

STRUCTURE OF CERTAIN WEYL MODULES FOR THE SPIN GROUPS

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ABSTRACT. Let K be an algebraically closed field of characteristic $p \geq 0$ and let W be a finite-dimensional K -space of dimension greater than or equal to 5. In this paper, we give the structure of certain Weyl modules for $G = \text{Spin}(W)$ in the case where $p \neq 2$, as well as the dimension of the corresponding irreducible, finite-dimensional, rational KG -modules. In addition, we determine the composition factors of the restriction of certain irreducible, finite-dimensional, rational $K\text{SL}(W)$ -modules to $\text{SO}(W)$.

1. INTRODUCTION

Let K be an algebraically closed field of characteristic $p \geq 0$, and let G be a simply connected, simple algebraic group over K . Fixing a Borel subgroup B of G containing a maximal torus T of G , one obtains an associated set of dominant weights for T , denoted by $X^+(T)$. It is well-known that for each $\omega \in X^+(T)$, there exists a unique (up to isomorphism) finite-dimensional, irreducible, rational KG -module $L_G(\omega)$ having highest weight ω . In other words, the isomorphism classes of finite-dimensional, irreducible, rational modules for G are in one-to-one correspondence with the aforementioned dominant weights for T .

In characteristic zero, the dimension of each $L_G(\omega)$ is known, and is given by the well-known Weyl's degree formula [Hum78, Corollary 24.3]. Also weight multiplicities in $L_G(\omega)$ can be recursively computed using Freudenthal's formula [Fre54], or one of the many variants developed in the last decades. (We refer the reader to [MP82], [Bre86], [dG00], [Sah00], [CT04], [Sch12], or [Cav17] for a few examples.) Closed formulas can also be used to obtain information on weight multiplicities, or even on the so-called character of a given irreducible (see [dG00] or [Kos59], for instance). Observe, however, that those methods are often quite demanding in terms of complexity.

In positive characteristic, not much is known about irreducible KG -modules in general. However, following the construction in [Ste68, Section 2], one obtains a universal highest weight module $V_G(\omega)$ of highest weight ω , for every $\omega \in X^+(T)$, by finding an appropriate \mathbb{Z} -form in a suitable irreducible module for the corresponding complex Lie algebra, and then tensoring it by K . The KG -module $V_G(\omega)$ is called the *Weyl module* of highest weight ω , and has the property that its quotient by its unique maximal submodule $\text{rad}(\omega)$ is irreducible with highest weight ω . In other words, we have

$$L_G(\omega) \cong V_G(\omega) / \text{rad}(\omega).$$

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The formulas introduced above can be used to determine the dimension, the weight multiplicities, and the character of $V_G(\omega)$. The problem consisting in determining the composition factors of $V_G(\omega)$, on the other hand, is essentially equivalent to the determination of weight multiplicities in simple modules for G : no closed formula is known to this day, and there seems to be no expectation of finding one in the future. Although it is possible to proceed in a recursive fashion, by arguing on generating sets for weight spaces, those processes are again quite demanding in terms of complexity, and give no insight on the obtained values. For sufficiently large p and small enough ω , other tools are at our disposal, like Kazhdan-Lusztig polynomials [KL79], or the Jantzen p -sum formula [Jan03, Proposition 8.19]. The former allows one to compute weight multiplicities in a recursive fashion, inspired by the study of Verma modules in characteristic zero [Hum08, Chapter 8]. The latter provides a tool for computing all characters of irreducible modules, but generally only in small rank.

In this paper, we determine the structure of certain Weyl modules for G in the case where $\text{char } K \neq 2$ and $G = \text{Spin}(W)$, with W a K -space of dimension at least 5. In order to do so, we proceed in two steps: inspired by an idea of McNinch [McN98], we first determine the composition factors of a well-chosen tilting module for G , in order to reduce the list of possible composition factors for $V_G(\omega)$, thanks to a generalization of [McN98, Proposition 4.6.2], namely Proposition 3.2. Finally, a suitable use of a truncated version of the Jantzen p -sum formula (Theorem 3.8) yields the desired result. We then deduce the dimensions of the corresponding irreducible KG -modules, and conclude by proving a result on the composition factors of the restriction to $\text{SO}(W)$ of certain $\text{SL}(W)$ -modules.

1.1. Statements of results. Assume $\text{char } K \neq 2$, and let G be a simply connected, simple algebraic group of type B_n ($n \geq 2$) or D_n ($n \geq 3$). Fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U is the unipotent radical of B , let $\Pi = \{\beta_1, \dots, \beta_n\}$ denote a corresponding base of the root system Φ of G , and let $\{\omega_1, \dots, \omega_n\}$ be the set of fundamental dominant weights for T corresponding to our choice of base Π , ordered as in [Bou68]. Also adopt the notation

$$\Lambda(B_n) = \{\omega_i : 1 \leq i \leq n\} \cup \{2\omega_n\},$$

as well as

$$\Lambda(D_n) = \{\omega_i : 1 \leq i \leq n\} \cup \{2\omega_{n-1}, 2\omega_n, \omega_{n-1} + \omega_n\}.$$

Since we assumed $\text{char } K \neq 2$, any Weyl module for G having highest weight $\omega \in \Lambda(G)$ is irreducible (see Lemmas 2.4 and 2.6, for instance), and hence the dimension of $V_G(\omega)$, as well as its weight multiplicities, can be computed using the tools provided by the theory in characteristic zero. In this paper, we thus focus our attention on Weyl modules having slightly more complicated highest weights, namely weights belonging to the set

$$\Lambda_1(G) = \omega_1 + \Lambda(G).$$

The result of our investigation (which can be viewed as a generalization of [McN98, Lemma 4.9.2], in which the case $\omega = \omega_1 + \omega_2$ is dealt with) is recorded in the following theorem. For $\ell \in \mathbb{Z}_{\geq 0}$ a prime, we let $\epsilon_\ell : \mathbb{Z}_{\geq 0} \rightarrow \{0, 1\}$ be the map defined by

$$\epsilon_\ell(z) = \begin{cases} 1 & \text{if } \ell \text{ divides } z; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1

Assume $\text{char } K \neq 2$, and let G be a simply connected, simple algebraic group of type B_n , $n \geq 2$ (resp. D_n , $n \geq 3$), over K . Also let ω be as in the first column of Table 1 (resp. Table 2). Then the structure of the radical $\text{rad}(\omega)$ of $V_G(\omega)$ is given by the second column of the table.

ω	$\text{rad}(\omega)$
$2\omega_1$	$K^{\epsilon_p(2n+1)}$
$\omega_1 + \omega_j$ ($2 \leq j \leq n-2$)	$L_G(\omega_{j+1})^{\epsilon_p(j+1)} \oplus L_G(\omega_{j-1})^{\epsilon_p(2n-j+2)}$
$\omega_1 + \omega_{n-1}$	$L_G(2\omega_n)^{\epsilon_p(n)} \oplus L_G(\omega_{n-2})^{\epsilon_p(n+3)}$
$\omega_1 + \omega_n$	$L_G(\omega_n)^{\epsilon_p(2n+1)}$
$\omega_1 + 2\omega_n$	$L_G(2\omega_n)^{\epsilon_p(n+1)} \oplus L_G(\omega_{n-1})^{\epsilon_p(n+2)}$

TABLE 1. Structure of certain Weyl modules for G of type B_n ($n \geq 2$).

ω	$\text{rad}(\omega)$
$2\omega_1$	$K^{\epsilon_p(n)}$
$\omega_1 + \omega_j$ ($2 \leq j \leq n-3$)	$L_G(\omega_{j+1})^{\epsilon_p(j+1)} \oplus L_G(\omega_{j-1})^{\epsilon_p(2n-j+1)}$
$\omega_1 + \omega_{n-2}$	$L_G(\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)} \oplus L_G(\omega_{n-3})^{\epsilon_p(n+3)}$
$\omega_1 + \omega_n$	$L_G(\omega_{n-1})^{\epsilon_p(n)}$
$\omega_1 + \omega_{n-1} + \omega_n$	$L_G(2\omega_{n-1})^{\epsilon_p(n)} \oplus L_G(2\omega_n)^{\epsilon_p(n)} \oplus L_G(\omega_{n-2})^{\epsilon_p(n+2)}$
$\omega_1 + 2\omega_n$	$L_G(\omega_{n-1} + \omega_n)^{\epsilon_p(n+1)}$

TABLE 2. Structure of certain Weyl modules for G of type D_n ($n \geq 3$).

Remarks

The assumption on the characteristic of K in Theorem 1 ensures that Weyl modules for G having highest weights belonging to $\Lambda(G)$ (as defined above) are irreducible, thus allowing us to apply the aforementioned generalization of McNinch's result (see Proposition 3.2). Also observe that the case where G is of type C_n ($n \geq 3$) is not treated in this paper. The reason is that if G is of type C_n over K , then Weyl modules having fundamental weights as highest weights are not necessarily irreducible, this even if $p \neq 2$. (In fact, there is no bound to the possible number of composition factors for such modules, as n grows [PS83]). In particular, the method employed in this paper requiring of Weyl modules having highest weights ω_i , $1 \leq i \leq n$, to be irreducible, would fail to apply in this context. A similar result for G of type C_n would then require a lot more investigation and would probably lead to a much more complicated table. Finally, observe that a table similar to Tables 1 and 2 in the case where $G = D_n$ ($n \geq 3$) and $\omega \in \omega_2 + \Lambda(D_n)$ can be found in [Cav15, Theorem 7.3]. However, since it is incomplete, and since the techniques employed are identical to the ones introduced here, we decided not to include the result in this paper.

As seen above, each irreducible module KG -module appearing in the second column of Table 1 or Table 2 of Theorem 1 is isomorphic to its corresponding Weyl module since $p \neq 2$. In particular, the dimensions of those irreducibles are known, and so one can deduce the dimension of each irreducible KG -module having highest weight ω as in the first column of the aforementioned tables. We record our findings in the form of a corollary to Theorem 1. For simplicity purposes, we let $\delta : \mathbb{Z} \times \mathbb{Z} \rightarrow \{0, 1\}$ denote the standard Kronecker delta, that is,

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a = b; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2

Assume $\text{char } K \neq 2$, and let G be a simply connected, simple algebraic group of type B_n , $n \geq 2$ (resp. D_n , $n \geq 3$), over K . Also let ω be as in the first column of Table 3 (resp. Table 4). Then the dimension of $L_G(\omega)$ is given by the second column of the table.

ω	$\dim L_G(\omega)$
$\omega_1 + \omega_j + \delta_{j,n}\omega_n$	$\binom{2n+1}{j+1} \left(\frac{j(2n+3)}{2n-j+2} - \epsilon_p(j+1) - \frac{\epsilon_p(2n-j+2)j(j+1)}{(2n-j+2)(2n-j+1)} \right)$
$\omega_1 + \omega_n$	$2^n(2n - \epsilon_p(2n+1))$

TABLE 3. Dimension of certain irreducible KG -modules for G of type B_n ($n \geq 2$). Here $1 \leq j \leq n$.

ω	$\dim L_G(\omega)$
$\omega_1 + \omega_j + \delta_{j,n-1}\omega_n$	$\binom{2n}{j+1} \left(\frac{2j(n+1)}{2n-j+1} - \epsilon_p(j+1) - \frac{\epsilon_p(2n-j+1)j(j+1)}{(2n-j)(2n-j+1)} \right)$
$\omega_1 + \omega_n$	$2^n(2n - \epsilon_p(2n+1))$
$\omega_1 + 2\omega_n$	$(n - \epsilon_p(n+1)) \binom{2n}{n+1}$

TABLE 4. Dimension of certain irreducible KG -modules for G of type D_n ($n \geq 3$). Here $1 \leq j \leq n-1$.

Let W be a finite-dimensional K -space of dimension at least 5, and let $Y = \mathrm{SL}(W)$, that is, Y is a simply connected, simple algebraic group of type $A_{\dim W - 1}$ over K . Fix a Borel subgroup B_Y of Y , containing a maximal torus T_Y of Y , and let $\{\lambda_1, \dots, \lambda_{\dim W - 1}\}$ denote the corresponding fundamental weights, ordered as in [Bou68]. Also consider the maximal, closed, connected subgroup $G = \mathrm{SO}(W)$ of Y . Then G is of type B_n ($n \geq 2$) if $\dim W = 2n + 1$, and of type D_n ($n \geq 3$) if $\dim W = 2n$. Without loss of generality, we suppose that T , B , and hence $\{\omega_1, \dots, \omega_n\}$, are chosen in such a way that $\lambda_i|_T = \omega_i$ for $1 \leq i \leq n - 2$, $\lambda_{n-1}|_T = \omega_{n-1} + \epsilon_2(\dim W)\omega_n$, and $\lambda_n|_T = 2\omega_n$.

If $\mathrm{char} K \neq 2$ and if V is an irreducible KY -module having highest weight λ_i , $1 \leq i \leq \dim W - 1$, then the restriction of V to G is irreducible as well by [Sei87, Theorem 1, Table 1 (I₂, I₃, I₄, I₅)]. We thus conclude this paper by giving a description of the composition factors of the restriction to G of irreducible KY -modules having slightly more complicated highest weights, namely weights of the form $\lambda = \lambda_1 + \lambda_j$, where $1 \leq j \leq \dim W - 1$.

Proposition 3

Assume $\mathrm{char} K \neq 2$, and let $Y = \mathrm{SL}(W)$ and $G = \mathrm{SO}(W)$ be as above. Also let λ be as in the first column of Table 5 (resp. Table 6). Then the composition factors of the restriction of $L_Y(\lambda)$ to G is given by the second column of the table.

λ	$L_Y(\lambda) _G$
$\lambda_1 + \lambda_j$ ($1 \leq j \leq n$)	$\omega / \omega_{j-1} / \omega_{j-1}^{\epsilon_p(2n-j+2)}$
$\lambda_1 + \lambda_{n+1}$	$\omega / 2\omega_n / 2\omega_n^{\epsilon_p(n+1)}$
$\lambda_1 + \lambda_{n+2}$	$\omega / 2\omega_n / 2\omega_n^{\epsilon_p(n)}$
$\lambda_1 + \lambda_j$ ($n+3 \leq j \leq 2n$)	$\omega / \omega_{2n-j+2} / \omega_{2n-j+2}^{\epsilon_p(2n-j+2)}$

TABLE 5. KG -composition factors of certain irreducible KY -modules for $Y = \mathrm{SL}(W)$, $G = \mathrm{SO}(W)$, $\dim W = 2n + 1$ for some $n \geq 2$.

