

On the Structure of the Capacity Region of Asynchronous Memoryless Multiple-Access Channels

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joint work with
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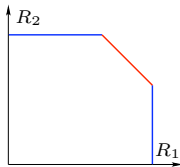
ITA, UCSD
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Motivating Examples: Rate Regions and Dominant Faces for M Users

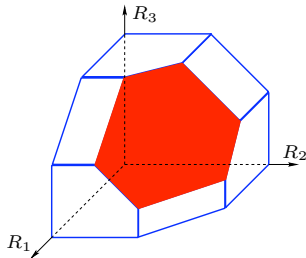
$M = 1$



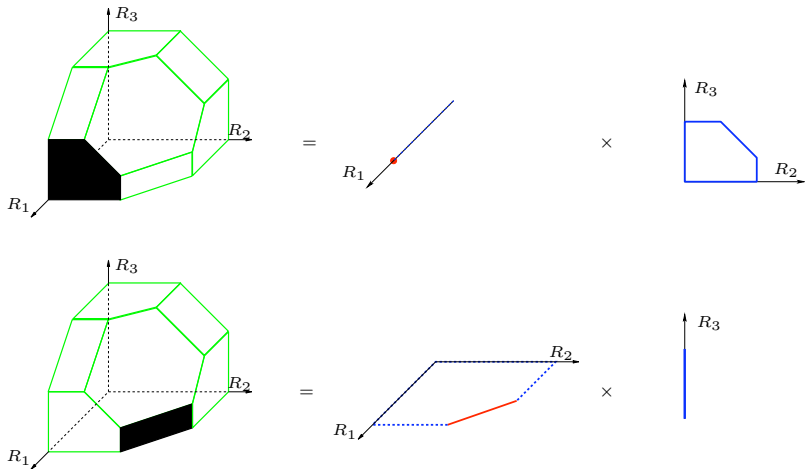
$M = 2$

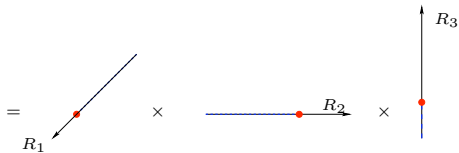
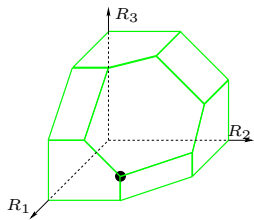
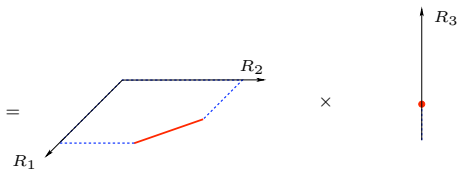
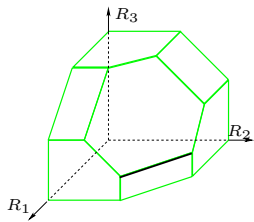


$M = 3$



Examples of Face Decomposition for $M = 3$





Motivating Observation

The examples suggest that an arbitrary face decomposes into a Cartesian product that comprises

- one or more **dominant faces**
- zero or one **rate region**

We will see how exactly and how this decomposition implies a decoding scheme that consists of successively decoding groups of users.

Goal

We classify all faces by means of an **informative label** from which we can:

- tell which faces **intersect** and which do not

- tell how faces **decompose**

- tell how to do **successive group-decoding**

- infer the **dimensionality** of a face

From the label we can deduce additional properties such as counting how many faces there are of a given dimensionality (not done here).

Formal Definition of the Rate Region \mathcal{R}

We use $[M]$ to denote $\{1, 2, \dots, M\}$ and for all $\mathcal{S} \subset [M]$,

$$R(\mathcal{S}) \triangleq \sum_{i \in \mathcal{S}} R_i \quad (\text{Example: } R(\{2, 5\}) = R_2 + R_5)$$

$$X_{\mathcal{S}} \triangleq (X_i : i \in \mathcal{S}) \quad (\text{Example: } X_{\{2, 5\}} = (X_2, X_5))$$

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Given a channel $P_{Y|X_{[M]}}$ and product input distribution $P_{[M]}$, we define \mathcal{R} to be

$$\mathcal{R} = \{R \in \mathbb{R}_+^M : R(\mathcal{S}) \leq I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq [M]\}.$$

Definitions: Faces and Facets of \mathcal{R}

A **face** is a non-empty intersection of \mathcal{R} with an hyperplane that keeps \mathcal{R} on one side. \mathcal{R} has dimension M .

