R(N)$ have the same asymptotic behaviour, as the following lemma shows.

Lemma 4.2.

$$\lim_{N\to\infty} R^*(N)N^{\frac{1}{d}} = \frac{\theta}{2K(\infty)}\,. \tag{4.30}$$

Proof. First notice that there exists an unbounded increasing sequence $N_1, N_2, \ldots$, such that $R^*(N_i) = R(N_i)$. This is a direct consequence of the bounds

$$R^*(N) \le R(N) \le \frac{\theta}{2K(\infty)N^{\frac{1}{d}}}\,. \tag{4.31}$$

Since $R(N)N^{\frac{1}{d}}$ increases, it is sufficient to show that $R^*(N)N^{\frac{1}{d}}$ is increasing. Consider the interval $[N, N+1]$. We have two possibilities: 1) $R(N+1) \ge R^*(N)$. In this case, $R^*(N+1)(N+1)^{\frac{1}{d}} = R^*(N)(N+1)^{\frac{1}{d}} \ge R^*(N)N^{\frac{1}{d}}$. 2) $R(N+1) \le R^*(N)$. In this case, $R^*(N+1)(N+1)^{\frac{1}{d}} = R(N+1)(N+1)^{\frac{1}{d}} \ge R(N)N^{\frac{1}{d}} \ge R^*(N)N^{\frac{1}{d}}$. $\square$

For $r > 0$, consider the strip

$$U(r) := \{z \in \mathbb{C} : |\mathrm{Re}\, z| < r\}\,. \tag{4.32}$$

Generally, we will restrict our attention to small magnetic fields, that is $h \in U_0 := U(h_0)$ where h_0 will be taken small enough. For instance, $h_0 < \frac{1}{16}$ so that the results on the restricted phases can be used in U_0.
We define the domain of analyticity for a contour:

$$U_\Gamma := U(R^*(V(\Gamma))) \cap U_0\,, \tag{4.33}$$

and for a chain X:

$$U_X := \bigcap_{\Gamma \in X} U_\Gamma\,. \tag{4.34}$$

That is, $U_X = U_{\Gamma^{\max}}$, where $\Gamma^{\max} \in X$ has the largest interior. Notice that the domains U_Γ, U_X depend on θ. Set $V(X) := V(\Gamma^{\max}) = \max\{V(\Gamma) : \Gamma \in X\}$. The main result of this section is the following.

Proposition 4.1. *Let $\theta \in (0,1)$, $\epsilon > 0$, $c > 0$ small enough. There exists $\beta_1 = \beta_1(\theta, \epsilon)$ such that for all $\beta \ge \beta_1$, the following holds. For each chain X, $h \mapsto \omega^+(X)$ is analytic in U_X. Moreover,*

$$\|\omega^+(X)\|_{U_X} < \omega_0(X)\,, \quad \Big\|\frac{\mathrm{d}}{\mathrm{d}h}\omega^+(X)\Big\|_{U_X} < \omega_0(X)\,, \tag{4.35}$$

where $\omega_0(X)$ is defined via the weights $\lambda_0(\hat{P})$ and $\rho_0(\Gamma)$ given in (4.37)-(4.38) *hereafter, and satisfies* (4.21)*.*

Before starting the proof of Proposition 4.1, we give explicitly the weights $\lambda_0(\hat{P})$ and $\rho_0(\Gamma)$. These weights are defined such that they can be used throughout the section, also when bounding the first derivative of $\omega^+(X)$. As will be seen, the non-trivial part of $\omega^+(\Gamma)$ will be bounded by:

$$\Big\|\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\Big\|_{U_\Gamma} \le e^{\beta\theta\|\Gamma\|} e^{\frac{2}{3}|\Gamma|}\,. \tag{4.36}$$

Using (4.11), $\|\rho_1(\Gamma)\|_{U_0} \le e^{-\beta\|\Gamma\|} e^{2\beta h_0|\Gamma|} e^{\frac{1}{3}|\Gamma|}$. This suggests to define the weight $\rho_0(\Gamma)$ in the following way:

$$\rho_0(\Gamma) := D_1 \beta |\Gamma|^{\frac{d}{d-1}} e^{-(1-\theta)\beta\|\Gamma\|} e^{2\beta h_0|\Gamma|} e^{|\Gamma|} . \tag{4.37}$$

The term $D_1\beta|\Gamma|^{\frac{d}{d-1}}$ has been added to take into account other contributions, especially when studying the first derivative. For clusters we get, using the definition of $\lambda^+(\hat{P})$ and (3.54),

$$\begin{aligned} \|\lambda^+(\hat{P})\|_{H_+} &\le \|\omega^+(\hat{P})\|_{H_+} e^{\|\omega^+(\hat{P})\|_{H_+}} \\ &\le |\omega_0(\hat{P})| e^{|\omega_0(\hat{P})|} \le |\omega_0(\hat{P})| e^{\epsilon_r} < D_2|\omega_0(\hat{P})| \equiv \lambda_0(\hat{P}) . \end{aligned} \tag{4.38}$$

The numerical constants D_1, D_2 are assumed to be fixed and sufficiently large, in order to cover all the cases that will appear in the sequel.

Lemma 4.3. *Let $\theta \in (0,1)$, $c > 0$, and $\epsilon > 0$ be small enough. Assume $2h_0 \le \frac{1}{2}(1-\theta)\rho$ (ρ is the Peierls constant). There exists $\beta_1 = \beta_1(\theta, \epsilon)$ such that for all $\beta \ge \beta_1$, the hypothesis* (4.20) *of Lemma 4.1 are satisfied.*

Proof. Define a new weight for polymers (see (3.46)):

$$\widetilde{\omega}_0(P) := \omega_0(P) e^{(c+\epsilon(2^d+1))|P|} . \tag{4.39}$$

If β is large enough, we can proceed as in (3.54) and get

$$\begin{aligned} \sum_{\hat{P}\ni 0} \lambda_0(\hat{P}) e^{(c+\epsilon(2^d+1))|\hat{P}|} &= D_2 \sum_{\hat{P}\ni 0} |\omega_0(\hat{P})| e^{(c+\epsilon(2^d+1))|\hat{P}|} \\ &\le D_2 \sum_{\hat{P}\ni 0} |\widetilde{\omega}_0(\hat{P})| \le \frac{\epsilon}{2} . \end{aligned} \tag{4.40}$$

This shows the first inequality of (4.20). For the second, we use the Peierls condition $\|\Gamma\| \ge \rho|\Gamma|$ (Proposition 2.2). This gives

$$\begin{aligned} \sum_{\Gamma:[\Gamma]_l \ni 0} \rho_0(\Gamma) e^{(c+\epsilon)|[\Gamma]_l|} &\le D_1\beta \sum_{\Gamma:[\Gamma]_l\ni 0} |\Gamma|^{\frac{d}{d-1}} e^{-(1-\theta)\beta\rho|\Gamma|} e^{2\beta h_0|\Gamma|} e^{|\Gamma|} e^{(c+\epsilon)|[\Gamma]_l|} \\ &\le D_1\beta \sum_{\Gamma:[\Gamma]_l\ni 0} |\Gamma|^{\frac{d}{d-1}} e^{-\frac{1}{2}(1-\theta)\beta\rho|\Gamma|} e^{|\Gamma|} e^{(c+\epsilon)|[\Gamma]_l|} . \end{aligned}$$

Since $|[\Gamma]_l| \le 3^d|\Gamma|$, a standard Peierls estimate allows to bound this sum by $\frac{\epsilon}{2}$ as soon as β is large enough. □

Until now we have denoted by $\epsilon_r = \epsilon_r(\beta)$ the small function appearing in the study of the restricted phases. Similarly, we denote by $\epsilon_c = \epsilon_c(\beta)$ the small function appearing in the study of chains. These two parameters are assumed to have a common bound $\max\{\epsilon_r, \epsilon_c\} \le \epsilon$, which is small.