λ	$L_Y(\lambda) _G$
$\lambda_1 + \lambda_j$ ($1 \leq j \leq n - 1$)	$\omega / \omega_{j-1} / \omega_{j-1}^{\epsilon_p(2n-j+1)}$
$\lambda_1 + \lambda_n$	$\omega / \omega_1 + 2\omega_{n-1} / \omega_1 + \omega_n / (\omega_1 + \omega_n)^{\epsilon_p(n+1)}$
$\lambda_1 + \lambda_{n+1}$	$\omega / 2\omega_{n-1}^{1+\epsilon_p(n)} / 2\omega_n^{1+\epsilon_p(n)} /$
$\lambda_1 + \lambda_{n+2}$	$\omega / \omega_{n-1} + \omega_n / (\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)}$
$\lambda_1 + \lambda_j$ ($n+3 \leq j \leq 2n$)	$\omega / \omega_{2n-j+1} / \omega_{2n-j+1}^{\epsilon_p(2n-j+1)}$

TABLE 6. KG -composition factors of certain irreducible KY -modules for $Y = \mathrm{SL}(W)$, $G = \mathrm{SO}(W)$, $\dim W = 2n$ for some $n \geq 3$.

2. PRELIMINARIES

Let K be an algebraically closed field having characteristic $p \geq 0$. In this section, we recall some elementary properties about representations of simple algebraic groups over K . Unless specified otherwise, most of the results presented here can be found in [Bou68], [Hum75], or [Hum78].

2.1. Notation. We first fix some notation that will be used for the rest of the paper. Let G be a simply connected, simple algebraic group over K . Also fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U denotes the unipotent radical of B . Let $n = \text{rank } G = \dim T$ and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a corresponding base of the root system $\Phi = \Phi^+ \sqcup \Phi^-$ of G , where Φ^+ and Φ^- denote the sets of positive and negative roots of G , respectively. Let

$$X(T) = \text{Hom}(T, K^*)$$

denote the character group of T , and set $X(T)_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Also, for $\alpha \in \Phi$, define the reflection $s_{\alpha} : X(T)_{\mathbb{R}} \rightarrow X(T)_{\mathbb{R}}$ relative to α by $s_{\alpha}(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$, where $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)(\alpha, \alpha)^{-1}$ for λ, α with $\alpha \neq 0$, and $(-, -)$ denotes the usual \mathscr{W} -invariant inner product on $X(T)_{\mathbb{R}}$. Denote by \mathscr{W} the finite group $\langle s_{\alpha_r} : 1 \leq r \leq n \rangle$, called the *Weyl group* of G . Recall the existence of a partial ordering on $X(T)_{\mathbb{R}}$, defined by $\mu \preceq \lambda$ if and only if $\lambda - \mu \in \Gamma$, where Γ denotes the monoid of $\mathbb{Z}_{\geq 0}$ -linear combinations of simple roots. (We also write $\mu \prec \lambda$ to indicate that $\mu \preceq \lambda$ and $\mu \neq \lambda$.) In addition, let $\{\lambda_1, \dots, \lambda_n\}$ be the set of fundamental weights for T corresponding to our choice of base Π , that is $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for every $1 \leq i, j \leq n$. Set

$$X^+(T) = \{\lambda \in X(T) : \langle \lambda, \alpha_r \rangle \geq 0 \text{ for every } 1 \leq r \leq n\}$$

and call a character $\lambda \in X^+(T)$ a *dominant character*. Every such character can be written as a $\mathbb{Z}_{\geq 0}$ -linear combination $\lambda = \sum_{r=1}^n a_r \lambda_r$, where $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$.

2.2. Rational representations. In this section, we recall some elementary properties of rational modules for semisimple algebraic groups, starting by investigating weights and their multiplicities. Unless specified otherwise, the results recorded here can be found in [Hum75, Chapter XI, Section 31]. Let V denote a finite-dimensional, rational KG -module. Then V can be decomposed into a direct sum of KT -modules

$$V = \bigoplus_{\mu \in X(T)} V_{\mu},$$

where $V_{\mu} = \{v \in V : t \cdot v = \mu(t)v \text{ for all } t \in T\}$ for $\mu \in X(T)$. A character $\mu \in X(T)$ with $V_{\mu} \neq 0$ is called a *T-weight* of V , and V_{μ} is said to be its corresponding *weight space*. The dimension of V_{μ} is called the *multiplicity of μ in V* and is denoted by $m_V(\mu)$. Write $\Lambda(V)$ to denote the set of *T-weights* of V , and set $\Lambda^+(V) = \Lambda(V) \cap X^+(T)$. Any weight in $\Lambda^+(V)$ is called *dominant*.

The natural action of the Weyl group \mathscr{W} of G on $X(T)$ induces an action on $\Lambda(V)$ and we say that $\lambda, \mu \in X(T)$ are *\mathscr{W} -conjugate* if there exists $w \in \mathscr{W}$ such that $w\lambda = \mu$. It is well-known (see [Hum78, Section 13.2, Lemma A], for example) that $X^+(T)$ is a fundamental domain for the latter action, that is, each weight in $X(T)$ is \mathscr{W} -conjugate to a unique dominant weight. Also, if $\lambda \in X^+(T)$, then $w\lambda \preceq \lambda$ for every $w \in \mathscr{W}$. Finally, $\Lambda(V)$ is a union of \mathscr{W} -orbits and all weights in a \mathscr{W} -orbit have the same multiplicity.

Now by the Lie-Kolchin Theorem ([Hum75, Theorem 17.6]), there exists $0 \neq v^+ \in V$ such that $\langle v^+ \rangle_K$ is invariant under the action of B . We call such a vector v^+ a *maximal vector in V for B* . Note that since $\langle v^+ \rangle_K$ is stabilized by any maximal torus of B , there exists $\lambda \in X(T)$ such that $v^+ \in V_\lambda$. In fact, one can show that $\lambda \in X^+(T)$. It is well-known that isomorphism classes of finite-dimensional, irreducible, rational modules are in one-to-one correspondence with dominant weights for T . In this paper, we shall write $L_G(\lambda)$ for the irreducible KG -module having highest weight λ , obtained as a quotient of the corresponding Weyl module $V_G(\lambda)$ by its unique maximal submodule $\text{rad}(\lambda)$, that is,

$$L_G(\lambda) = V_G(\lambda) / \text{rad}(\lambda).$$

Clearly each $\lambda \in X(T)$ determines a 1-dimensional KT -module K_λ on which every $t \in T$ acts as multiplication by $\lambda(t)$ and one observes that we get a KB -module structure on K_λ , given by $(ut)x = \lambda(t)x$, for every $ut \in B$ and $x \in K_\lambda$. For $r \geq 0$, we let $H^r(-) = H^r(G/B, -)$ denote the r^{th} derived functor of the left exact functor $\text{ind}_B^G(-)$ and write $H^r(\lambda) = H^r(K_\lambda)$. It turns out (see [Jan03, II, 2.13]) that if $\lambda \in X^+(T)$, then $H^0(\lambda) \cong V_G(-w_0\lambda)^*$, where w_0 denotes the longest element in the Weyl group of G . Consequently $L_G(\lambda) \cong L_G(-w_0\lambda)^*$ is the unique irreducible submodule of $H^0(\lambda)$ and hence is the *socle* of $H^0(\lambda)$, written $\text{soc}(\lambda)$. We refer the reader to [Jan03, Section 2.1] for more details. Finally, the following result makes it easier to compute weight multiplicities in certain situations.

Lemma 2.1

Let $V = L_Y(\lambda)$ be an irreducible KG -module having highest weight $\lambda \in X^+(T)$. Let $J \subset \Pi$ and $\mu \in \Lambda^+(V)$ be such that $\mu = \lambda - \sum_{\alpha \in J} c_\alpha \alpha$. Also write $H = \langle U_{\pm\alpha} : \alpha \in J \rangle$. Then $m_V(\mu) = m_{V'}(\mu')$, where $\mu' = \mu|_{T_H}$, $V' = L_H(\lambda|_{T_H})$.

Proof. Let P be the standard parabolic subgroup of G corresponding to the subset J , so that H is the derived subgroup of a Levi factor of P . The weight space V_μ lies in the fixed point space of the unipotent radical of P , which is isomorphic to $L_H(\lambda|_{T_H})$ by [Jan03, Proposition 2.11]. The result then follows. \square

2.3. Some dimension calculations. In this section, G denotes a simply connected, simple algebraic group of rank n over K and $V = L_G(\lambda)$ an irreducible KG -module having p -restricted highest weight $\lambda \in X^+(T)$. In general, the dimension of V is unknown, or at least there is no known formula holding for λ arbitrary. Nevertheless, the dimension of $V_G(\lambda)$ is given by the well-known *Weyl's dimension formula* (see [Hum78, Section 24.3], for instance). The following result consists in a slightly modified version of the latter formula, which allows one to compute the dimension of a given Weyl module recursively. The proof, being straightforward, is omitted here.

Theorem 2.2 (Weyl's degree formula)

Set $\Phi_1^+ = \{\alpha = \sum_{r=1}^n a_r \alpha_r \in \Phi^+ : a_1 > 0\}$ and let L denote a Levi subgroup of G corresponding to the simple roots $\alpha_2, \dots, \alpha_n$. Then

$$\dim V_G(\lambda) = \left(\prod_{\alpha \in \Phi_1^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \right) \dim V_L(\lambda|_{T \cap L}).$$

We now record some information on the dimension of various irreducible KG -modules for G of type A_n ($n \geq 1$), B_n ($n \geq 2$), and D_n ($n \geq 3$) over K , starting by dealing with the former case. We say that a dominant T -weight λ is p -restricted if either $p = 0$, or if $0 \leq \langle \lambda, \alpha \rangle < p$ for $\alpha \in \Pi$. Also, for $\ell \in \mathbb{Z}_{\geq 0}$ a prime, we let $\epsilon_\ell : \mathbb{Z}_{\geq 0} \rightarrow \{0, 1\}$ be the map defined by

$$\epsilon_\ell(z) = \begin{cases} 1 & \text{if } \ell \text{ divides } z; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3

Let G be a simple algebraic group of type A_n ($n \geq 2$) over K and consider an irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T)$. Then the following assertions hold.

1. If $\lambda = a\lambda_1$ for some $a \in \mathbb{Z}_{\geq 1}$, then $V = V_G(\lambda) \cong \text{Sym}^a W$, where $\text{Sym}^a W$ denotes the a^{th} symmetric power of the natural KG -module W .
2. If $\lambda = \lambda_i$ for some $1 \leq i \leq n$, then $V = V_G(\lambda) \cong \Lambda^i W$, where $\Lambda^i W$ denotes the i^{th} exterior power of the natural KG -module W .
3. If $\lambda = \lambda_1 + \lambda_j$ for some $2 \leq j \leq n$, then $V = V_G(\lambda)$ if and only if $p \nmid j + 1$.

Furthermore, if λ is as in 1, 2, or 3 above, then the dimension of V is given by the second column of Table 7.

λ	$\dim L_G(\lambda)$
$a\lambda_1$ ($a \geq 1$)	$\binom{a+n}{a}$
λ_i ($1 \leq i \leq n$)	$\binom{n+1}{i}$
$\lambda_1 + \lambda_j$ ($2 \leq j \leq n$)	$j \binom{n+2}{j+1} - \epsilon_p(j+1) \binom{n+1}{j+1}$

TABLE 7. Dimension of certain irreducible modules for G of type A_n ($n \geq 2$).

Proof. We refer the reader to [Sei87, Lemma 1.14] for a proof of 1, and assume λ is as in 2, in which case $\Lambda^+(\lambda) = \{\lambda\}$. Since $m_V(\lambda) = m_{V_G(\lambda)}(\lambda) = 1$, the weight λ cannot afford the highest weight of a second composition factor of $V_G(\lambda)$ by [Pre87]. Consequently $V = V_G(\lambda)$ as desired, and an application of Theorem 2.2 yields the assertion on the dimension of V . Now fix a K -basis $\{v_1, \dots, v_{n+1}\}$ for W , where $v_1 \in W_{\lambda_1}$, $v_{r+1} \in W_{\lambda_1 - (\alpha_1 + \dots + \alpha_r)}$ for $1 \leq r \leq n$. Then

$$\Lambda^i W = \langle v_{r_1} \wedge \dots \wedge v_{r_i} : 1 \leq r_1 < r_2 < \dots < r_i \leq n+1 \rangle_K$$

by definition, and one easily checks that $v_1 \wedge v_2 \wedge \dots \wedge v_i$ is a maximal vector of weight λ in $\Lambda^i W$. Hence $\Lambda^i W$ admits a composition factor isomorphic to V . An application of Theorem 2.2 then yields $\dim V = \dim \Lambda^i W$, thus showing that the second assertion holds as well.

Finally, let λ be as in 3, and observe that $\Lambda^+(\lambda) = \{\lambda, \lambda_{j+1}\}$, where we adopt the notation $\lambda_{n+1} = 0$. As above, applying [Pre87] shows that $V_G(\lambda)$ is reducible if and only if λ_{j+1} affords the highest weight of a composition factor of $V_G(\lambda)$. An application of [Sei87, Proposition 8.6] then shows that the latter assertion holds if and only if p divides $j+1$, in which case $m_V(\lambda_{j+1}) = m_{V_G(\lambda)}(\lambda_{j+1}) - 1$. Consequently $\dim V = \dim V_G(\lambda) - \epsilon_p(j+1) \dim L_G(\lambda_{j+1})$, and Theorem 2.2 together with our knowledge of the dimension of exterior powers allow us to conclude. \square

We next prove a result similar to Lemma 2.3, for certain irreducible KG -modules in the case where G is of type B_n ($n \geq 2$) and $\text{char } K \neq 2$.

Lemma 2.4

Assume $p \neq 2$, let G be a simple algebraic group of type B_n ($n \geq 2$) over K , and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda \in \{\lambda_i + \delta_{i,n}\lambda_n : 1 \leq i \leq n\} \cup \{\lambda_n\}$. Then $V = V_G(\lambda)$ and the dimension of V is given by the second column of Table 8.

λ	$\dim L_G(\lambda)$
$\lambda_i + \delta_{i,n}\lambda_n \ (1 \leq i \leq n)$	$\binom{2n+1}{i}$
λ_n	2^n

TABLE 8. Dimension of certain irreducible modules for G of type B_n ($n \geq 2$).

Proof. First consider a dominant T -weight $\lambda \in \{\lambda_i + \delta_{i,n}\lambda_n : 1 \leq i \leq n\}$, and embed G in a simply connected, simple algebraic group Y of type A_{2n} over K in the usual way. (Observe that this forces $G = \text{SO}_{2n+1}(K)$, that is, G is not simply connected. however, the proof does not rely on G being simply connected and so the argument remains valid.) By [Sei87, Theorem 1, Table 1 (I_2, I_3)], the irreducible module $L_G(\lambda)$ is isomorphic to the restriction to G of a suitable exterior power of the natural module for Y . Using this observation together with Lemma 2.3, one deduces the desired assertions on V in the situation where λ is as in the first row of the table. Finally, in the case where $\lambda = \lambda_n$, we get that $\Lambda^+(\lambda) = \{\lambda\}$ and hence $V_G(\lambda)$ is irreducible by [Pre87]. The assertion on the dimension of V then immediately follows from Theorem 2.2. \square

Remark 2.5

The structure of a Weyl module $V_G(\lambda)$ with highest weight λ as in the statement of Lemma 2.4 is more complex in the situation where $\text{char } K = 2$ (see [CP12], for instance). In particular $V_G(\lambda)$ is in general not irreducible and hence not tilting (see Definition 2.8).

