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# Examples of normal, not geometrically normal, projective Gorenstein del Pezzo surfaces with at most Du Val singularities

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## Abstract

While over fields of characteristic at least 5, a normal, projective and Gorenstein del Pezzo surface is geometrically normal, this does not hold for characteristic 2 and 3. There is no characterization of all such non-geometrically normal surfaces, but there is a complete characterization of all possible surfaces that can arise as the normalization of the base change to the algebraic closure of the base field. In four of these instances, for a given normalization, we describe the construction of such a non-geometrically normal surface.

## Keywords

# Abstract Deutsch

Eine Fläche in einem Körper von Charakteristik grösser als 5, welche die Eigenschaft hat, dass sie normal, projektiv, Gorenstein und del Pezzo ist, ist ebenfalls geometrisch normal. Dies bedeutet, dass für jede Körpererweiterung $k'$ des Ursprungskörpers $k$, und für jeden Punkt $x$ in der Fläche $X$ das folgende gilt: der lokale Ring $\mathscr{O}_{X_{k'},x'}$ ist normal für jedes $x' \in X_{k'}$, welches über $x$ liegt. Diese Tatsache gilt jedoch nicht in Körpern von Charakteristik kleiner oder gleich 3. Es existiert keine Beschreibung aller solcher Flächen, welche nicht geometrisch normal sind. Es gibt andererseits aber eine Beschreibung aller möglichen Flächen, welche in Frage kommen als Normalisierung des Basiswechsels der Ursprungsfläche zum algebraischen Abschluss des Ursprungskörpers. In vier solchen Fällen, mit gegebener Normalisierung haben wir eine zugehörige Fläche konstruiert, welche nicht geometrisch normal ist. Inhalt dieser Arbeit ist die Beschreibung dieser Flächen, inklusive der Methode, die wir zur Konstruktion verwenden.

## Schlüsselwörter

- Birationale Geometrie
- Del Pezzo Flächen
- Du Val Singularitäten
- Geometrisch Nicht-Normale Flächen
- Positive Charakteristik

# Contents

# Chapter 1

# Introduction

The motivation for the construction of an example as mentioned in the abstract comes from the article [PW17].

In general, the study of varieties over imperfect fields is motivated by fibrations of the Minimal Model Program, such as Mori fiber spaces and Iitaka fibrations. While over fields of characteristic zero, the general fiber of a fibration between smooth varieties is smooth, this does not hold in positive characteristic. The reason for this behavior is the fact that the generic fiber is a regular variety which is not necessarily geometrically regular. Over a perfect field, the properties -such as normality, regularity and reducedness- of a general fiber correspond to those of the geometric generic fiber. On the other hand if the above properties hold for the total space, that implies the same only for the generic fiber. The disparity between the two causes the bad behavior of general fibers. However, this bad behavior for terminal Mori fiber spaces of relative dimension $\leq 2$ is restricted to small primes according to [PW17, Theorem 4.1].

The theorem states that for a field $k$ of characteristic $p \geq 5$, a normal, projective and Gorenstein del Pezzo surface $X$ over $k$ is geometrically normal. For smaller characteristics, this does not hold. However, the theorem gives a list of possibilities that arise as the normalization $Y$ of the reduced subscheme of $X_{\overline{k}}$, where $X_{\overline{k}}$ is the base change of $X$ to the algebraic closure of $k$.

**Theorem 1.0.1** ( [PW17, Theorem 4.1])**.** *Let $X$ be a normal, projective and Gorenstein surface over a field $k$ of characteristic $p > 0$, with $k$ algebraically closed in $K(X)$, the function field of $X$, and such that $-K_X$ is ample. If $p > 3$ then $X$ is geometrically normal. Furthermore, if $Y$ is the normalization of the reduced subscheme of $X_{\overline{k}}$, then there is an integral divisor $C$ on $Y$ such that $K_Y + (p-1)C \sim \phi^* K_X$, where $\phi : Y \to X$ is the natural map, and $(p, Y, C)$ is one of the following:*

*(1)* $(3, \mathbb{P}^2, L)$

*(2)* $(3, S_d, F)$ *for* $d \geq 2$

*(3)* $(2, \mathbb{P}^2, L)$

*(4)* $(2, \mathbb{P}^2, C \in |2L|)$

*(5)* $(2, \mathbb{P}^1 \times \mathbb{P}^1, C \in |F_1 + F_2|)$

*(6)* $(2, \mathbb{P}^1 \times \mathbb{P}^1, F_i)$

*(7)* $(2, H_d, C \in |D + F|)$ *for* $d \geq 1$

*(8)* $(2, H_d, D)$ *for* $d \geq 1$

*(9)* $(2, S_d, 2F)$ *for* $d \geq 2$

*(10)* $(2, S_2, C)$, *where* $C_{H_2} \in |D + 2F|$

*where $H_d$ denotes the Hirzebruch surface of degree $d$, defined by $H_d = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d))$, $F$ and $D$ denote its fiber and its exceptional section, and $S_d$ denotes the surface obtained by the contraction of the exceptional section. Further, $L$ is a line and $F_1$ and $F_2$ are fibers in the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. Lastly, for the tenth triple, $C_{H_2}$ denotes the birational transform of $C$.*

*Remark* 1.0.2. If $X$ is geometrically reduced, then the divisor $C$ is equal to the conductor of the normalization $Y \to X_{\overline{k}}$, as remarked after [PW17, Theorem 1.1].

While this theorem does give a list of possible schemes $Y$, and corresponding divisors $C$ that can arise, it does not give any insight on whether such a surface $X$ actually exists in each of these ten cases.

The content of my thesis is the study of three of these cases. These three cases from Theorem 1.0.1 are listed in the table below.

Table 1.1: Cases of Theorem 1.0.1 studied

| Case in Theorem 1.0.1 | $(p, Y, C)$ |
|---|---|
| (6) | $(2, \mathbb{P}^1 \times \mathbb{P}^1, F_i)$ |
| (3) | $(2, \mathbb{P}^2, L)$ |
| (8) | $(2, H_2, D)$ $(2, H_3, D)$ |

For the four triples above, we are able to state that such a corresponding surface $X$ does exist, by constructing an explicit surface.

Examples of this type that have previously been published are listed in Table 1.2. For two of the four triples that we study, all previously known examples are not geometrically reduced, contrary to our examples that are geometrically reduced. Hence those two examples are new. For one of the triples, no previously known examples exist. For the last triple, there exists an example, [Tan19, Example 6.5] , that is geometrically reduced as well.

The first of four such results states that in the case of characteristic 2, with $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and the divisor $C = F_i$, where $F_i$ is the fiber of one of the projections to $\mathbb{P}^1$, such a surface exists.

**Theorem 1.0.3.** *Let $(p, Y, C) = (2, \mathbb{P}^1 \times \mathbb{P}^1, F_i)$. Then there is a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface $X$ has one singular point, which is a Gorenstein $A_2$-singularity.*

This theorem is proved in Chapter 3.

The second main result states that in the case of characteristic 2, with $Y = \mathbb{P}^2$ and the divisor $C = L$, a line, such a surface exists.

**Theorem 1.0.4.** *Let $(p, Y, C) = (2, \mathbb{P}^2, L)$. There exists a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface has one singular point, which is a Gorenstein $A_3$-singularity.*

The proof of this theorem is illustrated in Chapter 4.

The third main result states that in the case of characteristic 2, with $Y$ being a Hirzebruch surface of degree 2, a surface satisfying the properties exists.

**Theorem 1.0.5.** *Let $(p, Y, C) = (2, H_2, D)$. There exists a regular, geometrically reduced, projective surface $X$ that satisfies the properties of Theorem 1.0.1.*

This main theorem is proved in Section 5.4.1.

Lastly, the fourth main theorem states that in the case of characteristic 2, with $Y$ being a Hirzebruch surface of degree 3, a surface satisfying the properties exists.

**Theorem 1.0.6.** *Let $(p, Y, C) = (2, H_3, D)$. There exists a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface has two singular points.*

The proof of this theorem is found in Section 5.5.2.

In the following chapters, we will describe the construction of these surfaces and prove their properties.

## 1.1   Related Results

Del Pezzo surfaces with mild singularities over imperfect fields have been studied previously both by [Sch07] and by [BT20]. Additionally, [Tan20] studies invariants of algebraic varieties over

imperfect fields, measuring geometric non-normality or geometric non-reducedness. These results are then applied to curves over imperfect fields.

A classification similar to Theorem 1.0.1 has been established in [FS20]. Over fields of $p$-degree one, the authors eliminate some possibilities of surfaces of Picard rank one appearing in Theorem 1.0.1. Further results such as improving Theorem 1.0.1 and finding examples of surfaces corresponding to Theorem 1.0.1 include the following results by [JW19]. The main theorem, [JW19, Theorem 1.1] exhibits a fixed part and a movable part of a canonically determined linear system $\mathfrak{C}$ of Weil divisors, for which

$$K_Y + (p-1)\mathfrak{C} \sim \varphi^* K_X$$

holds, with $\varphi : Y \to X$. Here, $X/K$ denotes a projective, normal variety, $L$ denotes a field lying between $K$ and $K^{1/p}$ and $Y$ is the normalization of $(X \otimes_K L)_{red}$.

As a consequence of this theorem, in the case of a regular surface $X$, with $X$ being geometrically non-reduced over $K$, the authors restrict Theorem 1.0.1 in the following sense:

**Corollary 1.1.1** ( [JW19, Corollary 1.6]). *Let $X$ be a regular surface, which is geometrically non-reduced over $K$. Then $(Y, (p-1)C)$ cannot be of the type $(H_d, D)$.*

**Corollary 1.1.2** ( [JW19, Corollary 1.8]). *Let $X$ be a regular surface, which is geometrically non-reduced over $K$. Suppose that $(Y, (p-1)C)$ is of the type $(\mathbb{P}^1 \times \mathbb{P}^1, F)$. Then there exists a contraction $X \to V$ to a geometrically non-reduced curve $V$ such that over the generic point of $V$*

$$Y = X \times_V ((V \otimes_K \overline{K})^{\nu}_{red}).$$

A classification similar to Theorem 1.0.1 has been established in [Tan19, Theorem 4.6], under the assumption that $X$ is regular. The theorem states a list of possibilities arising as characteristic $p$, normalization $Y$ and the value of $(\phi^* K_X)^2$, where $\phi : Y \to X$ is the natural induced map. In particular, the theorem excludes the case of $Y$ being a Hirzebruch surface of degree 3. Hence in that case, no corresponding regular surface $X$ exists. This further highlights the importance of our four main theorems, especially Theorem 1.0.6.

An other point of view that highlights the importance of the examples we construct is the birational classification of Fano threefolds. A classification of Fano threefolds in positive characteristic has been established in [SB97]. The examples of surfaces we construct are the generic fibers of fibrations from threefolds to curves, over perfect fields. By comparing Table 1.2 of previously known examples to Table 1.3 containing the examples we construct, one sees that our examples are significantly different to the previously known examples, in the sense that they are geometrically reduced, and with the exception of one examples they are not regular. Hence our examples, and the idea behind their construction give a new insight into the classification of Fano threefolds over algebraically closed fields.

The examples mentioned above of regular del Pezzo surfaces which are not geometrically normal are constructed by [Mad16] and [Tan19]. Table 1.2 lists these surfaces $X$.

Table 1.2: Regular del Pezzo surfaces which are not geometrically normal

| Example $X$ | $(p, Y, C)$ | $K_X^2$ | geometrically reduced? |
|---|---|---|---|
| [Mad16, Example 3.2] | $(2, \mathbb{P}^2, C \in \lvert 2L \rvert)$ | 1 | Yes |
| [Mad16, Example 3.3] | $(2, \mathbb{P}^2, C \in \lvert 2L \rvert)$ | 2 | No |
| [Tan19, Example 6.2] | $(2, \mathbb{P}^2, C \in \lvert 2L \rvert)$ | 4 | No |
| [Tan19, Example 6.1] | $(2, \mathbb{P}^2, ?)$ | 8 | No |
| [Tan19, Example 6.1] | $(3, \mathbb{P}^2, L)$ | 3 | No |
| [Tan19, Example 6.3] | $(2, \mathbb{P}^1 \times \mathbb{P}^1, ?)$ | 4 | No |
| [Tan19, Example 6.4] | $(2, H_1, D)$ | 6 | No |
| [Tan19, Example 6.5] | $(2, H_2, D)$ | 6 | Yes |

Additionally to that, [FS20, Theorem 14.8] constructs a regular and geometrically non-normal del Pezzo surface with Picard number two.