A **facet** is a face of \mathcal{R} of dimension $M - 1$.

Facets of \mathcal{R}

For every $i \in [M]$, there is a **back facet** of the form

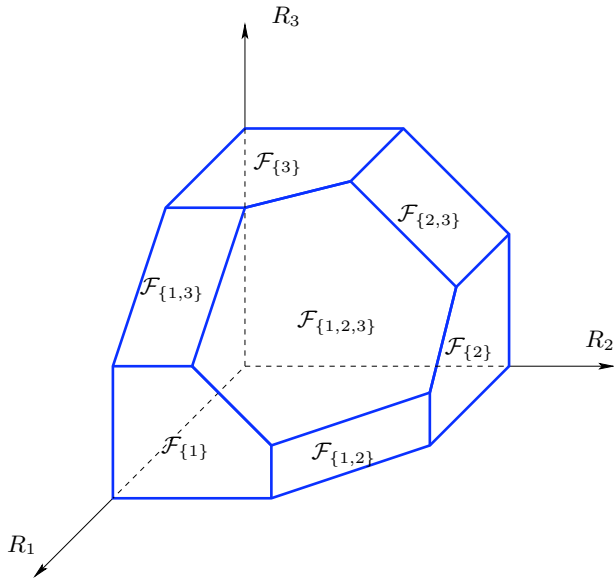
$$\mathcal{B}_i = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R_i = 0\}$$

and for every $\mathcal{S} \subseteq [M]$, $\mathcal{S} \neq \emptyset$, there is a **front facet**

$$\mathcal{F}_{\mathcal{S}} = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R(\mathcal{S}) = I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c})\}.$$

There are no other facets.

Example: The Front Facets for $M = 3$



Faces

Non-empty intersections of facets lead to faces. We define:

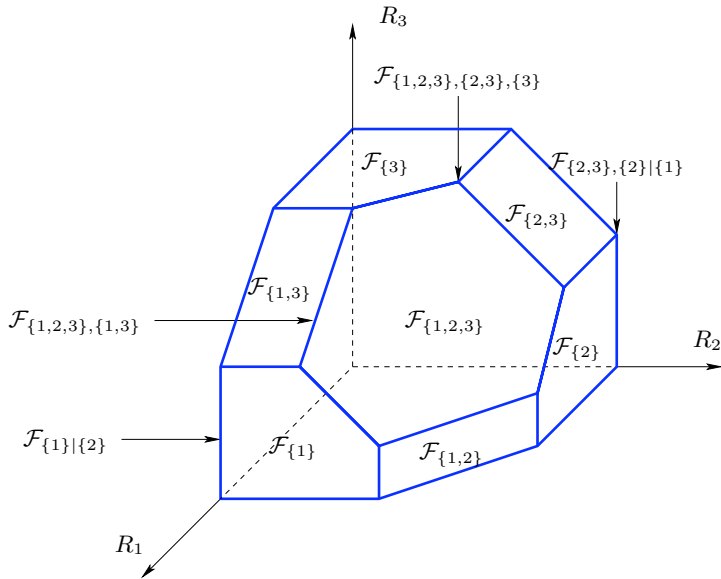
$$\mathcal{B}_{\mathcal{A}} = \bigcap_{i \in \mathcal{A}} \mathcal{B}_i, \text{ with } \mathcal{B}_{\emptyset} = \mathcal{R} \text{ by convention ,}$$

$$\mathcal{F}_{S_1, S_2, \dots, S_m} = \bigcap_{j=1}^m \mathcal{F}_{S_j}, \text{ with } \mathcal{F}_{\emptyset} = \mathcal{R} \text{ by convention ,}$$

$$\mathcal{F}_{S_1, S_2, \dots, S_m | \mathcal{A}} = \mathcal{F}_{S_1, S_2, \dots, S_m} \cap \mathcal{B}_{\mathcal{A}}.$$

Not all of the above are faces. (Some intersections are empty.)