We next prove a result similar to Lemmas 2.3 and 2.4 for certain irreducible KG -modules in the case where G is of type D_n ($n \geq 3$) and $\text{char } K \neq 2$.

Lemma 2.6

Assume $p \neq 2$, let G be a simple algebraic group of type D_n ($n \geq 3$) over K , and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda \in \{\lambda_i + \delta_{i,n-1}\lambda_n : 1 \leq i < n\} \cup \{2\lambda_{n-1}\}$. Then $V = V_G(\lambda)$ and the dimension of V is given by the second column of Table 9.

λ	$\dim L_G(\lambda)$
$\lambda_i + \delta_{i,n-1}\lambda_n$	$\binom{2n}{i}$
λ_{n-1}	2^{n-1}
$2\lambda_{n-1}$	$\frac{1}{2} \binom{2n}{n}$

TABLE 9. Dimension of certain irreducible modules for G of type D_n ($n \geq 3$).

Proof. First consider a dominant T -weight $\lambda \in \{\lambda_i + \delta_{i,n-1}\lambda_n : 1 \leq i < n\}$, and embed G in a simply connected, simple algebraic group Y of type A_{2n-1} over K , as in the proof of Lemma 2.4. (Again, this yields $G = \mathrm{SO}_{2n}(K)$.) By [Sei87, Theorem 1, Table 1 (I_4, I_5)], the irreducible module $L_G(\lambda)$ is isomorphic to the restriction to G of a suitable exterior power of the natural module for Y . Using this observation together with Lemma 2.3, one checks that the assertions on V hold in this situation. Next assume $\lambda = \lambda_{n-1}$, in which case $\Lambda^+(\lambda) = \{\lambda\}$ and so $V_G(\lambda)$ is irreducible by [Pre87]. Finally, we refer the reader to [BGT16, Lemma 2.3.6] for a proof of the assertions in the situation where $\lambda = 2\lambda_{n-1}$. \square

2.4. Formal character and dot action. Let $\{e^\mu\}_{\mu \in X(T)}$ denote the standard basis of the group ring $\mathbb{Z}[X(T)]$ over \mathbb{Z} . The Weyl group \mathscr{W} of G acts on $\mathbb{Z}[X(T)]$ by $we^\mu = e^{w\mu}$, $w \in \mathscr{W}$, $\mu \in X(T)$, and we write $\mathbb{Z}[X(T)]^\mathscr{W}$ to denote the set of fixed points. The *formal character* of a given KG -module V is defined by

$$\mathrm{ch} V = \sum_{\mu \in X(T)} m_V(\mu) e^\mu \in \mathbb{Z}[X(T)]^\mathscr{W}.$$

Formal characters are a valuable tools to study finite-dimensional, rational modules. Following the ideas in [Jan03, Section II.5.5], we also associate to every T -weight $\lambda \in X(T)$ the linear polynomial

$$\chi(\lambda) = \sum_{r \geq 0} (-1)^r \mathrm{ch} H^r(\lambda).$$

If $\lambda \in X^+(T)$, Kempf's vanishing Theorem [Jan03, II, 4.5] shows that $H^r(\lambda) = 0$ for $r > 0$ and hence $\chi(\lambda) = \mathrm{ch} H^0(\lambda)$ in this case. In addition, recall from [Jan03, II, 2.13] that if $\lambda \in X^+(T)$, then $\chi(\lambda) = \mathrm{ch} V_G(\lambda)$ as well. One shows (see [Jan03, II, 5.8]) that each of $\{\chi(\lambda)\}_{\lambda \in X^+(T)}$ and $\{\mathrm{ch} L_G(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$.

For a T -weight $\mu \in X(T)$ with $\mu \prec \lambda$, we also introduce a “truncated” version of $\chi(\lambda)$, which shall prove useful later on in the paper:

$$\chi_\mu(\lambda) = \text{ch } L_G(\lambda) + \sum_{\substack{\eta \in X^+(T) \\ \mu \preceq \eta \prec \lambda}} [V_G(\lambda), L_G(\eta)] \text{ch } L_G(\eta).$$

Let ρ denote the half-sum of all positive roots in Φ , or equivalently, the sum of all fundamental weights. The *dot action* of \mathcal{W} on $X(T)$ is given by the formula $w \cdot \lambda = w(\lambda + \rho) - \rho$, for $w \in \mathcal{W}$ and $\lambda \in X(T)$. One easily sees that

$$\mathcal{D} = \{\lambda \in X(T) : \langle \lambda + \rho, \alpha \rangle \geq 0 \text{ for every } \alpha \in \Phi^+\}$$

is a fundamental domain for the dot action of \mathcal{W} on $X(T)$: for every $\mu \in X(T)$, there exist $w \in \mathcal{W}$ and a unique $\lambda \in \mathcal{D}$ such that $w \cdot \mu = \lambda$. This observation, together with the next result, provide the necessary tools to compute $\chi(\lambda)$ for any given $\lambda \in X(T)$, without having to consider higher homology groups. For $w \in \mathcal{W}$, we write $\det(w)$ for the determinant of w as an invertible linear transformation of $X(T)_{\mathbb{R}}$.

Lemma 2.7

Let $\lambda \in X(T)$ and $w \in \mathcal{W}$. Then $\chi(w \cdot \lambda) = \det(w)\chi(\lambda)$. Moreover, if $\lambda \in \mathcal{D}$ is not in $X^+(T)$, then $\chi(\lambda) = 0$.

Proof. The first assertion immediately follows from [Jan03, II, 5.9 (1)] and we refer the reader to [Jan03, II, 5.5] for a proof of the second. \square

2.5. Filtrations and extensions of modules. In this section, we introduce some notation and recall a few basic results concerning filtrations and extensions of KG -modules. For such a module V and for $\mu \in X^+(T)$ a dominant weight, we write $[V, L_G(\mu)]$ to denote the number of times the irreducible $L_G(\mu)$ occurs as a composition factor of V . Also, we adopt the notation $V = \mu_1^{m_1}/\mu_2^{m_2}/\dots/\mu_s^{m_s}$ to indicate that V is a KG -module with same composition factors as $L_G(\mu_1)^{m_1} \oplus \dots \oplus L_G(\mu_s)^{m_s}$, where $m_1, \dots, m_s \in \mathbb{Z}_{>0}$.

Definition 2.8

A filtration $V = V^0 \supseteq V^1 \supseteq \dots \supseteq V^r \supseteq V^{r+1} = 0$ of V is called a *Weyl filtration* if for every $0 \leq i \leq r$, there exists a weight $\mu_i \in X^+(T)$ with $V^i/V^{i+1} \cong V_G(\mu_i)$. Similarly, such a filtration is called a *good filtration* if for every $0 \leq i \leq r$, there exists a weight $\mu_i \in X^+(T)$ with $V^i/V^{i+1} \cong H^0(\mu_i)$. Finally, we call a KG -module *tilting* if it admits both a good and a Weyl filtration.

Modules with filtrations as above behave nicely with respect to tensor products and exterior (respectively, symmetric) powers, as recorded in the following result.

Proposition 2.9

If U, V are two KG -modules admitting good (respectively, Weyl) filtrations, then $U \otimes V$ also admits a good (respectively, Weyl) filtration. In addition, if W is a KG -module affording a good (respectively, Weyl) filtration, then each of $\text{Sym}^r W$ and $\Lambda^r W$ admits a good (respectively, Weyl) filtration as well, for any $r \in \mathbb{Z}_{>0}$.

Proof. The first general proof of the first assertion was given in [Mat90], but it had already been proven in most cases in [Don85]. We refer to [HM13, Proposition 2.2.5] for a proof of the second assertion. \square

For V_1, V_2 two KG -modules, we identify $\text{Ext}_G^1(V_2, V_1)$ with the set of equivalence classes of all short exact sequences $0 \rightarrow V_1 \hookrightarrow V \twoheadrightarrow V_2 \rightarrow 0$ of KG -modules. To conclude this section, we record a result on the possible extensions between irreducible modules for G .

Proposition 2.10

Let $\lambda, \mu \in X^+(T)$, with $\mu \prec \lambda$, and suppose that $[V_G(\lambda), L_G(\mu)] = 0$. Then $\text{Ext}_G^1(L_G(\lambda), L_G(\mu)) = 0$.

Proof. Let $\lambda, \mu \in X^+(T)$ be such that $\text{Ext}_G^1(L_G(\lambda), L_G(\mu)) \neq 0$. By [Jan03, II, Proposition 2.14], this translates to $\text{Hom}_{KG}(\text{rad}(\lambda), L_G(\mu)) \neq 0$. Consequently, there exists a non-zero surjective morphism of KG -modules $\phi : \text{rad}(\lambda) \twoheadrightarrow L_G(\mu)$, so that $\ker \phi$ is maximal in $\text{rad}(\lambda)$. Finding a composition series of $\ker(\phi)$ then yields a composition series for $V_G(\lambda)$, say $V_G(\lambda) \supseteq \text{rad}(\lambda) \supseteq \ker(\phi) \supseteq V_3 \supseteq \dots \supseteq V_r \supseteq 0$. As $\text{rad}(\lambda)/\ker(\phi) \cong L_G(\mu)$, we get that $[V_G(\lambda), L_G(\mu)] \neq 0$, thus completing the proof. \square

3. MAIN TECHNIQUES

In this section, we introduce two techniques (namely Proposition 3.2 and Theorem 3.8 below) that shall be used in order to prove the three main results of the paper. The first result provides us with an upper bound (equal to zero for most dominant weights) for the number of times certain composition factors appear in a given Weyl module for G .

3.1. Extending a result of McNinch. Following the idea of [McN98], we first investigate pairs (V, τ) , where V is a finite-dimensional, rational KG -module and $\tau \in \Lambda^+(V)$ satisfy a certain set of properties.

Proposition 3.1

Let V be a finite-dimensional, rational KG -module, and let $\tau \in \Lambda^+(V)$ be a dominant weight of V . Assume in addition that V is tilting, that τ is the unique highest weight of V , and that $m_V(\tau) = 1$. Then there exists $\iota \in \text{Hom}_{KG}(V_G(\tau), V)$ injective and $\phi \in \text{Hom}_{KG}(V, H^0(\tau))$ surjective. Furthermore, under those hypotheses, we have $\iota(\text{rad}(\tau)) \subseteq \ker(\phi)$.

Proof. We refer the reader to [McN98, Proposition 4.6.2] for a proof of the existence of ι and ϕ as in the statement of the proposition. For simplicity, we identify $V_G(\tau)$ with $\iota(V_G(\tau))$ in the remainder of the proof. Also write $N = \ker(\phi) \cap V_G(\tau)$, and denote by $\bar{\phi} : V_G(\tau)/N \hookrightarrow H^0(\tau)$ the injective morphism of KG -modules induced by $\phi \circ \iota$. As $\text{rad}(\tau)$ is the unique maximal submodule of $V_G(\tau)$, we have $N \subseteq \text{rad}(\tau)$, and if $N \subsetneq \text{rad}(\tau)$, then we get $0 \subsetneq \bar{\phi}(\text{rad}(\tau)/N) \subseteq \text{Im}(\bar{\phi}) \subseteq H^0(\lambda)$, a contradiction with $\text{soc}(H^0(\tau)) = L_G(\tau)$, as $\tau \notin \Lambda(\text{rad}(\tau))$. Therefore $N = \text{rad}(\tau)$ and the proof is complete. \square

Proposition 3.2

Let V be a KG -module as in the statement of Proposition 3.1, and let $(a_\mu)_{\mu \in X^+(T)} \subset \mathbb{Z}_{\geq 0}$ be such that $\text{ch } V = \chi(\tau) + \sum_{\mu \in X^+(T)} a_\mu \text{ch } L_G(\mu)$. Then for every $\mu \in X^+(T)$ different from τ , we have

$$[V_G(\tau), L_G(\mu)] \leq a_\mu.$$

Proof. Let $\phi : V \rightarrow H^0(\tau)$ be as in Proposition 3.1, with $\text{rad}(\tau) \subseteq \ker \phi$, and write $M = \ker(\phi)/\text{rad}(\tau)$. Also consider the short exact sequence $0 \rightarrow \ker(\phi) \hookrightarrow V \rightarrow H^0(\tau) \rightarrow 0$. Then one easily checks that $\chi(\tau) = \text{ch } L_G(\tau) + \sum_{\mu \in X^+(T)} a_\mu \text{ch } L_G(\mu) - \text{ch } M$, so that $[V_G(\tau), L_G(\mu)] = a_\mu - [M, L_G(\mu)] \leq a_\mu$ as desired. \square

We next illustrate Proposition 3.2 with a concrete example, which shall prove useful later on in the paper. The result is somehow standard. (An alternative proof can be found in [Sei87, Lemma 8.6], for instance.)

Lemma 3.3

Let G be a simple algebraic group of type A_n ($n \geq 1$) over K . Also fix $1 \leq j \leq n$, and write $\lambda = \lambda_1 + \lambda_j$. Then $V_G(\lambda_1) \otimes V_G(\lambda_j)$ is tilting, and adopting the notation $\lambda_{n+1} = 0$, we have

$$\text{ch } V_G(\lambda_1) \otimes V_G(\lambda_j) = \chi(\lambda) + \chi(\lambda_{j+1}).$$

In addition, if $\mu \in X^+(T)$ affords the highest weight of a composition factor of $V_G(\lambda)$, then $\mu = \lambda$ or λ_{j+1} , and $[V_G(\lambda), L_G(\mu)] = 1$.

Proof. First observe that each of $V_G(\lambda_1)$ and $V_G(\lambda_j)$ is irreducible by Lemma 2.3, and hence both KG -modules are tilting. The first assertion then follows from Proposition 2.9. Also writing $T(\lambda)$ for $V_G(\lambda_1) \otimes V_G(\lambda_j)$, we observe that $\text{ch } T(\lambda)$ is independent of p and thus we may and shall assume K has characteristic zero in the remainder of the argument. An application of the Littlewood-Richardson formula [Jam78, 16.4] then yields the desired assertion on the character of $T(\lambda)$. Finally, as λ is the highest weight of $T(\lambda)$ and since $m_{T(\lambda)}(\lambda) = 1$, an application of Proposition 3.2 completes the proof. \square

3.2. A truncated version of the Jantzen p -sum formula. In this section, we introduce a few tools which shall be of use in order to better understand the composition factors of a given Weyl module for G . Most of the underlying theory can be found in [Jan03, II, Sections 4, 5, or 8], to which we refer the reader for more details. Let ρ denote the half-sum of all positive roots in Φ , or equivalently, the sum of all fundamental weights. Also for $\lambda, \mu \in X^+(T)$ such that $\mu \prec \lambda$, define

$$d(\lambda, \mu) = 2(\lambda + \rho, \lambda - \mu) - (\lambda - \mu, \lambda - \mu),$$

as in [Sei87, Section 6]. The following corollary to the strong linkage principle [And80] provides some insight on the possible composition factors of a given Weyl module for G , in the case where G is not of type G_2 and $p > 2$. We refer the reader to [Sei87, Proposition 6.2] for a proof.

