Wireless Operators in a Shared Spectrum

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Abstract— So far, cellular networks have been operated in "private" frequency bands. But recently, several researchers and legislators have argued in favor of a more flexible and more efficient management of the spectrum, leading to the possible coexistence of several network operators in a *shared frequency band*. In this paper, we study this situation in detail, assuming that mobile devices can freely roam among the various operators. Free roaming means that the mobile devices measure the signal strength of the *pilot signals* (i.e., beacon signals) of the base stations and attach to the base station with the strongest pilot signal. We model the behavior of the network operators in a game theoretic setting in which each operator decides about the power of the pilot signal of its base stations. We first identify possible Nash equilibria in the theoretical setting in which all base stations are located on the vertices of a two-dimensional lattice. We then prove that a socially optimal Nash equilibrium exists and that it can be enforced by using punishments. Finally, we relax the topological assumption and show that, in the more general case, finding the Nash equilibria is an NP-complete problem.

Index Terms— Wireless networks, shared spectrum, pilot power control, cooperation, game theory, Nash equilibrium, NP-completeness

I. INTRODUCTION

Cellular networks are notoriously difficult to design and operate; in particular, defining the optimal location of the base stations and fine tuning their configuration parameters is very challenging. For this reason, government agencies (such as the FCC in the US) have sold or rented, for example by auction, each operator a frequency band for its exclusive usage in a given country or region. Only a small part of the whole spectrum is allocated as a *shared spectrum*, in which networks function in the same (unlicensed) frequency band.

With the progress of technology and the fast growing demand for ubiquitous high-speed wireless services, it is clear that the pressure towards more flexibility of the usage of the spectrum will only increase. Therefore, the government agencies are likely to adapt the current regulations in order to increase the proportion of the unlicensed spectrum as discussed in [3], [7].

The evolution towards unlicensed frequency bands can lead to a better usage of the spectrum. Yet, it would also create a novel situation, in which the base stations of different operators would interfere with each other. An operator may be tempted to let its base stations transmit at the maximum authorized level. But by doing this, it would maximize interference not only to its own base stations, but also to the base stations of the other operators, and to all mobile devices in the power range of its base stations; in addition, it would face the danger that the other operators retaliate by behaving in the same way.

In our paper, we assume that mobile users can *freely roam* across the base stations located in their neighborhood, attaching to the one offering the most favorable signal quality (i.e., the base station with the strongest pilot signal) and bandwidth, irrespectively of the operator to which the base station belongs[1]. From the interference perspective, this operating principle is much more efficient than the current practice, because it enables mobile devices to find the "closest" base station in the area and hence mobile devices and base stations can significantly decrease their transmission power. This free roaming could be beneficial for both operators and users, because the former could serve an increased set of users, and the latter could enjoy various services across several operators.

We also assume that each operator wants to cover the largest possible area by increasing the transmission range of its base stations. At the same time, it wants to minimize interference. These two contradictory goals correspond to the willingness to maximize the number of users who can attach to its base stations. We model this situation as a game between operators in terms of power control of the base stations. We believe our paper to be one of the first steps towards a deeper understanding of the trade-offs of operating cellular networks in shared spectrum.

Note that the general problem of power control of base stations is hard to solve (i.e., NP-complete); it is characterized by the following dimensions: (i) the size of the base station sets, (ii) the geographic locations of the base stations and (iii) their possible radio ranges.

Game theory is used to study the power control of user devices in wireless networks, notably in cellular systems as studied in [1], [9], [12], [13], [14], [17], [27] and [29]. Game theory is also used to study cooperation in wireless ad hoc networks, for example in [5], [15] and [24], in particular for cooperative power control [18]. A general framework for resource allocation in wireless network is addressed in [6].

Recently, the coexistence of multiple Internet Service Providers (ISPs) was studied by Shakottai and Srikant in [23]. They consider both transit and customer prices for the ISPs. They show that if the number of ISPs competing for the same customers is large, then it can lead to price wars. In addition to this work in wired networks, the coexistence of wireless

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[1]The users might have other attachment preferences based on subscription type, past experience, etc. We will consider the extension of user attachment behavior in our future work.

operators in a non-shared spectrum is addressed in two contributions. In [10], Halldórsson *et al.* study channel assignment strategies for Wi-Fi operators. They use the maximum graph coloring problem to identify Nash equilibria and they also provide a bound on the price of anarchy of these equilibria. In another paper [28], Zemlianov and de Veciana consider the scenario, in which users are able to choose between a cellular network and a Wi-Fi network. They show that congestion sensitive strategies are better than proximity-based strategies. None of these works considers the power control of the base stations.

Our paper addresses the problem of pilot signal power control in shared spectrum networks. Haykin provides a comprehensive overview [11] of the current tendencies and research challenges in shared spectrum communications in general. One of the challenges, namely opportunistic spectrum access, is addressed in the paper of Wang *et al.* [26].

This paper is organized in the following way. In Section II, we describe the system model and the corresponding power control game. We solve this game on a two-dimensional lattice topology in Section III. In Section V, we present our results in the case of a general topology of base stations. We extend our study with a repeated game model in Section IV. Finally, we conclude in Section VI.

II. MODEL

A. System Model

We make the following assumptions with respect to the communication network. We assume two wireless communication networks, each operated by an *operator* and we call the operators A and B. Operator $i \in \{A, B\}$ controls a set of *base stations (BS-s)* denoted by $\mathcal{B}_i$. We denote the union of all base stations by $\mathcal{B}$. There exist no BS-s that are located at the same place and belong to different operators[2]. We also assume several *users* equipped with *mobile devices* to access the communication network. The networks reside in a given *service area*, where the operators want to provide wireless access for the users. We restrict ourselves to two operators in order to provide an insight in the basic principles of cooperation in a multi-operator context. Note that the problem is hard to solve for a general network topology as presented in Sections V.

We assume that the radios of the base stations and the mobile devices are compatible, meaning that any user is able to access the network via any of the base stations. Base stations and mobiles operate on the same unlicensed band of the frequency spectrum. Each of these devices might perform power control to optimize its transmission power and reduce interference. This optimization can be realized in three ways: the power control of the pilot signal of the BS-s, of the downlink (BS to mobile) and of the uplink (mobile to BS). In this paper, we focus on the first technique and we postpone the investigation of the other two techniques to our future work. To mitigate interference, the shared frequency band is usually

split up into channels (i.e., separated frequency sub-bands), but the pilot signal is typically emitted on a single shared channel for all the base stations, which results in mutual interference of the pilot signals (in CDMA networks, the interference of the pilot signals is referred to as the *pilot pollution* [22]).