Table 1.3: Specifications for the four surfaces constructed

| Example $X$ | $(p, Y, C)$ | $K_X^2$ | geometrically reduced? | regular? |
|---|---|---|---|---|
| Theorem 1.0.3 | $(2, \mathbb{P}^1 \times \mathbb{P}^1, F_i)$ | 4 | Yes | No |
| Theorem 1.0.4 | $(2, \mathbb{P}^2, L)$ | 4 | Yes | No |
| Theorem 1.0.5 | $(2, H_2, D)$ | 6 | Yes | Yes |
| Theorem 1.0.6 | $(2, H_3, D)$ | 9 | Yes | No |

For comparison, we list in Table 1.3 the same specifications for the four examples we construct, as stated in Theorem 1.0.3, Theorem 1.0.4, Theorem 1.0.5 and Theorem 1.0.6.

Table 1.2 lists previously known examples. The way these examples are stated in the original articles only includes the scheme $Y$, but not the corresponding divisor $C$. We describe in the subsection below how to identify the divisor $C$, if only $Y$ is known. We do so in both the geometrically reduced and non-reduced case. For two of the examples, the divisor is marked with a question mark. In those instances, our method does not identify $C$.

### 1.1.1   Geometrically reduced case

**Example 1.1.3.** First, we consider the examples that are geometrically reduced. We choose the example [Tan19, Example 6.5] to illustrate this. The possible divisors that can occur in the case of characteristic 2, with $Y = H_2$ are either $C = D$ or $C \in |D + F|$, where $F$ and $D$ denote the fiber and exceptional section of the Hirzebruch surface. The intersection numbers of $F$ and $D$ are

$$F^2 = 0, \quad D.F = 1, \quad D^2 = -2.$$

As [Tan19] states the value of $K_X^2$, we compute $K_X^2$ for both possibilities of $C$ to conclude which divisor arises in this example by comparing it to the stated value of $K_X^2$.

First, we assume $C = D$. In this case, $K_Y + (p-1)C$ is of the form $K_Y + D = aF + bD$ for some $a, b \in \mathbb{Z}$. In order to compute the values of $a$ and $b$, we compare the following intersections, where the first equality holds due to the adjunction formula. On one hand, we get

$$(K_Y + D).F = \deg K_F + F.D = -2 + 1 = -1,$$

and on the other hand, we get

$$(K_Y + D).F = (aF + bD).F = aF^2 + bD.F = b.$$

From this comparison, it follows that $b = -1$.

To obtain the value of $a$, we now compare the following intersections, again using the adjunction formula. On one hand,

$$(K_Y + D).D = \deg K_D = -2$$

and on the other hand,

$$(K_Y + D).D = (aF - D).D = aF.D - D^2 = a + 2.$$

From this comparison, it follows that $a = -4$. Therefore, $K_Y + D = -4F - D$, and

$$(K_Y + D)^2 = (-4F - D)^2 = 16F^2 + 8F.D + D^2 = 8 - 2 = 6.$$

Let $\phi$ denote the morphism $\phi : Y \to X$, and let $\varphi : Y \to X_{\overline{k}}$ and $\psi : X_{\overline{k}} \to X$. It holds that $\phi^* K_X = K_Y + (p-1)D$. In order to calculate $K_X^2$, we consider

$$\omega_{X/k}^2 = \omega_{X_{\overline{k}}/\overline{k}}^2 = (\varphi^* \omega_{X_{\overline{k}}/\overline{k}})^2, \quad \text{since } \varphi \text{ is birational}$$
$$= (\omega_Y + (p-1)D)^2.$$

From this, it follows that $K_X^2 = (K_Y + D)^2 = 6$.

Now assume that $C = D + F$. It holds that $K_Y + D + F = (K_Y + D) + F = (-4F - D) + F = -3F - D$. And hence

$$(K_Y + D + F)^2 = (-3F - D)^2 = 9F^2 + 6D.F + D^2 = 6 - 2 = 4.$$

Similar to the first case, it follows that $K_X^2 = (K_Y + D + F)^2 = 4$.

For the example [Tan19, Example 6.5], a value of $K_X^2 = 6$ is obtained. Comparing this to the two possibilities above, we see that the divisor $C$ needs to be of the form $C = D$.

## 1.1.2  Geometrically non-reduced case

In the geometrically non-reduced case, the scheme $X_{\overline{k}}$ is not reduced. In this case, we apply an iterative process described in [Tan18, Lemma 4.1]. The procedure described there will ensure that after a certain number of steps, we do obtain a scheme that is geometrically reduced. The lemma states the following.

**Lemma 1.1.4.** *( [Tan18, Lemma 4.1]) Let $k$ be a field of characteristic $p > 0$. Let $X$ be a normal variety over $k$. Then, there exist sequences*

- $k =: k_1^0 \subset k_1 \subset k_2^0 \subset k_2 \subset \ldots \subset k_n^0 \subset k_n \subset k^{p^{-\infty}}$,

- $X =: X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_n \leftarrow ((X \otimes_k k^{p^{-\infty}})_{red})^N$, and

- $X_i \to \operatorname{Spec} k_i \to \operatorname{Spec} k_i^0$,

*where $((X \otimes_k k^{1/p^\infty})_{red})^N$ is the normalization of $(X \otimes_k k^{1/p^\infty})_{red}$ which satisfy the following properties.*

(a) *The field extension $k_n/k$ is a finite purely inseparable extension.*

(b) *For every $i$, $X_i$ is a normal variety over $k_i$, and the morphism $X_i \to X_{i-1}$ is a finite surjective purely inseparable morphism.*

(c) *For every $i$, $k_i$ is the purely inseparable closure of $k_i^0$ in $K(X_i)$.*

(d) *For every $i$, $[k_{i+1}^0 : k_i] = p$.*

(e) *For every $i$, $X_{i-1} \otimes_{k_{i-1}} k_i^0$ is integral and $X_i$ is the normalization of $X_{i-1} \otimes_{k_{i-1}} k_i^0$.*

(f) *$X_n$ is geometrically reduced over $k_n$.*

(g) *The induced morphism $((X \otimes_k k^{1/p^\infty})_{red})^N \to X_n \otimes_{k_n} k^{1/p^\infty}$ is the normalization of $X_n \otimes_{k_n} k^{1/p^\infty}$.*

Applied to the present situation, the scheme $Y$ is obtained from $X$ after a finite number of iterative steps, as described in Lemma 1.1.4. In order to express the self-intersection of the canonical divisor $K_X$ in terms of that one of $K_Y$, these iterative steps need to be taken into consideration. In the geometrically reduced case, the self-intersection of the canonical divisor $K_X$ is equal to $K_X^2 = (K_Y + (p-1)C)^2$. However, in the geometrically non-reduced case, for each iterative step, the right hand side needs to be multiplied by the degree of the field extension, which according to Lemma 1.1.4(d) is equal to the characteristic $p$. This results in the following formula,

$$K_X^2 = (K_Y + (p-1)C)^2 \cdot p^i,$$

where $i$ denotes the number of iterative steps that have been performed.

**Example 1.1.5.** We first study the examples [Mad16, Example 3.2 and 3.3] and [Tan19, Example 6.1 and 6.2] with $p = 2$ and $Y = \mathbb{P}^2$.

[Tan19, Example 6.1] is defined as follows. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$ and let $k := \mathbb{F}(s_0, s_1, s_2, s_3)$ be the purely transcendental extension over $\mathbb{F}$ of degree four. Set

$$X := \operatorname{Proj} k[x_0, x_1, x_2, x_3]/(s_0 x_0^p + s_1 x_1^p + s_2 x_2^p + s_3 x_3^p).$$

[Tan19, Example 6.2] is defined in the following way. Let $\mathbb{F}$ be an algebraically closed field of characteristic two. Let

$$k := \mathbb{F}\left(\{s_i \mid 0 \le i \le 4\} \cup \{t_i \mid 0 \le i \le 4\}\right)$$

be the purely transcendental extension over $\mathbb{F}$ of degree ten. Set

$$X := \text{Proj}\left(\frac{k[x_0, x_1, x_2, x_3, x_4]}{\left(\sum_{i=0}^{4} s_i x_i^2, \sum_{i=0}^{4} t_i x_i^2\right)}\right)$$

The possible divisors are $C = L$ a line, or $C = C \in |2L|$ a conic, according to Theorem 1.0.1. In the case of the line, the associated divisor to $K_Y + (p-1)L = \mathscr{O}(-3) + (p-1)\mathscr{O}(1) = \mathscr{O}(-3+1) = \mathscr{O}(-2)$. This follows from the fact that the canonical divisor of the projective space $\mathbb{P}^n$ is equal to $\mathscr{O}(-n-1)$, hence for $\mathbb{P}^2$, the associated divisor is $\mathscr{O}(-3)$. Furthermore, due to the fact that the line $L$ is a hypersurface of degree 1 in $\mathbb{P}^2$, its associated divisor is equal to $\mathscr{O}(1)$, according to [Har77, Proposition II.6.4]. For the conic, which is defined as the zero set of a polynomial of degree 2, its associated divisor is equal to $\mathscr{O}(2)$, according to the same proposition. Hence for the conic, the associated divisor to $K_Y + (p-1)C = \mathscr{O}(-3+2) = \mathscr{O}(-1)$. By the formula obtained above, $K_X^2$ is equal to

$$K_X^2 = (K_Y + (p-1)C)^2 \cdot 2^i$$

for some power $i$ that corresponds to the amount of iterations that have been performed to reach the scheme $Y$ with divisor $C$ on $Y$. In the case of the divisor $C$ being a line, $(K_Y + (p-1)C)^2 = 4$, and in the case of a $C$ being a conic, $(K_Y + (p-1)C)^2 = 1$. The values that $K_X^2$ obtains in the examples [Mad16, Example 3.2 and 3.3] and [Tan19, Example 6.1 and 6.2] are $1, 2, 4, 8$. We list below how these values can be expressed as a product, with one of the terms being of the form $2^i$ for some $i \in \mathbb{N}$.

| 1 | 1 |
|---|---|
| 2 | $2 \cdot 1$ or $1 \cdot 2$ |
| 4 | $2^2 \cdot 1$ or $2 \cdot 2$ or $1 \cdot 2^2$ |
| 8 | $2^3 \cdot 1$ or $2^2 \cdot 2$ or $2 \cdot 2^2$ or $1 \cdot 2^3$ |

Table 1.4: Values of $K_X^2$

The value $K_X^2 = 1$ can only be expressed in the form $1 = 1 \cdot 2^0$, which indicates that only geometrically reduced examples can exist, due to the fact that the number of iterations is zero. The first product for the value $K_X^2 = 2$ is of the form $2 = 2 \cdot 2^0$, which again indicates that this corresponds to the geometrically reduced case. The same argument holds for the first product in the table for $K_X^2 = 4$ and $K_X^2 = 8$.

In the case of $K_X^2 = 2$, if the example is not geometrically reduced, we are in the case where the product appears as $1 \cdot 2$ which indicates that $i = 1$ and $(K_Y + (p-1)C)^2 = 1$, which holds if the divisor $C$ is a conic.

In the case of $K_X^2 = 4$, if the example is not geometrically reduced, the product can appear as $2 \cdot 2$ and $1 \cdot 2^2$. But the first case does not exist, since $(K_Y + (p-1)C)^2 \ne 2$, for either choice of the divisor $C$. Hence the only case that can occur is if $i = 2$ and $(K_Y + (p-1)C)^2 = 1$, which means that the divisor $C$ is a conic.

In the case of $K_X^2 = 8$, if the example is not geometrically reduced, the product can appear as $2^2 \cdot 2$, $2 \cdot 2^2$ or $1 \cdot 2^3$. But the case $2 \cdot 2^2$ does not exist, since $(K_Y + (p-1)C)^2 \ne 2$, for either choice of divisor. Hence the possible cases are $2^2 \cdot 2$, in the case where $i = 1$ and $(K_Y + (p-1)C)^2 = 4$, which means that $C$ is a line. The other possible case is $1 \cdot 2^3$, where $i = 3$ and $(K_Y + (p-1)C)^2 = 1$, which means that $C$ is a conic. So for [Tan19, Example 6.1] this method of finding the divisor $C$ associated to $Y$ is inconclusive.

**Example 1.1.6.** [Tan19, Example 6.1] has been defined in Example 1.1.5. With $Y = \mathbb{P}^2$ and $p = 3$, there is only one possible divisor $C = L$ a line, according to Theorem 1.0.1. The self-intersection of

$K_X$ in the example is $K_X^2 = 3$. With $C$ being a line $L$, we obtain that the divisor associated to $K_Y + (p-1)L$ is

$$K_Y + (p-1)L = \mathscr{O}(-3) + (3-1)\mathscr{O}(1) = \mathscr{O}(-3+2) = \mathscr{O}(-1).$$

It follows that $(K_Y + 2L)^2 = 1$. By the formula obtained in the geometrically non-reduced case, $K_X^2 = (K_Y + (p-1)L)^2 \cdot p^i = (K_Y + 2L)^2 \cdot 3^i$ for some $i \in \mathbb{N}$. The value for $K_X^2$ is stated to be 3. There are only two possibilities to write this as a product where one of the terms is a power of 3. Those are either $3 = 3 \cdot 1$ or $3 = 1 \cdot 3$. The first product corresponds to the geometrically reduced case. The second product corresponds to the example [Tan19, Example 6.1], with one iterative step, and $(K_Y + 2L)^2 = 1$. This verifies that in the example, $L$ is indeed a line.