Consider the weight $\omega^+(\Gamma)$ given (4.12). We can use the linking procedure for the partition functions $\Theta^\pm(\mathrm{int}\Gamma;\mp\sigma_\Gamma)$, yielding

$$\omega^+(\Gamma)=\rho_1(\Gamma)\frac{e^{-\beta hV(\Gamma)}\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{e^{+\beta hV(\Gamma)}\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\,. \tag{4.41}$$

Proof of Proposition 4.1: The proof will be done by induction. We say a contour Γ is of class n if $V(\Gamma)=n$. A chain is of class n if $V(X)=n$. Consider a contour Γ of small class (say, of class smaller than l^d). Then the last ratio appearing in (4.41) equals 1. We bound $\omega^+(\Gamma)$ at $h=x+iy\in U_\Gamma$. First,

$$|e^{-2\beta hV(\Gamma)}|\leq e^{2\beta|x|V(\Gamma)}\leq e^{2\beta R^*(V(\Gamma))V(\Gamma)}\leq e^{2\beta R(V(\Gamma))V(\Gamma)}\leq e^{\theta\beta\|\Gamma\|}\,, \tag{4.42}$$

where we used the definition of the isoperimetric constants $K(\cdot)$ given in (2.44). Then, write

$$\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h}=\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}}\frac{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} \tag{4.43}$$

The middle term has modulus 1 by symmetry (see (3.32)). The two other terms can be treated as follows:

$$\Big|\log\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}\Big|=\Big|\int_0^x\mathrm{d}s\frac{\mathrm{d}}{\mathrm{d}s}\log\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{s+iy}\Big|\leq|x|\epsilon_rV(\Gamma)\,. \tag{4.44}$$

We used Theorem 3.1. Proceeding as in (4.42), we get

$$\Big\|\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big\|_{U_\Gamma}\leq e^{\theta\epsilon_r\|\Gamma\|}\leq e^{\frac{1}{3}|\Gamma|}\,, \tag{4.45}$$

when β is large enough. Altogether this gives

$$\|\omega^+(\Gamma)\|_{U_\Gamma}\leq\|\rho_1(\Gamma)\|_{U_\Gamma}e^{\theta\beta\|\Gamma\|}e^{\frac{1}{3}|\Gamma|}\leq e^{-(1-\theta)\beta\|\Gamma\|}e^{2\beta h_0|\Gamma|}e^{2\frac{1}{3}|\Gamma|}<\rho_0(\Gamma)\,. \tag{4.46}$$

Since $\|\lambda^+(\hat{P})\|_{U_0}<\lambda_0(\hat{P})$, we have shown the first inequality of (4.35) for chains of small class. For the derivative, a Cauchy estimate (any disc centered at $h\in U_0$ with radius $\frac{1}{16}$ is contained in H_+) gives

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\lambda^+(\hat{P})\Big\|_{U_0}\leq 16\|\lambda^+(\hat{P})\|_{H_+}\,. \tag{4.47}$$

For contours,

$$\frac{\mathrm{d}}{\mathrm{d}h}\omega^+(\Gamma)=\omega^+(\Gamma)\frac{\mathrm{d}}{\mathrm{d}h}\log\omega^+(\Gamma)=$$

$$\omega^+(\Gamma)\Big(-\beta\frac{\mathrm{d}}{\mathrm{d}h}H_\Gamma(\sigma[\Gamma])-\beta|\Gamma|+\frac{\mathrm{d}}{\mathrm{d}h}E^+_\Gamma-2\beta V(\Gamma)+\frac{\mathrm{d}}{\mathrm{d}h}\log\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big)$$

Using $V(\Gamma) \le |\Gamma|^{\frac{d}{d-1}}$ (this is a consequence of Lemma 2.10) and

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \log \frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))} \Big\|_{U_\Gamma} \le 2\epsilon_r V(\Gamma)\,, \tag{4.48}$$

this gives the upper bound

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \omega^+(\Gamma) \Big\|_{U_\Gamma} \le 6\beta |\Gamma|^{\frac{d}{d-1}} \|\omega^+(\Gamma)\|_{U_\Gamma}\,, \tag{4.49}$$

which implies, as can be seen easily, that

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \omega^+(X) \Big\|_{U_X} < \omega_0(X)\,. \tag{4.50}$$

With Lemma 4.1, this shows the proposition for chains of small class. Suppose it has been shown for chains of class $\le n$. By this induction hypothesis, (4.21) and Lemma A.1, a cluster expansion can be used for the partition functions containing chains. Let X be a chain of class $n+1$, and consider $\Gamma \in X$. The treatment of the restricted phases is the same, and we must study the ratio

$$\frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} = \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}} \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}} \frac{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h}\,. \tag{4.51}$$

Again the middle term has modulus 1 and the rest is treated using the induction hypothesis.

$$\Big| \log \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}} \Big| = \Big| \int_0^x \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \log \Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{s+iy} \Big| \le |x| \epsilon_c V(\Gamma)\,. \tag{4.52}$$

This implies

$$\Big\| \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))} \Big\|_{U_\Gamma} \le e^{\theta\epsilon_c \|\Gamma\|} \le e^{\frac{1}{3}|\Gamma|}\,. \tag{4.53}$$

For the weight of Γ, we thus have (compare with (4.46)):

$$\|\omega^+(\Gamma)\|_{U_\Gamma} \le e^{-(1-\theta)\beta\|\Gamma\|} e^{2\beta h_0 |\Gamma|} e^{3\frac{1}{3}|\Gamma|} < \rho_0(\Gamma)\,. \tag{4.54}$$

For the derivative, use again the induction hypothesis, and bound

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \log \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))} \Big\|_{U_\Gamma} \le 2\epsilon_c V(\Gamma)\,. \tag{4.55}$$

It is easy to check that (4.49) still holds which, in turn, implies (4.50). This shows the proposition. □

4.4 Pure Phases

In the last section we gave for each chain X a domain U_X in which the weight $\omega^+(X)$ behaves analytically. The size of the domain U_X shrinks to $\{\mathrm{Re}\, h = 0\}$ when the size of the largest contour of X increases. In the present section we show that the weights $\omega^+(X)$ can actually be controlled when $0 < \mathrm{Re}\, h < h_+$ where h_+ is fixed, independently of the size of X. This treatment is standard and was first introduced by Zahradník [Z1].
We consider only chains of type $+$, the case $-$ being similar by symmetry. Define

$$U_+ := \{z \in \mathbb{C} : 0 < \mathrm{Re}\, h < h_+\}\,, \tag{4.56}$$

where $0 < h_+ \leq \min\{\frac{1}{16}, \frac{\rho}{2}\}$ is fixed (ρ is the Peierls constant). In Section 5, domains will have to be made optimal, with θ close to 1, but here we choose $\theta := \frac{1}{2}$. The main result of this section is the following

Proposition 4.2. *Let $\epsilon, c > 0$ be small enough. There exists $\beta_2 = \beta_2(\epsilon)$ such that for all $\beta \geq \beta_2$, the following holds. For each chain X of type $+$, $h \mapsto \omega^+(X)$ is analytic in U_+, and*

$$\|\omega^+(X)\|_{U_+} \leq \omega_0(X)\,, \tag{4.57}$$

where $\omega_0(X)$ satisfies (4.21).

Proof. Since $U_+ \subset H_+$, clusters $\hat{P}$ and restricted phases are under control. For each Γ, we use the representation (4.12) (rather than (4.41)). The main ingredient of the proof is the following lemma, whose proof is standard and can be found, e.g. in [Z1] or [FP] (with minor modifications due to the fact that we are working with analytic restricted phases rather than ground states).

Lemma 4.4. *Let β be large enough. Then for each contour Γ of type $+$, we have $\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma) \neq 0$ on U_+ and*

$$\left\| \frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)} \right\|_{U_+} \leq e^{\frac{2}{3}|\Gamma|}\,. \tag{4.58}$$

The proof of Proposition 4.2 finishes by using Lemma 4.1. $\square$

5 Derivatives of the Pressure

In this section we prove Theorem 1.3, adapting the mechanism used by S.N. Isakov for the Ising model. Although estimates of Theorem 1.3 hold for the pressure density p_γ, we will always work in a finite volume Λ, and obtain bounds on the derivatives of the pressure that are uniform in the volume. As in the preceding section, we assume $\gamma \in (0, \gamma_0)$ is fixed.

We consider a box $\Lambda = [-M, +M]^d \cap \mathbf{Z}^d$, with M large, chosen so that $\Lambda \in \mathcal{C}^{(l)}$. Outside Λ we fix the spins to the value $+1$, i.e. we consider the set Ω^+_Λ, defined in (4.2) and the associated partition function $Z^+(\Lambda)$ defined in (4.1). The finite volume pressure $p^+_{\gamma,\Lambda}$ is defined by

$$p^+_{\gamma,\Lambda} := \frac{1}{\beta|\Lambda|} \log Z^+(\Lambda)\,. \tag{5.1}$$

Clearly, this function equals the pressure density of (1.16) in the thermodynamic limit. Consider the set $\mathcal{C}^+(\Lambda)$ of *all possible* external contours of type $+$ associated to the set Ω^+_Λ. That is, each contour of $\mathcal{C}^+(\Lambda)$ appears in at least one configuration $\sigma_\Lambda \in \Omega^+_\Lambda$. Remember that $V(\Gamma) = |\mathrm{int}\Gamma|$, where $\mathrm{int}\Gamma$ denotes the union of all components of Γ^c with label $-$. The family $\mathcal{C}^+(\Lambda)$ can be totally ordered, with an order relation denoted $\preceq$, such that $V(\Gamma') \le V(\Gamma)$ when $\Gamma' \preceq \Gamma$. When Γ is not the smallest contour we denote its predecessor (w.r.t. $\preceq$) by $i(\Gamma)$.
For a given external contour $\Gamma \in \mathcal{C}^+(\Lambda)$, consider the set

$$\Omega^+_\Lambda(\Gamma) := \{\sigma_\Lambda \in \Omega^+_\Lambda : \Gamma' \preceq \Gamma \text{ for all external contour } \Gamma' \text{ of } \sigma_\Lambda +_{\Lambda^c}\}\,,$$

and define the partition function

$$\Theta^+_\Gamma(\Lambda) := \sum_{\sigma_\Lambda \in \Omega^+_\Lambda(\Gamma)} \exp\big(-\beta H_\Lambda(\sigma_\Lambda +_{\Lambda^c})\big)\,. \tag{5.2}$$