According to the *physical model* of signal propagation [22], the pilot signal of a base station $b_i \in \mathcal{B}$ can be received by a user device u if its *signal-to-interference-noise ratio (SINR)* exceeds a reception threshold β:

$$\frac{P_i \cdot g_{iu}}{N_0 + \sum_{j \in \mathcal{B}, j \neq i} P_j \cdot g_{ju}} \geq \beta \qquad (1)$$

where P_i is the transmission power of BS b_i, g_{iu} is the channel gain between the BS b_i and user device u and N_0 is the Gaussian thermal noise. We assume that the channel gain depends only on the distance of the transmitter and the receiver we normalize the effect of the antenna characteristics, thus we have $g_{iu} = d_{iu}^{-\alpha}$ between the BS b_i and user device u, where $2 \leq \alpha \leq 5$ is the path loss exponent that characterizes the radio signal propagation properties of the environment. Hence, (1) corresponds to the Friis free space radio signal propagation equation (see [22] Equation (4.1)). It captures how the reception power depends on the most important factors, namely on the transmission power and the distance between transmitter and receiver. Note that we consider the local average of the received pilot signal as described in [22]. In reality, on a small time scale, the pilot power signals have a time-varying property due to fading. In our future work, we will consider a more realistic radio signal modelling that incorporates fading and more realistic path loss models.

We assume that (1) holds for every point in the service area for at least one base station and that the user device u attaches to the base station b_i with the best SINR. Thus, we can write that:

$$\frac{P_i \cdot d_{iu}^{-\alpha}}{N_0 + \sum_{j \in \mathcal{B}, j \neq i} P_j \cdot d_{ju}^{-\alpha}} \geq \frac{P_l \cdot d_{lu}^{-\alpha}}{N_0 + \sum_{m \in \mathcal{B}, l \neq m} P_m \cdot d_{mu}^{-\alpha}} \qquad (2)$$

for any other base station b_l.

We abstract away the mobiles and assume that their expected position is uniformly distributed over the service area. Note that this also means a balanced load on the base stations (i.e., no users have to switch base stations due to the lack of available bandwidth). We leave the topic of other user distributions for future work.

Let us assume that the pilot signals propagate in an open area, meaning $\alpha = 2$. Then (2) defines a Multiplicatively Weighted Voronoi power diagram (MW power diagram) [20], which determines the set of points in the service area (potential places of user devices) that are attached to a given base station. In the MW power diagram, a point belongs to a base station if it is "closer" to it than to any other base station, where the distance is defined as follows:

Definition 1: The *multiplicatively weighted power distance* between the points u and b_i is defined as:

$$d_{mpw}(u, b_i; w_i) = \frac{d_{iu}^2}{w_i} \qquad (3)$$

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[2]To reduce operating costs, operators of current cellular networks often share the same site. However, if users can freely roam, then this site sharing does not make sense anymore.

where d_{iu} is the Euclidian distance between the points u and b_i and w_i is a weight assigned to point i.

We can define the *Voronoi region* $V(b_i)$ around a base station $b_i \in \mathcal{B}$ as the set of points u that are "closer" to point b_i than to any other point b_j (i.e., $b_i \neq b_j$). Hence, we can write $V(b_i)$ as:

$$V(b_i) = \{u | d_{mpw}(u, b_i; w_i) \leq d_{mpw}(u, b_j; w_j) \ for \ i \neq j\} \tag{4}$$

We can write the *Voronoi diagram* $\mathcal{V}(\mathcal{B})$ of all base stations $\mathcal{B}$ as:

$$\mathcal{V}(\mathcal{B}) = \bigcup V(b_i) \tag{5}$$

where $b_i \in \mathcal{B}$.

Due to the complex shape of the Voronoi diagram with multiplicatively weighted distances, it is difficult to derive analytical solutions for the pilot power control problem. Hence, we apply a *radio range model* that is widely used in the literature. We will show in Section III-D that the principles derived from the range model hold for the physical model as well.

Let us derive from (1) the *radio range* of the pilot signal of the BS b_i as the Euclidian distance within which the users are able to attach to this base station if there is no interference from other devices:

$$r_i = \sqrt[\alpha]{\frac{P_i}{\beta N_0}} \tag{6}$$

According to the radio range model, we can define the Additively Weighted Voronoi power diagram (AW power diagram) [20]. In the AW power diagram, the distance is defined as follows:

Definition 2: The *additively weighted power distance* between the points u and b_i is defined as:

$$d_{apw}(u, b_i; w_i) = d_{iu}^2 - w_i \tag{7}$$

where d_{iu} is the Euclidian distance between the points u and b_i and w_i is a weight assigned to point b_i.

In this paper, we substitute $w_i = r_i^2$ and hence we obtain a *Voronoi diagram in the Laguerre geometry* [20]. This model corresponds to a Voronoi diagram, where the distance is defined as a tangential Euclidean distance to circles centered at the base stations' locations and radii corresponding to their radio ranges.

We assume that the base stations are placed on the vertices of a two-dimensional lattice in an alternating way such that any BS that belongs to operator A has four neighboring BS-s that belong to operator B (a small part of the network is shown in Figure 1). Let us call d the smallest Euclidian distance between base stations. In Section V and IV, we will extend our model to general network topologies.

To further specify our model, we assume that:

- **A1:** Operators want to provide wireless access service everywhere. Thus, no place remains uncovered in the service area.
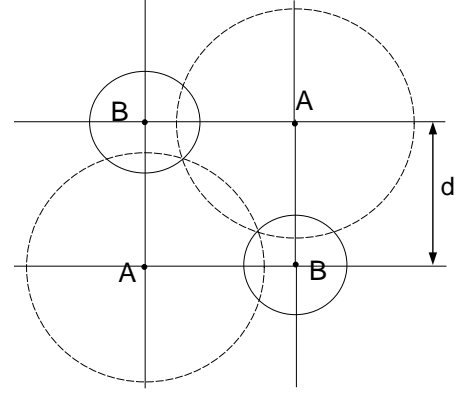- **A2:** Operators can estimate or measure their coverage including the action of other operators.

Fig. 1. Base stations on the vertices of a two-dimensional lattice. Here A is the operator with a larger radio range.

- **A3:** Each BS belonging to the same operator has the same radio range. We show in Section V that relaxing this assumption makes the power control problem NP-complete.
- **A4:** There exists a limitation P_{MAX} on the transmission power of any base station, which is defined by the regulator of the wireless spectrum. Then, the maximum radio range R_{MAX} can be derived from (6) by substituting $P_i = P_{MAX}$. Furthermore, if the radio ranges of all base stations $b_m \in \mathcal{B}$ are equal, we denote the minimum radio range for which **A1** holds by $R_{MIN} = \frac{\sqrt{2}}{2}d$.
- **A5:** The users can freely roam between any of the base stations (i.e., the operators do not forbid roaming between their networks).
- **A6:** Users are uniformly distributed over the area and hence, the expected load is the same on every base station.
- **A7:** The base stations and the mobile devices have omnidirectional antennae. The investigation of the effect of directional antennas is part of our future work.

These assumptions ensure an open spectrum environment, in which users enjoy ubiquitous wireless connectivity. In particular, we make Assumption **A3**, as well as the assumption that the base stations are placed on the vertices of a grid, to make the model tractable. This special scenario is reasonable for a small number of base stations, such as for a small city network. We will show stability points for this special model. We are motivated to study this special model, because we wanted to provide some quantitative insights into the power control problem. The general problem is very involved: We show that if operators can set an arbitrary radio range for their base stations (i.e., **A3** does not hold), then the power control problem is hard to solve.