**Example 1.1.7.** We now consider [Tan19, Example 6.4]. The divisor associated to this example in characteristic 2, with $Y = H_1$, is either $C = D$ or $C \in |D + F|$, where $F$ is the fiber and $D$ is the exceptional section of the Hirzebruch surface $H_1$. The intersection numbers are

$$F^2 = 0, \quad D.F = 1, \quad D^2 = -1.$$

Firstly, we calculate $K_X^2$ for the divisor $C = D$. Similarly to the calculations is Example 1.1.3 we let $K_Y + (p-1)D = K_Y + D = aF + bD$, for some $a, b \in \mathbb{Z}$. In order to determine these values, we compare the following intersections, using the adjunction formula for the first equality. On one hand,

$$(K_Y + D).F = \deg K_F + D.F = -2 + 1 = -1,$$

and on the other hand,

$$(K_Y + D).F = (aF + bD).F = aF^2 + bD.F = b.$$

From this comparison, it follows that $b = -1$. To obtain the value of $a$, we compare the following intersections, again using the adjunction formula. On one hand,

$$(K_Y + D).D = \deg K_D = -1$$

and on the other hand,

$$(K_Y + D).D = (aF - D).D = aF.D - D.D = a + 1.$$

From this comparison, we get $a = -2$. Therefore, $(K_Y + D) = -2F - D$, and

$$(K_Y + D)^2 = (-2F - D)^2 = 4F^2 + 4F.D + D^2 = 4 - 1 = 3.$$

If we consider the divisor $C = D + F$, we get that

$$(K_Y + (p-1)(D+F)) = (K_Y + (D+F)) = (K_Y + D) + F = -2F - D + F = -F - D.$$

From this, it follows that

$$(K_Y + (D+F))^2 = (-F - D)^2 = F^2 + 2F.D + D^2 = 2 - 1 = 1.$$

In [Tan19, Example 6.4], the value of $K_X^6$ is equal to 6. According to the formula for the geometrically non-reduced case, $K_X^2$ is of the from $K_X^2 = (K_Y + C)^2 \cdot 2^i$. We conclude that we obtain $K_X^2 = 6$ only in the case where $(K_Y + C)^2 = 3$, with $i = 1$. This corresponds to the divisor $C$ being equal to the exceptional section $D$.

# Chapter 2

# General Analysis

The content of my thesis consists of the construction of explicit examples. To motivate how these examples are being obtained, we first describe a general framework that reduces the construction of varieties over imperfect fields to the construction of fibrations over finite fields together with foliations on the total space of these fibrations. Although our specific construction could be presented without explaining this general framework we decided to include the latter for two reason. Firstly, we do so in order to show that our specific construction arises naturally. Secondly, it allows us to introduce notation used in later parts of the thesis.

We describe the process of transforming the question of finding a surface $X$ as in Theorem 1.0.1 into a question about finding foliations on certain varieties defined over finite fields. Most of the chapter will be about explaining the construction of these varieties and also about explaining why finding foliations with adequate properties on these varieties is equivalent to the original question. Furthermore, we will explain the precise construction of the divisor $C$ appearing in the formula for purely inseparable base change, which is crucial for obtaining the examples.

We start with the general construction. So let $(p, Y, C)$ be one of the triples in Theorem 1.0.1, and let $k$ be a non-perfect field of characteristic $p$ over which $X$ is defined. In order to apply techniques known to work for foliations, we transform the setup as described below.

Since $X$ is by assumption a projective surface, the morphism $X \to \operatorname{Spec} k$ factors into a closed immersion $\iota : X \to \mathbb{P}^n$, followed by the projection $\mathbb{P}^n \to \operatorname{Spec} k$, for some $n \in \mathbb{N}$. Let $(f_1, \ldots, f_r) \subseteq k[x_0, \ldots, x_n]$ be the homogeneous polynomials defining $X$. Let $A$ be the finite set of the coefficients of the $f_i$'s. As a set, $A$ is contained in the field $k$, which contains the field $\mathbb{F}_p$. Adjoining $A$ to $\mathbb{F}_p$, we get a subring $\mathbb{F}_p[A] \subseteq k$. Define $T := \operatorname{Spec}(\mathbb{F}_p[A])$, an integral affine scheme of finite type over $\mathbb{F}_p$. Let $X_{K(T)}$ be the scheme defined over $\operatorname{Spec} K(T)$ that is defined by the same equations $f_i$ that define $X$. The scheme $X_{K(T)}$ is defined over $\operatorname{Spec} K(T) = \operatorname{Spec} k(\eta_T)$, the spectrum of the residue field of the generic point $\eta_T$ of $T$.

In order to transform the setup to work over perfect fields, we define $\mathfrak{X}$ to be a scheme over $T$ such that $X_{K(T)}$ corresponds to the fiber of the morphism $\mathfrak{X} \to T$ over $\eta_T : X_{K(T)} = \mathfrak{X} \times_T \operatorname{Spec} k(\eta_T) = \mathfrak{X}_{\eta_T}$. Hence $\mathfrak{X}$ is defined by the polynomials $f_i$, which define $X$, in $\mathbb{P}^n_T$. It holds that $X_{\overline{k}}$ is the geometric generic fiber of the morphism $\mathfrak{X} \to T$. The advantage of this approach is that $\mathfrak{X}$ is defined over $T$, which is defined over the perfect field $\mathbb{F}_p$. Hence we can use the correspondence between finite purely inseparable morphisms of height one and foliations, which holds over perfect fields, stated in Proposition 2.0.6. We have constructed the following schemes, as illustrated in Figure 2.1.

$$
\begin{array}{ccccccc}
Y & \xrightarrow{\text{norm.}} & X_{\overline{k}} & \longrightarrow & X & \longrightarrow & X_{K(T)} & \longrightarrow & \mathfrak{X} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} K(T) & \longrightarrow & T
\end{array}
$$

Figure 2.1: Construction of schemes

In order to use the correspondence between finite purely inseparable morphisms of height one and foliations, we further construct the following schemes, as in Figure 2.2.
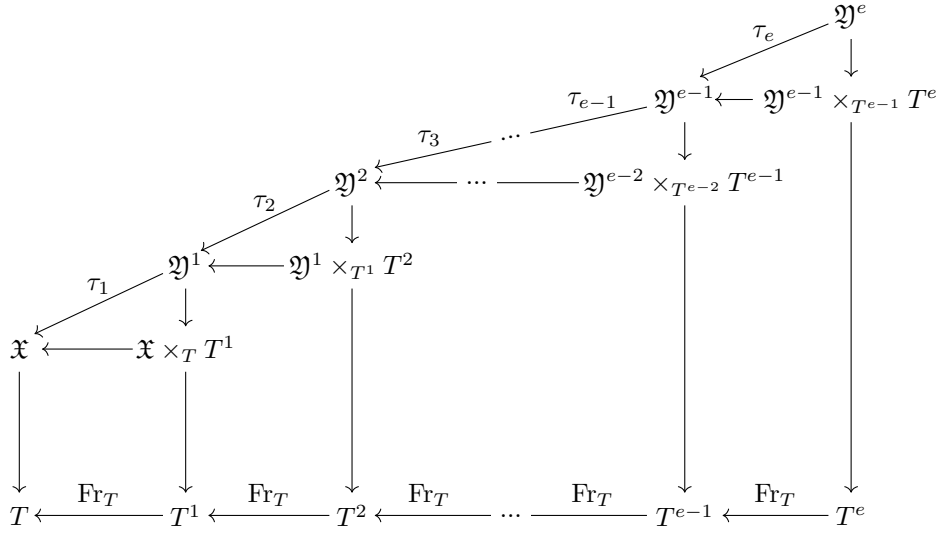


Figure 2.2: Construction of schemes

We let $T^i = T$, where the index $i$ is added to keep track of the number of times the absolute Frobenius morphism has been performed. The scheme $\mathfrak{Y}^i$ is defined to be the normalization of $\mathfrak{Y}^{i-1} \times_{T^{i-1}} T^i$ for each $i = 2, \ldots, e$. The morphisms between the schemes $\mathfrak{Y}^i$ are denoted by $\tau_i : \mathfrak{Y}^i \to \mathfrak{Y}^{i-1}$. The number $e$ which indicates how many Frobenius base changes have been performed is defined as follows.

**Definition 2.0.1.** We define $e$ to be the smallest integer $e \gg 0$ such that

$$\left( \begin{array}{c} \mathfrak{Y}^e \longrightarrow \mathfrak{X} \times_T T^e \\ \searrow \qquad \downarrow \\ T^e \end{array} \right) \times_{T^e} \operatorname{Spec} \overline{k} = \left( \begin{array}{c} Y \longrightarrow X_{\overline{k}} \\ \searrow \qquad \downarrow \\ \operatorname{Spec} \overline{k} \end{array} \right)$$

holds.

For each of these Frobenius base changes, we can apply the following adjunction formula, stated in [PW17, Theorem 3.1].

**Theorem 2.0.2** ( [PW17, Theorem 3.1])**.** *Let $\mathfrak{X}$ be a normal variety over a perfect field $k$ of characteristic $p > 0$, and let $f : \mathfrak{X} \to T$ be a morphism to a normal variety over $\mathbb{F}_p$. Let $\tau : T' \to T$ be a finite purely inseparable height one $k$-morphism from a normal variety and let $\mathfrak{Y}$ be the normalization of (the reduced subscheme associated to) $\mathfrak{X} \times_T T'$. Then the following statements hold:*

(a) *$K_{\mathfrak{Y}/\mathfrak{X}} \sim (p-1)\mathfrak{D}$ for some Weil divisor $\mathfrak{D}$ on $\mathfrak{Y}$.*

(b) *There is a non-empty open set $U \subseteq T'$ and an effective divisor $\mathfrak{C}$ on $g^{-1}U$ satisfying $-\mathfrak{C} \sim \mathfrak{D}\big|_{g^{-1}U}$, where $g : \mathfrak{Y} \to T'$ is the induced morphism.*

We denote the arising morphism at each step of the Frobenius base change as described in the following diagram.
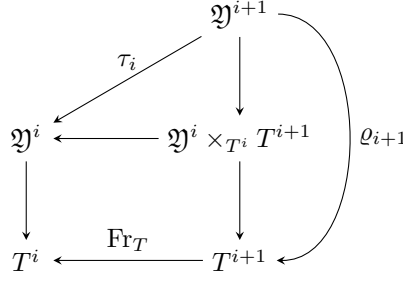
Figure 2.3: Notation for the Frobenius base change

We shrink the base $T$ and assume that $T$ is regular, and that it is small enough for Theorem 2.0.2(b) to work with $U = T$. For each $i$ there exists an effective divisor $\mathfrak{C}^{i+1}$ on $\mathfrak{Y}^{i+1}$, such that

$$K_{\mathfrak{Y}^{i+1}/\mathfrak{Y}^i} \sim -(p-1)\mathfrak{C}^{i+1}$$

on $\mathfrak{Y}^{i+1}$, according to Theorem 2.0.2.

Inductively, this leads to the following formula for the canonical divisor of $\mathfrak{Y}^e$ relative to $\mathfrak{X}$.

**Claim 2.0.3.**

$$K_{\mathfrak{Y}^e/\mathfrak{X}} \sim -(p-1)\sum_{i=1}^{e}(\mathfrak{C}^i)_{\mathfrak{Y}^e},$$

*where the index $(\cdot)_{\mathfrak{Y}^e}$ denotes the pull back of the divisor to $\mathfrak{Y}^e$.*

*Proof.* Taking the sum of

$$K_{\mathfrak{Y}^{i+1}/\mathfrak{Y}^i} \sim -(p-1)\mathfrak{C}^{i+1}$$

and the pullback via $\tau_{i+1}$ of

$$K_{\mathfrak{Y}^i/\mathfrak{Y}^{i-1}} \sim -(p-1)\mathfrak{C}^i$$

we get

$$K_{\mathfrak{Y}^{i+1}} - (\tau_{i+1})^* K_{\mathfrak{Y}^i} + (\tau_{i+1})^*\left(K_{\mathfrak{Y}^i} - (\tau_i)^* K_{\mathfrak{Y}^{i-1}}\right) \sim -(p-1)\mathfrak{C}^{i+1} - (p-1)(\tau_i)^*\mathfrak{C}^i$$
$$K_{\mathfrak{Y}^{i+1}} - (\tau_{i+1})^*\left((\tau_i)^* K_{\mathfrak{Y}^{i-1}}\right) \sim -(p-1)\left(\mathfrak{C}^{i+1} + (\tau_i)^*\mathfrak{C}^i\right)$$
$$K_{\mathfrak{Y}^{i+1}/\mathfrak{Y}^{i-1}} \sim -(p-1)\left(\mathfrak{C}^{i+1} + (\tau_i)^*\mathfrak{C}^i\right).$$

From this, the claim follows inductively.                                                     $\square$

In order to characterize the surface $X$, we consider the setting of the schemes $\mathfrak{X}$ and $\mathfrak{Y}$, because these are defined over a perfect field which allows us to use the theory of foliations. To do so, we exhibit the correspondence between the divisor $\mathfrak{C} := \sum_{i=1}^{e}(\mathfrak{C}^i)_{\mathfrak{Y}^e}$ on $\mathfrak{Y}^e$ and the divisor $C$ on $Y$.