When Γ is the largest contour then clearly $\Theta^+_\Gamma(\Lambda) = Z^+(\Lambda)$ and when Γ is the smallest contour, we define $\Theta^+_{i(\Gamma)}(\Lambda) := Z^+_r(\Lambda)$. We also introduce the following set in which the presence of Γ is *forced*:

$$\begin{aligned}\Omega^+_\Lambda[\Gamma] := \{\sigma_\Lambda \in \Omega^+_\Lambda :& \Gamma' \preceq \Gamma \text{ for all external contour } \Gamma' \text{ of } \sigma_\Lambda +_{\Lambda^c} \\ & \text{and } \Gamma \text{ is a contour of } \sigma_\Lambda +_{\Lambda^c}\}\,.\end{aligned} \tag{5.3}$$

The partition function $\Theta^+_{[\Gamma]}(\Lambda)$ is defined as (5.2), with $\Omega^+_\Lambda[\Gamma]$ in place of $\Omega^+_\Lambda(\Gamma)$. We have the following fundamental identity:

$$\Theta^+_\Gamma(\Lambda) = \Theta^+_{i(\Gamma)}(\Lambda) + \Theta^+_{[\Gamma]}(\Lambda)\,. \tag{5.4}$$

A crucial idea of Isakov is to consider the following identity.

$$Z^+(\Lambda) = Z^+_r(\Lambda) \prod_{\Gamma \in \mathcal{C}^+(\Lambda)} \frac{\Theta^+_\Gamma(\Lambda)}{\Theta^+_{i(\Gamma)}(\Lambda)}\,. \tag{5.5}$$

Then, the logarithm is written as a *finite* sum:

$$\log Z^+(\Lambda) = \log Z^+_r(\Lambda) + \sum_{\Gamma \in \mathcal{C}^+(\Lambda)} u^+_\Lambda(\Gamma)\,, \tag{5.6}$$

where

$$u_\Lambda^+(\Gamma) := \log \frac{\Theta_\Gamma^+(\Lambda)}{\Theta_{i(\Gamma)}^+(\Lambda)}\,. \tag{5.7}$$

Using (5.4) we can write $u_\Lambda^+(\Gamma) = \log(1 + \varphi_\Lambda^+(\Gamma))$, where

$$\varphi_\Lambda^+(\Gamma) := \frac{\Theta_{[\Gamma]}^+(\Lambda)}{\Theta_{i(\Gamma)}^+(\Lambda)}\,. \tag{5.8}$$

Non-analyticity of the pressure is examined by studying high order derivatives of the functions $\varphi_\Lambda^+(\Gamma)$ at $h = 0$, using Cauchy's formula

$$\varphi_\Lambda^+(\Gamma)^{(k)}(0) = \frac{k!}{2\pi i}\int_C \frac{\varphi_\Lambda^+(\Gamma)(z)}{z^{k+1}}\mathrm{d}z\,. \tag{5.9}$$

To obtain bounds on $\varphi_\Lambda^+(\Gamma)^{(k)}(0)$, we exponentiate $\varphi_\Lambda^+(\Gamma)$ and use a stationary phase analysis to estimate the integral. The contour C will be chosen in a k-dependent way. If the domain $U_\Gamma \ni 0$ in which $\varphi_\Lambda^+(\Gamma)$ is analytic is too small, then no information (not even the sign!) can be given about $\varphi_\Lambda^+(\Gamma)^{(k)}(0)$.

For a while, consider the structure of the partition function $\Theta_{[\Gamma]}^+(\Lambda)$. We write $\Lambda = \mathrm{ext}_\Lambda\Gamma \cup \Gamma \cup \mathrm{int}\Gamma$, where $\mathrm{ext}_\Lambda\Gamma := \mathrm{ext}\Gamma \cap \Lambda$. By construction, $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ are at distance at least $l > 2R$. We will therefore consider $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ as independent systems (see Figure 6). The sums over

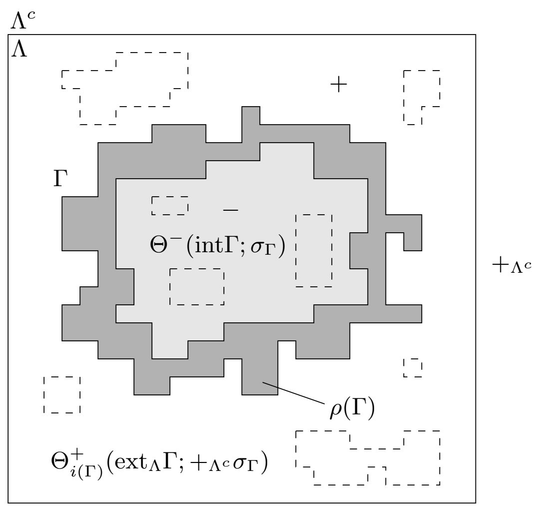

Figure 6: The decomposition (5.10) of the partition function $\Theta_{[\Gamma]}^+(\Lambda)$.

configurations on $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ can be done separately, yielding

$$\Theta_{[\Gamma]}^+(\Lambda) = \rho(\Gamma)\Theta_{i(\Gamma)}^+(\mathrm{ext}_\Lambda\Gamma;+_{\Lambda^c}\sigma_\Gamma)\Theta^-(\mathrm{int}\Gamma;\sigma_\Gamma)\,. \tag{5.10}$$

All the contours of these partition functions are at distance larger than l from Γ, and have an interior smaller than $V(\Gamma)$. The point is that we control these functions for $h \in U_\Gamma$, where $U_\Gamma \subset \mathbb{C}$ is a domain that depends *only on the volume of* Γ.

The program for the rest of the section is the following. In Section 5.1 we show that $\varphi_\Lambda^+(\Gamma)$ can be exponentiated, using the results of Section 4. We then use a stationary phase analysis and obtain upper and lower bounds on some derivatives of $\varphi_\Lambda^+(\Gamma)$ and $u_\Lambda^+(\Gamma)$ at $h=0$. In Section 5.2 we fix k and take the box Λ large enough. For a class of contours called k-large and thin, the k-th derivative of $u_\Lambda^+(\Gamma)$ can be estimated from below, using the results of Section 5.1. This gives a lower bound on $p_{\gamma,\Lambda}^{+(k)}(0)$. In Section 5.3 we show that for $p_{\gamma,\Lambda}^+$, the operations $\lim_\Lambda$ and $(\cdot)^{(k),\leftarrow}(0)$ commute, leading to the proof of our main results.

5.1 Study of the Functions $\varphi_\Lambda^+(\Gamma)$

The proof of the following lemma requires the main results of Sections 3 and 4. After that, the proof of non-analyticity of the pressure will essentially follow the argument of Isakov (see [I1], [I2] or [FP]).

Lemma 5.1. *Let $\theta \in (0,1)$, β large enough. Then the following holds. For all contour $\Gamma \in \mathcal{C}^+(\Lambda)$ with $V(\Gamma) \neq 0$ there exists a map $h \mapsto g_\Lambda^+(\Gamma)(h)$ analytic in the strip U_Γ, such that for all $h \in U_\Gamma$, $\varphi_\Lambda^+(\Gamma)$ can be exponentiated:*

$$\varphi_\Lambda^+(\Gamma) = \exp\big(-\beta\|\Gamma\| - 2\beta h V(\Gamma) + 2\beta V(\Gamma) g_\Lambda^+(\Gamma)\big)\,. \tag{5.11}$$

Moreover, we have the following local estimate

$$2\beta V(\Gamma)|g_\Lambda^+(\Gamma)(0)| \leq \delta_1(\beta)\beta\|\Gamma\|\,, \tag{5.12}$$

and a uniform bound on the first derivative

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h} g_\Lambda^+(\Gamma)\Big\|_{U_\Gamma} \leq \delta_2(\beta) + 2\frac{|\Gamma|}{V(\Gamma)}\,. \tag{5.13}$$

The functions δ_i are such that $\lim_{\beta\nearrow\infty}\delta_i = 0$.