B. Power Control Game

We model the power control problem with two operators as a two-player, nonzero-sum game. We refer to the two operators as *players* A and B, respectively. Due to **A3**, we designate the radio ranges of the pilot signal of the players by r_A and r_B. The *strategy* of the players defines their best radio range. The goal of the players is to maximize the area they cover
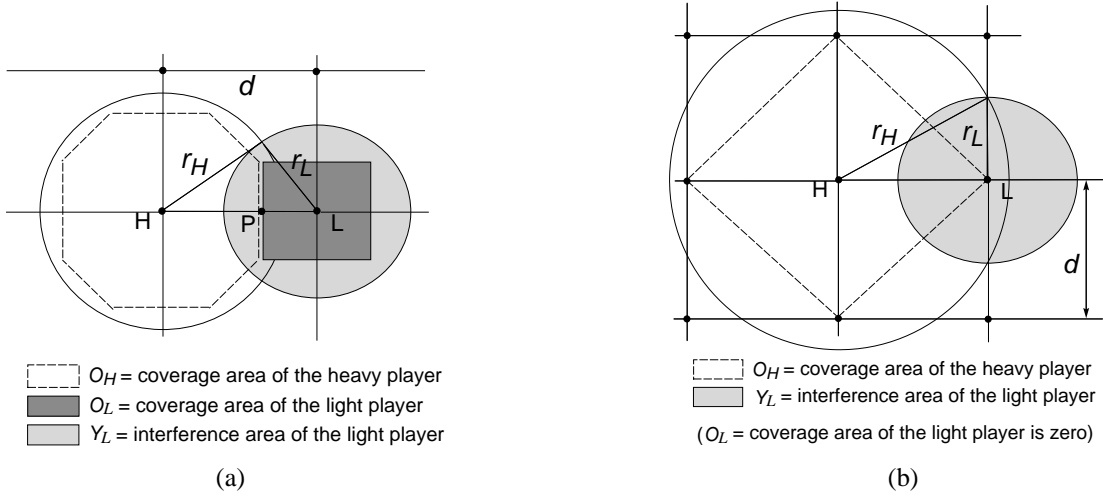
(a)

(b)

Fig. 2.   Coverage and interference area of a base station, illustrated with two base stations: (a) both BS-s have a coverage area; (b) the BS-s of the light player are overwhelmed by the BS-s of the heavy player and thus the light player has no coverage area at all.

with their pilot signal as expressed by their *utility* function. To express the utility of the players formally, let us introduce the following concepts.

Assume that the two players choose a different radio range. Let us call the player with the larger radio range *heavy* and the player with the smaller range *light*. Let us denote the radio range of the heavy player by r_H and the one of the light player by r_L (recall from **A3** that a player has the same range for all of its base stations). In this section, we assume that A is the heavy player and B is the light player (i.e., $r_A = r_H$ and $r_B = r_L$); note, however, that it can be the opposite due to the symmetric situation. Since the placement of the BS-s is symmetric and the players apply the same radio range to all of their BS-s, we can analyze the game considering two neighboring base stations, as shown in Figure 2.

We define the *coverage area (O_i)* for any BS b_i as its Voronoi region $V(b_i)$ in the radio range model (i.e., in the Voronoi diagram in the Laguerre geometry). We define the *interference area (Y_i)* for a BS b_i as:

$$Y_i = T_i - O_i = r_i^2 \cdot \pi - O_i \qquad (8)$$

where T_i is the total area covered by the radio range and r_i denotes the radio range of BS b_i.

Note that the coverage area of a player depends on the radio range of the other player. Accordingly, we can distinguish two cases as follows.

In the first case, both players have a non-empty coverage area as presented in Figure 2a. For this case, the following condition holds:

$$r_H < \sqrt{r_L^2 + d^2} \qquad (\mathbf{C1})$$

We can express the coverage area of the heavy player by calculating the area of the octagon. As shown in Figure 2, this area can be calculated based on the distance d of the two base stations, the distances $\overline{HP}$ and $\overline{LP}$ and the ranges r_H and r_L. Thus, we can write the coverage area as follows:

$$O_H = \frac{d^4 + 2d^2(r_H^2 - r_L^2) - (r_H^2 - r_L^2)^2}{d^2} \qquad (9)$$

The coverage area of the light player is as follows:

$$O_L = \frac{(d^2 - r_H^2 + r_L^2)^2}{d^2} \qquad (10)$$

If Condition **C1** does not hold, then the light player is overwhelmed by the heavy player, meaning that the pilot signal of the heavy player is the strongest everywhere (as presented in Figure 2b). If the heavy player overwhelms the light player, the coverage area functions are as follows:

$$O_H = (\sqrt{2}d)^2 = 2d^2 \qquad (11)$$

$$O_L = 0 \qquad (12)$$

In addition to **C1**, we can derive a condition for the radio ranges of the two players from **A1** from the geometry presented in Figure 1 as follows:

$$r_H^2 \geq (\frac{\sqrt{2}}{2}d - r_L)^2 + (\frac{\sqrt{2}}{2}d)^2 = d^2 - \sqrt{2}dr_L + r_L^2 \quad (\mathbf{C2})$$

In the limit case, in which the equality holds in (**C2**), they just cover the service area (as shown in Figure 1).

From (**C2**) and **A4**, we can derive the definition interval for r_H:

$$\sqrt{d^2 - \sqrt{2}dr_L + r_L^2} \leq r_H \leq R_{MAX} \qquad (13)$$

Similarly, from (**C2**) we get the bounds on r_L knowing that it is positive and smaller than r_H:

$$\max\{0, \frac{\sqrt{2}}{2}(d - \sqrt{-d^2 + 2r_H^2})\} \leq r_L \leq r_H \qquad (14)$$

The upper bound comes from the fact that $r_H \leq \frac{\sqrt{2}}{2}(d + \sqrt{-d^2 + 2r_H^2})$ for all values of r_H. Note that the expressions in (13) and (14) always take real values.

We assume that the goal of the players is to maximize their utility, in other words to maximize their coverage area while minimizing their interference area (i.e., the area, which is in their radio range, but they do not cover eventually). We define
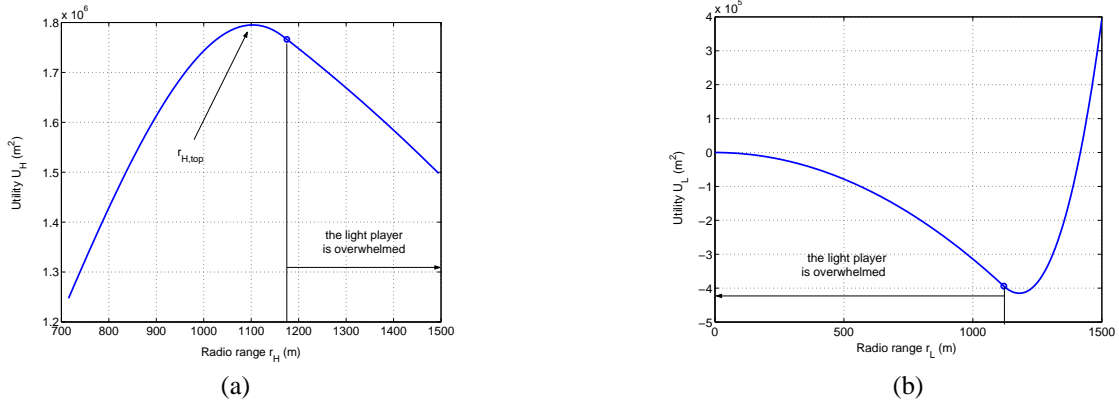
(a)