On one hand, as seen above, the canonical divisor of $\mathfrak{Y}^e$ relative to $\mathfrak{X}$ is linearly equivalent to $K_{\mathfrak{Y}^e/\mathfrak{X}} \sim -(p-1)\sum_{i=1}^{e}(\mathfrak{C}^i)_{\mathfrak{Y}^e}$. On the other hand, by Theorem 1.0.1, the canonical divisor of $Y$ relative to $X$ is linearly equivalent to $K_{Y/X} \sim -(p-1)C$. Comparing these two divisors and using the fact that the base changes of $\mathfrak{Y}^e$ and $\mathfrak{X}$ over $T^e$ to $\operatorname{Spec}\overline{k}$ are $Y$, and $X_{\overline{k}}$ respectively, we get

$$\left((p-1)\sum_{i=1}^{e}(\mathfrak{C}^i)_{\mathfrak{Y}^e}\right) \times_{T^e} \operatorname{Spec}\overline{k} \sim -\left(K_{\mathfrak{Y}^e/\mathfrak{X}}\right) \times_{T^e} \operatorname{Spec}\overline{k}$$
$$= -K_{Y/X_{\overline{k}}} = -K_{Y/X} \sim (p-1)C.$$

**Claim 2.0.4.** *It holds that $K_{Y/X_{\overline{k}}} = K_{Y/X}$.*

*Proof.* This holds because

$$K_{Y/\overline{k}}\big/_{X/k} = K_{Y/\overline{k}}\big/_{X_{\overline{k}}/\overline{k}} - \psi^* \left( K_{X_{\overline{k}}/\overline{k}} - \varphi^* K_{X/k} \right),$$

where $\psi$ and $\varphi$ denote the morphisms $\psi : Y \to X_{\overline{k}}$ and $\varphi : X_{\overline{k}} \to X$. The fact that $K_{X_{\overline{k}}/\overline{k}} - \varphi^* K_{X/k} = 0$ follows from the base change properties of the sheaf of relative differentials, stated in [Har77, Proposition II.8.10]. $\square$

In addition to being linearly equivalent, the divisor $C$ is indeed equal to $\left( \sum_{i=1}^{e} (\mathfrak{C}^i)_{\mathfrak{Y}^e} \right) \times_{T^e} \operatorname{Spec} \overline{k}$. This is due to the construction of $C$, described in the proof of [PW17, Theorem 3.1 (c)].

The change of setup from the original question of finding a surface $X$, given a corresponding scheme $Y$ and divisor $C$ into the setup involving the schemes $\mathfrak{X}$ and $\mathfrak{Y}^e$, and divisors $\mathfrak{C}^i$ is motivated by the idea of using foliations. The schemes $\mathfrak{X}$ and $\mathfrak{Y}$ are defined over the base $T$, which is defined over the perfect field $\mathbb{F}_p$. This allows us to use a correspondence between foliations and purely inseparable morphisms of hight one which holds over perfect fields. Via this correspondence, stated in [PW17, Proposition 2.9], we can characterize the scheme $\mathfrak{X}$ by characterizing the foliation $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ for which $\mathfrak{X} = \mathfrak{Y}/\mathcal{F}$ holds.

A foliation is by definition a subsheaf of the tangent sheaf that satisfies two additional conditions.

**Definition 2.0.5.** Let $\mathfrak{Y}$ be a normal variety over a perfect field of positive characteristic $p > 0$. A foliation on $\mathfrak{Y}$ is a subsheaf $\mathcal{F} \subset \mathcal{T}_{\mathfrak{Y}}$ which is saturated and closed under $p$-powers. In the literature, this is sometimes referred to as a $p$-closed foliation.

The correspondence mentioned above is stated in the following proposition.

**Proposition 2.0.6** ( [PW17, Proposition 2.9]). *Let $\mathfrak{Y}$ be a normal variety over a perfect field $k$ of characteristic $p > 0$. Then, there is a 1-to-1 correspondence between*

*(1) finite purely inseparable $k$-morphisms $f : \mathfrak{Y} \to \mathfrak{X}$ of height 1 with $\mathfrak{X}$ normal, and*

*(2) foliations $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$.*

*This correspondence is given by:*

*(1) $\to$ (2) $\mathcal{F}$ is the sheaf of derivations that vanish on $\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{O}_{\mathfrak{Y}}$ ($\mathfrak{Y}$ and $\mathfrak{X}$ have the same topological space, so this containment does make sense), and*

*(2) $\to$ (1) $\mathfrak{X} = \operatorname{Spec}_{\mathfrak{Y}} \mathcal{A}$, where $\mathcal{A}$ is the subsheaf of $\mathcal{O}_{\mathfrak{Y}}$ that is taken to zero by all the sections of $\mathcal{F}$.*

*Furthermore, morphisms of degree $p^m$ correspond to foliations of rank $m$.*

## 2.1 General Construction with Specific Choices

In the study above, we have described the change of setup from the original question into a setup that allows us to use the theory of foliations, characterizing the scheme $\mathfrak{X}$ by a foliation $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$. We have done so while maintaining the largest generality possible, and without any additional assumptions on the schemes we have constructed. From now on, as we are aiming to explicitly construct specific surfaces, we are no longer maintaining this generality. Motivated by the analysis of the first part of this chapter, we now explicitly choose certain parameters for the construction of all examples. These choices limit the possible surfaces we can obtain. On the other hand, some choices are necessary in order to create workable condition and find explicit examples. Every time we set a parameter, we motivate the choice and the thought process that justifies the choice.

We remark that for all examples we construct in Chapter 3, Chapter 4 and Chapter 5, the characteristic of the field $k$, over which the surface $X$ is defined, is equal to 2. The first specific choice we make is choosing the base $T$ of $\mathfrak{X}$ to be a curve, more specifically $T = \mathbb{A}^1_{\mathbb{F}_2}$. This way, $k = \mathbb{F}_2(x)$. Furthermore, as we will describe in more detail in the corresponding chapter, for every example, the integer $e$ defined in Definition 2.0.1 is equal to one. This means that only one Frobenius base change is applied to obtain the scheme $\mathfrak{Y}^1$ as the normalization of $\mathfrak{X} \times_T T^1$. The divisor $C$ is

equal to $C = \left((\mathfrak{C}^1)_{\mathfrak{Y}^1}\right) \times_T \operatorname{Spec} \overline{k} = \mathfrak{C}^1 \times_T \operatorname{Spec} \overline{k}$, where $\mathfrak{C}^1$ is a divisor on $\mathfrak{Y}^1$. For simplicity, we denote $\mathfrak{Y} := \mathfrak{Y}^1$ and $\mathfrak{C} := \mathfrak{C}^1$.

As a second specific choice, we may fix for $\mathfrak{Y}$ any $Y$ family freely, assuming eventually we may find an adequate foliation on it. Hence we let $\mathfrak{Y}$ be the fiber product of $T$ and $Y$ over $\operatorname{Spec} \mathbb{F}_2$. With this choice, the correspondence between $\mathfrak{Y}$ and $Y$ is satisfied, meaning that

$$\mathfrak{Y} \times_T \operatorname{Spec} \overline{k} = Y.$$

Based on these two explicit choices, the construction of all examples follows the same general strategy. We start with a fibration $\varrho : (\mathfrak{Y}, \mathfrak{C}) = (Y \times_{\operatorname{Spec} \mathbb{F}_2} T, C \times_{\operatorname{Spec} \mathbb{F}_2} T) \to T^1$ over the perfect field $\mathbb{F}_2$. This fibration is illustrated in the figure below, with $\nu$ denoting the normalization of $\mathfrak{X} \times_T T^1$.
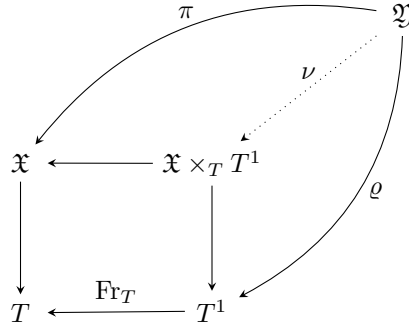


Figure 2.4: Setup for General Strategy

Using the correspondence stated in Proposition 2.0.6, we construct the scheme $\mathfrak{X}$ via a foliation $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}} = \mathcal{T}_{Y \times T}$. This foliation is constructed with certain specific properties that ensure that the corresponding scheme $\mathfrak{X}$ and consequently the surface $X$ is of the required form. These properties depend on $Y$ and $C$ in each specific construction, and are stated more precisely in the following chapters for each of the examples. The scheme $\mathfrak{X}$ is then defined by the quotient of the foliation, fitting the diagram in Figure 2.4, with $\mathfrak{C}$ being the conductor of the normalization $\nu$. Lastly, $X$ is obtained by taking the generic fiber of the map $\mathfrak{X} \to T$.

The divisor $\mathfrak{C}$, respectively $C$ has a special role in the construction of the examples. The support of $\mathfrak{C}$ on $\mathfrak{Y}$ is equal to the locus where the natural map $\mathcal{F} \to \varrho^* \mathcal{T}_{T^1}$ is not surjective. This is stated and proved in the proposition below. In the construction of the explicit examples in the following chapters, this proposition is a key result, due to the fact that it restricts the form of a possible foliation.

**Proposition 2.1.1.** *Assume that the scheme $T$ is smooth. The support of the divisor $\mathfrak{C}$ on $\mathfrak{Y}$ is equal to the locus where the map*

$$\gamma : \mathcal{F} \hookrightarrow \varrho^* \mathcal{T}_{T^1}$$

*is not surjective, including multiplicity. The map $\varrho$ denotes the map from $\mathfrak{Y}$ to $T^1$, as illustrated in Figure 2.4.*

*Remark* 2.1.2. The proposition above is stated in larger generality than needed for our application. The proposition holds for the base $T$ of $\mathfrak{X}$ being smooth, and hence in particular also holds for our explicit choice of $T$ being $\mathbb{A}^1_{\mathbb{F}_2}$.

*Proof.* In order to prove this, we consider the proof of [PW17, Theorem 3.1], which states that with the above requirements for $\mathfrak{X}$ and $\mathfrak{Y}$, there exist two divisors $\mathfrak{D}$ and $\mathfrak{C}$ with the following properties.

(a) $K_{\mathfrak{Y}/\mathfrak{X}} \sim (p-1)\mathfrak{D}$, where $\mathfrak{D}$ is a Weil divisor on $\mathfrak{Y}$

(b) there is a nonempty open subset $U \subseteq T^1$ and an effective divisor $\mathfrak{C}$ on $\varrho^{-1}U$ satisfying
$$-\mathfrak{C} \sim \mathfrak{D}\big|_{\varrho^{-1}U}.$$

Statement (a) is proven by taking any Weil divisor in the class of $\det\mathcal{F}$, for which the formula

$$\omega_{\mathfrak{Y}/\mathfrak{X}} \cong (\det\mathcal{F})^{[p-1]}$$

holds.