Proof. Consider $\Theta_{[\Gamma]}^+(\Lambda)$. We have seen how to re-sum over configurations on $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$. We write

$$\varphi_\Lambda^+(\Gamma) = \rho(\Gamma)\frac{\Theta_{i(\Gamma)}^+(\mathrm{ext}_\Lambda\Gamma; +_{\Lambda^c}\sigma_\Gamma)\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}{\Theta_{i(\Gamma)}^+(\Lambda)}\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\,. \tag{5.14}$$

All the volume contributions coming from the first quotient will be shown to vanish. The partition functions $\Theta_{i(\Gamma)}^+(\mathrm{ext}_\Lambda\Gamma; +_{\Lambda^c}\sigma_\Gamma)$ and $\Theta^\pm(\mathrm{int}\Gamma; \mp\sigma_\Gamma)$

are of the type (4.5). We can therefore apply the linking procedure and obtain a representation of the form (4.18) for each of them:

$$\Theta^{+}_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma;+_{\Lambda^c}\sigma_\Gamma)=e^{\beta h|\mathrm{ext}_\Lambda\Gamma|}\mathcal{Z}_r(\mathcal{P}^{+}_{\mathrm{ext}_\Lambda\Gamma})\Xi(\mathcal{X}^{+}_{\mathrm{ext}_\Lambda\Gamma})\,, \tag{5.15}$$

$$\Theta^{\pm}(\mathrm{int}\Gamma;\mp\sigma_\Gamma)=e^{\pm\beta hV(\Gamma)}\mathcal{Z}_r(\mathcal{P}^{\pm}_{\mathrm{int}\Gamma})\Xi(\mathcal{X}^{\pm}_{\mathrm{int}\Gamma})\,, \tag{5.16}$$

where we omitted, in the notation, to mention that the families of polymers and chains always depend on the boundary conditions specified by $+_{\Lambda^c}$ and σ_Γ. Moreover, the family $\mathcal{X}^{+}_{\mathrm{ext}_\Lambda\Gamma}$ contains chains X that satisfy $V(X)\leq V(\Gamma)$. In the same way:

$$\Theta^{+}_{i(\Gamma)}(\Lambda)=e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^{+}_{\Lambda})\Xi(\mathcal{X}^{+}_{\Lambda})\,, \tag{5.17}$$

where the families $\mathcal{P}^{+}_{\Lambda}$ and $\mathcal{X}^{+}_{\Lambda}$ depend only on the boundary condition $+_{\Lambda^c}$. Using the definition of $\rho(\Gamma)$, it is easy to see that $\varphi^{+}_{\Lambda}(\Gamma)$ has the form (5.11), where $g^{+}_{\Lambda}(\Gamma)$ is defined by

$$2\beta V(\Gamma)g^{+}_{\Lambda}(\Gamma):=-\beta\sum_{i\in\Gamma}u((\sigma_\Gamma)_i)-\beta h|\Gamma|+\log Q_r+\log Q_c\,, \tag{5.18}$$

where $u(\sigma_i)=-h\sigma_i$, and the quotients Q_r,Q_c are defined by

$$Q_r(h):=\frac{\mathcal{Z}_r(\mathcal{P}^{+}_{\mathrm{ext}_\Lambda\Gamma})\mathcal{Z}_r(\mathcal{P}^{+}_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^{+}_{\Lambda})}\frac{\mathcal{Z}_r(\mathcal{P}^{-}_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^{+}_{\mathrm{int}\Gamma})}\,, \tag{5.19}$$

$$Q_c(h):=\frac{\Xi(\mathcal{X}^{+}_{\mathrm{ext}_\Lambda\Gamma})\Xi(\mathcal{X}^{+}_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^{+}_{\Lambda})}\frac{\Xi(\mathcal{X}^{-}_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^{+}_{\mathrm{int}\Gamma})}\,. \tag{5.20}$$

Since all the families of chains involved contain contours with an interior smaller than Γ, $h\mapsto g^{+}_{\Lambda}(\Gamma)$ is analytic in the strip U_Γ (by Proposition 4.1). Rearranging the terms of the cluster expansions for Q_r leads to

$$\log Q_r=\log\frac{\mathcal{Z}_r(\mathcal{P}^{-}_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^{+}_{\mathrm{int}\Gamma})}+\sum_{\substack{\hat P\in\hat{\mathcal{P}}^{+}_{\mathrm{ext}_\Lambda\Gamma}\\ \hat P\cap[\Gamma]_R\neq\emptyset}}\omega^{+}(\hat P)+\sum_{\substack{\hat P\in\hat{\mathcal{P}}^{+}_{\mathrm{int}\Gamma}\\ \hat P\cap[\Gamma]_R\neq\emptyset}}\omega^{+}(\hat P)-\sum_{\substack{\hat P\in\hat{\mathcal{P}}^{+}_{\Lambda}\\ \hat P\cap[\Gamma]_R\neq\emptyset}}\omega^{+}(\hat P)$$

Notice that the volume contributions from $\mathrm{ext}_\Lambda\Gamma$ cancelled, and that the three sums are boundary terms. By symmetry, the quotient equals 1 at $h=0$, and so

$$|\log Q_r(0)|\leq 3\epsilon_r|[\Gamma]_R|\,. \tag{5.21}$$

For the derivative, using (3.56) gives

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\log Q_r\Big\|_{\tilde H_+}\leq 2\epsilon_r V(\Gamma)+3\epsilon_r|[\Gamma]_R|\,. \tag{5.22}$$

The same computations can be done for Q_c. Clusters of chains are denoted $\hat{X}$. The contributions from $\mathrm{ext}_\Lambda\Gamma$ also cancel. Indeed, consider the difference

$$\sum_{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}} \omega^+(\hat{X}) - \sum_{\hat{X}\in\hat{\mathcal{X}}^+_\Lambda} \omega^+(\hat{X})\,. \tag{5.23}$$

Using Lemma 2.5, there exists for all $\hat{X}_1 \in \hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}$ with $d(\hat{X}_1,\Gamma) > R$, a cluster $\hat{X}_2 \in \hat{\mathcal{X}}^+_\Lambda$, $\hat{X}_2 \cap \mathrm{ext}_\Lambda\Gamma \neq \emptyset$, $d(\hat{X}_2,\Gamma) > R$, such that $\omega^+(\hat{X}_1) = \omega^+(\hat{X}_2)$. We are thus left with

$$\log Q_c = \log\frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma})} + \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}} \omega^+(\hat{X}) + \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{int}\Gamma}\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}} \omega^+(\hat{X}) - \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_\Lambda\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}} \omega^+(\hat{X})\,.$$

Using symmetry,

$$|\log Q_c(0)| \leq 3\epsilon_c|[\Gamma]_R|\,. \tag{5.24}$$

For the derivative, a similar treatment gives

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\log Q_c\Big\|_{U_\Gamma} \leq 2\epsilon_c V(\Gamma) + 3\epsilon_c|[\Gamma]_R|\,. \tag{5.25}$$

Estimates (5.21) and (5.24) yield

$$2\beta V(\Gamma)|g^+_\Lambda(\Gamma)(0)| \leq 3(\epsilon_r+\epsilon_c)|[\Gamma]_R| \leq \delta_1(\beta)\beta\|\Gamma\|\,, \tag{5.26}$$

where $\delta_1(\beta) := 3^{d+1}\beta^{-1}(\epsilon_r+\epsilon_e)\rho^{-1}$ (ρ is the Peierls constant). We get (5.13) by setting $\delta_2(\beta) := \beta^{-1}(\epsilon_r+\epsilon_e)$. □

We are now in position of computing derivatives of the functions $\varphi^+_\Lambda(\Gamma)$. The main ingredient is the following theorem, which appeared in [I2]. The proof can be obtained by following the Appendix of [I1], which is nothing but a stationary phase analysis applied to the Cauchy integral giving the k-th derivative at $z=0$ of a function of the type $e^{-cz+bf(z)}$.

Theorem 5.1. *Let $r>0$, $F(z)=\exp(-cz+bf(z))$ where $1\leq b\leq c$, and f is analytic in a disc $\{|z|<r\}$, taking real values on the real line, with a uniformly bounded derivative:*

$$\sup_{|z|<r}|f'(z)| \leq A < \frac{1}{25}\,. \tag{5.27}$$

There exists $k_0=k_0(A)$ such that the following holds: define $k_+ = r(c-2b\sqrt{A})$. For all integer $k\in[k_0,k_+]$ there exists $r_k\in(0,r)$ and $c_k>0$ satisfying

$$\frac{k}{c+bA} \leq r_k \leq \frac{k}{c-bA}\,,\quad \frac{3}{10}\frac{1}{\sqrt{2\pi c r_k}} < c_k < \frac{1}{\sqrt{c r_k}}\,, \tag{5.28}$$

such that

$$F^{(k)}(0) = \frac{k!}{2\pi i}\int_{|z|=r_k} \frac{F(z)}{z^{k+1}}\mathrm{d}z = k!\frac{c_k}{(-r_k)^k}F(-r_k)\,. \tag{5.29}$$

In particular, $(-1)^k F^{(k)}(0) > 0$*. Moreover, if* f *satisfies the local condition*

$$bf(0) \leq -\alpha r c\,, \tag{5.30}$$

with $\alpha \in (\log 2, 1)$*, then for all* $k \in [k_0, k_+]$ *and* A *sufficiently small,*

$$\big(\log(1+F)\big)^{(k)}(0) = (1 + a \cdot e^{-\frac{1}{2}\zeta k})F^{(k)}(0)\,, \tag{5.31}$$

where a *is a bounded function of* k, c, b *and* $\zeta = \zeta(\alpha) > 0$*.*

In Lemma 5.1, we have put $\varphi^+_\Lambda(\Gamma)$ in the form $e^{-cz+bf(z)}$. In order to satisfy (5.27), we must introduce a distinction among the contours. Consider the function $\delta_2(\beta)$ of (5.13).