(b)

Fig. 3. Utility function (a) of the heavy player for $d = 1km$, $\gamma_H = 0.1$, $r_L = 0.6km$ and $R_{MAX} = 1.5km$ (defined by the regulator); and (b) of the light player for $d = 1km$, $\gamma_L = 0.1$ and $r_H = 1.5km$.

the utility per base station for player i playing r_i given that the other player j plays r_j at its BS-s as follows[3]:

$$U_i(r_i, r_j) = O_i - \gamma_i \cdot Y_i = (1 + \gamma_i) \cdot O_i - \gamma_i \cdot r_i^2 \cdot \pi \quad (15)$$

where $\gamma_i \geq 0$ is a *cooperation* parameter that defines how much player i cares about the size of its interference area. Note that the cooperation parameter provides a general method to model both internal considerations of the operator such as cooperativeness, as well as external cooperation enforcement mechanisms such as an agreement between operators or a *power price* induced by the regulator of the spectrum.

Let us graphically present the utilities of the players based on expression (15). Figure 3a presents an example for the utility of the heavy player for a fixed value of r_L and Figure 3b presents the utility of the light player for a fixed value of r_H. In the next section, we derive stability points in the game using these utility functions.

III. SINGLE-STAGE GAME

In this section, we consider a *single-stage game*, where both players simultaneously choose their radio range once and for all. This corresponds to the case in which the base stations are not able to perform power control during the operation of the network, thus the radio power has to be set manually at the installation of the base stations. We use this basic scenario to study the basic equilibria of the power control game. We extend our investigation to more complex scenarios in the following sections.

We make use of the concept of Nash equilibrium [8], [19], [21] to show stability points in the game. Let us denote the strategy of player i by $s_i \in S$ and the strategy of the other player by $s_j \in S$, where S is the set of strategies (i.e., the set of possible radio ranges). Then, we can define the *best response function* of player i as follows (as presented in [21] Equation (15.1)):

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[3]Note that due to the specific scenario, the utility of player i can be calculated by multiplying U_i with the number of its base stations. In this scenario, we refer to the utility per base station as the utility of the player.

Definition 3: For any $s_j \in S$, define $BR_i(s_j)$ to be the set of player i's best strategies given s_j.

$$BR_i(s_j) = \{s_i \in S : U_i(s_i, s_j) \geq U_i(s_i', s_j), \ \forall s_i' \in S\}$$

Based on this definition, we can formulate the Nash equilibria as follows (corresponds to Equation (15.2) in [21]):

Definition 4: In a Nash equilibrium, in which the players play $\hat{s}_i$ and $\hat{s}_j$, we have:

$$\hat{s}_i \in BR_i(\hat{s}_j), \ i \in \{A, B\}$$

Hence, in a Nash equilibrium, none of the players is motivated to change its strategy. This formulation shows us a method to find Nash equilibria: we first find the best response function for each player, then we identify a set of strategies for which Definition 4 holds.

A. Best Response Function

We derive the best response function for the heavy player from the utility functions presented in Figures 3a. For the heavy player, its utility is a concave function with a maximum point $r_{H,top}$ as shown in Figure 3a. We can derive $r_{H,top}$ by maximizing (15) with the coverage area defined in (9).

$$r_{H,top} = \frac{\sqrt{2(1 + \gamma_H)(d^2 + r_L^2) - d^2 \gamma_H \pi}}{\sqrt{2}\sqrt{1 + \gamma_H}} \quad (16)$$
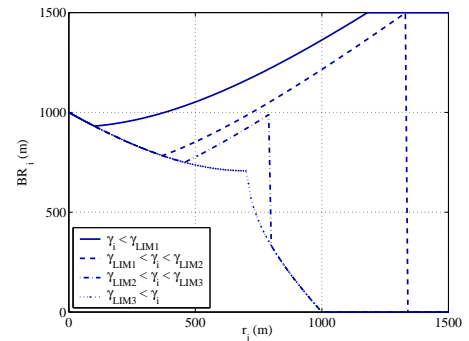

Fig. 4. Best response function of player i for various γ_i values for $d = 1km$ and $R_{MAX} = 1.5km$

We can identify different best response strategies for this case, corresponding to the definition interval of the utility function, as follows.

1) If $r_{H,top} < \sqrt{d^2 - \sqrt{2}dr_L + r_L^2}$, then the best response of the heavy player is the lower bound in (13), because the utility function is strictly decreasing:

$$BR_H(r_L) = \sqrt{d^2 - \sqrt{2}dr_L + r_L^2} \qquad (17)$$

2) If $\sqrt{d^2 - \sqrt{2}dr_L + r_L^2} < r_{H,top} < R_{MAX}$, then the best response is (this corresponds to Figure 3a).

$$BR_H(r_L) = r_{H,top} \qquad (18)$$

3) Finally, if $r_{H,top} > R_{MAX}$, then the best response of the heavy player is:

$$BR_H(r_L) = R_{MAX} \qquad (19)$$

For the light player, the best response strategy should be one of the bounds as defined in (14) as shown in Figure 3b. If the upper bound in (14) applied, then the light player would have reason to become a heavy player (i.e., apply a radio range larger than the upper bound in (14)). Hence, it is enough to compare only the lower bound with the best response solutions derived in (17), (18) and (19). The lower bound has two cases as expressed in (14).

Let us now define the *critical range* of player j as the range r_j for which the utility of player i, whether the heavy or the light player, is equal. We denote the critical radio range by r_j^*. If player j plays a radio range larger than the critical range of player i, then player i should be the light player.

We can now derive the best response function of the players by substituting (17) through (19) in the utility function and compare them with the utility playing $r_i = 0$. The result is shown in Figure 4. We can notice that the critical range (identified by the vertical lines) decreases as the cooperation parameter γ_i increases.

Table I presents the critical ranges for player i as a function of γ_i, and we show the numerical values of the limits in Table II. If $\gamma_i < \frac{d^2}{\pi \cdot R_{MAX}^2 - d^2}$, then the critical range is larger than the maximum range R_{MAX} and hence, the best response is never $r_i = 0$. Note that $\gamma_{LIM1} = \frac{d^2}{\pi \cdot R_{MAX}^2 - d^2} \le \frac{1}{\pi - 1}$ for any $R_{MAX} > d$. If $\gamma_i \ge \gamma_{LIM3}$, then the critical range $r_j^* \le R_{MIN}$ and the player necessarily plays R_{MIN}.