Examples



We Study Only Non-Degenerated Cases

A region \mathcal{R} is called **non-degenerated** if the following two conditions hold

1. $I(X_{\mathcal{S}}; Y) > 0$ for all non-empty sets $\mathcal{S} \subseteq [M]$,
2. $I(X_{\mathcal{S}}; Y | X_{\mathcal{A}}) < I(X_{\mathcal{S}}; Y | X_{\mathcal{B}})$ for all $\emptyset \subset \mathcal{S} \subset [M]$,
 $\mathcal{A} \subset \mathcal{B} \subseteq [M]$, and $\mathcal{S} \cap \mathcal{B} = \emptyset$.

Nonempty Intersections

Lemma

$\mathcal{F}_S \cap \mathcal{B}_A \neq \emptyset$ iff $\mathcal{A} \cap S = \emptyset$.

Proof of “ \Rightarrow ”:

Let $R \in \mathcal{F}_S$ and let \mathcal{M} and \mathcal{N} be a partition of S with $\mathcal{M} \neq \emptyset$.

$$\begin{aligned} R(\mathcal{M}) + R(\mathcal{N}) &= R(S) = I(X_S; Y | X_{S^c}) \\ &= I(X_{\mathcal{M}}; Y | X_{S^c}) + I(X_{\mathcal{N}}; Y | X_{\mathcal{N}^c}) \end{aligned}$$

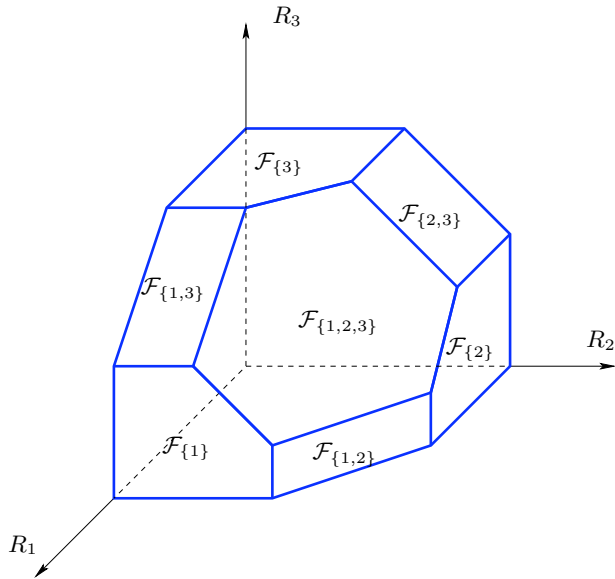
From $R(\mathcal{N}) \leq I(X_{\mathcal{N}}; Y | X_{\mathcal{N}^c}) \Rightarrow R(\mathcal{M}) \geq I(X_{\mathcal{M}}; Y | X_{S^c})$.

Hence $R(\mathcal{M}) > 0$ for all $\mathcal{M} \subseteq S$. In particular $R(\mathcal{A}) = 0$ implies $\mathcal{A} \cap S = \emptyset$.

Lemma

$\mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$ is not empty iff $\mathcal{S}_1 \subseteq \mathcal{S}_2$ or $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

Sanity Check



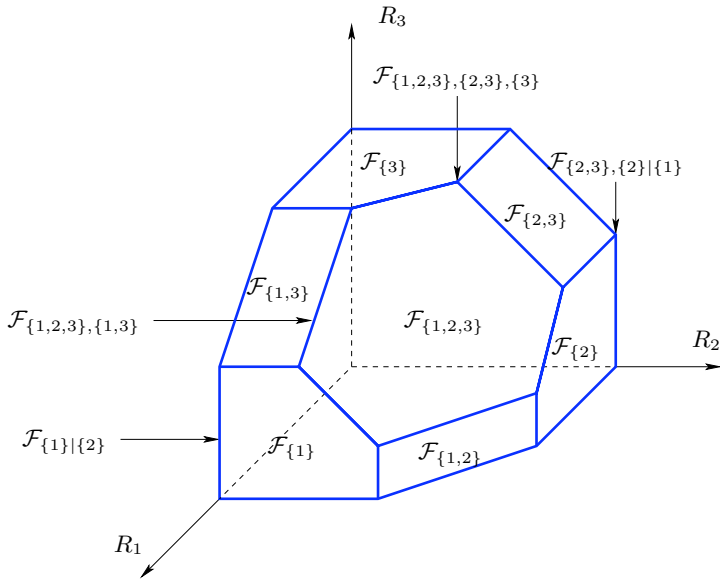
Corollary

The intersection $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}$ is not empty iff the following two conditions are simultaneously satisfied:

1. $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m$ may be ordered into a telescopic sequence
2. $\mathcal{A} \cap \mathcal{S}_i = \emptyset$ for $i = 1, \dots, m$.