Definition 5.1. *A contour* $\Gamma \in \mathcal{C}^+(\Lambda)$ *is* thin *if* $|\Gamma| \leq \frac{\delta_2(\beta)}{2}V(\Gamma)$*, and* fat *if it is not thin.*

Now, any thin contour Γ satisfies, when β is large enough,

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} g^+_\Lambda(\Gamma)\Big\|_{U_\Gamma} \leq 2\delta_2(\beta) \equiv A(\beta) < \frac{1}{25}\,. \tag{5.32}$$

Lemma 5.2. *There exists* k_0 *such that when* β *is sufficiently large, the following holds. For all thin contour* Γ*, define*

$$k_+(\Gamma) := 2\beta V(\Gamma) R^*(V(\Gamma))(1 - 2\sqrt{A})\,. \tag{5.33}$$

Then for all integer $k \in [k_0, k_+(\Gamma)]$*, we have*

$$(-1)^k u^+_\Lambda(\Gamma)^{(k)}(0) \geq \frac{1}{10}\big(2\beta V(\Gamma) D_-\big)^k e^{-(1+\delta_1(\beta))\|\Gamma\|}\,, \tag{5.34}$$

$$(-1)^k u^+_\Lambda(\Gamma)^{(k)}(0) \leq 20\big(2\beta V(\Gamma) D_+\big)^k e^{-(1-\delta_1(\beta))\|\Gamma\|}\,, \tag{5.35}$$

where $\lim_{\beta\to\infty} D_\pm = 1$*.*

Proof. Let Γ be a thin contour. Consider $\varphi^+_\Lambda(\Gamma)$ in its exponentiated form (5.11). We apply Theorem 5.1 with $c = b = 2\beta V(\Gamma)$, $f = g^+_\Lambda(\Gamma) - \frac{1}{2}\frac{\|\Gamma\|}{V(\Gamma)}$, $r = R^*(V(\Gamma))$, and $A = A(\beta)$. (5.32) guarantees (5.27). There exists $r_k = r_k(\Gamma)$ and $c_k = c_k(\Gamma)$ such that

$$(-1)^k \varphi^+_\Lambda(\Gamma)^{(k)}(0) = k!\frac{c_k}{(r_k)^k}\varphi^+_\Lambda(\Gamma)(-r_k)\,. \tag{5.36}$$

Using the analyticity of $g_\Lambda^+(\Gamma)$ in U_Γ, we have with (5.28)

$$\begin{aligned}\varphi_\Lambda^+(\Gamma)(-r_k) &= e^{-\beta\|\Gamma\|}e^{cr_k}e^{cg_\Lambda^+(\Gamma)(0)}e^{c(g_\Lambda^+(\Gamma)(-r_k)-g_\Lambda^+(\Gamma)(0))}\\ &\geq e^{-\beta\|\Gamma\|}e^{\frac{k}{1+A}}e^{-\delta_1\beta\|\Gamma\|}e^{-\frac{A}{1-A}k}\\ &= e^{-(1+\delta_1)\beta\|\Gamma\|}e^{k}e^{-\frac{2A}{1-A^2}k}\,.\end{aligned}$$

Using Stirling's formula and the estimates for r_k, c_k, we get

$$(-1)^k\varphi_\Lambda^+(\Gamma)^{(k)}(0) \geq \frac{1}{5}\big(2\beta V(\Gamma)D_-\big)^k e^{-(1+\delta_1)\beta\|\Gamma\|}\,, \tag{5.37}$$

where

$$D_-(\beta) = (1-A)e^{-\frac{2A}{1-A^2}}\,. \tag{5.38}$$

Using (5.12) we can satisfy (5.30):

$$\begin{aligned}bf(0) = 2\beta V(\Gamma)g_\Lambda^+(\Gamma)(0) - \beta\|\Gamma\| &\leq -(1-\delta_1)\beta\|\Gamma\|\\ &\leq -(1-\delta_1)2\beta V(\Gamma)R^*(V(\Gamma)) \qquad (5.39)\\ &= -(1-\delta_1)rc\,. \qquad (5.40)\end{aligned}$$

In (5.39) we used

$$\|\Gamma\| \geq \frac{1}{K(V(\Gamma))}V(\Gamma)^{\frac{d}{d-1}} \geq 2V(\Gamma)\frac{\theta}{2K(V(\Gamma))V(\Gamma)^{\frac{1}{d}}} \geq 2V(\Gamma)R^*(V(\Gamma))\,.$$

We can thus use (5.31) once β is large enough. This gives the lower bound (5.34). The upper bound is obtained similarly. □

5.2 Derivatives in a Finite Volume

In this section, we *fix* k large enough. When a thin contour satisfies $[k_0, k_+(\Gamma)] \ni k$ then $u_\Lambda^+(\Gamma)^{(k)}(0)$ can be estimated with Lemma 5.2. To characterize this class of contours, we introduce a k-dependent notion of size.

Definition 5.2. *Let $k \in \mathbb{N}$, $\epsilon' > 0$ small enough. A contour Γ is k-large if $V(\Gamma) \geq V_0(k)$ where*

$$V_0(k) := \Big(\frac{K(\infty)(1+\epsilon')}{\theta\beta(1-2\sqrt{A})}k\Big)^{\frac{d}{d-1}}\,, \tag{5.41}$$

where $K(\infty)$ was defined in Lemma 2.9. Γ is k-small if $V(\Gamma) < V_0(k)$.

Let $N_0(\epsilon')$ be such that for all $N \geq N_0(\epsilon')$ (see Lemma 4.2),

$$\frac{1}{(1+\epsilon')}\frac{\theta}{2K(\infty)N^{\frac{1}{d}}} \leq R^*(N) \leq \frac{\theta}{2K(\infty)N^{\frac{1}{d}}}\,. \tag{5.42}$$

Let $k_- = k_-(\epsilon', \gamma)$ be such that when $k \geq k_-$ then $V_0(k) \geq N_0(\epsilon')$. This definition implies that when $k \geq k_-$, we have for all k-large contour Γ

$$k_+(\Gamma) = 2\beta V(\Gamma)(1 - 2\sqrt{A})R^*(V(\Gamma)) \geq \frac{\theta\beta(1 - 2\sqrt{A})}{K(\infty)(1 + \epsilon')} V(\Gamma)^{\frac{d-1}{d}} \geq k\,. \tag{5.43}$$

That is, the k-th derivative of a k-large thin contour can be studied with Lemma 5.2. The dependence of k_- on γ comes from the bound $K(\infty) \geq c_-\gamma$. We therefore have $\lim_{\gamma \searrow 0} k_- = +\infty$.

Proposition 5.1. *Let θ be close to 1, β large enough. There exists a constant $C_1 > 0$ and an unbounded increasing sequence of integers $k_1, k_2, \dots$ such that for large N, we have whenever Λ is sufficiently large,*

$$\frac{(-1)^{k_N}}{|\Lambda|} \frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}} \sum_{\Gamma \in \mathcal{C}^+(\Lambda)} u_\Lambda^+(\Gamma)\Big|_{h=0} \geq \big(C_1 K(\infty)^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}}\,. \tag{5.44}$$

Proof. Fix $\epsilon > 0$ small and consider the sequence $(\Gamma_N)_{N \geq 1}$ of Lemma 2.9. We have $\lim_{N\to\infty} V(\Gamma_N) = +\infty$ and when N is large enough,

$$(1 - \epsilon)K(\infty) \leq \frac{V(\Gamma_N)^{\frac{d-1}{d}}}{\|\Gamma_N\|} \leq (1 + \epsilon)K(\infty)\,. \tag{5.45}$$

The sequence $(k_N)_{N\geq 1}$ is defined such that the contribution from the contour Γ_N to $p_{\gamma,\Lambda}^{+(k_N)}(0)$ is close to maximal. Let

$$k_N := \Big\lfloor \frac{d-1}{d} \beta \|\Gamma_N\| \Big\rfloor\,. \tag{5.46}$$

Since $\lim_{N\to\infty} V(\Gamma_N) = +\infty$, we have $\lim_{N\to\infty} k_N = +\infty$. From now on we consider N large enough so that (5.45) and (5.48) hold and $k_N \geq \max\{k_0, k_-\}$. When considering the k_N-th derivative, we use the following decomposition:

$$\sum_{\Gamma \in \mathcal{C}^+(\Lambda)} = \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k_N-\text{large, thin}}} + \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k_N-\text{small, thin}}} + \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ \text{fat}}} \tag{5.47}$$