TABLE I

CRITICAL RADIO RANGES DEPENDING ON THE VALUE OF γ_i

Value of γ_i	Critical range: r_j^*
$\gamma_i < \gamma_{LIM1}$	$r_j^* > R_{MAX}$, no critical range exists
$\gamma_{LIM1} \le \gamma_i < \gamma_{LIM2}$	$d < r_j^* < R_{MAX}$
$\gamma_{LIM2} \le \gamma_i < \gamma_{LIM3}$	$R_{MIN} < r_j^* < d$
$\gamma_i \ge \gamma_{LIM3}$	$r_j^* \le R_{MIN}$, no critical range exists

B. Nash Equilibria in the Single Stage Game

Let us designate the stability point in the game as follows. NE_{MIN} denotes a Nash equilibrium in which the players play the radio ranges (R_{MIN}, R_{MIN}). Similarly, we define the Nash

TABLE II

NUMERICAL VALUES OF THE LIMITS OF γ_i

γ_{LIM1}	$\frac{d^2}{\pi \cdot R_{MAX}^2 - d^2} < \frac{1}{\pi - 1} \approx 0.46$
γ_{LIM2}	$\frac{2(-4 + 2\pi - \sqrt{2}\pi)}{8 - 8\pi + \pi^2} \approx 0.59$
γ_{LIM3}	$\frac{2}{\pi - 2} \approx 1.75$

equilibrium NE_{MAX} for the joint action (R_{MAX}, R_{MAX}). We write $NE_{MIN,i,j}$ if the players just cover the service area, but they have a different radio range (i.e., their radio ranges define the limit case in (**C2**) as shown in Figure 1). In the subscript, i refers to the player with the larger radio range.

We can identify Nash equilibria in the single stage game based on Definition 4 by searching for the intersections of the possible best response functions shown in Figure 4 using the corresponding equations (14), (17), (18) and (19).

1) The radio ranges of the players just cover the service area. They play one of the minimum Nash equilibria (meaning NE_{MIN} or $NE_{MIN,i,j}$). This case holds for:
   a) $\gamma_A > \gamma_{LIM2}$ or;
   b) $\gamma_B > \gamma_{LIM2}$.
2) There is a unique Nash equilibrium NE_{MAX} if $\gamma_A < \gamma_{LIM1}$ and $\gamma_B < \gamma_{LIM1}$;
3) No Nash equilibrium exists if $\gamma_{LIM1} < \gamma_A < \gamma_{LIM2}$ and $\gamma_B < \gamma_{LIM1}$ or vice versa.

Table III shows the types of different Nash equilibria as a function of the cooperation values of the players.

C. Equilibrium Selection

From Table III, we can observe that there exist a variety of Nash equilibria depending on the parameters (i.e., cooperation, maximum radio range) in the power control game. In order to assess the success of the players in these Nash equilibria, we use the concept of *Pareto-optimality*.

Definition 5: A pair of radio ranges is Pareto-optimal (or socially optimal), if none of the players can increase its utility unless the utility of another player decreases.

In order to assess the feasible region (all possible values of the utilities) of the radio ranges, we show the utilities for each possible values of r_A and r_B for $\gamma_A = \gamma_B = 0.1$ in Figure 5. Furthermore, let us distinguish the $NE_{MIN,i,j0}$ state in which the player with the larger radio range plays $r_i = d$ and the other player plays $r_j = 0$.

The following theorem shows that, depending on the parameter values, the minimum Nash equilibria can be socially optimal.

Theorem 1: If several $NE_{MIN,i,j}$ Nash-equilibria exist in the grid scenario, then *each of them is Pareto-optimal*, except for the following cases:

1) If $\gamma_A > \gamma_{LIM3}$ and $\gamma_B < \gamma_{LIM3}$, then only $NE_{MIN,B,A0}$ is Pareto-optimal.
2) If $\gamma_A < \gamma_{LIM3}$ and $\gamma_B > \gamma_{LIM3}$, then only $NE_{MIN,A,B0}$ is Pareto-optimal.

We provide the proof of the theorem in Appendix I-A.

TABLE III

NASH EQUILIBRIA (NE) IN THE SINGLE STAGE GAME AS A FUNCTION OF THE COOPERATION VALUES

	$\gamma_B < \gamma_{LIM1}$	$\gamma_{LIM1} < \gamma_B < \gamma_{LIM2}$	$\gamma_{LIM2} \leq \gamma_B < \gamma_{LIM3}$	$\gamma_B \geq \gamma_{LIM3}$
$\gamma_A < \gamma_{LIM1}$	NE_{MAX}	no NE	$NE_{MIN,A,B}$	$NE_{MIN,A,B}$
$\gamma_{LIM1} < \gamma_A < \gamma_{LIM2}$	no NE	no NE	$NE_{MIN,A,B}$	$NE_{MIN,A,B}$
$\gamma_{LIM2} \leq \gamma_A < \gamma_{LIM3}$	$NE_{MIN,B,A}$	$NE_{MIN,B,A}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$
$\gamma_A \geq \gamma_{LIM3}$	$NE_{MIN,B,A}$	$NE_{MIN,B,A}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$; NE_{MIN}

TABLE IV

THE BEST (PARETO-OPTIMAL) NASH EQUILIBRIA IN THE REPEATED GAME AS A FUNCTION OF THE COOPERATION VALUES

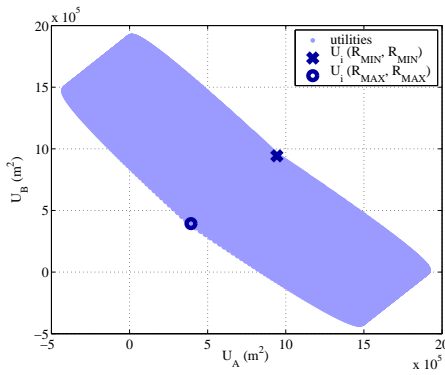|	$\gamma_B < \gamma_{LIM1}$	$\gamma_{LIM1} < \gamma_B < \gamma_{LIM2}$	$\gamma_{LIM2} \leq \gamma_B < \gamma_{LIM3}$	$\gamma_B \geq \gamma_{LIM3}$
$\gamma_A < \gamma_{LIM1}$	NE_{MAX}	no NE	$NE_{MIN,A,B}$	$NE_{MIN,A,B0}$
$\gamma_{LIM1} < \gamma_A < \gamma_{LIM2}$	no NE	no NE	$NE_{MIN,A,B}$	$NE_{MIN,A,B0}$
$\gamma_{LIM2} \leq \gamma_A < \gamma_{LIM3}$	$NE_{MIN,B,A}$	$NE_{MIN,B,A}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$	$NE_{MIN,A,B0}$
$\gamma_A \geq \gamma_{LIM3}$	$NE_{MIN,B,A0}$	$NE_{MIN,B,A0}$	$NE_{MIN,B,A0}$	$NE_{MIN,A,B}$; $NE_{MIN,B,A}$; NE_{MIN}

Fig. 5. The utilities for the possible values of r_A and r_B if $d = 1km$ and $\gamma_A = \gamma_B = 0.1$.

Fig. 6. Discrete area model with points in ϵ distance.

Based on Theorem 1, we can identify the most beneficial Nash equilibria from Table III. We express this modified solution in Table IV.

Table IV shows that if the operators are cooperative, then they should play the minimum radio range with which they are able to cover the service area. Furthermore in a fair solution, they should both play R_{MIN}. However, if one of the players does not cooperate and the other does, then the non-cooperative player can increase its radio range to force the cooperative player out of the game. If none of the players cooperate, then they will end up in both playing the maximum radio range R_{MAX}.

D. Discussion

Our model based on the Voronoi diagram in the Laguerre geometry results in coverage areas with straight separation lines. We adopted this model, because if we applied the physical radio model based on (1), it would be difficult to derive a closed-form expression for the coverage and hence for the utility of the players. We now use a numerical method to compare the radio range model to the physical model and to show that the principles derived in our model hold for the physical model as well.