To prove statement (b), one wants to show that after shrinking $T^1$, $H^0(\mathfrak{Y}, (\det\mathcal{F})^*) \neq 0$ holds. For this, it is enough to exhibit an embedding $\mathcal{F} \hookrightarrow \mathscr{O}_{\mathfrak{Y}}$. We note that since by assumption, $T$ is smooth, $\mathcal{T}_{T^1} \cong \mathscr{O}_{T^1}$ holds. Using the two morphisms $\varrho : \mathfrak{Y} \to T^1$ and the morphism $T^1 \to k$, there exists an exact sequence of sheaves on $\mathfrak{Y}$ according to [Har77, Proposition II.8.11],

$$\varrho^*\Omega_{T^1/k} \to \Omega_{\mathfrak{Y}/k} \to \Omega_{\mathfrak{Y}/T^1} \to 0.$$

Dualizing yields

$$0 \to \mathscr{H}om(\Omega_{\mathfrak{Y}/T^1}, \mathscr{O}_{\mathfrak{Y}}) \to \underbrace{\mathscr{H}om(\Omega_{\mathfrak{Y}/k}, \mathscr{O}_{\mathfrak{Y}})}_{=\mathcal{T}_{\mathfrak{Y}}} \to \underbrace{\mathscr{H}om(\varrho^*\Omega_{T^1/k}, \mathscr{O}_{\mathfrak{Y}})}_{=\varrho^*\mathcal{T}_{T^1} \cong \varrho^*\mathscr{O}_{T^1} \cong \mathscr{O}_{\mathfrak{Y}}}$$

The foliation $\mathcal{F}$ is embedded in $\mathcal{T}_{\mathfrak{Y}}$, hence there is a map $\gamma : \mathcal{F} \to \mathscr{O}_{\mathfrak{Y}}$, which is shown to be an embedding in the proof of [PW17, Theorem 3.1(b)]. We denote its dual by $\gamma' : \mathscr{O}_{\mathfrak{Y}} \to \mathcal{F}^*$.

Via the correspondence between line bundles and Cartier divisors, there exists a divisor $\mathfrak{C}$ on $\mathfrak{Y}$ such that $\mathscr{O}_{\mathfrak{Y}}(\mathfrak{C}) \sim \mathcal{F}^*$. We claim that $\gamma'$ is an isomorphism exactly where $\gamma$ is surjective. From this it follows that the support of $\mathfrak{C}$ is the locus where $\gamma$ is not surjective.

The fact that $\gamma'$ is an isomorphism exactly where $\gamma$ is surjective can be shown in two steps.

In the first step we show that the locus where $\gamma$ is not surjective is purely of codimension 1. For this, assume that $\xi$ is a point of codimension $\geq 2$ in $\mathfrak{Y}$. From the sequence

$$0 \to \mathcal{F} \to \mathscr{O}_{\mathfrak{Y}} \to \mathscr{O}_{\mathfrak{Y}}/\mathcal{F} \to 0$$

we get the following long exact sequence by taking cohomology

$$0 \to \underbrace{H^0_\xi(\mathfrak{Y}, \mathcal{F})}_{=0} \to \underbrace{H^0_\xi(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})}_{=0} \to H^0_\xi(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathcal{F}) \to \underbrace{H^1_\xi(\mathfrak{Y}, \mathcal{F})}_{=0} \to \cdots$$

The local cohomologies $H^i_\xi(\mathfrak{Y}, \mathcal{F})$ for $i = 0, 1$ are zero because the reflexive sheaf $\mathcal{F}$ satisfies the $S_2$ condition by [Har94, Theorem 1.9]. Hence by the Grothendieck vanishing Theorem, the first and second local cohomology vanishes for the point $\xi$ of codimension $\geq 2$. It follows that $H^0_\xi(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathcal{F}) = 0$, which means that $\mathscr{O}_{\mathfrak{Y}}/\mathcal{F}$ does not contain any points of codimension $\geq 2$. The locus where the map $\gamma$ is not surjective is the locus where $\mathscr{O}_{\mathfrak{Y}}/\mathcal{F}$ is not zero, hence the first claim follows.

For the second step, it suffices to exhibit the correspondence between the surjectivity of $\gamma$ and the bijectivity of $\gamma'$ at points of codimension 1. For this, let $\zeta$ be a point of $\mathfrak{Y}$ of codimension 1. We denote by $R$ the discrete valuation ring $\mathscr{O}_{\mathfrak{Y},\zeta}$. The foliation $\mathcal{F}$ is of rank 1, hence $\mathcal{F} \cong \mathscr{O}_{\mathfrak{Y}}$, and so at the stalk $\zeta$, $\mathcal{F}_\zeta = f \cdot \mathscr{O}_{\mathfrak{Y},\zeta} = f \cdot R =: M$, for some $f \in R$. Let $\alpha$ denote the morphism $\mathcal{F}_\zeta = M \to R = \mathscr{O}_{\mathfrak{Y},\zeta}$. The induced morphism $\gamma_\zeta : R \to R$ as a morphism of $R$-modules is defined by the image of 1. Denote by $t$ the generator of the maximal ideal of $R$. Then $\gamma_\zeta$ is defined as

$$\gamma_\zeta : \begin{cases} R_1 & \to & R_2 \\ 1 & \mapsto & \alpha(f) = u \cdot t^n \end{cases}$$

for some unit $u$ and $n \in \mathbb{N}$. The indices are added to simplify reading. This uniquely defines the morphism, since any $r \in R$ can be expressed as $r = r \cdot 1$, and hence $\gamma_\zeta(r) = \gamma_\zeta(r \cdot 1) = r \cdot \gamma_\zeta(1) = r \cdot \alpha(f) = r \cdot u \cdot t^n$.

This morphism is surjective if and only if $n = 0$. For $n \geq 1$, the image of $\gamma_\zeta$ does not contain the units of $R$. In order to prove this, assume that $n \geq 1$. Let $r \in R$, which can be uniquely expressed as $r = v \cdot t^m$, for some $m \in \mathbb{N}$. Then $\gamma_\zeta(r) = r \cdot \gamma_\zeta(1) = v \cdot t^m \cdot u \cdot t^n = u \cdot v \cdot t^{m+n}$, with $m + n \geq 1$. Hence independently of $r$, units can not be expressed as images of $r \in R$ via $\gamma_\zeta$, for $n \geq 1$.

On the dualized side, the induced morphism $\gamma'_\zeta$ corresponds to

$$\gamma'_\zeta : \begin{cases} \mathrm{Hom}(R_2, R) & \overset{-\circ\gamma_\zeta}{\to} & \mathrm{Hom}(R_1, R) \\ \mathrm{Id}_R & \mapsto & \mathrm{Id}_R \circ \gamma_\zeta \end{cases}$$

If $\gamma_\zeta$ is surjective, and hence $\gamma_\zeta$ is defined by the image of 1, which is equal to a unit $u$, then the dual $\gamma'_\zeta$ is an isomorphism. Therefore, at codimension 1 points, the required correspondence is satisfied. But by the first step, it suffices to consider codimension 1 points. Therefore, the support of the divisor $\mathfrak{C}$ which defines $\mathcal{F}^*$ is chosen such that $\gamma'$ is not an isomorphism. This corresponds to the locus where the morphism $\gamma$ is not surjective.

$\square$

# Chapter 3

# $\mathbb{P}^1 \times \mathbb{P}^1$ Example

The goal of this chapter is to prove the first main theorem.

**Main Theorem** (Theorem 1.0.3). *Let $(p, Y, C) = (2, \mathbb{P}^1 \times \mathbb{P}^1, F_i)$. Then there is a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface $X$ has one singular point, which is a Gorenstein $A_2$-singularity.*

In the previous chapter, we described the general setup as well as the approach of the proof of Theorem 1.0.1, for any of the triples in Theorem 1.0.1.Based on this approach, we construct a specific surface $X$ in one of the cases of Theorem 1.0.1. This construction follows the general strategy introduced above, given the two specific choices of the base $T$ and the form of the scheme $\mathfrak{Y}$. The desired properties of the surface $X$ are translated into conditions that the corresponding foliation needs to satisfy.

## 3.1 Notation and Assumptions

We now consider the following setup, stated in Assumption 3.1.1.

*Assumption* 3.1.1. We use the notations and constructions of Chapter 2. In what follows, we consider the triple

$$(p, Y, C) = \left(2, \mathbb{P}^1_{\overline{k}} \times_{\operatorname{Spec} \overline{k}} \mathbb{P}^1_{\overline{k}}, F\right)$$

from Theorem 1.0.1, with $F$ being one of the fibers of the projection of $\mathbb{P}^1_{\overline{k}} \times_{\operatorname{Spec} \overline{k}} \mathbb{P}^1_{\overline{k}}$ to $\mathbb{P}^1_{\overline{k}}$. Additionally, for the construction of an explicit example of a corresponding surface $X$, we have chosen the base $T$ of $\mathfrak{X}$ to be $T = \mathbb{A}^1_{\mathbb{F}_2}$. This way, $k = \mathbb{F}_2(x)$.

As stated in Section 2.1, it holds that the Frobenius step number $e = 1$. This holds due to the fact that the divisor $C$ on $Y$ is equal to $\left(\sum_{i=1}^{e}(\mathfrak{C}^i)_{\mathfrak{Y}^e}\right) \times_{T^e} \operatorname{Spec} \overline{k}$. But by Assumption 3.1.1, $C = F$ is the fiber of one of the projections of $\mathbb{P}^1_{\overline{k}} \times_{\operatorname{Spec} \overline{k}} \mathbb{P}^1_{\overline{k}}$ to $\mathbb{P}^1_{\overline{k}}$, which can not be expressed as a sum of more than one summands. Hence we see that $e = 1$. Only one Frobenius base change is applied to obtain the scheme $\mathfrak{Y}$ as the normalization of $\mathfrak{X} \times_T T^1$. The divisor $C$ is equal to $C = \mathfrak{C} \times_T \operatorname{Spec} \overline{k}$, where $\mathfrak{C}$ is a divisor on $\mathfrak{Y}$.

Under Assumption 3.1.1, we want to construct a surface $X$ corresponding to $(p, Y, C)$ fixed there. For this construction, the desired properties of the surface $X$ should be translated to the properties of $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ via Proposition 2.0.6 and the construction of Chapter 2.

*Properties* 3.1.2. The restrictions that ensure that the surface obtained by this construction is of the required form are the following, possibly allowing shrinking $T$.

*Property 1* The subsheaf $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ is a foliation. This property is necessary to obtain $\mathfrak{X}$ via the correspondence in Proposition 2.0.6.

*Property 2* $(\mathfrak{Y}, \mathfrak{C}) = (\mathbb{P}^1 \times \mathbb{P}^1 \times U, F_i \times U)$, for $U \subseteq T$ non-empty open. The choice of $\mathfrak{Y}$ we have made above states that $\mathfrak{Y}$ is of this form.

*Property 3* The surface $X$ is Gorenstein. Equivalently, $\mathfrak{X}$ is Gorenstein over the generic point of $T$.

*Property 4* Lastly, the anticanonical divisor of $X$ is ample. Equivalently, the anticanonical divisor of $\mathfrak{X}$ over $T$ is ample. This means that $K_{\mathfrak{Y}} + (p-1)\mathfrak{C}$ is ample over some open subset $U \subseteq T$. This is automatic if $(p, Y, C)$ is of the chosen type.

## 3.2   Restrictions

*Remark* 3.2.1. In the remainder of this chapter, we use the notation introduced in Section 3.1.

Let $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ be the foliation on $\mathfrak{Y}$ such that $\mathfrak{X} = \mathfrak{Y}/\mathcal{F}$. In order for the foliation $\mathcal{F}$ to correspond to a threefold $\mathfrak{X}$, and hence to a surface $X$ of the required form, the Properties 3.1.2 need to be satisfied. Let $U_{x,y}$ be the affine chart of $\mathfrak{Y}$, defined by

$$U_{x,y} := \mathbb{A}^1_x \times \mathbb{A}^1_y \times T \subseteq \mathbb{P}^1_{x,x'} \times \mathbb{P}^1_{y,y'} \times T = \mathfrak{Y}.$$

Firstly, we study what restrictions Property 1, the fact that $\mathcal{F}_{U_{x,y}} := \mathcal{F}\big|_{U_{x,y}}$ is indeed a foliation poses. The tangent sheaf $\mathcal{T}_{U_{x,y}}$ is reflexive and a locally free sheaf of rank equal to $\dim U_{x,y} = 3$. Since $\mathcal{F}_{U_{x,y}} \subseteq \mathcal{T}_{U_{x,y}}$ is by definition saturated and $\mathcal{T}_{U_{x,y}}$ is reflexive, so is $\mathcal{F}_{U_{x,y}}$. Assume that $\mathcal{F}_{U_{x,y}}$ is of rank 1. According to [Har80, Proposition 1.9], in this case, the rank 1 reflexive sheaf $\mathcal{F}_{U_{x,y}}$ is invertible. Furthermore, since the divisor class group $\mathrm{Cl}(U_{x,y}) = 0$ and $\mathrm{Pic}(U_{x,y}) \cong \mathrm{Cl}(U_{x,y})$, every invertible sheaf on $U_{x,y}$ is isomorphic to $\mathscr{O}_{U_{x,y}}$.