We show that the dominant term comes from Γ_N, which belongs to the first sum, and that the two other sums are negligible. To see that Γ_N appears in the first sum, we first show that Γ_N is k_N-large. Indeed, if θ is close to 1 and $\epsilon, \epsilon', A(\beta)$ are small,

$$\begin{aligned} V_0(k_N) &\leq \Big(\frac{K(\infty)(1 + \epsilon')}{\theta(1 - 2\sqrt{A})} \frac{d-1}{d} \|\Gamma_N\|\Big)^{\frac{d}{d-1}} \\ &\leq \Big(\frac{1}{\theta(1 - 2\sqrt{A})} \frac{1 + \epsilon'}{1 - \epsilon} \frac{d-1}{d}\Big)^{\frac{d}{d-1}} V(\Gamma_N) \leq V(\Gamma_N)\,. \end{aligned}$$

Then we show that Γ_N is thin:

$$\frac{|\Gamma_N|}{V(\Gamma_N)} \leq \frac{1}{\rho}\frac{\|\Gamma_N\|}{V(\Gamma_N)} \leq \frac{1}{\rho K(\infty)(1-\epsilon)}\frac{1}{V_0(k_N)^{\frac{1}{d}}} \leq \frac{1}{2}\delta_2(\beta)\,. \tag{5.48}$$

Finally, we assume Λ is large enough in order to contain at least $a|\Lambda|$ translates of Γ_N, $a > 0$. Then we apply Lemma 5.2 to $u^+_\Lambda(\Gamma_N)$. Using (5.45),

$$\begin{aligned} V(\Gamma_N)^{k_N} e^{-(1+\delta_1)\beta\|\Gamma_N\|} &\geq \big((1-\epsilon)K(\infty)\|\Gamma_N\|\big)^{\frac{d}{d-1}k_N} e^{-(1+\delta_1)\beta\|\Gamma_N\|} \\ &\geq \Big((1-\epsilon)K(\infty)\frac{d}{d-1}\frac{1}{\beta}k_N\Big)^{\frac{d}{d-1}k_N} e^{-(1+\delta_1)\frac{d}{d-1}(k_N+1)} \\ &\geq c(k_N)K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{d}{d-1}k_N}\Big[\frac{d}{d-1}(1-\epsilon)e^{-\delta_1}\Big]^{\frac{d}{d-1}k_N} k_N!^{\frac{d}{d-1}}\,, \end{aligned} \tag{5.49}$$

where $c(k_N) \geq C_3 k_N^{-\frac{1}{2}}$ and we used Stirling's formula. Since

$$(-1)^{k_N} u^+_\Lambda(\Gamma)^{(k_N)}(0) \geq 0 \tag{5.50}$$

for all k_N-large thin contour, we can bound the first sum from below using only the contributions coming from the translates of Γ_N. We get

$$\begin{aligned} &\frac{(-1)^{k_N}}{|\Lambda|}\frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}} \sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k_N-\text{large, thin}}} u^+_\Lambda(\Gamma)\Big|_{h=0} \geq \\ &\frac{c(k_N)}{20}2^{k_N}K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{1}{d-1}k_N}\Big[\frac{d}{d-1}(1-\epsilon)e^{-\delta_1}D_-\Big]^{\frac{d}{d-1}k_N} k_N!^{\frac{d}{d-1}}\,, \end{aligned} \tag{5.51}$$

Consider now a k_N-small thin contour, i.e. $R^*(V(\Gamma)) \geq R^*(V_0(k_N))$. Using the Cauchy formula with a disc of radius $R^*(V_0(k_N))$ centered at $h = 0$,

$$|u^+_\Lambda(\Gamma)^{(k_N)}(0)| \leq k_N!\Big(\frac{1}{R^*(V_0(k_N))}\Big)^{k_N}\|u^+_\Lambda(\Gamma)\|_{U_\Gamma}\,. \tag{5.52}$$

Lemma 5.3. *Setting $\alpha_1 = \alpha_1(\theta,\beta) := \rho^{-1}(1-\theta(1+A(\beta))-\delta_1(\beta))$. If β is large enough, we have $\alpha_1 > 0$ and the bound*

$$\|u^+_\Lambda(\Gamma)\|_{U_\Gamma} \leq \frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}} \tag{5.53}$$

Proof. Using (5.11), (5.12) and (5.32),

$$\|\varphi^+_\Lambda(\Gamma)\|_{U_\Gamma} \leq \sup_{h\in U_\Gamma} e^{-\beta(1-\delta_1)\|\Gamma\|}e^{2\beta(1+A)|\mathrm{Re}\,h|V(\Gamma)} \leq e^{-\alpha_1\beta|\Gamma|} < 1\,, \tag{5.54}$$

where we used the definition of the radius of analyticity:

$$\sup_{h\in U_\Gamma}|h|V(\Gamma) \leq R^*(V(\Gamma))V(\Gamma) \leq R(V(\Gamma))V(\Gamma) \leq \frac{\theta}{2}\|\Gamma\|\,. \tag{5.55}$$

The proof finishes by using the Taylor expansion of $\log(1+x)$. $\square$

A standard Peierls estimate implies, when β is large, the existence of a number C_4 such that

$$\sum_{\Gamma\in\mathcal{C}^+(\Lambda)} e^{-\beta\alpha_1|\Gamma|} \le C_4|\Lambda|\,. \tag{5.56}$$

Using the Stirling formula, it easy to see that $k_N!k_N^{\frac{1}{d-1}k_N} \le k_N!^{\frac{d}{d-1}}e^{\frac{1}{d-1}k_N}$. The contribution from the k_N-small contours is then bounded by

$$\frac{1}{|\Lambda|}\Big|\frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k_N-\text{small, thin}}} u_\Lambda^+(\Gamma)\Big|_{h=0} \le$$

$$C_5 2^{k_N}K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{1}{d-1}k_N}\Big[e^{\frac{1}{d-1}}\Big(\frac{1+\epsilon'}{\theta}\Big)^{\frac{d}{d-1}}\Big(\frac{1}{1-2\sqrt{A}}\Big)^{\frac{1}{d-1}}\Big]^{k_N}k_N!^{\frac{d}{d-1}} \tag{5.57}$$

Since $\frac{d}{d-1} > e^{\frac{1}{d}}$, the comparison of the square brackets of (5.57) with those of (5.51) shows that if θ is close to 1, if ϵ,ϵ' are small, and if β is large enough, then the contribution from the k_N-small contours is negligible in comparison to the k_N-large ones.
We are then left with the contribution of the fat contours. We can use a Cauchy bound

$$\begin{aligned}\Big|\frac{\mathrm{d}^k}{\mathrm{d}h^k}u_\Lambda^+(\Gamma)\Big|_{h=0} &\le k!\Big(\frac{1}{R^*(V(\Gamma))}\Big)^k\|u_\Lambda^+(\Gamma)\|_{U_\Gamma}\\ &\le k!\Big(\frac{2K(1)}{\theta}\Big)^k V(\Gamma)^{\frac{k}{d}}\frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}}\\ &\le k!\Big(\frac{2K(1)}{\theta}\Big(\frac{2}{\delta_2}\Big)^{\frac{1}{d}}\Big)^k|\Gamma|^{\frac{k}{d}}\frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}}\end{aligned}$$

Then a Peierls estimate leads to

$$\sum_{\Gamma\in\mathcal{C}^+(\Lambda)}|\Gamma|^{\frac{k}{d}}e^{-\alpha_1\beta|\Gamma|} \le |\Lambda|\sum_{L\ge1}L^{\frac{k}{d}}e^{-\alpha_1'\beta L} \le |\Lambda|(\alpha_1'\beta)^{-\frac{k}{d}}\Gamma\Big(\frac{k}{d}+1\Big)\,, \tag{5.58}$$

where $\Gamma(x)$ is the Gamma-function. Using the Stirling formula, it is then easy to show that the contribution from the fat contours is bounded by

$$\frac{1}{|\Lambda|}\Big|\frac{\mathrm{d}^k}{\mathrm{d}h^k}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ \text{fat}}}u_\Lambda^+(\Gamma)\Big|_{h=0} \le \big(K(1)\beta^{-\frac{1}{d}}D(k)\big)^k k!^{\frac{d}{d-1}}\,, \tag{5.59}$$

where $\lim_{k\to\infty}D(k) = 0$. The fat contours can thus always be ignored. This finishes the proof of Proposition 5.1 $\square$

With (3.57), we get the lower bound, for a large enough box Λ,

$$|p_{\gamma,\Lambda}^{+(k_N)}(0)| \geq \big(C_1 K(\infty)^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}} - C_r^{k_N} k_N! \tag{5.60}$$

$$\geq \big(C_- \gamma^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}} - C_r^{k_N} k_N!\,. \tag{5.61}$$

We used the lower bound $K(\infty) \geq c_-\gamma$ from Lemma 2.9. Notice that we could extract the contribution of the translates of Γ_N to $p_{\gamma,\Lambda}^{+(k_N)}(0)$ *without* knowing its explicit shape. This is where our formulation of the isoperimetric problems differs from the one of Isakov. Notice also that the lower bound (5.61) shows how non-analyticity is detected in *finite* volumes.