Proposition 3.4

Assume $p > 2$ and let G be a simple algebraic group of type different from G_2 . Also let λ and μ be as above, and assume the inner product on $\mathbb{Z}\Phi$ is normalized so that long roots have length 1. If μ affords the highest weight of a composition factor of $V_G(\lambda)$, then

$$2d(\lambda, \mu) \in p\mathbb{Z}.$$

For $r \in \mathbb{Z}$ and $\alpha \in \Phi$, we denote by $s_{\alpha,r} : X(T) \rightarrow X(T)$ the affine reflection on $X(T)$ defined by $s_{\alpha,r}(\lambda) = s_\alpha(\lambda) + r\alpha$, $\lambda \in X(T)$. Also for ℓ a prime, set \mathcal{W}_ℓ equal to the subgroup of $\text{Aff}(X(T))$ generated by all $s_{\alpha,n\ell}$, with $\alpha \in \Phi$, $n \in \mathbb{Z}$, and call \mathcal{W}_ℓ the *affine Weyl group* associated to G and ℓ .

The dot action introduced in Section 2.4 can be extended to an action of \mathscr{W}_ℓ on $X(T)$ and $X(T)_\mathbb{R}$ in the obvious way, setting $w \cdot \lambda = w(\lambda + \rho) - \rho$, $w \in \mathscr{W}_\ell$, $\lambda \in X(T)$. Finally, for ℓ a prime number and $m \in \mathbb{Z}$, we write $\nu_\ell(m)$ to denote the greatest integer r such that ℓ^r divides m (adopting the notation $\nu_0(m) = 0$ for every $m \in \mathbb{Z}$). The following result provides a powerful tool for understanding Weyl modules.

Proposition 3.5 (The Jantzen p -sum formula)

Let $\lambda \in X^+(T)$ be a dominant weight. Then there exists a filtration $V_G(\lambda) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\lambda)$ such that $V^0/V^1 \cong L_G(\lambda)$ and

$$\sum_{i=1}^k \text{ch } V^i = - \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \det(w_{\alpha,r}) \chi(\xi_{\alpha,r}), \quad (1)$$

where for $\alpha \in \Phi^+$ and $1 < r < \langle \lambda + \rho, \alpha \rangle$, $\xi_{\alpha,r}$ denotes the unique weight in $\mathscr{W} \cdot (\lambda - r\alpha) \cap \mathscr{D}$ and $w_{\alpha,r}$ is an element in \mathscr{W} satisfying $w_{\alpha,r} \cdot (\lambda - r\alpha) = \xi_{\alpha,r}$.

Proof. By [Jan03, II, 8.19], there exists a filtration $V_G(\lambda) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\lambda)$ such that $V^0/V^1 \cong L_G(\lambda)$ and

$$\sum_{i=1}^k \text{ch } V^i = \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \chi(s_{\alpha,r} \cdot \lambda).$$

Fix $\alpha \in \Phi^+$, $1 < r < \langle \lambda + \rho, \alpha \rangle$, and let $w_{\alpha,r} \in \mathscr{W}$ and $\xi_{\alpha,r} \in \mathscr{D}$ be such that $w_{\alpha,r} \cdot \xi_{\alpha,r} = \lambda - r\alpha$. (Such elements exist, since \mathscr{D} is a fundamental domain for the dot action.) A straightforward calculation yields $s_{\alpha,r} \cdot \lambda = s_\alpha \cdot (\lambda - r\alpha)$, from which one deduces that $\chi(s_{\alpha,r} \cdot \lambda) = \chi((s_\alpha w_{\alpha,r}) \cdot \xi_{\alpha,r})$. An application of Lemma 2.7 then completes the proof. \square

We shall call a filtration of $V_G(\lambda)$ as in Proposition 3.5 a *Jantzen filtration* of $V_G(\lambda)$. Let us then fix such a filtration $V_G(\lambda) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ in the remainder of the section. Also, following [Jan03, II, 8.14], we write $\nu^c(T_\lambda)$ to denote the expression (1). As $\{\chi(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$ (see [Jan03, Remark II.5.8], for instance), there exists $(a_\nu)_{\nu \in X^+(T)} \subset \mathbb{Z}$ such that

$$\nu^c(T_\lambda) = \sum_{\nu \in X^+(T)} a_\nu \chi(\nu). \quad (2)$$

Consider a T -weight $\mu \in X(T)$ with $\mu \prec \lambda$. In what follows, we introduce a “truncated” version of the character $\nu^c(T_\lambda)$, which shall prove useful in computations. Define

$$\nu_\mu^c(T_\lambda) = \sum_{\substack{\nu \in X^+(T) \\ \mu \preceq \nu \prec \lambda}} a_\nu \chi(\nu), \quad (3)$$

where the a_ν ($\nu \in X^+(T)$) are as in (2). Finally, the latter decomposition yields

$$\nu_\mu^c(T_\lambda) = \sum_{\substack{\xi \in X^+(T) \\ \mu \preceq \xi \prec \lambda}} b_\xi \text{ch } L_G(\xi), \quad (4)$$

for some $b_\xi \in \mathbb{Z}$, $\xi \in X^+(T)$.

The following proposition provides some insight on how the truncated p -sum formula (4) can be used in order to determine the possible composition factors of $V_G(\lambda)$, together with an upper bound for their multiplicity.

Proposition 3.6

Let $\lambda \in X^+(T)$ and consider a T -weight $\mu \prec \lambda$. Also let $\xi \in X^+(T)$ be a dominant weight such that $\mu \preceq \xi \prec \lambda$. Then ξ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $b_\xi \neq 0$ in (4). Also $[V_G(\lambda), L_G(\xi)] \leq b_\xi$.

Proof. By definition, we have $\nu^c(T_\lambda) = \text{ch } V$, where $V = V^1 \oplus \cdots \oplus V^k$. We first claim that ξ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $[V, L_G(\xi)] \neq 0$. Indeed, $V^i \subsetneq V_G(\lambda)$ for $1 \leq i \leq k$, and since $V_G(\lambda)/V^1 \cong L_G(\lambda)$, we have

$$0 \leq [V_G(\lambda), L_G(\xi)] \leq [V, L_G(\xi)].$$

In particular if ξ affords the highest weight of a composition factor $V_G(\lambda)$, then $[V, L_G(\xi)] \neq 0$. Conversely, assume $[V, L_G(\xi)] \neq 0$. Since $V^0/V^1 \cong L_G(\lambda)$ and $\xi \neq \lambda$, there exists $1 \leq i \leq k$ such that $[V^i, L_G(\xi)] \neq 0$, and as $V^i \subseteq V_G(\lambda)$, the claim holds. In order to conclude, it remains to show that $[V, L_G(\xi)] = b_\xi$, where b_ξ is as in (4). Since every T -weight of $V_G(\lambda)$ (and hence of V) is under λ , there exist integers c_ν such that

$$\nu_\mu^c(T_\lambda) = \nu^c(T_\lambda) - \sum_{\substack{\nu \in X^+(T) \\ \mu \not\prec \nu \prec \lambda}} c_\nu \text{ch } L_G(\nu).$$

In particular, this shows that $b_\xi = [V, L_G(\xi)]$ as desired, thus completing the proof. \square

Fix $\mu \in X^+(T)$ with $\mu \prec \lambda$. For $\nu \in X^+(T)$, we call the coefficient a_ν in (3) the *contribution* of ν to $\nu_\mu^c(T_\lambda)$, and we say that ν *contributes* to $\nu_\mu^c(T_\lambda)$ if its contribution is non-zero. Now applying Proposition 3.6 to a given triple $\mu \preceq \xi \prec \lambda$ requires the knowledge of the contribution of ν to $\nu_\mu^c(T_\lambda)$ for each dominant T -weight $\xi \preceq \nu \prec \lambda$. In certain cases, knowing whether or not a given T -weight contributes to $\nu_\mu^c(T_\lambda)$ can be easily determined, as the following result shows.

Lemma 3.7

Let λ, μ and ν be as above, with ν maximal with respect to the partial order \preceq , such that ν contributes to $\nu_\mu^c(T_\lambda)$. Then ν affords the highest weight of a composition factor of $V_G(\lambda)$.

Proof. Write $\Lambda^+ = \{\xi \in X^+(T) : \mu \preceq \xi \prec \lambda, \nu \not\prec \xi\}$. By maximality of ν , there exists $a_\nu \in \mathbb{Z}^*$ and $(a_\xi)_{\xi \in \Lambda^+} \subset \mathbb{Z}$ such that

$$\nu_\mu^c(T_\lambda) = a_\nu \chi_\mu(\nu) + \sum_{\xi \in \Lambda^+} a_\xi \chi_\mu(\xi). \quad (5)$$

As $\sigma \preceq \xi$ for all $\sigma \in \Lambda(\xi)$, $\xi \in \Lambda^+$, we get that ν is not a weight in $\Lambda(\xi)$, $\xi \in \Lambda^+$, and hence cannot afford the highest weight of a composition factor of $V_G(\xi)$, $\xi \in \Lambda^+$. Therefore, rewriting $\chi_\mu(\nu)$ and each $\chi_\mu(\xi)$ of (5) in terms of characters of irreducibles (recall that $[V_G(\nu), L_G(\nu)] = 1$) yields the existence of a tuple $(b_\xi)_{\xi \in \Lambda^+} \subset \mathbb{Z}$ such that $\nu_\mu^c(T_\lambda) = a_\nu \text{ch } L_G(\nu) + \sum_{\xi \in \Lambda^+} b_\xi \text{ch } L_G(\xi)$. An application of Proposition 3.6 then completes the proof. \square

Fix $\nu \in X^+(T)$, and recall from [Bou68] the description of the simple roots and fundamental weights for T in terms of a basis $\{\varepsilon_1, \dots, \varepsilon_{d_\Phi}\}$ for a Euclidean space E of dimension d_Φ . Following the idea of [McN98], for $\alpha \in \Phi^+$ and $r \in \mathbb{Z}_{\geq 0}$ such that $1 < r < \langle \lambda + \rho, \alpha \rangle$, we write $\lambda + \rho - r\alpha = a_1\varepsilon_1 + \dots + a_{d_\Phi}\varepsilon_{d_\Phi}$, as well as $\nu + \rho = b_1\varepsilon_1 + \dots + b_{d_\Phi}\varepsilon_{d_\Phi}$. Also, we set $A_{\alpha,r} = (a_j)_{j=1}^{d_\Phi} \in \mathbb{Q}^{d_\Phi}$ and $B_\nu = (b_j)_{j=1}^{d_\Phi} \in \mathbb{Q}^{d_\Phi}$. The action of the Weyl group \mathcal{W} of G on the basis $\{\varepsilon_1, \dots, \varepsilon_{d_\Phi}\}$ is described in [Bou68], and extends to an action of \mathcal{W} on \mathbb{Q}^{d_Φ} in the obvious way. (We write $w \cdot A$ for $w \in \mathcal{W}$, $A \in \mathbb{Q}^{d_\Phi}$.) Define the *support* of an element $z \in \mathbb{Z}\Phi$ to be the subset $\text{supp}(z)$ of Π consisting of those simple roots α such that $c_\alpha \neq 0$ in the decomposition $z = \sum c_\alpha \alpha$. Also for $w \in \mathcal{W}$, we write $\det(w)$ for the determinant of w as an invertible linear transformation of $X(T)_\mathbb{R}$. The following result is our main tool for determining the contribution of ν to $\nu_\mu^c(T_\lambda)$, for each weight $\nu \in X^+(T)$ with $\mu \preccurlyeq \nu \prec \lambda$.

Theorem 3.8

Let $\lambda \in X^+(T)$, and fix a Jantzen filtration $V_G(\lambda) \supseteq V^1 \supseteq V^2 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\lambda)$. Also consider a weight $\mu \in X(T)$ with $\mu \prec \lambda$, and let $\nu \in X^+(T)$ be such that $\mu \preccurlyeq \nu \prec \lambda$. Finally, write $I_\nu = \{(\alpha, r) \in \Phi^+ \times [2, \langle \lambda + \rho, \alpha \rangle] : \text{supp}(\alpha) = \text{supp}(\lambda - \nu), B_\nu \in \mathcal{W} \cdot A_{\alpha,r}\}$, and for each pair $(\alpha, r) \in I_\nu$, choose $w_{\alpha,r} \in \mathcal{W}$ such that $w_{\alpha,r} \cdot A_{\alpha,r} = B_\nu$. Then the contribution of ν to $\nu_\mu^c(T_\lambda)$ is given by

$$- \sum_{(\alpha,r) \in I_\nu} \nu_p(r) \det(w_{\alpha,r}).$$

Proof. By Proposition 3.5, the contribution of ν to the truncated Jantzen p -sum formula $\nu_\mu^c(T_\lambda)$ is given by

$$- \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \det(w_{\alpha,r}),$$

where for $\alpha \in \Phi^+$ and $2 \leq r \leq \langle \lambda + \rho, \alpha \rangle - 1$, the element $w_{\alpha,r}$ is either zero (if $\nu \notin \mathcal{W} \cdot (\lambda - r\alpha)$), or a chosen element in \mathcal{W} satisfying $w_{\alpha,r} \cdot (\lambda - r\alpha) = \nu$. In addition, observe that by [McN98, Lemma 4.5.6], the latter can only occur if $\lambda - r\alpha$ and ν have the same support. Finally, identifying the action of \mathcal{W} on $X(T)$ with that of \mathcal{W} on \mathbb{Q}^{d_Φ} as above completes the proof. \square

4. THE B_n -CASE ($n \geq 2$)

Let K be an algebraically closed field of characteristic $p \neq 2$, and let Y a simply connected, simple algebraic group of type A_{2n} ($n \geq 2$) over K . Consider a subgroup G of type B_n , embedded in the usual way, as the stabilizer of a non-degenerate quadratic form on the natural module for Y . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_{2n}\}$ denote a corresponding base of the root system $\Phi(Y) = \Phi^+(Y) \sqcup \Phi^-(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_{2n}\}$ be the set of fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. Also set $T = T_Y \cap G$, $B = B_Y \cap G$, so that T is a maximal torus of G , and B is a Borel subgroup of G containing T . Let $\Pi(G) = \{\beta_1, \dots, \beta_n\}$ be the corresponding base for the root system $\Phi(G) = \Phi^+(G) \sqcup \Phi^-(G)$ of G , and let $\omega_1, \dots, \omega_n$ denote the associated fundamental weights. Here the root restrictions are given by $\alpha_i|_T = \alpha_{2n-i+1}|_T = \beta_i$ for $1 \leq i \leq n$. Finally, using [Hum78, Table 1, p.69] and the fact that $\lambda_1|_T = \omega_1$ yields

$$\lambda_i|_T = \lambda_{2n-i+1}|_T = \omega_i, \quad \lambda_n|_T = \lambda_{n+1}|_T = 2\omega_n, \quad 1 \leq i \leq n-1. \quad (6)$$

In this section, we show how to obtain the tables 1, 3, and 5 in Theorem 1, Corollary 2, and Proposition 3, respectively. In order to do so, we rely as much as possible on the embedding of G in Y described above, proceeding in the following steps: we start by computing the formal character of the restriction to G of certain Weyl modules for Y (see Proposition 4.2), and then deduce the character of certain tensor products of irreducible Weyl modules for G (see Lemmas 4.3 and 4.4). Applying Proposition 3.2 shall then yield an upper bound for the multiplicities of the possible composition factors of the Weyl modules $V_G(\omega)$, in the case where ω is as in the statement of Theorem 1. Finally, using Theorem 3.8, we compute various contributions to the Jantzen p -sum formula in each case, and we conclude using Proposition 3.6. The proofs of Corollary 2 and Proposition 3 are also given at the end of the section.