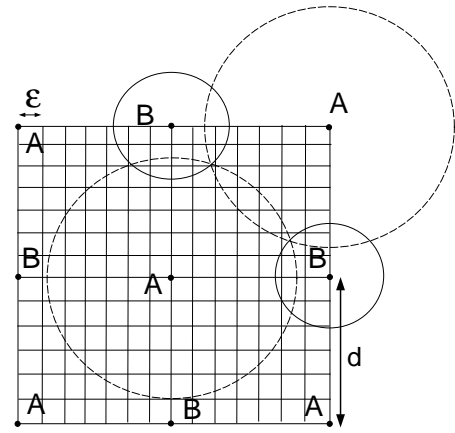We compare the coverage areas in both models as follows. We transform the continuous area into a discrete area by substituting it with a grid of interval ϵ as shown in Figure 6. In our numerical study we use a grid of 100x100 points. For a given set of radio ranges, we determine the number of points that belong to the base station in the middle of the considered area in each of the radio models. This results in an empirical value of the coverage area. We substitute this coverage area value into (15) to obtain the utility of the players in both cases and then we calculate the best responses from the utility function. Figure 7 shows an example of the best response of player i who controls the base station in the middle of Figure 6 for each of the radio models.

We can observe that the best response functions are very similar for the two models. We performed our numerical analysis for various values of γ_i and γ_j and it resulted in the same conclusion. Hence, the conclusions about the Nash equilibria in the radio range model hold for the physical model as well. However, the derivation of the precise values of γ_i and γ_j requires an extensive set of numerical calculations. This fact motivated us to study the problem based on the radio range model.

IV. REPEATED GAME

In the previous section, we assumed that the radio range of the base stations has to be set in advance and no power adjust-
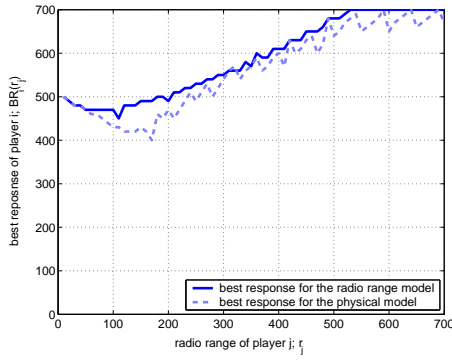
Fig. 7.   Best response function of player i for $\gamma_i = \gamma_j = 0.1$, $d = 500m$ and $R_{MAX} = 700m$.

ment is possible. In this section, we consider the possibility of an iterative power control in a *repeated game*. We assume that the operators do not know the end of the game, hence we study the problem in an infinite repeated game model with discounting [2], [8]. We will show that cooperation (i.e., both players playing R_{MIN}) can be enforced in the cases in which no cooperative equilibrium exists.

We extend the single-stage game as follows: We assume that the game is split up into steps denoted by t. In each step, player $i \in \{A, B\}$ adjusts the radio range of its base stations according to its strategy s_i.

Furthermore, let us define the discounted cumulative utility in $k < \infty$ time steps as:

$$\bar{U}_i(k) = \sum_{t=0}^{k} U_i(t) \cdot \omega^t \qquad (20)$$

where $0 < \omega < 1$ is the *discounting factor*, which expresses the value of future utilities for the players. The discounting factor is sometimes interpreted as a value related to the probability that the game ends in the subsequent time slot[4].

We now prove a theorem to show that non-cooperation based on R_{MAX} is a Nash equilibrium.

Theorem 2: Both players playing R_{MAX} in all time steps is a Nash equilibrium if $\gamma_i < \gamma_{LIM1}$ and $\gamma_j < \gamma_{LIM1}$ holds.

Proof: Let us assume that player i plays R_{MAX} all the time. Since the decision of the other player does not affect player i's radio range, we can analyze the game by time steps. In any time step, player i necessarily becomes the heavy player (or they are of equal weight). If $\gamma_i < \gamma_{LIM1}$ and $\gamma_j < \gamma_{LIM1}$, then the best strategy of the other player is $r_j = r_i = R_{MAX}$ in every time step.   ∎

In this case, the players are in a socially non-optimal equilibrium. We have seen in Theorem 1 that for high γ_A and γ_B values, cooperation does not need to be enforced. We will now prove conditions that enable the players to enforce cooperation for other cooperation values. We prove in this section that they can do better, by applying a strategy called *Punisher*.

Definition 6:   If player i plays the *Punisher* strategy, it plays R_{MIN} in the first time step. For any further time steps, it plays:

- R_{MIN} in the next time step if the other player played R_{MIN} in the previous time step, or
- R_{MAX} for the next k_i time steps, if the other player played anything else.

The parameter k_i (also called the *punishment interval*) defines the number of time steps for which player i punishes the other player. Note that the Punisher strategy is similar to the well-known *Tit-For-Tat (TFT)* strategy [2][5]. The major difference is that it retaliates any defection by playing R_{MAX} instead of copying the same behavior. Furthermore, the Punisher strategy is different from the *Trigger* strategy defined in [16], because the Punisher strategy imposes a punishment that is comparable to the amount of misbehavior and thus it is able to recover from erroneous defections.

If both players cooperate, they both play R_{MIN}. In this case they both have the *cooperative utility* $C_i = U_H(R_{MIN}, R_{MIN})$[6]. If player i defects, while the other player does not, the defecting player has a *cheating gain* $G_i = U_H(BR_i(R_{MIN}), R_{MIN})$. Substituting γ_j to γ_H, we can obtain the best response value $BR_j(R_{MIN})$ from (16):

$$BR_j(R_{MIN}) = \frac{d^2(3 - (\pi - 3)\gamma_j)}{2(1 + \gamma_j)} \qquad (21)$$

If we substitute $r_i = r_L = R_{MIN}$ into (15), we get a cheating gain $G_j = U_H(BR_j(R_{MIN}), R_{MIN})$:

$$G_j = \frac{d^2(8 + (16 - 6\pi)\gamma_j + (8 - 6\pi + \pi^2)\gamma_j^2)}{4(1 + \gamma_j)} \qquad (22)$$

After the defection, player i retaliates by playing R_{MAX} and player j plays its best response to this radio range. In this case player j has the *defection utility* $D_i = U_H(R_{MAX}, BR_j(R_{MAX}))$. If $BR_j(R_{MAX}) = R_{MAX}$, then its utility for the next k_i time slots is the defection utility D_j':

$$D_j' = (1 + \gamma_j)d^2 - \gamma_j R_{MAX}^2 \pi \qquad (23)$$

If $BR_j(R_{MAX}) = 0$, then its utility for the next k_i time slots is the defection utility D_j'':

$$D_j'' = 0 \qquad (24)$$

Otherwise, if player j played R_{MIN}, it would have a cooperation utility C_j for all the k_i time slots:

$$C_j = \frac{d^2(2 - (\pi - 2)\gamma_j)}{2} \qquad (25)$$

Cooperation can be enforced using the Punisher strategy as proven in the following theorem.