5.3 Thermodynamic Limit; Proofs of Theorems 1.2 and 1.3

To extend the bounds we have on $p_{\gamma,\Lambda}^{+(k_N)}(0)$ to the infinite volume limit, we first show that in the strip U_+ the derivatives of the pressure are uniformly bounded.

Lemma 5.4. *Let β be large enough. There exists $C_+ > 0$ such that for all $k \geq 2$,*

$$\sup_{\Lambda} \|p_{\gamma,\Lambda}^{+(k)}\|_{U_+} \leq \big(C_+ \gamma^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^{k} k!^{\frac{d}{d-1}} + C_r^{k} k!\,. \tag{5.62}$$

Proof. Like in Section 4.4, we can fix $\theta := \frac{1}{2}$. The term $C_r^k k!$ comes from (3.57). Consider $u_\Lambda^+(\Gamma)$ and the representation (5.14) of $\varphi_\Lambda^+(\Gamma)$. From Lemma 5.1, $\varphi_\Lambda^+(\Gamma)$ is analytic in U_Γ. From Proposition 4.2 and Lemma 4.4, it is also analytic in U_+, i.e. in $U_+ \cup U_\Gamma$. Proceeding like in the proof of Lemma 5.1, we get

$$\Big\| \frac{\Theta_{i(\Gamma)}^{+}(\mathrm{ext}_\Lambda\Gamma;\sigma_\Gamma)\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{\Theta_{i(\Gamma)}^{+}(\Lambda)} \Big\|_{U_+} \leq \sup_{h\in U_+} e^{-\beta \mathrm{Re}\, h|\Gamma|} e^{3(\epsilon_r+\epsilon_c)|[\Gamma]_R|}$$
$$= e^{3(\epsilon_r+\epsilon_c)|[\Gamma]_R|}\,.$$

Assume $3^{d+1}(\epsilon_r+\epsilon_c) \leq \frac{1}{3}$. Using (4.58),

$$\|\varphi_\Lambda^+(\Gamma)\|_{U_+} \leq e^{-\beta\|\Gamma\|} e^{\beta h_+|\Gamma|} e^{|\Gamma|} \leq e^{-\alpha_2\beta|\Gamma|} < 1\,. \tag{5.63}$$

Notice that unlike in (5.54), α_2 in independent of θ. This implies that $u_\Lambda^+(\Gamma)$ is also analytic in $U_+ \cup U_\Gamma$. Set $\alpha_3 = \min\{\alpha_1, \alpha_2\}$. Using a disc of radius $R^*(V(\Gamma))$ around each $h \in U_+$, we have

$$\|u_\Lambda^+(\Gamma)^{(k)}\|_{U_+} \leq k!\Big(\frac{1}{R^*(V(\Gamma))}\Big)^k \|u_\Lambda^+(\Gamma)\|_{U_+\cup U_\Gamma}$$
$$\leq k!\Big(\frac{2K(1)}{\theta}\Big)^k V(\Gamma)^{\frac{k}{d}} \frac{e^{-\beta\alpha_3|\Gamma|}}{1-e^{-\beta\alpha_3|\Gamma|}}$$
$$\leq k!\Big(\frac{2K(1)}{\theta l^{\frac{1}{d-1}}}\Big)^k |\Gamma|^{\frac{k}{d-1}} \frac{e^{-\beta\alpha_3|\Gamma|}}{1-e^{-\beta\alpha_3|\Gamma|}}$$

We used the isoperimetric inequality of Lemma 2.10. Remember that $K(1) \leq c_+\gamma$ (Lemma 2.9), and that $l = \nu\gamma^{-1}$. The proof finishes like for the upper bound on fat contours. $\square$

Corollary 5.1. *For all $h' \in U_+ \cup \{\mathrm{Re}\, h = 0\}$ and for all $k \in \mathbb{N}$,*

$$p_\gamma^{(k),\leftarrow}(h') = \lim_{\Lambda \nearrow \mathbf{Z}^d} p_{\gamma,\Lambda}^{+(k)}(h') = \lim_{h \searrow h'} p_\gamma^{(k)}(h)\,. \tag{5.64}$$

Proof. We show (5.64) for $k = 1$. By definition,

$$\begin{aligned} p_\gamma^{(1),\leftarrow}(h') &= \lim_{\delta \searrow 0} \frac{p_\gamma(h'+\delta) - p_\gamma(h')}{\delta} \\ &= \lim_{\delta \searrow 0} \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{p_{\gamma,\Lambda}^{+}(h'+\delta) - p_{\gamma,\Lambda}^{+}(h')}{\delta} \\ &= \lim_{\delta \searrow 0} \lim_{\Lambda \nearrow \mathbf{Z}^d} \big(p_{\gamma,\Lambda}^{+(1)}(h') + \frac{1}{2!} p_{\gamma,\Lambda}^{+(2)}(h(\delta))\delta\big)\,, \end{aligned}$$

where $\lim_{\delta \searrow 0} h(\delta) = h'$. The following lemma will allow to permute the limits $\lim_{\delta \searrow 0}$ and $\lim_{\Lambda \nearrow \mathbf{Z}^d}$.

Lemma 5.5. *Let, for all $N \in \mathbb{N}$, $\delta > 0$, $b_N(\delta) = a_N + c_N(\delta)$, such that $|c_N(\delta)| \leq D\delta$ uniformly in N, and $\lim_{N\to\infty} b_N(\delta) = b(\delta)$ exists. Then $\lim_{N\to\infty} a_N$ and $\lim_{\delta \searrow 0} b(\delta)$ exist and are equal.*

Proof. We first show that $\lim_{\delta \searrow 0} b(\delta)$ exists. Let (δ_k) be any sequence $\delta_k > 0$ such that $\lim_{k\to\infty} \delta_k = 0$. Then we have

$$|b(\delta_k) - b(\delta_{k'})| = |\lim_{N\to\infty} (c_N(\delta_k) - c_N(\delta_{k'}))| \leq D(\delta_k + \delta_{k'})\,, \tag{5.65}$$

and so $\lim_{k\to\infty} b(\delta_k)$ exists. Fix $\epsilon > 0$. There exists $N_{\epsilon,\delta}$ such that if $N \geq N_{\epsilon,\delta}$ then $|b_N(\delta) - b(\delta)| \leq \epsilon$. We then have

$$b(\delta) - \epsilon - D\delta \leq \liminf_{N\to\infty} a_N \leq \limsup_{N\to\infty} a_N \leq b(\delta) + \epsilon + D\delta\,, \tag{5.66}$$

which finishes the proof, once we take $\epsilon \to 0$, $\delta \to 0$. $\square$

Using the fact that the second derivative is uniformly bounded on U_+ (Lemma 5.4), this shows the first equality in (5.64). For the second, we only need to consider the case where $h' = 0$.

$$\begin{aligned} p_\gamma^{(1),\leftarrow}(0) &= \lim_{\delta \searrow 0} \frac{p_\gamma(\delta) - p_\gamma(0)}{\delta} \\ &= \lim_{\delta \searrow 0} \Big[\frac{p_\gamma(\delta) - p_\gamma(\frac{\delta}{2})}{\delta} + \frac{p_\gamma(\frac{\delta}{2}) - p_\gamma(0)}{\delta}\Big] \\ &= \big(\lim_{\delta \searrow 0} \frac{1}{2} p_\gamma^{(1)}(h(\delta))\big) + \frac{1}{2} p_\gamma^{(1),\leftarrow}(0)\,, \end{aligned}$$

where $h(\delta) \in [\frac{\delta}{2}, \delta]$ and $\lim_{\delta \searrow 0} h(\delta) = 0$. This shows

$$p_\gamma^{(1),\leftarrow}(0) = \lim_{\delta \searrow 0} p_\gamma^{(1)}(h(\delta)), \tag{5.67}$$

which extends easily to any sequence $h \searrow 0$, since derivatives of any order are uniformly bounded on U_+. $\square$

We can then complete the proofs of our main results.