4.1. Restriction of certain Weyl modules for Y . We start our investigation by showing that if V is a KY -module with unique highest weight $\lambda \in X^+(T_Y)$, then every T -weight of V is under the restriction of λ to T . Hence the ordering of T_Y -weights is preserved when restricting to G .

Lemma 4.1

Let $\lambda \in X^+(T_Y)$ be a dominant weight, and let V be a KY -module with unique highest weight λ . Then every T -weight ξ of V satisfies $\xi \preceq \lambda|_T$.

Proof. Let $\xi \in X(T)$ be a weight of $V|_G$. Then there exists a T_Y -weight μ of V such that $\mu|_T = \xi$. Since V has unique highest weight, there exist $c_1, \dots, c_{2n} \in \mathbb{Z}_{\geq 0}$ such that $\mu = \lambda - \sum_{r=1}^{2n} c_r \alpha_r$. Therefore $\xi = \lambda|_T - \sum_{r=1}^{2n} c_r \alpha_r|_T$ and the assertion follows from the root and weight restrictions in (6). \square

We next investigate the formal character of the restriction to G of the Weyl module $V_Y(\lambda_1 + \lambda_j)$, where $1 \leq j \leq 2n$.

Proposition 4.2

Let $1 \leq j \leq 2n$, and consider the dominant T_Y -weight $\lambda = \lambda_1 + \lambda_j \in X^+(T_Y)$. Also set $\omega = \lambda|_T$ and adopt the notation $\omega_0 = 0$. Then

$$\text{ch } V_Y(\lambda)|_G = \begin{cases} \chi(\omega) + \chi(\omega_{j-1}) & \text{if } 1 \leq j \leq n; \\ \chi(\omega) + \chi(2\omega_n) & \text{if } j = n+1, n+2; \\ \chi(\omega) + \chi(\omega_{2n-j+2}) & \text{otherwise.} \end{cases}$$

Proof. Write $V = V_Y(\lambda)$ and first notice that $\text{ch } V|_G$ is independent of p . Hence we may and shall assume K has characteristic zero in the remainder of the proof. In the case where $1 \leq j \leq n$, an application of [KT87, Proposition 1.5.3] yields $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_{j-1})$, from which the result follows in this situation. Next assume $j = n+1$. Here the weights $\lambda - (\alpha_1 + \dots + \alpha_n)$, $\lambda - (\alpha_1 + \dots + \alpha_r + \alpha_{n+1} + \dots + \alpha_{2n-r})$ ($1 \leq r \leq n-1$), and $\lambda - (\alpha_{n+1} + \dots + \alpha_{2n})$ all restrict to $\omega' = 2\omega_n \in X^+(T)$. Therefore the latter occurs in a second composition factor of $V_Y(\lambda)|_G$, whose highest weight $\mu \in X^+(T)$ satisfies $\omega' \preceq \mu \preceq \omega$ by Lemma 4.1. The only possibility is that ω' itself affords the highest weight of a composition factor. Applying Theorem 2.2, Table 7, and Table 8 then yields $\dim V_Y(\lambda) = \dim V_G(\omega) + \dim V_G(2\omega_n)$, from which the desired result follows in this case as well. The remaining cases can be dealt with in a similar fashion, hence the details are omitted here. \square

4.2. Formal character of various tensor products. We next determine the formal character of the tensor product $V_G(\omega_1) \otimes V_G(\omega_j)$ for $1 \leq j \leq n$, as well as of the formal character of the tensor product $V_G(\omega_1) \otimes V_G(2\omega_n)$. Observe that since $p \neq 2$, each of the considered Weyl modules is irreducible, and hence is tilting.

Lemma 4.3

Let $1 \leq j \leq n$, and consider the dominant T -weight $\omega = \omega_1 + \omega_j \in X^+(T)$. Also set $\omega_0 = \omega_{n+1} = 0$ and write $T(\omega)$ for the tensor product $V_G(\omega_1) \otimes V_G(\omega_j)$. Then $T(\omega)$ is tilting and its formal character is given by

$$\text{ch } T(\omega) = \chi(\omega) + (1 - \delta_{j,n})\chi(\omega_{j-1}) + (1 - \delta_{j,1})\chi(\omega_{j+1} + (\delta_{j,n-1} + \delta_{j,n})\omega_n).$$

Proof. By Lemma 2.4, both $V_G(\omega_1)$ and $V_G(\omega_j)$ are irreducible KG -modules, and hence $T(\omega)$ is tilting by Proposition 2.9. Also $\text{ch } T(\omega)$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. By [Sei87, Theorem 1, Table 1 (I_2, I_3)] together with Lemma 3.3, we successively get

$$T(\omega) \cong (V_Y(\lambda_1) \otimes V_Y(\lambda_j))|_G \cong V_Y(\lambda)|_G \oplus V_Y(\lambda_{j+1})|_G.$$

Now if $1 \leq j < n$, then applying (6) yields the restrictions $\lambda|_T = \omega$ and $\lambda_{j+1}|_T = \omega_{j+1} + \delta_{j,n-1}\omega_n$. Therefore Proposition 4.2 yields $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_{j-1})$, while $V_Y(\lambda_{j+1})|_G \cong V_G(\omega_{j+1} + \delta_{j,n}\omega_n)$ by [Sei87, Theorem 1, Table 1 (I_2, I_3)]. The assertion thus holds in this situation and so it remains to consider the case where $j = n$. Here applying Theorem 2.2 yields

$$\dim T(\omega) = \dim V_G(\omega) + 2^n > \dim V_G(\omega), \tag{7}$$

showing the existence of a second composition factor of $T(\omega)$. Now ω is the unique highest weight of $T(\omega)$ and so an application of Lemma 4.1 yields $\Lambda^+(T(\omega)) = \{\omega, \omega_n\}$. As $m_{T(\omega)}(\omega) = m_{V_G(\omega)}(\omega) = 1$, the weight ω cannot afford the highest weight of a second composition factor of $T(\omega)$, thus forcing $[T(\omega), V_G(\omega_n)] > 0$. Finally, $\dim V_G(\omega_n) = 2^n$ by Table 8, and hence (7) completes the proof. \square

Lemma 4.4

Consider the T -weight $\omega = \omega_1 + 2\omega_n \in X^+(T)$, and write $T(\omega) = V_G(\omega_1) \otimes V_G(2\omega_n)$. Then $T(\omega)$ is tilting and its formal character is given by

$$\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_{n-1}) + \chi(2\omega_n).$$

Proof. By Lemma 2.4, both $V_G(\omega_1)$ and $V_G(2\omega_n)$ are irreducible KG -modules, and hence $T(\omega)$ is tilting by Proposition 2.9. Also $\text{ch } T(\omega)$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. By [Sei87, Theorem 1, Table 1 (I_2, I_3)] together with Lemma 3.3, we successively get

$$T(\omega) \cong (V_Y(\lambda_1) \otimes V_Y(\lambda_n))|_G \cong V_Y(\lambda)|_G \oplus V_G(2\omega_n).$$

Now applying (6) yields the restriction $\lambda|_T = \omega$ and so $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_{n-1})$ by Proposition 4.2, thus completing the proof. \square

4.3. Various contributions to the truncated Jantzen p -sum formula. In this section, we compute certain contributions to the truncated Jantzen formula for some Weyl modules $V_G(\omega)$, where ω is as in the first column of Table 1, starting by the case where $\omega = 2\omega_1$.

Proposition 4.5

Assume $\omega = 2\omega_1$, and consider the zero weight $\mu = 0$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(2n+1) \operatorname{ch} L_G(\mu).$$

Proof. We start by computing all contributions to $\nu_\mu^c(T_\omega)$. Here the dominant T -weights $\nu \in X^+(T)$ such that $\mu \preccurlyeq \nu \prec \omega$ are $\omega - \beta_1$, $\omega - (\beta_1 + \dots + \beta_n)$, and μ itself. Now $\omega - \beta_1$ and $\omega - (\beta_1 + \dots + \beta_n)$ have multiplicity 1 in $V_G(\omega)$, and hence none of them can afford of $V_G(\omega)$ the highest weight of a composition factor by [Pre87]. Recursively applying Lemma 3.7 then shows that those same weights cannot contribute to $\nu_\mu^c(T_\omega)$. Therefore it remains to compute the contribution of μ , starting by determining all pairs $(\beta, r) \in I_\mu$ as in Theorem 3.8. A straightforward computation yields

$$B_\mu = \frac{1}{2}(2n-1, 2n-3, \dots, 3, 1),$$

and since $\omega - \mu$ has support Π , we get that $\beta \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n\}$ by definition of I_μ . Recall from [Bou68, Planche II] that \mathscr{W} acts by all permutations and sign changes of the ε_i . Also, one checks that for $2 \leq \ell \leq n$ and $r \in \mathbb{Z}$, we have

$$A_{\varepsilon_1 + \varepsilon_\ell, r} = B_\mu + (2-r, \underbrace{0, \dots, 0}_{\ell-2}, -r, \underbrace{0, \dots, 0}_{n-\ell}).$$

Consequently $B_\mu \in \mathscr{W} \cdot A_{\varepsilon_1 + \varepsilon_\ell, r}$ if and only if $\{|2(n-r)+3|, |2(n-\ell-r)+1|\} = \{2n-1, 2(n-\ell)+1\}$. We thus study each possibility separately and show that in each case, no weight in $\mathscr{W} \cdot (\omega - \mathbb{Z}(\varepsilon_1 + \varepsilon_\ell))$ can contribute to $\nu_\mu^c(T_\omega)$.

1. If $2(n-r)+3 = 2n-1$, then $r = 2$ and hence $\nu_p(r) = 0$, since $p \neq 2$ by assumption. Hence the weight $\omega - 2(\varepsilon_1 + \varepsilon_\ell)$ cannot contribute to $\nu_\mu^c(T_\omega)$ in this situation.
2. If $2(n-r)+3 = -2n+1$, then $r = 2n+1$, so that $|2(n-\ell-r)+1| = 2(n+\ell)+1$. As it is impossible for the latter to be equal to $2(n-\ell)+1$, we get the desired assertion in this case.
3. If $2(n-r)+3 = 2(n-\ell)+1$, then $r = \ell+1$ and $|2(n-\ell-r)+1| = |2(n-2\ell)-1|$. The latter cannot equal $2n-1$, thus showing the assertion in this case as well.
4. If $2(n-r)+3 = -2(n-\ell)-1$, then $r = 2n-\ell+2$, in which case $|2(n-\ell-r)+1| = 2n+3 \neq 2n-1$.

Therefore a contribution to $\nu_\mu^c(T_\omega)$ can only come from the situation where $\beta = \varepsilon_1$, which we assume holds in the remainder of the proof. Here for $r \in \mathbb{Z}$, we have

$$A_{\beta, r} = B_\mu + (2-r, 0, \dots, 0),$$

from which one deduces that $B_\mu \in \mathscr{W} \cdot A_{\beta, r}$ if and only if $|2(n-r)+3| = 2n-1$. The latter equality is satisfied if and only if $r = 2$ or $2n+1$, and since $p \neq 2$, the weight $\omega - 2\beta$ cannot contribute to the p -sum $\nu_\mu^c(T_\omega)$. Using the action of \mathscr{W} described above, one checks that $\mu = s_{\varepsilon_1} \cdot (\omega - (2n+1)\varepsilon_1)$, so that $\nu_\mu^c(T_\omega) = \nu_p(2n+1)\chi_\mu(\mu)$. Finally, as $\chi_\mu(\mu) = \operatorname{ch} L_G(\mu)$, the proof is complete. \square

Proceeding as in the proof of Proposition 4.5, we next compute the formal character $\nu_\mu^c(T_\omega)$ in the situation where $\omega = \omega_1 + \omega_j$ for some $1 < j < n$, and $\mu = \omega_{j-1}$. Recall that the case $j = 2$ was dealt with in [McN98, Lemma 4.5.7].

Proposition 4.6

Assume $\omega = \omega_1 + \omega_j$ for some $1 < j < n$, and write $\mu = \omega_{j-1}$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(j+1) \operatorname{ch} L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n) + \nu_p(2n-j+2) \operatorname{ch} L_G(\mu).$$

Proof. We refer the reader to [McN98, Lemma 4.5.7] for a proof in the case where $j = 2$ and hence assume $2 < j < n$ in what follows. Here the dominant T -weights $\nu \in X^+(T)$ such that $\mu \preccurlyeq \nu \prec \omega$ are $\nu_1 = \omega_{j+1} + \delta_{j,n-1}\omega_n$, $\nu_2 = \omega_1 + \omega_{j-2}$, and μ itself. We compute the contribution of each of those weights, starting by determining all pairs $(\beta, r) \in I_{\nu_1}$ as in Theorem 3.8. A straightforward computation yields

$$B_{\nu_1} = \frac{1}{2}(2n+1, 2n-1, \dots, \underbrace{2n-2j+1, 2n-2j-3, \dots}_{(j+1)^{\text{th}} \text{ coord.}}),$$

and since $\omega - \nu_1$ has support $\{\beta_1, \dots, \beta_j\}$, we get that $\beta = \varepsilon_1 - \varepsilon_{j+1}$ by definition of I_{ν_1} . Also, one easily checks that for $r \in \mathbb{Z}$, we have

$$A_{\beta,r} = \frac{1}{2}(2n+3-2r, 2n-1, \dots, \underbrace{2n-2j+3, 2n-2j-1+2r, 2n-2j-3, \dots}_{j^{\text{th}} \text{ coord.}}, 1).$$

Therefore $B_{\nu_1} \in \mathcal{W} \cdot A_{\beta,r}$ if and only if $\{|2(n-r)+3|, |2(n-j+r)-1|\} = \{2n+1, 2(n-j)+1\}$. Again, we deal with each possibility separately.

1. If $2(n-r)+3 = 2n+1$, then $r = 1$ and so $\nu_p(r) = 0$. Hence the weight $\omega - \beta$ cannot contribute to $\nu_\mu^c(T_\omega)$ in this situation.
2. If $2(n-r)+3 = -2n-1$, then $r = 2(n+1)$, so that $2(n-j+r)-1 = 6n-2j+3$. As it is impossible for the latter to be equal to $2(n-j)+1$, we get that $\omega - 2(n+1)\beta$ does not contribute to the truncated sum either.
3. If $2(n-r)+3 = 2(n-j)+1$, then $r = j+1$ and $2(n-j+r)-1 = 2n+1$, in which case one checks that $B_\mu = s_{\varepsilon_1 - \varepsilon_{j+1}} \cdot A_{\beta, j+1}$.
4. If $2(n-r)+3 = -2(n-j)-1$, then $r = 2n-j+2$, in which case $2(n-j+r)-1 = 6n-4j+3 \neq 2n+1$. Therefore $\omega - (2n-j+2)\beta$ does not contribute in this case.