Locally on the affine chart $U_{x,y}$, the foliation $\mathcal{F}$ is therefore of the form

$$\mathcal{F}_{U_{x,y}} = \mathscr{O}_{U_{x,y}} \cdot v,$$

where $v \in \mathrm{Der}_{\mathbb{F}_2}(\mathscr{O}_{U_{x,y}}, \mathscr{O}_{U_{x,y}}) = \mathbb{F}_2[x,y,t]\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)$, and $t$ denotes the coordinate of $T$. We let $v$ be defined by

$$v = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t},$$

with $f, g, h \in \mathbb{F}_2[x,y,t]$. Due to the fact that the subsheaf $\mathcal{F}_{U_{x,y}}$ is saturated, the polynomials $f, g$ and $h$ have no common divisor. Furthermore, we may assume that the polynomials $f, g$ and $h$ are all different from zero.

In the remainder of this section, we study the restrictions obtained by Properties 3.1.2. From these restrictions, we deduce as many conditions on the polynomials $f, g$ and $h$ as possible. The result of this study is summed up in Proposition 3.3.6. In that proposition, we are able to express $f$ and $g$ as the sum of monomials of low degree in $x$ and $y$, multiplied by polynomials in the variable $t$, which need to satisfy certain relations between them. The proposition is restrictive enough for us to create explicit examples by choosing the polynomials in the variables $t$, in accordance with the proposition. One such example is Example 3.4.1, constructed in Section 3.4. In the last section, Section 3.5, we verify that the surface obtained by Example 3.4.1 indeed satisfies all requirements of Theorem 1.0.1.

The proposition below studies the first conditions on $f, g$ and $h$, using the restrictions posed by Property 1.

**Proposition 3.2.2.** *The polynomials $f, g$ and $h \in \mathbb{F}_2[x,y,t]$ define a foliation on the affine chart $U_{x,y}$ if and only if they satisfy the equations*

$$gh(ff_x + gf_y + hf_t) = fh(fg_x + gg_y + hg_t) = fg(fh_x + gh_y + hh_t), \qquad (3.2.1)$$

*where $f_x$ denotes the derivation $f_x = \frac{\partial}{\partial x}f$, and similarly for the other polynomials and derivations.*

*Proof.* By definition, $\mathcal{F}_{U_{x,y}}$ is a foliation if $\mathbb{F}_2[x,y,t] \cdot v$ is closed under addition and multiplication by $\mathbb{F}_2[x,y,t]$, which is clearly satisfied, and additionally if it is closed under Lie brackets and $p$-th powers.

The closedness under Lie brackets poses no conditions on $\mathcal{F}_{U_{x,y}}$, as $\mathrm{rk}(\mathcal{F}_{U_{x,y}}) = 1$.

The closedness under $p$-th power states that

$$\forall q \in \mathbb{F}_2[x,y,t], \quad (qv)^2 \in \mathbb{F}_2[x,y,t]v.$$

Since $(qv)^2 = qv(q)v + q^2v^2$, and $qv(q)v$ and $q^2$ are contained in $\mathbb{F}_2[x,y,t]v$, this holds if $v^2 \in \mathbb{F}_2[x,y,t]v$. But $v^2$ is of the form

$$v^2 = \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t}\right)\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t}\right)$$
$$= (ff_x + gf_y + hf_t)\frac{\partial}{\partial x} + (fg_x + gg_y + hg_t)\frac{\partial}{\partial y} + (fh_x + gh_y + hh_t)\frac{\partial}{\partial t}.$$

In the remainder of the proof, we show that $v^2 \in \mathbb{F}_2[x,y,t]v$ if and only if (3.2.1) holds.

As seen above,

$$v^2 = (ff_x + gf_y + hf_t)\frac{\partial}{\partial x} + (fg_x + gg_y + hg_t)\frac{\partial}{\partial y} + (fh_x + gh_y + hh_t)\frac{\partial}{\partial t}.$$

We assume that $v^2 \in \mathbb{F}_2[x,y,t]v$, which means that there exists a polynomial $r(x,y,t) \in \mathbb{F}_2[x,y,t]$ such that $v^2 = rv = r\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t}\right)$. Comparing the coefficients of the derivations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}$ of $v^2$ and $rv$, we get that

- $rf = (ff_x + gf_y + hf_t)$
- $rg = (fg_x + gg_y + hg_t)$
- $rh = (fh_x + gh_y + hh_t)$

From here it follows that (3.2.1) holds.

On the other hand, suppose that (3.2.1) holds. Then

$$fghv^2 = f\underbrace{gh(ff_x + gf_y + hf_t)}_{fg(fh_x+gh_y+hh_t)}\frac{\partial}{\partial x} + g\underbrace{fh(fg_x + gg_y + hg_t)}_{fg(fh_x+gh_y+hh_t)}\frac{\partial}{\partial y}$$
$$+ hfg(fh_x + gh_y + hh_t)\frac{\partial}{\partial t}$$
$$= \underbrace{fg(fh_x + gh_y + hh_t)}_{\in\mathbb{F}_2[x,y,t]}\underbrace{\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t}\right)}_{=v}$$

and so $fghv^2 \in \mathbb{F}_2[x,y,t]v$. Using the fact that the foliation $\mathcal{F}_{U_{x,y}} = \mathcal{O}_{U_{x,y}} \cdot v$ is saturated inside $\mathcal{T}_{U_{x,y}}$ by definition, it follows that $v^2 \in \mathbb{F}_2[x,y,t]v$. $\square$

In order to restrict the the possibilities for $f, g$ and $h$ further, we exploit the role of the divisor $\mathfrak{C}$ on $\mathfrak{Y}$, respectively $C$ on $Y$. For this, we recall Proposition 2.1.1, which states that the support of $\mathfrak{C}$ on $\mathfrak{Y}$ is equal to the locus where the map $\gamma : \mathcal{F} \to \varrho^*\mathcal{T}_{T^1}$ is not surjective, with $\varrho$ denoting the map form $\mathfrak{Y}$ to $T^1$.

We can explicitly characterize the map $\gamma$. From this characterization we get an explicit characterization of $\mathfrak{C}$. Let $y \in \mathfrak{Y}$. The induced map $\gamma_y$ is defined as

$$\gamma_y : \begin{cases} \mathcal{F}_y & \to & \varrho^*\mathcal{T}_{T^1,\varrho(y)} \cong \mathcal{O}_{\mathfrak{Y},y} \\ \mathcal{O}_{\mathfrak{Y},y} \cdot (f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial t}) & \mapsto & h\frac{\partial}{\partial t}. \end{cases}$$

This map is not surjective where $h = 0$. It follows that the support of the divisor $\mathfrak{C}$ is equal to the vanishing locus of the polynomial $h$.

**Important Consequence.** *By assumption, $C = F$ is one of the fibers of the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. By symmetry, and since the multiplicity is one, we are free to choose the projection to the x-coordinate. Let $F = \{x = 0\}$ the fiber over the point 0. It then follows that $h = x$.*

Using this description of $h$, the equations (3.2.1) translate to

$$gx(ff_x + gf_y + xf_t) = fx(fg_x + gg_y + xg_t) = ffg. \tag{3.2.2}$$

We use these equations in the section below to describe the general construction of a corresponding surface.

## 3.3   Construction

In order to obtain restrictions on the forms of the polynomials $f$ and $g$, we study the divisibility conditions that need to be satisfied so that the equations (3.2.2) can hold. The equations (3.2.2) imply the conditions of the lemma below.

**Claim 3.3.1.** *The equations (3.2.2) imply that the following divisibility conditions need to hold for the polynomials $f$ and $g$.*

$$f \big| g(gf_y + xf_t) \tag{3.3.1a}$$

$$f \big| x(gf_y + xf_t) \tag{3.3.1b}$$

$$g \big| f(fg_x + xg_t) \tag{3.3.1c}$$

$$g \big| x(fg_x + xg_t) \tag{3.3.1d}$$

$$x \big| f \cdot f \tag{3.3.1e}$$

$$x \big| g \cdot f \tag{3.3.1f}$$

*Proof.* These conditions follow immediately from the equations (3.2.2).                    □

**Claim 3.3.2.** *The conditions 3.3.1a to 3.3.1f are equivalent to the following conditions,*

$$f' \big| (gf_y' + xf_t') \tag{3.3.2a}$$

$$g \big| (f'g_x + g_t) \tag{3.3.2b}$$

*where $f'$ is the polynomial such that $f = x \cdot f'$.*

*Remark* 3.3.3. These two equations (3.3.2a) and (3.3.2b) describe conditions that are necessary for the equations (3.2.2) to hold, but they are not sufficient. The fact that these conditions are not sufficient is exhibited in Example 3.3.4. Hence in order to find an example that satisfies Property 1, we will use the conditions (3.3.2a) and (3.3.2b). However, for each example found this way, we need to verify that it additionally also satisfies the equations (3.2.2).

*Proof.* From (3.3.1e) it follows that $x \big| f$. Let $f = x \cdot f'$. Then the partial derivatives are

$$f_x = \frac{\partial}{\partial x} f = \frac{\partial}{\partial x}(x \cdot f') = f' + x \cdot f_x'$$

$$f_y = \frac{\partial}{\partial y} f = \frac{\partial}{\partial y}(x \cdot f') = x \cdot f_y'$$

$$f_t = \frac{\partial}{\partial t} f = x \cdot f_t'$$

Since by assumption, $f, g$ and $h$ do not share a common divisor, it follows that $x \nmid g$. From this it follows that $g$ and $x$ are coprime.

With $x \mid f$, the condition (3.3.1f) is satisfied. Replacing $f$ with $x \cdot f'$ in the remaining conditions, we get

$$x \cdot f' \big| g(gxf_y' + x^2 f_t') \tag{3.3.3a}$$

$$x \cdot f' \big| x(gxf_y' + x^2 f_t') \tag{3.3.3b}$$

$$g \big| xf'(xf'g_x + xg_t) \tag{3.3.3c}$$

$$g \big| x(xf'g_x + xg_t) \tag{3.3.3d}$$

Condition (3.3.3d) implies (3.3.3c), so we can dismiss condition (3.3.3c). Furthermore, since $x$ and $g$ are coprime, it follows from condition (3.3.3d) that $g \mid f'g_x + g_t$. From condition (3.3.3a) it follows that $f' \mid g(gf_y' + xf_t')$, and similarly for (3.3.3b). Hence we get

$$f' \mid g(gf'_y + xf'_t) \tag{3.3.4a}$$

$$f' \mid x(gf'_y + xf'_t) \tag{3.3.4b}$$

$$g \mid (f'g_x + g_t) \tag{3.3.4c}$$

Combining condition (3.3.4a) and (3.3.4b), we see that there exist $\alpha, \beta \in k[x, y, t]$ such that $f'\alpha = g(gf'_y + xf'_t)$ and $f'\beta = x(gf'_y + xf'_t)$. It follows that

$$\frac{f'\alpha}{g} = gf'_y + xf'_t = \frac{f'\beta}{x} \Rightarrow \frac{\alpha}{g} = \frac{\beta}{x} \Rightarrow \beta g = \alpha x.$$

Hence $x \mid \beta g$, and since $g$ and $x$ are coprime, it follows that $x \mid \beta$. So there exists $\beta' \in k[x, y, t]$ such that $\beta = \beta'x$. Condition (3.3.4b) changes to $f'\beta'x = x(gf'_y + xf'_t) \Rightarrow f' \mid (gf'_y + xf'_t)$. Similarly condition (3.3.4a) changes to the same, which leads to the two remaining conditions (3.3.2a) and (3.3.2b).

$\square$

**Example 3.3.4.** There are examples of polynomials $f$ and $g$ that satisfy the equations (3.3.2a) and (3.3.2b), but not the equation (3.2.2). One such case is if we let $f' = g = x + t^2$, and $h = x$. Then $f = x^2 + xt^2$. The equations (3.3.2a) and (3.3.2b) are satisfied, as

- $f' \mid gf'_y + xf'_t$, since $gf'_y + xf'_t = 0$ and $f' \mid 0$ and

- $g \mid f'g_x + g_t$, since $f'g_x + g_t = f'$ and $g = f'$.

But equation (3.2.2) is not satisfied, since

- $ffg = (x^2 + xt^2)(x + t^2) = (x^2 + xt^2)x + (x^2 + xt^2)t^2$, but

- $fx(fg_x + gg_y + hg_t) = fxf = (x^2 + xt^2)^2x$

which is not equal. This example shows that the conditions in Claim 3.3.2 are indeed only necessary conditions.

This concludes the study of the restrictions that Property 1 in Properties 3.1.2 poses.

To pose further restrictions on the polynomials $f$ and $g$, we use the fact that the anticanonical divisor $-K_X$ of the desired surface $X$ is ample, as stated as Property 4 in Properties 3.1.2. The divisors $C$ that appear in Theorem 1.0.1 are chosen in such a way that this condition is satisfied. Since $\mathcal{F}^* \cong \mathscr{O}_{\mathfrak{Y}}(\mathfrak{C})$, the foliation $\mathcal{F} \cong \mathscr{O}_{\mathfrak{Y}}(-\mathfrak{C})$. Furthermore, the divisor $C$ is defined by the fiber over the point $0$ of the projection $\mathbb{P}^1 \times \mathbb{P}^1$ to the $x$-coordinate. Hence $C = \{x = 0\}$, and

$$\mathcal{F} \cong \mathscr{O}_{\mathfrak{Y}}(-\mathfrak{C}) \cong \mathscr{O}_{\mathfrak{Y}}(-1, 0).$$

Knowing the degree of the foliation, we can further restrict the exponents of the variables $x$ and $y$ in $f$ and $g$.