Theorem 3: A Nash equilibrium NE_{MIN} based on R_{MIN} is enforceable with the *Punisher* strategy (i.e., player i is able to punish the defection of the other player j) if

1) $\gamma_j < \frac{2}{\pi - 2}$ and
2)

$$\frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) < 1 \qquad (26)$$

---

[4]Based on this interpretation, we assume that the discounting factor is the same for both players.

[5]TFT defines the choice of a given player in the next time slot, whereas the Punisher strategy defines the punishment interval as a set of subsequent time slots.

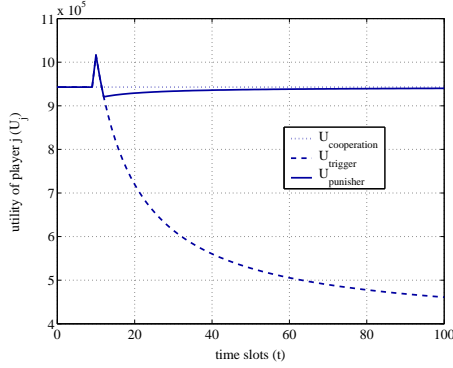[6]Note that for $r_A = r_B$, $U_H(r_A, r_B) = U_L(r_A, r_B)$. Hence we can apply any of the two utility functions.

Fig. 8. Average utility of player i for $d = 1km$, $\gamma_j = 0.1$, $R_{MAX} = 1.5km$ and $\omega = 0.1$ if player j applies a punishment. One-time defection is quickly retaliated and hence cooperation is the best choice. The Trigger strategy stabilizes in infinite punishment, and the Punisher strategy returns to the cooperative state.

where $\omega \leq 1$, $\gamma_i \neq 0$ and $\gamma_j \neq 0$.

If the above condition holds, the punishment interval is defined by:

$$k_i \geq \log_\omega \left(1 - \frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) \right) - 1 \qquad (27)$$

We refer to Appendix I-B for the proof of the theorem. Note that for $\omega = 1$, cooperation can always be enforced using the Punisher strategy. This principle is expressed in general by the Nash folk theorem [8].

The typical value of k_i is small (for $d = 1km$, $\gamma_j = 0.1$, $R_{MAX} = 1.5km$ and $\omega = 0.1$, the value is $k_i = \lceil 1.23 \rceil = 2$). For higher values of γ_j, R_{MAX} and ω, the punishment interval is one time slot (i.e., there is an immediate punishment). Figure 8 illustrates the average per time slot utility of a player for both cooperation and defection. One can observe that cooperation is more beneficial, because defection is quickly retaliated by the other player.

Based on Theorem 3, we state the following result.

Corollary 1: If both players play the *Punisher* strategy and the conditions of Theorem 3 hold, then it results in a Nash equilibrium.

V. NP-COMPLETENESS OF THE GENERAL PROBLEM

In this section, we analyze the power control problem for general network topologies and for general values of radio ranges in the single stage game.

The goal of player i is to allocate the radio ranges such that their overall utility $U_i = \sum_{b_m=1}^{|\mathcal{B}_i|} U_m$ is maximized, where $|\mathcal{B}_i|$ is the number of bases stations that belong to player i and the utility per base station U_m is as follows (derived from (15)).

$$U_i = \sum_{m=1}^{|\mathcal{B}_i|} \left[(1 + \gamma_i) \cdot O_m - \gamma \cdot r_m^2 \cdot \pi \right] \qquad (28)$$

where O_m is the coverage area and r_m is the radio range of base station b_m.

We can now formulate the following theorem.

Theorem 4: Finding the maximum utility of player i for general values of radio ranges is NP-complete.

We provide the proof in Appendix I-C.

Since finding the maximum utility for an operator is NP-complete in general, it is impossible to calculate the best responses for a given player in polynomial time. Thus, we can state the following result.

Corollary 2: Finding Nash equilibria in the power control game for general values of radio ranges is NP-complete.

VI. CONCLUSION

In this paper, we have investigated the problem of co-existing wireless operators in a shared spectrum. We have assumed that the operators apply power control at the base stations to mitigate interference, while providing a permanent service to the users. To the best of our knowledge, our paper is the first to investigate this problem.

The contribution of this paper is threefold. First, we have shown that Nash equilibria exist if the operators set the power of their base stations at the beginning of the operation of the network. We have identified different equilibrium situations that depend on the cooperativeness of the operators. Second, we have proved a condition for which a socially optimal Nash equilibrium exists and that it can be enforced using punishments. Third, we have shown that the solution of the power control problem is NP-complete for a general topology of base stations. In general, our results show which operation points are beneficial for the players and how these should be achieved.

In terms of future work, we will solve the power control problem by designing an approximation algorithm that converges to a desirable equilibrium situation for a general set of radio ranges; in particular, we will study the properties of the convergence by simulations. Furthermore, we will consider power control on the data channels as well. We also intend to study the effect of other techniques to mitigate interference, such as directional antennae and mobile devices with multiple antennae.

REFERENCES

[1] T. Alpcan, T. Basar, R. Srikant, and E. Altman. CDMA uplink power control as a noncooperative game. *Wireless Networks, The Journal of Mobile Communication, Computation and Information, Kluwer Academic Publishing*, 8:659–670, Nov. 2002.

[2] R. Axelrod. *The Evolution of Cooperation*. Basic Books, New York, 1984.

[3] Y. Benkler. Some economics of wireless communication. *Harvard Journal of Law and Technology*, 16(1), Fall 2004.

[4] B. Chamaret, S. Josselin, P. Kuonen, M. Pizarroso, B. Salas-Manzanedo, S. Ubeda, and D. Wagner. Radio network optimization with maximum independent set search. In *Proceedings of IEEE Vehicular Technology Conference '97*, pages 770–774, May 1997.

[5] J. Crowcroft, R. Gibbens, F. Kelly, and S. Östring. Modelling incentives for collaboration in mobile ad hoc networks. In *Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt'03)*, March 3–5 2003.

[6] M. Dramitinos, C. Courcoubetis, and G. D. Stamoulis. Auction-based resource reservation in 2.5/3g networks. *Kluwer/ACM Mobile Networks and Applications (MONET)*, 2003.

[7] G. Faulhaber and D. Farber. Spectrum management: Property rights, markets, and the commons. In *Telecommunications Policy Research Conference Proceedings*, 2003.

[8] D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, 1991.

[9] D. Goodman and N. Mandayam. Network assisted power control for wireless data. *Mobile Networks and Applications (MONET)*, 6:409–415, 2001.

[10] M. M. Halldórsson, J. Y. Halpern, L. (Erran) Li, and V. S. Mirrokni. On spectrum sharing games. In *Proceedings of the 23rd annual ACM symposium on Principles of Distributed Computing (PODC 2004)*, pages 107–114, July 25–28 2004.

[11] S. Haykin. Cognitive radio: Brain-empowered wireless communications. *IEEE Journal on Selected Areas in Communications (JSAC)*, 23(2):201–220, Feb. 2005.

[12] J. Huang, R. Berry, and M. Honig. Auction-based spectrum sharing. *to appear in ACM/Kluwer Journal of Mobile Networks and Applications (MONET) special issue on WiOpt'04*, 2004.