Consequently $I_{\nu_1} = \{(\varepsilon_1 - \varepsilon_{j+1}, j+1)\}$ and hence an application of Theorem 3.8 (where we take $w_{\varepsilon_1 - \varepsilon_{j+1}, j+1} = s_{\varepsilon_1 - \varepsilon_j}$) yields the contribution $\nu_p(j+1)$ for ν_1 to $\nu_\mu^c(T_\omega)$.

Next applying Lemma 2.4 to the B_{n-j+2} -Levi subgroup of G corresponding to the simple roots $\beta_{j-1}, \dots, \beta_n$, we get that ν_2 does not afford the highest weight of a composition factor of $V_G(\omega)$. Therefore ν_2 cannot contribute to $\nu_\mu^c(T_\omega)$ by Lemma 3.7 and it only remains to consider the dominant T -weight μ . Here we have

$$B_\mu = \frac{1}{2}(2n+1, 2n-1, \dots, \underbrace{2(n-j)+5, 2(n-j)+1, 2(n-j)-1, \dots}_{(j-1)^{\text{th}} \text{ coord.}}, 3, 1),$$

and since $\omega - \mu$ has support Π , we get that $\beta \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n\}$. Also for $r \in \mathbb{Z}$, we have

$$A_{\varepsilon_1, r} = B_\mu + (1 - r, \underbrace{0, \dots, 0}_{j-2}, 1, 0, \dots, 0).$$

Hence $|(A_{\varepsilon_1, r})_j|$ is distinct from $(B_\mu)_t$ for all $1 \leq t \leq n$, showing that $\mu \notin \mathcal{W} \cdot (\omega - \mathbb{Z}\varepsilon_1)$. Arguing in a similar fashion, one checks that if $\beta = \varepsilon_1 + \varepsilon_\ell$ for some $2 \leq \ell \leq n$ different from j , then $\mu \notin \mathcal{W} \cdot (\omega - \beta\mathbb{Z})$. Finally, consider $\beta = \varepsilon_1 + \varepsilon_j$, in which case

$$A_{\beta, r} = B_\mu + (1 - r, \underbrace{0, \dots, 0}_{j-2}, 1 - r, 0, \dots, 0), \quad r \in \mathbb{Z}.$$

Therefore $B_\mu \in \mathcal{W} \cdot A_{\beta, r}$ if and only if $\{|2(n-r)+3|, |2(n-j-r)+3|\} = \{2n+1, 2(n-j)+1\}$. Now clearly $2(n-r)+3 \neq 2n+1$, for it would force $r=1$. Also if $2(n-r)+3 = -2n-1$, then $r = 2(n+1)$, in which case $|2(n-j-r)+3| = 2(n+j)+1 \neq 2(n-j)+1$. If on the other hand $2(n-r)+3 = 2(n-j)+1$, then $r = j+1$, and again one checks that $|2(n-j-r)+3| \neq 2n+1$ in this situation. Finally, arguing in a similar fashion shows that $B_\mu \in \mathcal{W} \cdot A_{\beta, r}$ if and only if $r = 2n-j+2$, in which case $\mu = (s_{\varepsilon_1} s_{\varepsilon_j} s_{\varepsilon_1 - \varepsilon_j}) \cdot (\omega - r\beta)$. Consequently

$$\nu_\mu^c(T_\omega) = \nu_p(j+1)\chi_\mu(\omega_{j+1} + \delta_{j, n-1}\omega_n) + \nu_p(2n-j+2)\chi_\mu(\mu).$$

In order to conclude, first observe that $\chi_\mu(\mu) = \text{ch } L_G(\mu)$ by definition, while Lemma 2.4 yields $\chi_\mu(\omega_{j+1} + \delta_{j, n-1}\omega_n) = \text{ch } L_G(\omega_{j+1} + \delta_{j, n-1}\omega_n)$. The proof is complete. \square

Proceeding as in the proofs of Proposition 4.5, we next compute the formal character $\nu_\mu^c(T_\omega)$ in the situation where $\omega = \omega_1 + \omega_n$ and $\mu = \omega_n$.

Proposition 4.7

Assume $\omega = \omega_1 + \omega_n$, and write $\mu = \omega_n$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(2n+1) \text{ch } L_G(\mu).$$

Proof. We proceed as in the proof of Propositions 4.5 and 4.6. Here $\Lambda^+(\omega) = \{\omega, \mu\}$, thus we only need to compute the contribution of all pairs $(\beta, r) \in I_\mu$ as in Theorem 3.8. We have

$$B_\mu = (n, n-1, \dots, 2, 1),$$

and since $\omega - \mu = \beta_1 + \dots + \beta_n$ has support Π , we get that $\beta \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n\}$ by definition of I_μ . Let then $2 \leq \ell \leq n$ be fixed and assume $\beta = \varepsilon_1 + \varepsilon_\ell$, in which case one checks that we have

$$A_{\beta, r} = (n+1-r, n-1, n-2, \dots, 2, 1) + (\underbrace{0, \dots, 0}_{\ell-1}, -r, 0, \dots, 0), \quad r \in \mathbb{Z}.$$

Consequently $B_\mu \in \mathcal{W} \cdot A_{\beta, r}$ if and only if $\{|n+1-r|, |n+1-\ell-r|\} = \{n, n+1-\ell\}$. Proceeding exactly as in the proofs of Propositions 4.5 and 4.6, one easily shows that the latter set equality never holds. Therefore $\mu \notin \mathcal{W} \cdot (\omega - r\beta)$ for $r \in \mathbb{Z}$ in this situation. Finally, we also leave to the reader to check that $B_\mu \in \mathcal{W} \cdot A_{\varepsilon_1, r}$ if and only if $r = 2n+1$, in which case $\mu = s_{\varepsilon_1} \cdot (\omega - r\varepsilon_1)$. Hence $\nu_\mu^c(T_\omega) = \nu_p(2n+1)\chi(\mu)$, and since $V_G(\mu)$ is irreducible by Lemma 2.4, we get that $\chi(\mu) = \text{ch } L_G(\mu)$, from which the result follows. \square

We conclude this section by computing the character $\nu_\mu^c(T_\omega)$ in the situation where $\omega = \omega_1 + 2\omega_n$ and $\mu = \omega_{n-1}$. Again, we proceed as in the proofs of Propositions 4.5, 4.6, and 4.7.

Proposition 4.8

Assume $\omega = \omega_1 + 2\omega_n$, and write $\mu = \omega_{n-1}$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(n+1) \operatorname{ch} L_G(2\omega_n) + \nu_p(n+2) \operatorname{ch} L_G(\mu).$$

Proof. We proceed as usual, starting by observing that the dominant weights $\nu \in X^+(T)$ satisfying $\mu \preccurlyeq \nu \prec \omega$ are $\omega - \beta_n$, and $\nu_1 = 2\omega_n$. Since $\omega - \beta_n$ has multiplicity 1 in $V_G(\omega)$, it cannot afford the highest weight of a composition factor of $V_G(\omega)$ by [Pre87], and hence does not contribute to $\nu_\mu^c(T_\omega)$ by Lemma 3.7. We next compute the contribution of all pairs $(\beta, r) \in I_{\nu_1}$ as in Theorem 3.8. Here

$$B_{\nu_1} = \frac{1}{2}(2n+1, 2n-1, \dots, 3),$$

and since $\omega - \nu_1 = \beta_1 + \dots + \beta_n$ has support Π , we get that $\beta \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n\}$ by definition of I_{ν_1} . Now for any pair (β, r) in I_{ν_1} , we have

$$A_{\beta,r} = \frac{1}{2}(2n+3, 2n-1, 2n-3, \dots, 3) - r\beta.$$

One checks that B_{ν_1} and $A_{\beta,r}$ are \mathscr{W} -conjugate for the dot action if and only if $\beta = \varepsilon_1$ and $r = 2(n+1)$, in which case $\nu_1 = s_{\varepsilon_1} \cdot (\omega - 2(n+1)\beta)$. Hence the T -weight ν_1 contributes by $\nu_p(n+1)$ to $\nu_\mu^c(T_\omega)$ by Theorem 3.8. We now compute the contribution of all pairs $(\beta, r) \in I_\mu$ as in Theorem 3.8. Here we have

$$B_\mu = \frac{1}{2}(2n+1, 2n-1, \dots, 5, 1).$$

Again, since $\omega - \mu$ has support Π , we get that $\beta \in \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_n\}$ by definition of I_μ . Now for any pair (β, r) in I_μ , we have

$$A_{\beta,r} = \frac{1}{2}(2n+3, 2n-1, 2n-3, \dots, 3) - r\beta,$$

and hence if B_μ and $A_{\beta,r}$ are \mathscr{W} -conjugate for the dot action, then $\beta = \varepsilon_1 + \varepsilon_n$, which we shall assume holds in what follows. One then checks that $B_\mu \in \mathscr{W} \cdot A_{\beta,r}$ if and only if $r = n+2$, and that $\mu = (s_{\varepsilon_1} s_{\varepsilon_n} s_{\varepsilon_1 - \varepsilon_n}) \cdot (\omega - (n+2)\beta)$ in this situation. In particular, we get that μ contributes by $\nu_p(n+2)$ to $\nu_\mu^c(T_\omega)$ by Theorem 3.8. Therefore $\nu_\mu^c(T_\omega) = \nu_p(n+1)\chi_\mu(\nu_1) + \nu_p(n+2)\chi_\mu(\mu)$, and since $\operatorname{m}_{V_G(\nu_1)}(\mu) = 1$, an application of [Pre87] yields $\chi_\mu(\nu_1) = \operatorname{ch} L_G(\nu_1)$. Also we clearly have $\chi_\mu(\mu) = \operatorname{ch} L_G(\mu)$, thus completing the proof. \square

4.4. Conclusion. We conclude this section by giving proofs of Theorem 1, Corollary 2, and Proposition 3, in the situation where G is of type B_n ($n \geq 2$) and Y is of type A_{2n} .

Proof of Theorem 1 (The B_n -case). We deal with each row of Table 1 separately, starting with the case where $\omega = 2\omega_1$. Here by Lemma 4.3, we have $\operatorname{ch} T(\omega) = \chi(\omega) + \operatorname{ch} L_G(0)$, and an application of Proposition 3.2 yields $[V_G(\omega), L_G(\mu)] = 0$ for $\mu \in X^+(T)$ different from ω or 0. Moreover, the same proposition also shows that if 0 affords the highest weight of a composition factor of $V_G(\omega)$, then $[V_G(\omega), L_G(0)] = 1$. Now

$$\nu_0^c(T_\omega) = \nu_p(2n+1) \operatorname{ch} L_G(0)$$

by Proposition 4.5, and so the zero weight affords the highest weight of a composition factor if and only if p divides $2n+1$ by Proposition 3.6. The desired assertion thus holds in this situation.

We next consider the case where $\omega = \omega_1 + \omega_j$ for some $1 < j < n$. Here again, an application of Lemma 4.3 yields $\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_{j-1}) + \chi(\omega_{j+1} + \delta_{j,n-1}\omega_n)$, so that $[V_G(\omega), L_G(\mu)] = 0$ for $\mu \in X^+(T)$ different from ω , ω_{j-1} , or $\omega_{j+1} + \delta_{j,n-1}\omega_n$, by Proposition 3.2. The latter also shows that if $\mu \in \{\omega_{j-1}, \omega_{j+1} + \delta_{j,n-1}\omega_n\}$ affords the highest weight of a composition factor of $V_G(\omega)$, then $[V_G(\omega), L_G(\mu)] = 1$. Applying Proposition 4.6 yields

$$\nu_{\omega_{j-1}}^c(T_\omega) = \nu_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n) + \nu_p(2n-j+2) \text{ch } L_G(\omega_{j-1}).$$

Arguing as in the previous case, one gets that $V_G(\omega) = \omega/(\omega_{j+1} + \delta_{j,n-1}\omega_n)^{\epsilon_p(j+1)}/\omega_{j-1}^{\epsilon_p(2n-j+2)}$ by Proposition 3.6. Finally, both $V_G(\omega_{j+1} + \delta_{j,n-1}\omega_n)$ and $V_G(\omega_{j-1})$ are irreducible by Lemma 2.4, and hence $\text{Ext}_G^1(L_G(\omega_{j-1}), L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n)) = 0$ by Proposition 2.10. The desired result thus holds in this case as well.

Next consider the dominant T -weight $\omega = \omega_1 + \omega_n$. Here $\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_n)$ by Lemma 4.3, so that $[V_G(\omega), L_G(\mu)] = 0$ for $\mu \in X^+(T)$ different from ω or ω_n by Proposition 3.2. The latter also shows that if ω_n affords the highest weight of a composition factor of $V_G(\omega)$, then $[V_G(\omega), L_G(\omega_n)] = 1$. Applying Proposition 4.7 yields

$$\nu_{\omega_n}^c(T_\omega) = \nu_p(2n+1) \text{ch } L_G(\omega_n),$$

and so ω_n affords the highest weight of a composition factor of $V_G(\omega)$ if and only if p divides $2n+1$ by Proposition 3.6. Hence the assertion holds as desired.

Finally, we consider the situation where $\omega = \omega_1 + 2\omega_n$. Here an application of Lemma 4.4 yields $\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_{n-1}) + \chi(2\omega_n)$, while $[V_G(\omega), L_G(\mu)] = 0$ for $\mu \in X^+(T)$ different from ω , ω_{n-1} , or $2\omega_n$ by Proposition 4.6. The latter also shows that if $\mu = 2\omega_n$ affords the highest weight of a composition factor of $V_G(\omega)$, then $[V_G(\omega), L_G(\mu)] = 1$. Applying Proposition 4.8 yields

$$\nu_{2\omega_n}^c(T_\omega) = \nu_p(n+1) \text{ch } L_G(\omega_{n-1}) + \nu_p(n+2) \text{ch } L_G(2\omega_n).$$

Using Proposition 3.6, we then get that $V_G(\omega) = \omega/(\omega_{j+1} + \delta_{j,n-1}\omega_n)^{\epsilon_p(j+1)}/\omega_{j-1}^{\epsilon_p(2n-j+2)}$. Here again, a suitable application of Proposition 2.10 allows us to conclude, thus completing the proof. \square

The proof of Corollary 2 in the B_n -case simply consists in computing the dimension of $V_G(\omega)$ and $\text{rad}(\omega)$, this for each ω appearing in the first column of Table 1. The details are thus left to the reader. We conclude with a proof of Proposition 3 in the case where G is of type B_n .