From now on we will use the notation $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}$ only if $\mathcal{S}_1 \supset \mathcal{S}_2, \dots, \mathcal{S}_{m-1} \supset \mathcal{S}_m$ and $\mathcal{A} \cap \mathcal{S}_1 = \emptyset$.

Sanity Check



PART II

All faces are now labeled.

Next we look into the structure of individual faces.

Implied Channels And Regions

So far we have dealt with a single channel $P_{Y|X_{[M]}}$ and input distribution $P_{[M]}$. Let them be fixed and consider the following new channels and input distributions.

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Decomposition and Group Decoding

Lemma

$$R \in \mathcal{F}_S \quad \text{iff} \quad \begin{aligned} R_{S^c} &\in \mathcal{R}_{Y|X_{S^c}} \text{ and} \\ R_S &\in \mathcal{D}_{YX_{S^c}|X_S}. \end{aligned}$$

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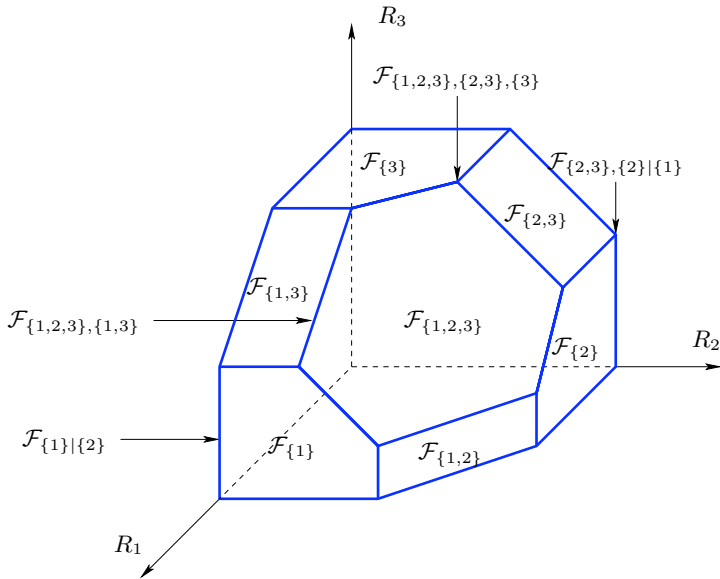
$$R \in \mathcal{F}_{\mathcal{S}} \quad \text{iff} \quad \begin{aligned} R_{\mathcal{S}^c} &\in \mathcal{R}_{Y|X_{\mathcal{S}^c}} \text{ and} \\ R_{\mathcal{S}} &\in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}. \end{aligned}$$

Corollary

if $R \in \mathcal{F}_{\mathcal{S}}$ then we can

1. decode as a group the users indexed by \mathcal{S}^c and subsequently
2. use the estimates obtained so far to decode the users indexed by \mathcal{S} .

Sanity Check



Lemma

$R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}}$ iff

1. $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$
2. $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c} | X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ for $i = 1, \dots, m$ ($\mathcal{S}_{m+1} \triangleq \emptyset$).
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If $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}}$ then we can decode groups of users sequentially the following order:

$$\mathcal{S}_1^c, \quad \mathcal{S}_1 \setminus \mathcal{S}_2, \quad \dots \quad \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \quad \mathcal{S}_m$$

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Note: Sequential decoding of dominant-face vertices is just a special case in which $\mathcal{S}_1 = [M]$, $|\mathcal{S}_i \setminus \mathcal{S}_{i+1}| = 1$, $i = 1, 2, \dots, m-1$, and $\mathcal{A} = \emptyset$.

Corollary

$$\begin{aligned}\dim(\mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m} | \mathcal{A}) &= \dim(\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}) - |\mathcal{A}| \\ &= \dim(\mathcal{R}_{Y|X_{\mathcal{S}_1^c}}) + \sum_{i=1}^m \dim(\mathcal{D}_{YX_{\mathcal{S}_i^c} | X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}) - |\mathcal{A}| \\ &= M - |\mathcal{S}_1| + \sum_{i=1}^m (|\mathcal{S}_i| - |\mathcal{S}_{i+1}| - 1) - |\mathcal{A}| \\ &= M - m - |\mathcal{A}|.\end{aligned}$$

THANK YOU!