[13] H. Ji and C.-Y. Huang. Non-cooperative uplink power control in cellular radio systems. *Wireless Networks (WINET)*, 46(3):233–240, 1998.

[14] A. B. MacKenzie and S. B. Wicker. Game theory and the design of self-configuring, adaptive wireless networks. *IEEE Communications Magazine*, Nov. 2001.

[15] P. Marbach and Y. Qiu. Cooperation in wireless ad hoc networks: A market-based approach. *to appear in IEEE/ACM Transactions on Networking*.

[16] A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford Univ. Press, 1995.

[17] F. Meshkati, M. Chiang, H. V. Poor, and S. Schwartz. A non-cooperative power control game for multi-carrier CDMA systems. In *Proc. IEEE Wireless Communications and Networking Conference (WCNC)*, March 2005.

[18] A. Muqattash and M. Krunz. POWMAC: A single-channel power-control protocol for throughput enhancement in wireless ad hoc networks. *IEEE Journal on Selected Areas in Communications (JSAC) - Special Issue on Advances in Military Wireless Communications*, 23(5):1067–1084, May 2005.

[19] J. Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.

[20] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tessellations, Concepts and Applications of Voronoi Diagrams (2nd edition)*. John Wiley and Sons Ltd., 2000.

[21] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, Cambridge, MA, 1994.

[22] T. S. Rappaport. *Wireless Communications: Principles and Practice (2nd Edition)*. Prentice Hall, 2002.

[23] S. Shakkottai and R. Srikant. Economics of network pricing with multiple ISPs. In *Proceedings of the IEEE Conference on Computer Communications (INFOCOM '05)*, March 13–17 2005.

[24] V. Srinivasan, P. Nuggehalli, C. F. Chiasserini, and R. R. Rao. Co-operation in wireless ad hoc networks. In *Proceedings of the IEEE Conference on Computer Communications (INFOCOM '03)*, March 30 – April 3 2003.

[25] P. Värbrand and D. Yuan. A mathematical programming approach for pilot power optimization in WCDMA networks. In *Proceedings of the Australian Telecommunications, Networks and Applications Conference (ATNAC '03)*, December 2003.

[26] W. Wang, X. Liu, and Hong Xiao. Exploring opportunistic spectrum availability in wireless communication networks. In *IEEE Vehicular Technology Conference (VTC '05)*, Sep. 25–28 2003.

[27] M. Xiao, N. B. Schroff, and E. K. P. Chong. A utility-based power control scheme in wireless cellular systems. *IEEE/ACM Trans. on Networking*, 11(10):210–221, March 2003.

[28] A. Zemlianov and G. de Veciana. Cooperation and decision making in wireless multi-provider setting. In *Proceedings of the IEEE Conference on Computer Communications (INFOCOM '05)*, March 13–17 2005.

[29] C. Zhou, P. Zhang, M. L. Honig, and S. Jordan. Two-cell power allocation for downlink CDMA. *IEEE Transactions on Wireless Communications*, 3(6):2256–2266, Nov. 2004.

APPENDIX I
PROOFS

A. Proof of Theorem 1

Suppose that in any state of the game player i increases its radio range. It is easy to see that the utility of player increases if $\gamma_i < \frac{2}{\pi-2}$. Similarly, the utility of the other player increases if its range increases and we have $\gamma_j < \frac{2}{\pi-2}$. Hence, if both $\gamma_i < \frac{2}{\pi-2}$ and $\gamma_j < \frac{2}{\pi-2}$, then the increase of the utility of one of the players results in the decrease of the utility of the other player. Hence, any $NE_{MIN,i,j}$ state is Pareto-optimal. However, if player j has $\gamma_j > \frac{2}{\pi-2}$, then decreasing its range increases its utility and hence the only Pareto-optimal state is $NE_{MIN,i,j0}$.

B. Proof of Theorem 3

Let us assume that player j deviates in time step t_0. Let us assume that it applies the best option, hence it plays $BR_j(R_{MIN})$. The Punisher strategy played by player i reduces the discounted cumulative utility of player j for the time interval from t_0 to $t_0 + k_i$ if:

$$G_j + D_j \cdot \sum_{t=1}^{k_i} \omega^t \le C_j \cdot \sum_{t=0}^{k_i} \omega^t \qquad (29)$$

If $\omega = 1$, we can write (29) as follows:

$$G_j + D_j \cdot k_i \le C_j \cdot (k_i + 1) \qquad (30)$$

Hence, we obtain the following bound on the punishment interval:

$$k_i \ge \frac{G_j - C_j}{C_j - D_j} \qquad (31)$$

Note that if $\omega = 1$, then cooperation is always enforceable.

Now if $\omega < 1$, we can transform the sums in (29) to the same intervals:

$$G_j - D_j + D_j \cdot \sum_{t=0}^{k_i} \omega^t \le C_j \cdot \sum_{t=0}^{k_i} \omega^t \qquad (32)$$

Since the sums are geometric sequences, we can write that:

$$G_j - D_j \le (C_j - D_j) \cdot \frac{1 - \omega^{k_i+1}}{1 - \omega} \qquad (33)$$

If $\gamma_j < \frac{2}{\pi-2}$, then $C_j - D_j > 0$. Furthermore, we have $1 - \omega > 0$, and we can rewrite the inequality:

$$\frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) \le 1 - \omega^{k_i+1} \qquad (34)$$

Reordering the inequality gives us:

$$\omega^{k_i+1} \le 1 - \frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) \qquad (35)$$

This gives the condition on k_i, because the left side is strictly positive. Thus the inequality cannot be fulfilled if the right side is non-positive, meaning that:

$$\frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) \leq 1 \tag{36}$$

If the condition in (36) holds, we can take the logarithm of both sides in (35). Since $\omega < 1$, the logarithm function is strictly decreasing and hence the direction of the inequality changes.

$$k_i \geq \log_\omega \left(1 - \frac{G_j - D_j}{C_j - D_j} \cdot (1 - \omega) \right) - 1 \tag{37}$$

Due to the symmetric situation, the same arguments apply for the opposite case that defines the punishment interval for player j.

C. Proof of Theorem 4

To prove the theorem, let us consider the special case of finding the optimal radio range allocation in the presence of a single operator. In this case, operator i has the utility:

$$U_i = \sum_{b_m=1}^{|\mathcal{B}_i|} \left[(1 + \gamma_i) \cdot O_m - \gamma \cdot r_m^2 \cdot \pi \right] \tag{38}$$

Let us denote the whole service area by $O_{tot} = \sum_{b_m=1}^{|\mathcal{B}_i|} O_m$. Since the γ_i values are the same for all base stations, we can reformulate the utility as:

$$U_i = (1 + \gamma) \cdot O_{tot} - \gamma_i \cdot \pi \sum_{b_m=1}^{|\mathcal{B}_i|} r_m^2 \tag{39}$$

Under the assumption that $\alpha = 2$, the power is proportional to the square of the radio range. Chamaret *et al.* [4] as well as Värbrand and Yuan [25] have proven that finding the minimum power allocation in the network of a cellular operator while maintaining the total coverage is NP-complete. Hence, the minimum value of U_i cannot be determined in polynomial time. Because the problem is NP-complete for the special case of one operator, we conclude that it is NP-complete in the general game as well.