Proof of Proposition 3 (The B_n -case). We give a proof in the situation where $1 \leq j \leq n-1$, and omit the other cases, as they can be dealt with in a similar fashion. First observe that by Lemma 3.3, we have

$$\text{ch } V|_G = \text{ch } V_Y(\lambda)|_G - \epsilon_p(j+1) \text{ch } L_Y(\lambda_{j+1})|_G.$$

Also, an application of Proposition 4.2 yields $\text{ch } V_Y(\lambda)|_G = \chi(\omega) + \text{ch } L_G(\omega_{j-1})$, while by [Sei87, Theorem 1, Table 1 (I_2, I_3)], we have $\text{ch } L_Y(\lambda_{j+1})|_G = \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n)$. Therefore

$$\text{ch } V|_G = \chi(\omega) + \text{ch } L_G(\omega_{j-1}) - \epsilon_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n).$$

Now $\chi(\omega) = \text{ch } L_G(\omega) + \epsilon_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n) + \epsilon_p(2n-j+2) \text{ch } L_G(\omega_{j-1})$ by Theorem 1, and so the result follows. \square

5. THE D_n -CASE ($n \geq 4$)

Let K be an algebraically closed field of characteristic $p \neq 2$, and let Y a simply connected simple algebraic group of type A_{2n-1} ($n \geq 3$) over K . Consider a subgroup G of type D_n , embedded in the usual way, as the stabilizer of a non-degenerate quadratic form on the natural module for Y . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_{2n-1}\}$ denote a corresponding base of the root system $\Phi(Y) = \Phi^+(Y) \sqcup \Phi^-(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_{2n-1}\}$ be the set of fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. Also set $T = T_Y \cap G$, $B = B_Y \cap G$, so that T is a maximal torus of G , and B is a Borel subgroup of G containing T . Let $\Pi(G) = \{\beta_1, \dots, \beta_n\}$ be the corresponding base for the root system $\Phi(G) = \Phi^+(G) \sqcup \Phi^-(G)$ of G , and let $\omega_1, \dots, \omega_n$ denote the associated fundamental weights. The A_{n-1} -parabolic subgroup of G corresponding to the simple roots $\{\beta_1, \dots, \beta_{n-1}\}$ embeds in an $A_{n-1} \times A_{n-1}$ -parabolic subgroup of Y , and up to conjugacy, we may assume that this gives $\alpha_i|_T = \alpha_{2n-1-i}|_T = \beta_i$ for $1 \leq i \leq n-1$. By considering the action of the Levi factors of these parabolics on the natural KY -module $L_Y(\lambda_1)$, we can deduce that $\alpha_n|_T = \beta_n - \beta_{n-1}$. Finally, using [Hum78, Table 1, p.69] and the fact that $\lambda_1|_T = \omega_1$ yields

$$\lambda_i|_T = \lambda_{2n-i}|_T = \omega_i, \quad \lambda_{n-1}|_T = \lambda_{n+1}|_T = \omega_{n-1} + \omega_n, \quad \lambda_n|_T = 2\omega_n \quad \text{for } 1 \leq i \leq n-2. \quad (8)$$

In this section, we show how to obtain the tables 2, 4, and 6 in Theorem 1, Corollary 2, and Proposition 3, respectively. We proceed as in Section 4, relying as much as possible on the embedding of G in Y described above.

5.1. Restriction to G of certain Weyl modules for Y . Let V be a KY -module with unique highest weight $\lambda \in X^+(T_Y)$. We start by proving a result similar to Lemma 4.1. Contrary to what we had in the latter result, the T -weight $\lambda|_T$ is not necessarily the unique highest weight of $V|_G$. (For example, if $\langle \lambda, \alpha_n \rangle \neq 0$, then the weight $\lambda - \alpha_n$ restricts to $\lambda|_T + \beta_{n-1} - \beta_n$, which is neither under nor above $\lambda|_T$.) Notice that according to the restrictions to T of the simple roots for T_Y stated above, we have that the restriction to T of a given T_Y -weight $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r$ is given by

$$\mu|_T = \lambda|_T - \sum_{r=1}^{n-2} (c_r + c_{2n-r})\beta_r - (c_{n-1} - c_n + c_{n+1})\beta_{n-1} - c_n\beta_n. \quad (9)$$

Lemma 5.1

Let $\lambda \in X^+(T_Y)$ be a dominant weight, and let V be a KY -module with unique highest weight λ . Then the following assertions hold.

1. If $\langle \lambda, \alpha_n \rangle = 0$, then every T -weight ξ of V satisfies $\xi \preceq \lambda|_T$.
2. If $\langle \lambda, \alpha_n \rangle = 1$, then each of $\lambda|_T$ and $(\lambda - \alpha_n)|_T$ affords the highest weight of a KG -composition factor of V . Furthermore, every T -weight ξ of V either satisfies $\xi \preceq \lambda|_T$ or $\xi \preceq (\lambda - \alpha_n)|_T$.

Proof. Write $\omega = \lambda$ and $\omega' = (\lambda - \alpha_n)|_T$. First suppose that $\langle \lambda, \alpha_n \rangle = 0$ and, seeking a contradiction, assume the existence of non-negative integers $c_1, \dots, c_{2n-1} \in \mathbb{Z}_{\geq 0}$ such that $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r$ satisfies $\xi = \mu|_T \not\preceq \omega$. By (9), we get that $c_n > c_{n-1} + c_{n+1}$. In particular, we get that $\langle \mu, \alpha_n \rangle < -c_n$, from which one deduces that $s_{\alpha_n}(\mu)$ is a T_Y -weight of V that is not under λ , a contradiction. Therefore 1 holds as desired, and we assume $\langle \lambda, \alpha_n \rangle = 1$ in the remainder of the proof.

We first show that ω and ω' are highest weights of $V|_G$, and hence each of them affords the highest weight of a KG -composition factor of V . Let $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r$ be a T_Y -weight of V such that $\omega' \preceq \mu|_T$. By (9), we get $c_r = 0$ for every $1 \leq r \leq 2n-1$ different from $n-1, n, n+1$, as well as $c_n = 1$, and hence $c_{n-1} + c_{n+1} = 0$. Therefore $c_{n-1} = c_{n+1} = 0$ and thus $\mu|_T = \omega'$ as desired. In order to prove the last assertion, assume for a contradiction the existence of a T_Y -weight $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r$ of V such that neither $\mu|_T \preceq \omega$ nor $\mu|_T \preceq \omega'$. By (9) again, we have $c_{n-1} - c_n + c_{n+1} < -1$. In particular $\langle \mu, \alpha_n \rangle < -c_n$, showing that $s_{\alpha_n}(\mu)$ is a T_Y -weight of V that is not under λ , a contradiction. \square

Corollary 5.2

Let $\lambda = \lambda_n \in X^+(T_Y)$, and consider the Weyl module $V_Y(\lambda)$ having highest weight λ . Then the formal character of the restriction $V_Y(\lambda)$ to G is given by

$$\text{ch } V_G(\lambda)|_G = \chi(2\omega_{n-1}) + \chi(2\omega_n).$$

Proof. As usual, we assume K has characteristic zero. Now by part 2 of Lemma 5.1, each of $\omega = 2\omega_n$ and $\omega' = 2\omega_{n-1}$ affords the highest weight of a composition factor of $V_Y(\lambda)|_G$. An application of Tables 7 and 9 then yields $\dim V_Y(\lambda) = \dim V_G(2\omega_{n-1}) + \dim V_G(2\omega_n)$, thus completing the proof. \square

We next investigate the formal character of the restriction to G of the Weyl module $V_Y(\lambda_1 + \lambda_j)$, where $1 \leq j \leq 2n-1$.

Proposition 5.3

Let $1 \leq j \leq 2n-1$ and consider the dominant T_Y -weight $\lambda = \lambda_1 + \lambda_j \in X^+(T_Y)$. Also set $\omega = \lambda|_T$ and adopt the notation $\omega_0 = 0$. Then

$$\text{ch } V_Y(\lambda)|_G = \begin{cases} \chi(\omega) + \chi(\omega_{j-1}) & \text{if } 1 \leq j \leq n-1; \\ \chi(\omega) + \chi(\omega_1 + 2\omega_{n-1}) + \chi(\omega_{n-1} + \omega_n) & \text{if } j = n; \\ \chi(\omega) + \chi(2\omega_{n-1}) + \chi(2\omega_n) & \text{if } j = n+1; \\ \chi(\omega) + \chi(\omega_{n-1} + \omega_n) & \text{if } j = n+2; \\ \chi(\omega) + \chi(\omega_{2n-j+1}) & \text{otherwise.} \end{cases}$$

Proof. Write $V = V_Y(\lambda)$ and first observe that $\text{ch } V|_G$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. In the case where $1 \leq j \leq n-1$, an application of [KT87, Proposition 1.5.3] yields $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_{j-1})$, from which the result follows.

Next assume $j = n$, and write $\omega' = \omega_1 + 2\omega_{n-1}$, $\omega'' = 2\omega_{n-1} + \omega_n$. By Part 2 of Lemma 5.1, each of ω and ω' affords the highest weight of a composition factor of $V|_G$. Observe that the only T_Y -weights restricting to $\omega'' = \omega_{n-1} + \omega_n$ are $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_n + \cdots + \alpha_{2n-r-1})$ (for $1 \leq r \leq n-1$) and $\lambda - (\alpha_n + \cdots + \alpha_{2n-1})$. An application of Lemma 3.3 then yields $m_{V|_G}(\omega'') = 2n-1$ as well as $m_{V_G(\omega)}(\omega'') = m_{V_G(\omega')}(\omega'') = n-1$, thus showing that ω'' occurs in a third KG -composition factor of V . Now one easily checks that every T -weight $\nu \in \Lambda^+(V|_G)$ such that $\omega'' \prec \nu \prec \omega$ or $\omega'' \prec \nu \prec \omega'$ satisfies $m_{V|_G}(\nu) = m_{V_G(\omega)}(\nu) + m_{V_G(\omega')}(\nu)$, showing that ω'' affords the highest weight of a third KG -composition factor of V by Part 2 of Lemma 5.1. As in the proof of Proposition 4.2, an application of Theorem 2.2 then yields the desired result.

Next suppose that $j = n+1$ and consider the dominant T -weight $\omega' = \omega - (\beta_1 + \cdots + \beta_{n-1}) \in X^+(T)$. Then the T_Y -weights restricting to ω' are $\lambda - (\alpha_1 + \cdots + \alpha_{n-1})$, $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_{n+1} + \cdots + \alpha_{2n-r-1})$ ($1 \leq r < n-1$), and $\lambda - (\alpha_{n+1} + \cdots + \alpha_{2n-1})$. Therefore $m_{V|_G}(\omega') = n$, while $m_{V_G(\omega)}(\omega') = n-1$ by Lemma 3.3, showing that ω' occurs in a second KG -composition factor of V . Since there is no dominant weight $\nu \in X^+(\omega)$ such that $\omega' \prec \nu \prec \omega$, an application of Lemma 5.1 yields $[V|_G, L_G(\omega')] = 1$ as desired. We leave to the reader to show that $[V|_G, L_G(2\omega_{n-1})] = 1$ as well, from which one easily concludes thanks to Theorem 2.2, for example.

Next consider $n+1 < j \leq 2n-1$ and let $\omega' = \omega - (\beta_1 + \cdots + \beta_{2n-j}) \in X^+(T)$. Then one easily checks that the T_Y -weights restricting to ω' are $\lambda - (\alpha_j + \cdots + \alpha_{2n-1})$, $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_{2n-r-1})$ ($1 \leq r \leq 2n-j-1$), and $\lambda - (\alpha_1 + \cdots + \alpha_{2n-j})$. Therefore $m_{V|_G}(\omega') = 2n-j+1$, while an application of Lemma 3.3 yields $m_{V_G(\omega)}(\omega') = 2n-j$, showing that ω' occurs in a second KG -composition factor of V . As above, there is no dominant weight $\nu \in X^+(\omega)$ such that $\omega' \prec \nu \prec \omega$ and thus $[V|_G, L_G(\omega')] = 1$ by Lemma 5.1. Again, applying Theorem 2.2 completes the proof. \square

5.2. Formal character of certain tensor products. We next determine the formal character of the tensor product $V_G(\omega_1) \otimes V_G(\omega_j + \delta_{j,n}\omega_n)$ for $1 \leq j \leq n$, as well as of the formal character of the tensor product $V_G(\omega_1) \otimes V_G(\omega_{n-1} + \omega_n)$. Observe that since $p \neq 2$, each of the Weyl modules considered is irreducible, and hence is tilting.

Lemma 5.4

Let $1 \leq j \leq n-1$, and consider the dominant T -weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n \in X^+(T)$. Also set $\omega_0 = \omega_{n+1} = 0$ and write $T(\omega)$ for the tensor product $V_G(\omega_1) \otimes V_G(\omega_j + \delta_{j,n-1}\omega_n)$. Then $T(\omega)$ is tilting and its formal character is given by

$$\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_{j-1}) + (1 - \delta_{j,n-1})\chi(\omega_{j+1} + \delta_{j,n-2}\omega_n) + \delta_{j,n-1}(\chi(2\omega_{n-1}) + \chi(2\omega_n)).$$

Proof. By Lemma 2.6, both $V_G(\omega_1)$ and $V_G(\omega_j + \delta_{j,n-1}\omega_n)$ are irreducible KG -modules, and hence $T(\omega)$ is tilting by Proposition 2.9. Also $\text{ch } T(\omega)$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. By [Sei87, Theorem 1, Table 1 (I_4, I_5)] together with Lemma 3.3, we successively get

$$T(\omega) \cong (V_Y(\lambda_1) \otimes V_Y(\lambda_j))|_G \cong V_Y(\lambda)|_G \oplus V_Y(\lambda_{j+1})|_G.$$

Now if $1 \leq j \leq n-2$, then applying (6) yields the restrictions $\lambda|_T = \omega$ and $\lambda_{j+1}|_T = \omega_{j+1} + \delta_{j,n-2}\omega_n$. Therefore Proposition 5.3 yields $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_{j-1})$, while $V_Y(\lambda_{j+1})|_G \cong V_G(\omega_{j+1} + \delta_{j,n-2}\omega_n)$ by [Sei87, Theorem 1, Table 1 (I_4, I_5)]. The assertion thus holds in this situation and so it remains to consider the case where $j = n-1$. Here we have $\lambda|_T = \omega$, $\lambda_n|_T = 2\omega_n$, and applying Corollary 5.2 and Proposition 5.3 yields $T(\omega) \cong V_G(\omega) \oplus V_G(\omega_{n-1} + \omega_n) \oplus V_G(2\omega_{n-1}) \oplus V_G(2\omega_n)$ as desired. The proof is complete. \square