Proof of Theorem 1.3: The bounds on $p_{\gamma,\Lambda}^{(k)}(0)$ of (5.61) and Lemma 5.4 extend to the thermodynamic limit using Corollary 5.1. $\square$

Proof of Theorem 1.2: Using the symmetry $p_\gamma(h) = p_\gamma(-h)$, we can write, for $m \geq 0$,

$$f_\gamma(m) = \sup_{h \geq 0} \big(hm - p_\gamma(h)\big). \tag{5.68}$$

By the Theorem of Lee and Yang, $h \mapsto p_\gamma(h)$ and $m \mapsto m_\gamma(h) := p_\gamma^{(1)}(h)$ are analytic in $\{\mathrm{Re}\, h > 0\}$. If $m^* := p_\gamma^{(1),\leftarrow}(0)$, then for all $m \in (m^*, 1)$,

$$f_\gamma(m) = h(m)m - p_\gamma(h(m)), \tag{5.69}$$

where $h_\gamma(m)$ is the unique solution of the equation $m = m_\gamma(h)$. The GKS inequality (see [GKS]) allows to obtain, for all $h \neq 0$,

$$p_\gamma^{(2)}(h) \geq \beta\big(1 - \tanh(\beta(h+1))^2\big) > 0, \quad \lim_{h \searrow 0} p_\gamma^{(2)}(h) > 0 \tag{5.70}$$

Since $p_\gamma^{(2)}(h) \neq 0$ for all $h > 0$, the biholomorphic mapping theorem [10] implies that $m \mapsto h_\gamma(m)$ is analytic in a complex neighbourhood of each $m \in (m^*, 1)$. So f_γ, which is a composition of analytic maps, is analytic on $(m^*, 1)$.
We now show that f_γ has no analytic continuation at m^*. Assume this is wrong. We compute

$$h_\gamma^{(1)}(m^*) = \lim_{m \searrow m^*} h_\gamma^{(1)}(m) = \lim_{h \searrow 0} m_\gamma^{(1)}(h)^{-1} = \lim_{h \searrow 0} p_\gamma^{(2)}(h)^{-1} \neq 0. \tag{5.71}$$

We used the fact that $p_\gamma^{(2),\leftarrow}(0)$ is bounded at $h = 0$. Again, (5.71) implies that the inverse of $h_\gamma = h_\gamma(m)$ can be inverted in a neighbourhood of m^* and that the inverse, $m_\gamma = m_\gamma(h)$, is analytic at $h = 0$. This is a contradiction with Theorem 1.3. $\square$

[10]Let $g : D \to \mathbb{C}$ be an analytic function, $z_0 \in D$ be a point such that $g'(z_0) \neq 0$. Then there exists a domain $V \subset D$ containing z_0, such that the following holds: $V' = g(V)$ is a domain, and the map $g : V \to V'$ has an inverse $g^{-1} : V' \to V$ which is analytic, and which satisfies, for all $\omega \in V'$, ${g^{-1}}'(\omega) = \big(g'(g^{-1}(\omega))\big)^{-1}$. The proof of this result can be found in [Rem], pp. 281-282.

6 Conclusion

Our analysis has lead to the following representation of the pressure for $h \geq 0$:

$$p_\gamma(h) = p^+_{r,\gamma}(h) + s^+_\gamma(h)\,, \tag{6.1}$$

where $p^+_{r,\gamma}$ is the restricted pressure. As we have seen in Section 3, $p^+_{r,\gamma}$, which describes a homogeneous phase with positive magnetization, behaves analytically at $h = 0$. On the other side, s^+_γ contains the contributions from droplets (contours) of any possible sizes, and is responsible for the non-analytic behaviour of the pressure at $h = 0$. Non-analyticity can be detected only in the very high order derivatives of s^+_γ, although s^+_γ contributes essentially nothing to the pressure when γ is small. Indeed, s^+_γ can be expressed as a sum over clusters of chains, and each chain contains at least one contour. Since the length $|\Gamma|$ of a contour is bounded below by the size of a cube $C^{(l)}$, we have

$$\|s^+_\gamma\|_{U_+} \leq a e^{-b\beta\gamma^{-d}}\,, \tag{6.2}$$

where $a, b > 0$ are constants.
For the pressure, the Lebowitz-Penrose Theorem takes the form (see [Pr]):

$$p_0(h) := \lim_{\gamma \searrow 0} p_\gamma(h) = \sup_{m \in [-1,+1]} \big(hm - f_{MF}(m)\big)\,, \tag{6.3}$$

where the mean field free energy f_{MF} was defined in (1.4). The bound (6.2) implies, for $h \geq 0$,

$$p_0(h) = \lim_{\gamma \searrow 0} p^+_{r,\gamma}(h) = \sup_{m \geq 0} \big(hm - f_{MF}(m)\big)\,. \tag{6.4}$$

From this last expression, the analytic continuation of the pressure, in the van der Waals limit, at $h = 0$, can be understood easily: for $h > 0$, $hm - f_{MF}(m)$ has a unique global maxima at $m^*(h, \beta) > 0$. When $h < 0$ this maxima is only local, but provides the analytic continuation at $h = 0$. The identity (6.4) shows that the constraint on the local magnetization, in $p^+_{r,\gamma}$, has the effect of always selecting the maxima $m^*(h, \beta)$, which is global when $h > 0$ and local when $h < 0$. When $\gamma > 0$, this scenario breaks down: droplets are well defined, and they are all stable at $h = 0$, creating arbitrarily large fractions of the $-$ phase. As we saw, this gives a contribution $k!^{\frac{d}{d-1}}$ to the k-th derivative of the pressure.

A Cluster Expansion

Consider a countable set $\mathcal{D}$ whose elements are called animals, and denoted $\gamma \in \mathcal{D}$. To each animal γ is associated a finite subset of $\mathbf{Z}^d$, called the

support of γ. Usually we also denote the support by γ. In the cases we consider, the support is always an R-connected set. Assume we are given a symmetric binary relation on $\mathcal{D}$, denoted $\sim$. We say two animals γ, γ' are compatible if $\gamma \sim \gamma'$. When γ and γ' are not compatible we write $\gamma \not\sim \gamma'$. We assume that the following condition is necessary to characterize incompatibility: for each each animal γ, there exists a set $b(\gamma) \subset \mathbf{Z}^d$ such that if $\gamma \not\sim \gamma'$, then $b(\gamma) \cap b(\gamma') \neq \emptyset$.
To each animal $\gamma \in \mathcal{D}$ we associate a complex weight $\omega(\gamma) \in \mathbb{C}$. The partition function is defined by

$$\Xi(\mathcal{D}) := \sum_{\substack{\{\gamma\} \subset \mathcal{D} \\ \text{compat.}}} \prod_{\gamma \in \{\gamma\}} \omega(\gamma)\,, \tag{A.1}$$

where the sum extends over all sub-families of $\mathcal{D}$ of pairwise compatible animals (we assume this sum exists, which is the case in every concrete situation). When $\{\gamma\} = \emptyset$, we define the product over γ as equal to 1. We are interested in studying the logarithm of the partition function. To this end, we define the family $\hat{\mathcal{D}}$ of all maps $\hat{\gamma} : \mathcal{D} \to \{0, 1, 2, \dots\}$. The support of $\hat{\gamma}$ is the set $\{\gamma \in \mathcal{D} : \hat{\gamma}(\gamma) \geq 1\}$. Usually we also denote the support of $\hat{\gamma}$ by $\hat{\gamma}$. We will also write $\hat{\gamma} \ni x$ if the support of $\hat{\gamma}$ contains an animal whose support contains x. A map $\hat{\gamma} \in \hat{\mathcal{D}}$ is a cluster of animals if its support can't be decomposed into a disjoint union $S_1 \cup S_2$ such that each $\gamma_1 \in S_1$ is compatible with each $\gamma_2 \in S_2$. Formally, the logarithm of the partition function has the form (see e.g [Pf])

$$\log \Xi(\mathcal{D}) = \sum_{\hat{\gamma} \in \hat{\mathcal{D}}} \omega(\hat{\gamma})\,, \tag{A.2}$$

where the weight of $\hat{\gamma}$ equals

$$\omega(\hat{\gamma}) = a^T(\hat{\gamma}) \prod_{\gamma \in \mathcal{D}} \omega(\gamma)^{\hat{\gamma}(\gamma)}\,. \tag{A.3}$$

The functions $a^T(\hat{\gamma})$ are purely combinatorial factors. They equal zero if $\hat{\gamma}$ is not a cluster. The following is the technical lemma that gives explicit conditions for the convergence of the development (A.2). The proof is standard and can be adapted from [Pf].

Lemma A.1. *Let $\omega_0(\gamma)$ be a positive weight such that*

$$\sup_{x \in \mathbf{Z}^d} \sum_{\gamma : b(\gamma) \ni x} \omega_0(\gamma) e^{|b(\gamma)|} \leq \epsilon\,, \tag{A.4}$$

where $0 < \epsilon < 1$. Define $\omega_0(\hat{\gamma})$ as in (A.3) with $\omega_0(\gamma)$ in place of $\omega(\gamma)$. Then there exists a function $\eta(\epsilon)$, $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ such that

$$\sup_{x \in \mathbf{Z}^d} \sum_{\hat{\gamma} \ni x} |\omega_0(\hat{\gamma})| \leq \eta(\epsilon)\,. \tag{A.5}$$

Typically, in the cases we consider, the weights are maps $z \mapsto \omega(\gamma; z)$, analytic in a domain $A \subset \mathbb{C}$, and there exists a positive weight $\omega_0(\gamma)$ such that $\|\omega(\gamma; \cdot)\|_A \leq \omega_0(\gamma)$ for all γ. Lemma A.1 thus implies that the series (A.2) is normally convergent on A. This guarantees analyticity of the logarithm of $\Xi(\mathcal{D})$, by a standard Theorem of Weierstrass (see e.g. [Rem]).

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