**Claim 3.3.5.** *The exponents of the variables $x$ and $y$ in $f$ and $g$ are limited as follows:*

- *The exponent of the $x$-variable in $f$ is at most 3, and hence the exponent of the $x$-variable in $f'$ is at most 2.*

- *The exponent of the $y$-variable in $f$ is 0, and equally for $f'$.*

- *The exponent of the $x$-variable in $g$ is at most 1.*

- *The exponent of the $y$-variable in $g$ is at most 2.*

*Proof.* In the first variable, the foliation is of degree $-1$, meaning that $v$ has poles of order 1 in $x$. This holds on all affine charts, hence we first describe $\mathcal{F}$ on the charts $U_{x',y} = \operatorname{Spec} \mathbb{F}_2[x', y, t], U_{x,y'}$ and $U_{x',y'}$, where $x' = \frac{1}{x}$ and $y' = \frac{1}{y}$.

The foliation $\mathcal{F}$ on the chart $U_{x,y}$ is described by $\mathscr{O}_{U_{x,y}} \cdot v$, where $v = f\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}$ with $f, g \in \mathbb{F}_2[x, y, t]$. Restricted to the other affine charts, $\mathcal{F}$ is of the following form.

*Chart $U_{x',y}$*

$$v(x,y,t) = v\left(\frac{1}{x'}, y, t\right) = f\left(\frac{1}{x'}, y, t\right)\frac{\partial}{\partial x} + g\left(\frac{1}{x'}, y, t\right)\frac{\partial}{\partial y} + h\left(\frac{1}{x'}, y, t\right)\frac{\partial}{\partial t}$$

$$= f\left(\frac{1}{x'}, y, t\right)(x')^2\frac{\partial}{\partial x'} + g\left(\frac{1}{x'}, y, t\right)\frac{\partial}{\partial y} + \frac{1}{x'}\frac{\partial}{\partial t}.$$

*Chart $U_{x,y'}$*

$$v(x,y,t) = v\left(x, \frac{1}{y'}, t\right) = f\left(x, \frac{1}{y'}, t\right)\frac{\partial}{\partial x} + g\left(x, \frac{1}{y'}, t\right)(y')^2\frac{\partial}{\partial y'} + x\frac{\partial}{\partial t}.$$

*Chart $U_{x',y'}$*

$$v(x,y,t) = v\left(\frac{1}{x'}, \frac{1}{y'}, t\right) = f\left(\frac{1}{x'}, \frac{1}{y'}, t\right)(x')^2\frac{\partial}{\partial x'} + g\left(\frac{1}{x'}, \frac{1}{y'}, t\right)(y')^2\frac{\partial}{\partial y'} + \frac{1}{x'}\frac{\partial}{\partial t}.$$

It suffices to consider the chart $U_{x',y'}$. If the variable $\frac{1}{x'}$ appears to the power $k$ in the polynomial $f$, then after multiplication with $(x')^2$, this results in $(\frac{1}{x'})^{k-2}$ as coefficient in front of $\frac{\partial}{\partial x'}$. Since poles of order one in this variable may appear, the first variable can appear with exponent at most three. Due to similar arguments, we get the restrictions as stated. $\qquad\square$

Hence Property 4 in Properties 3.1.2 is satisfied by the restrictions posed in Claim 3.3.5.

We now combine the restrictions from Claim 3.3.2 and Claim 3.3.5, and get the following result.

**Proposition 3.3.6.** *The restrictions posed by Claim 3.3.2 and Claim 3.3.5 are combined in this proposition. The restrictions of those two claims combined are equivalent to the polynomials $f, g$ and $h \in \mathbb{F}_2[x,y,z]$ being of the form*

- $f = x \cdot f'(x,y,t)$, *with* $f'(x,y,t) = x^2 f_1(t) + f_2(t)$

- $g = xg_1(y,t) + g_2(y,t)$ *with*

  - $g_1(y,t) = y^2\alpha(t) + y\beta(t) + \gamma(t)$,
  - $g_2(y,t) = y^2\mu(t) + y\eta(t) + \epsilon(t)$

- $h = x$,

*where the polynomials $f_1, f_2, \alpha, \beta, \gamma, \mu, \eta, \epsilon \in \mathbb{F}_2[t]$ satisfy the following conditions*

$$f_2(t) = \eta(t), \text{ with the exponent of } t \text{ a multiple of } 2 \tag{3.3.5a}$$
$$\alpha(t)\eta(t) + \mu(t)\beta(t) + \mu_t(t) = 0 \tag{3.3.5b}$$
$$\gamma(t)\eta(t) + \epsilon(t)\beta(t) + \epsilon_t(t) = 0 \tag{3.3.5c}$$
$$f_1(t)\mu(t) + \alpha(t)\beta(t) + \alpha_t(t) = 0 \tag{3.3.5d}$$
$$f_1(t)\eta(t) + (\beta(t))^2 + \beta_t(t) = 0 \tag{3.3.5e}$$
$$f_1(t)\epsilon(t) + \gamma(t)\beta(t) + \gamma_t(t) = 0. \tag{3.3.5f}$$

*Proof.* We recall the restrictions (3.3.2a) and (3.3.2b), given by Claim 3.3.2:

$$f' \mid (gf'_y + xf'_t)$$
$$g \mid (f'g_x + g_t).$$

To these restrictions on $f'$ and $g$, we add the restrictions on the exponents appearing, given by Claim 3.3.5.

Since $f'$ does not contain the variable $y$, it holds that $f'_y = 0$. This transforms condition (3.3.2a) into

$$f' \mid (gf'_y + xf'_t) \Leftrightarrow f' \mid xf'_t. \tag{3.3.7}$$

Since the degree of $f'_t$ in the variable $t$ is lower than the degree of $f'$ in the variable $t$, there are two possibilities. Either $xf'_t = f'$, or $f'_t = 0$. From $f' = xf'_t$ it follows that $f' = 0$. This follows since the multiplication by the variable $x$ can not make up for the lower degree of $f'_t$ in the $t$-variable. But this is a case we excluded.

Hence assume that $f'_t = 0$. This means that the power of the variable $t$ in $f'$ is a multiple of 2. The equations (3.2.2) translate to

$$gx(xf'(f' + xf'_x) + gx\cancel{f'_y} + x^2\cancel{f'_t}) = x^2 f'(xf'g_x + gg_y + xg_t) = x^2(f')^2 g$$
$$\Leftrightarrow gx(x(f')^2 + x^2 f' f'_x) = x^2 f'(xf'g_x + gg_y + xg_t) = x^2(f')^2 g.$$

We get the following two equations

$$x^2(f')^2 g = gx(x(f')^2 + x^2 f' f'_x) \tag{3.3.8a}$$
$$x^2(f')^2 g = x^2 f'(xf'g_x + gg_y + xg_t) \tag{3.3.8b}$$

Considering equation (3.3.8a)), we get

$$(f')^2 = (f')^2 + xf' f'_x$$
$$\Leftrightarrow xf' f'_x = 0.$$

Since $f'$ is non-zero by assumption, it holds that $f'_x = 0$, which means that the exponent of the variable $x$ in $f'$ is divisible by 2. Together with the fact that $f'_t = 0$, and the fact that the exponent of the variable $x$ in $f'$ is at most 2, and the exponent of the variable $y$ in $f'$ is zero, it follows that $f'$ is a polynomial in the variables $x$ and $t$, where $x$ has exponent 2 or 0 and $t$ appears to any power divisible by 2. Hence $f'$ is of the following form

$$f'(x, y, t) = x^2 f_1(t) + f_2(t),$$

with $f_1$ and $f_2 \in \mathbb{F}_2[t]$ not both zero, such that the exponent of the variable $t$ is a multiple of 2.

The polynomial $g$ is of the following form. The exponent of the variable $x$ is at most 1, and by the coprime assumption, $g$ is not divisible by $x$. Hence we can write $g$ as

$$g(x, y, t) = xg_1(y, t) + g_2(y, t)$$

with $g_1, g_2 \in \mathbb{F}_2[y, t]$ such that the exponent of $y$ is at most 2, and $g_2 \neq 0$. The partial derivations of $g$ are

$$g_x(x, y, t) = g_1(y, t)$$
$$g_y(x, y, t) = xg_{1,y}(y, t) + g_{2,y}(y, t)$$
$$g_t(x, y, t) = xg_{1,t}(y, t) + g_{2,t}(y, t),$$

where $g_{1,y}$ denotes $\frac{\partial}{\partial y} g_1$, and similarly for the remaining derivations. Using these descriptions of $f'$ and $g$, equation (3.3.8b) translates to

$$f'g = xf'g_x + gg_y + xg_t$$
$$\Leftrightarrow \quad (x^2 f_1 + f_2)(xg_1 + g_2) = x(x^2 f_1 + f_2)g_1 + (xg_1 + g_2)(xg_{1,y} + g_{2,y}) + x(xg_{1,t} + g_{2,t})$$
$$\Leftrightarrow \quad 0 = x^3(f_1 g_1 + f_1 g_1) + x^2(f_1 g_2 + g_1 g_{1,y} + g_{1,t}) + x(f_2 g_1 + f_2 g_1 + g_1 g_{2,y} + g_2 g_{1,y} + g_{2,t})$$
$$\qquad + (f_2 g_2 + g_2 g_{2,y}).$$

This is satisfied if and only if the following equations hold

$$f_1 g_1 + f_1 g_1 = 0, \text{ which is automatically satisfied in characteristic 2,} \tag{3.3.9a}$$
$$f_1 g_2 + g_1 g_{1,y} + g_{1,t} = 0, \tag{3.3.9b}$$
$$f_2 g_1 + f_2 g_1 + g_1 g_{2,y} + g_2 g_{1,y} + g_{2,t} = 0 \Rightarrow g_1 g_{2,y} + g_2 g_{1,y} + g_{2,t} = 0, \tag{3.3.9c}$$
$$f_2 g_2 + g_2 g_{2,y} = 0. \tag{3.3.9d}$$

From equation (3.3.9d) if follows that $g_2(f_2 + g_{2,y}) = 0$. By the coprime assumption, $g_2 \neq 0$, and so $f_2(t) + g_{2,y}(y,t) = 0$.

Let $g_1(y,t)$ and $g_2(y,t)$ be of the following forms

$$g_1(y,t) = y^2\alpha(t) + y\beta(t) + \gamma(t)$$
$$g_2(y,t) = y^2\mu(t) + y\eta(t) + \epsilon(t),$$

with $\alpha, \beta, \gamma, \mu, \eta, \epsilon \in \mathbb{F}_2[t]$.

Equation (3.3.9d) results in $f_2(t) = g_{2,y}(y,t)$. With the above description of $g_2$, it follows that $f_2(t) = g_{2,y}(y,t) = \eta(t)$. Note that since the exponent of the variable $t$ in $f_2$ is divisible by 2, it holds that $\eta_t(t) = 0$.