Lemma 5.5

Consider the dominant T -weight $\omega = \omega_1 + \omega_{n-1} \in X^+(T)$, and write $T(\omega)$ for the tensor product $V_G(\omega_1) \otimes V_G(\omega_{n-1})$. Then $T(\omega)$ is tilting and its formal character is given by

$$\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_n).$$

Proof. By Lemma 2.6, both $V_G(\omega_1)$ and $V_G(\omega_{n-1})$ are irreducible, thus showing that $T(\omega)$ is tilted by Proposition 2.9. Also $\text{ch } T(\omega)$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. Applying Theorem 2.2 and Table 9 then yields

$$\dim T(\omega) = \dim V_G(\omega) + 2^{n-1} > \dim V_G(\omega),$$

showing the existence of a second composition factor of $T(\omega)$. Now $\Lambda^+(T(\omega)) = \{\omega, \omega_n\}$, and since $m_{T(\omega)}(\omega) = m_{V_G(\omega)}(\omega) = 1$, we deduce that $[T(\omega), V_G(\omega_n)] > 0$. Finally, an application of Table 9 yields $\dim V_G(\omega_n) = 2^{n-1}$, thus completing the proof. \square

Lemma 5.6

Consider the dominant T -weight $\omega = \omega_1 + 2\omega_n \in X^+(T)$, and write $T(\omega)$ for the tensor product $V_G(\omega_1) \otimes V_G(2\omega_n)$. Then $T(\omega)$ is tilting and its formal character is given by

$$\text{ch } T(\omega) = \chi(\omega) + \chi(\omega_{n-1} + \omega_n).$$

Proof. By Lemma 2.6, both $V_G(\omega_1)$ and $V_G(2\omega_n)$ are irreducible KG -modules, and hence $T(\omega)$ is tilting by Proposition 2.9. Also $\text{ch } T(\omega)$ is independent of p , so we may and shall assume K has characteristic zero in the remainder of the proof. Observe that by Corollary 5.2, we have

$$(V_Y(\lambda_1) \otimes V_Y(\lambda_n))|_G \cong T(\omega) \oplus V_G(\omega_1) \otimes V_G(2\omega_{n-1}),$$

while on the other hand, Lemma 3.3 yields $(V_Y(\lambda_1) \otimes V_Y(\lambda_n))|_G \cong V_Y(\lambda)|_G \oplus V_Y(\lambda_{n+1})|_G$. Now Proposition 5.3 yields $V_Y(\lambda)|_G \cong V_G(\omega) \oplus V_G(\omega_1 + 2\omega_{n-1}) + V_G(\omega_{n-1} + \omega_n)$, while applying [Sei87, Theorem 1, Table 1 (I₅)] gives $V_Y(\lambda_{n+1})|_G \cong V_G(\omega_{n+1} + \omega_n)$. Consequently we have

$$T(\omega) \oplus V_G(\omega_1) \otimes V_G(2\omega_{n-1}) \cong V_G(\omega) \oplus V_G(\omega_1 + 2\omega_{n-1}) \oplus V_G(\omega_{n-1} + \omega_n)^2.$$

Clearly $\dim T(\omega) = \dim V_G(\omega_1) \otimes V_G(2\omega_{n-1})$, as ω_n and ω_{n-1} are conjugate under the action of the graph automorphism of order 2 of G , from which one deduces the desired result. \square

5.3. Various contributions to the truncated Jantzen p -sum formula. In this section, we compute certain contributions to the truncated Jantzen formula for some Weyl modules $V_G(\omega)$, where ω is as in the first column of Table 2, starting by the case where $\omega = 2\omega_1$.

Proposition 5.7

Assume $\omega = 2\omega_1$, and write $\mu = 0$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(n) \text{ch } L_G(\mu).$$

Proof. We proceed as in the proof of Proposition 4.5, starting by computing all contributions to $\nu_\mu^c(T_\omega)$. Here the only dominant T -weights $\nu \in X^+(T)$ such that $\mu \preccurlyeq \nu \prec \omega$ are $\omega - \beta_1$ and μ itself. By [Pre87], the former cannot afford the highest weight of a composition factor of $V_G(\omega)$ and so Lemma 3.7 shows that $\omega - \beta_1$ cannot contribute to $\nu_\mu^c(T_\omega)$. We now determine all pairs $(\beta, r) \in I_\mu$ as in Theorem 3.8. A straightforward computation yields

$$B_\mu = (n - 1, \dots, 1, 0),$$

and since $\omega - \mu$ has support Π , we get that $\beta \in \{\varepsilon_1 + \varepsilon_\ell : 2 \leq \ell \leq n\}$ by definition of I_μ . Recall from [Bou68, Planche II] that \mathscr{W} acts by all permutations and even number of sign changes of the ε_i . Also, one checks that for $2 \leq \ell \leq n$, we have

$$A_{\varepsilon_1 + \varepsilon_\ell, r} = B_\mu + (2 - r, \underbrace{0, \dots, 0}_{\ell-2}, -r, \underbrace{0, \dots, 0}_{n-\ell}), \quad r \in \mathbb{Z}.$$

Consequently $B_\mu \in \mathscr{W} \cdot A_{\varepsilon_1 + \varepsilon_\ell, r}$ if and only if $\{|n+1-r|, |n-\ell-r|\} = \{n-1, n-\ell\}$. We now study each possibility separately.

1. If $n+1-r = n-1$, then $r = 2$ and hence $\nu_p(r) = 0$, since $p \neq 2$ by assumption. Hence the weight $\omega - 2(\varepsilon_1 + \varepsilon_\ell)$ cannot contribute to $\nu_\mu^c(T_\omega)$ in this situation.
2. If $n+1-r = n-\ell$, then $r = \ell+1$, and $|n-\ell-r| = |n-2\ell-1|$. One checks that the latter is equal to $n-\ell$ if and only if $\ell = n-1$, in which case $r = n$.
3. If $n+1-r = -n+\ell$, then $r = 2n-\ell+1$, and $|n-\ell-r| = n+1 \neq n-1$.
4. If $n+1-r = -n+1$, then $r = 2n$, and $|n-\ell-r| = n+\ell \neq n-\ell$.

Consequently, the only possible contribution to $\nu_\mu^c(T_\omega)$ can only come from the situation where $\ell = n-1$ and $r = n$, in which case we have

$$A_{\varepsilon_1 + \varepsilon_{n-1}, n} = (1, n-2, \dots, 2, 1-n, 0).$$

Using the action of \mathscr{W} described above, one deduces that $\mu = s_{\varepsilon_1 + \varepsilon_n} s_{\varepsilon_1 - \varepsilon_n} s_{\varepsilon_1 - \varepsilon_{n-1}} \cdot (\omega - r(\varepsilon_1 + \varepsilon_{n-1}))$. Therefore $\nu_\mu^c(T_\omega) = \nu_p(n) \chi_\mu(\mu)$, and since $\chi_\mu(\mu) = \text{ch } L_G(\mu)$, the proof is complete. \square

Proceeding as in the proof of Proposition 5.7, we next compute the formal character $\nu_\mu^c(T_\omega)$ in the situation where $\omega = \omega_1 + \omega_j$ for some $1 < j < n-1$, and $\mu = \omega_{j-1}$. Observe that the case $j = 2$ was dealt with in [McN98, Lemma 4.5.7].

Proposition 5.8

Assume $\omega = \omega_1 + \omega_j$ for some $2 \leq j \leq n-2$, and write $\mu = \omega_{j-1}$. Let $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ be a Jantzen filtration of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-2} \omega_n) + \nu_p(2n-j+1) \text{ch } L_G(\mu).$$

Proof. We refer the reader to [McN98, Lemma 4.5.7] for a proof in the case where $j = 2$ and hence assume $j > 2$ in what follows. Write $\nu_1 = \omega_{j+1} + \delta_{j,n-2} \omega_n$. For simplicity, we also assume $j \leq n-3$ (so that $\nu_1 = \omega_{j+1}$) and leave the case where $j = n-2$ to the reader. Here the dominant T -weights $\nu \in X^+(T)$ satisfying $\mu \preceq \nu \prec \omega$ are ν_1 , $\omega_1 + \omega_{j-1}$, and μ . We next determine all pairs $(\beta, r) \in I_{\nu_1}$ as in Theorem 3.8. A straightforward computation yields

$$B_{\nu_1} = (n, n-1, \dots, n-j, n-j-2, \dots, 1, 0),$$

and since $\omega - \nu_1$ has support $\{\beta_1, \dots, \beta_j\}$, we get that $\beta = \varepsilon_1 - \varepsilon_{j+1}$ by definition of I_{ν_1} . Also, one easily checks that for $r \in \mathbb{Z}$, we have

$$A_{r, \varepsilon_1 - \varepsilon_{j+1}} = B_{\nu_1} + (\underbrace{1-r, 0, \dots, 0}_{j-1}, \underbrace{r-1, 0, \dots, 0}_{n-j-1}).$$

We leave it to the reader to show that $B_{\nu_1} \in \mathscr{W} \cdot A_{r, \varepsilon_1 - \varepsilon_{j+1}}$ if and only if $r = j+1$, and then to deduce that μ contributes to $\nu_\mu^c(T_\omega)$ by $\nu_p(j+1) \chi_\mu(\nu_1)$.

Finally, we determine all pairs $(\beta, r) \in I_\mu$ as in Theorem 3.8. Here again, a straightforward computation yields

$$B_\mu = (n, n-1, \dots, n-j+2, n-j, \dots, 1, 0),$$

and since $\omega - \mu$ has support Π , we get that $\beta \in \{\varepsilon_1 + \varepsilon_\ell : 2 \leq \ell \leq n\}$ by definition of I_μ . Also, one easily checks that $B_\mu \notin \mathcal{W} \cdot A_{\varepsilon_1 + \varepsilon_\ell, r}$ if $\ell \neq j$, and hence we assume $\ell = j$ in the remainder of the argument. Setting $\beta = \varepsilon_1 + \varepsilon_j$, we then get

$$A_{\beta, r} = B_\mu + (1-r, \underbrace{0, \dots, 0}_{j-2}, -1-r, \underbrace{0, \dots, 0}_{n-j}), r \in \mathbb{Z}.$$

As usual, a case by case analysis then shows that $B_\mu \in \mathcal{W} \cdot A_{\beta, r}$ if and only if $r = 2n - j + 1$, from which one deduces that μ contributes to $\nu_\mu^c(T_\omega)$ by $\nu_p(2n - j + 1)\chi_\mu(\mu)$. Consequently, we have

$$\nu_\mu^c(T_\omega) = \nu_p(j+1)\chi_\mu(\nu_1) + \nu_p(2n - j + 1)\chi_\mu(\mu).$$

Now $\chi_{\nu_1}(\mu) = \text{ch } L_G(\mu)$ by Lemma 2.6, while $\chi_\mu(\mu) = L_G(\mu)$ by definition, thus completing the proof. \square

The next result is very similar to Proposition 5.8, the difference residing in the apparition of two composition factors having highest weights conjugate under the action of the graph automorphism of order 2 of G . Since the proof is fairly identical, the details are omitted.

Proposition 5.9

Assume $\omega = \omega_1 + \omega_{n-1} + \omega_n$, and write $\mu = \omega_{n-2}$. Also consider a Jantzen filtration $V_G(\omega) = V^0 \supsetneq V^1 \supseteq \dots \supseteq V^k \supseteq 0$ of $V_G(\omega)$. Then

$$\nu_\mu^c(T_\omega) = \nu_p(n) \text{ch } L_G(2\omega_n) + \nu_p(n) \text{ch } L_G(2\omega_{n-1}) + \nu_p(n+2) \text{ch } L_G(\mu).$$

We conclude this section with a result describing the decomposition of $\chi_\mu(\omega)$ in terms of characters of irreducibles for well-chosen $\omega, \mu \in X^+(T)$. Even though the proof of the following proposition does not require the use of the Jantzen p -sum formula, we record it here for completeness.

Proposition 5.10

Let $\xi \in \{1, 2\}$, and let $\omega_\xi = \omega_1 + \xi\omega_n$. Also consider the dominant T -weight $\mu_\xi = \omega_{n-1} + (\xi - 1)\omega_n$. Then

$$\chi_{\mu_\xi}(\omega_\xi) = \text{ch } L_G(\omega_\xi) + \epsilon_p(n) \text{ch } L_G(\mu_\xi).$$

Proof. We give a proof of the Proposition in the situation where $\xi = 1$, and leave the case $\xi = 2$ to the reader. Here the only dominant T -weight $\nu \in X^+(T)$ satisfying $\mu_1 \preceq \nu \prec \omega$ is μ_1 itself. Also, an application of Lemma 3.3 to the Levi subgroup of type A_{n-1} corresponding to the simple roots $\beta_1, \dots, \beta_{n-1}$ shows that $[V_G(\omega), L_G(\mu_1)] = \epsilon_p(n)$, thus completing the proof. \square

5.4. Conclusion. We conclude by giving proofs of Theorem 1, Corollary 2, and Proposition 3, in the situation where G is of type D_n and Y is of type A_{2n+1} ($n \geq 3$).

Proof of Theorem 1 (The D_n -case). We proceed exactly as in the B_n -case, dealing with each row of Table 2 separately. Since the argument is identical to the one given in the proof in the situation where G is of type B_n , we consistently refer the reader to the proof of the latter for more details.

1. If $\omega = 2\omega_1$, then replacing Lemma 4.3 by Lemma 5.4 and Proposition 4.5 by Proposition 5.7 yields the desired assertion.
2. If $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$ for some $1 < j < n$, then replacing Lemma 4.3 by Lemma 5.4, Proposition 4.6 by Proposition 5.8, and Lemma 2.4 by Lemma 2.6 yields the result.
3. If $\omega = \omega_1 + \omega_{n-1}$, then replacing Lemma 4.3 by Lemma 5.5 and Proposition 4.7 by Proposition 5.10 yields the result.
4. If $\omega = \omega_1 + 2\omega_n$, then replacing Lemma 4.4 by Lemma 5.6 and Proposition 4.8 by Proposition 5.10 yields the result.

Therefore the result holds for each dominant T -weight ω as in the first column of Table 2, thus completing the proof. \square

The proof of Corollary 2 in the B_n -case simply consists in computing the dimension of $V_G(\omega)$ and $\text{rad}(\omega)$, this for each ω appearing in the first column of Table 1. The details are thus left to the reader. We conclude with a proof of Proposition 3 in the case where G is of type B_n .

Proof of Proposition 3 (The B_n -case). We give a proof in the situation where $1 \leq j \leq n-1$, and omit the other cases, as they can deal with in a similar fashion. First observe that by Lemma 3.3, we have

$$\text{ch } V|_G = \text{ch } V_Y(\lambda)|_G - \epsilon_p(j+1) \text{ch } L_Y(\lambda_{j+1})|_G.$$

Also, an application of Proposition 4.2 yields $\text{ch } V_Y(\lambda)|_G = \chi(\omega) + \text{ch } L_G(\omega_{j-1})$, while by [Sei87, Theorem 1, Table 1 (I_2, I_3)], we have $\text{ch } L_Y(\lambda_{j+1})|_G = \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n)$. Therefore

$$\text{ch } V|_G = \chi(\omega) + \text{ch } L_G(\omega_{j-1}) - \epsilon_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n).$$

Now $\chi(\omega) = \text{ch } L_G(\omega) + \epsilon_p(j+1) \text{ch } L_G(\omega_{j+1} + \delta_{j,n-1}\omega_n) + \epsilon_p(2n-j+2) \text{ch } L_G(\omega_{j-1})$ by Theorem 1, and so the result follows. \square

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