We will further evaluate the form of $f'$ and $g$ using equations (3.3.9c) and (3.3.9b). Equation (3.3.9c) states

$$g_1(y,t)g_{2,y}(y,t) + g_2(y,t)g_{1,y}(y,t) + g_{2,t}(y,t) = 0$$
$$\Leftrightarrow \quad (y^2\alpha(t) + y\beta(t) + \gamma(t))\eta(t) + (y^2\mu(t) + y\eta(t) + \epsilon(t))\beta(t) + (y^2\mu_t(t) + y\eta_t(t) + \epsilon_t(t)) = 0$$
$$\Leftrightarrow \quad y^2(\alpha(t)\eta(t) + \mu(t)\beta(t) + \mu_t(t)) + y(2\beta(t)\eta(t)) + (\gamma(t)\eta(t) + \epsilon(t)\beta(t) + \epsilon_t(t)) = 0,$$

where $\mu_t(t)$ denotes the partial derivation of the polynomial $\mu$ with respect to the variable $t$, and similarly for the remaining polynomials. Hence (3.3.9c) is satisfied if and only if

$$\alpha(t)\eta(t) + \mu(t)\beta(t) + \mu_t(t) = 0 \tag{3.3.10a}$$
$$\gamma(t)\eta(t) + \epsilon(t)\beta(t) + \epsilon_t(t) = 0. \tag{3.3.10b}$$

Equation (3.3.9b) states

$$f_1(t)g_2(y,t) + g_1(y,t)g_{1,y}(y,t) + g_{1,t}(y,t) = 0$$
$$\Leftrightarrow \quad f_1(t)(y^2\mu(t) + y\eta(t) + \epsilon(t)) + (y^2\alpha(t) + y\beta(t) + \gamma(t))\beta(t) + (y^2\alpha_t(t) + y\beta_t(t) + \gamma_t(t)) = 0$$
$$\Leftrightarrow \quad y^2(f_1(t)\mu(t) + \alpha(t)\beta(t) + \alpha_t(t)) + y(f_1(t)\eta(t) + \beta^2(t) + \beta_t(t))$$
$$+ (f_1(t)\epsilon(t) + \gamma(t)\beta(t) + \gamma_t(t)) = 0.$$

This is satisfied if and only if the following three equations hold

$$f_1(t)\mu(t) + \alpha(t)\beta(t) + \alpha_t(t) = 0 \tag{3.3.11a}$$
$$f_1(t)\eta(t) + (\beta(t))^2 + \beta_t(t) = 0 \tag{3.3.11b}$$
$$f_1(t)\epsilon(t) + \gamma(t)\beta(t) + \gamma_t(t) = 0. \tag{3.3.11c}$$

Summing up, we arrive to the conclusion that the polynomials $f, g$ and $h \in \mathbb{F}_2[x,y,t]$ need to be of the form

- $f = x \cdot f'(x,y,t)$, with $f'(x,y,t) = x^2 f_1(t) + f_2(t)$

- $g = xg_1(y,t) + g_2(y,t)$ with

    ○ $g_1(y,t) = y^2\alpha(t) + y\beta(t) + \gamma(t)$,
    ○ $g_2(y,t) = y^2\mu(t) + y\eta(t) + \epsilon(t)$

- $h = x$,

such that the polynomials $f_1, f_2, \alpha, \beta, \gamma, \mu, \eta, \epsilon$ satisfy the conditions (3.3.10a),(3.3.10b) and (3.3.11a) to (3.3.11c), and the additional condition

$$f_2(t) = \eta(t),$$

with the exponent of $t$ a multiple of 2. $\qquad\square$

Using Proposition 3.3.6 we are now able to choose some of the parameters $\alpha$ to $\epsilon$ that define the polynomials $f$ and $g$ in order to construct an explicit example of a foliation that eventually gives back a surface $X$ of the required form.

## 3.4  Example of a foliation

We construct an explicit example as follows, using Proposition 3.3.6.

**Example 3.4.1.** In order to explicitly construct a foliation $\mathcal{F}$ of the required form, we let $f_2(t) = 0, \beta(t) = 0$ and $f_1(t) = 1$. With these choices, the conditions (3.3.5a) to (3.3.5f) in Proposition 3.3.6 translate to

$$\eta(t) = f_2(t) = 0 \tag{3.4.1a}$$
$$\alpha(t)\eta(t) + \mu(t)\beta(t) + \mu_t(t) = 0 \Rightarrow \mu_t(t) = 0 \tag{3.4.1b}$$
$$\gamma(t)\eta(t) + \epsilon(t)\beta(t) + \epsilon_t(t) = 0 \Rightarrow \epsilon_t(t) = 0 \tag{3.4.1c}$$
$$f_1(t)\mu(t) + \alpha(t)\beta(t) + \alpha_t(t) = 0 \Rightarrow \mu(t) + \alpha_t(t) = 0 \tag{3.4.1d}$$
$$f_1(t)\eta(t) + (\beta(t))^2 + \beta_t(t) = 0 \tag{3.4.1e}$$
$$f_1(t)\epsilon(t) + \gamma(t)\beta(t) + \gamma_t(t) = 0 \Rightarrow \epsilon(t) + \gamma_t(t) = 0. \tag{3.4.1f}$$

Choosing $\alpha(t) = \gamma(t) = t$ and $\mu(t) = \epsilon(t) = 1$, these equations are satisfied. Furthermore, one can easily verify that the initial restrictions (3.2.2), which ensure that $\mathcal{F}$ is indeed a foliation are satisfied by this choice.

Hence we get an example of a foliation on $U_{x,y}$ with

$$f(x, y, t) = x(x^2 f_1(t) + f_2(t)) = x^3$$

and

$$g(x, y, t) = x(y^2\alpha(t) + y\beta(t) + \gamma(t)) + y^2\mu(t) + y\eta(t) + \epsilon(t) = xy^2 t + xt + y^2 + 1.$$

The resulting foliation on the chart $U_{x,y}$ is defined by $\mathcal{F}\big|_{U_{x,y}} = \mathcal{O}_{U_{x,y}} \cdot v$, with

$$v(x, y, t) = x^3\frac{\partial}{\partial x} + (xy^2 t + xt + y^2 + 1)\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}.$$

For the construction of this example we have not taken into consideration the remaining Property 3 in Properties 3.1.2. However, as we will see in Proposition 3.5.3, the singularities that arise for the surface constructed this way are in fact Gorenstein.

## 3.5  Properties of the Resulting Surface

After having constructed the surface $X$ above, we now study its properties, and verify that all requirements of Theorem 1.0.1 are satisfied.

We first study the foliation $\mathcal{F}$ on the affine chart $U_{x,y}$. There, it is defined via

$$v(x, y, t) = x^3\frac{\partial}{\partial x} + (xy^2 t + xt + y^2 + 1)\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}.$$

For $x = 0$ and $xy^2 t + xt + y^2 + 1 = 0 \Rightarrow y^2 + 1 = 0$, it is not regular.

**Proposition 3.5.1.** *The arising singularity on the chart $U_{x,y}$ is an $A_2$ surface singularity, after passing to the function field of the base.*

*Proof.* In order to study the arising singularity, we change coordinates to $\tilde{y} := y + 1$. Hence, there is a singular line at $x = 0, \tilde{y} = 0$, and $t \in T$. Since $\tilde{y}^2 = (y + 1)^2 = y^2 + 1$, the foliation is defined via

$$v(x, y, t) = x^3 \frac{\partial}{\partial x} + (xt(y^2 + 1) + (y^2 + 1)) \frac{\partial}{\partial y} + x \frac{\partial}{\partial t} = x^3 \frac{\partial}{\partial x} + (x\tilde{y}^2 t + \tilde{y}^2) \frac{\partial}{\partial \tilde{y}} + x \frac{\partial}{\partial t}.$$

To simplify the notation, we use the variable $y$ instead of $\tilde{y}$.

The schematic illustration below shows the two blow ups we have to perform. At the root of the two arrows we have the chart which is being blown up. Above it, connected with the two-branched arrow are the two charts of the blow up. This illustration helps to keep track of the notation of both charts and coordinates. We remark that we use the same notation for the coordinates of the two open charts of a blow up. This is indicated in the diagram below as well.
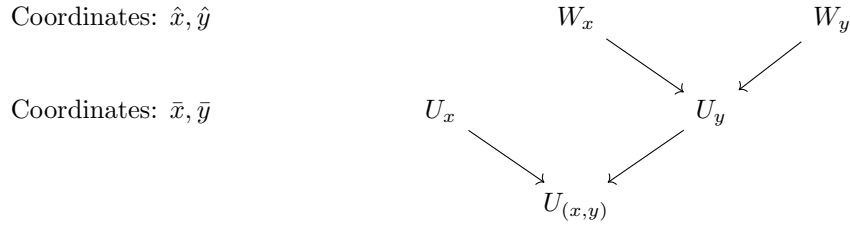


Figure 3.1: Schematic illustration of the two blow ups

Furthermore, we let $\mathbb{A}^3_{x,y,t} := \operatorname{Spec} \mathbb{F}_2[x, y, t]$, and $L := \{(0, 0, t) \,|\, t \in T\}$ be the line along which we blow up, with defining ideal $I(L) = (x, y)$. Consider the map

$$\begin{array}{rcl}
\mathbb{A}^3_{x,y,t} \setminus L & \to & \mathbb{A}^3 \times \mathbb{P}^1 \\
(x, y, t) & \mapsto & ((x, y, t), [x : y]).
\end{array}$$

The blow up is defined to be the closure in $\mathbb{A}^3 \times \mathbb{P}^1$ of the image of the above map. Hence $\operatorname{Bl}_L \mathbb{A}^3_{x,y,t} = \overline{\{((x, y, t), [u : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 \,|\, xv = uy\}}$. We denote the blow up by $\pi : \operatorname{Bl}_L \mathbb{A}^3_{x,y,t} \to \mathbb{A}^3_{x,y,t}$. As $\pi$ is an isomorphism when restricted to the preimage of $\mathbb{A}^3_{x,y,t} \setminus L$, it holds that $\pi^* \mathcal{F}|_{\mathbb{A}^3 \setminus L}$ is a foliation on this preimage and hence extends uniquely to a foliation $\mathcal{F}_{\operatorname{Bl}_L \mathbb{A}^3}$ on $\operatorname{Bl}_L \mathbb{A}^3$ by saturatedness.

$\operatorname{Bl}_L \mathbb{A}^3$ can be covered by two affine charts $U_x$ and $U_y$, which are given by

$$U_x = \{((x, y, t), [1 : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 \,|\, xv = y\},$$

$$U_y = \{((x, y, t), [u : 1]) \in \mathbb{A}^3 \times \mathbb{P}^1 \,|\, x = uy\}.$$

Both charts are isomorphic to $\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}]$ via the following isomorphisms

$$\begin{array}{rcl}
\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] & \to & U_x \\
(\overline{x}, \overline{y}, \overline{t}) & \mapsto & ((\overline{x}, \overline{x}\overline{y}, \overline{t}), [1 : \overline{y}])
\end{array}$$

and

$$\begin{array}{rcl}
\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] & \to & U_y \\
(\overline{x}, \overline{y}, \overline{t}) & \mapsto & ((\overline{x}\overline{y}, \overline{y}, \overline{t}), [\overline{x} : 1]).
\end{array}$$

The map $\pi$ restricted to these charts is

$$\begin{array}{rcl}
\mathbb{F}_2[x, y, t] & \to & \mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] \\
x & \mapsto & \overline{x} \\
y & \mapsto & \overline{x}\overline{y} \\
t & \mapsto & \overline{t}
\end{array}$$

on the chart $U_x$, and

$$
\begin{aligned}
\mathbb{F}_2[x,y,t] &\rightarrow \mathbb{F}_2[\overline{x},\overline{y},\overline{t}] \\
x &\mapsto \overline{x}\,\overline{y} \\
y &\mapsto \overline{y} \\
t &\mapsto \overline{t}
\end{aligned}
$$

on the chart $U_y$.

On the chart $U_x$, with the blow up defined as above, the derivations transform as follows

$$
x\frac{\partial}{\partial x} = \overline{x}\frac{\partial}{\partial \overline{x}} + \overline{y}\frac{\partial}{\partial \overline{y}}, \quad y\frac{\partial}{\partial y} = \overline{y}\frac{\partial}{\partial \overline{y}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.
$$

Hence the foliation $\mathcal{F}$ is defined by

$$
\begin{aligned}
v(x,y,t) &= x^3\frac{\partial}{\partial x} + (xy^2 t + y^2)\frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \\
&= x^2 x\frac{\partial}{\partial x} + (xyt + y)y\frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \\
&= \overline{x}^2\left(\overline{x}\frac{\partial}{\partial \overline{x}} + \overline{y}\frac{\partial}{\partial \overline{y}}\right) + (\overline{x}^2\overline{y}t + \overline{x}\,\overline{y})\overline{y}\frac{\partial}{\partial \overline{y}} + \overline{x}\frac{\partial}{\partial \overline{t}} \\
&= \overline{x}^3\frac{\partial}{\partial \overline{x}} + (\overline{x}^2\overline{y} + \overline{x}^2\overline{y}^2 t + \overline{x}\,\overline{y}^2)\frac{\partial}{\partial \overline{y}} + \overline{x}\frac{\partial}{\partial \overline{t}}
\end{aligned}
$$

Dividing by $\overline{x}$, we get

$$
\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3, U_x} = \overline{x}^2\frac{\partial}{\partial \overline{x}} + (\overline{x}\,\overline{y} + \overline{x}\,\overline{y}^2 t + \overline{y}^2)\frac{\partial}{\partial \overline{y}} + 1\frac{\partial}{\partial \overline{t}}
$$

on the chart $U_x$. Hence after blowing up, the foliation becomes regular. It remains to compute the discrepancy of this blow up. Consider the following diagram,

$$
\begin{array}{ccc}
\mathrm{Bl}_L\,\mathbb{A}^3 & \xrightarrow{\ \beta\ } & \mathrm{Bl}_L\,\mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
\mathbb{A}^3 & \xrightarrow[\ \alpha\ ]{} & \mathbb{A}^3/\mathcal{F}
\end{array}
$$

Figure 3.2: Notation for the blow up

where we denote by $E$ the exceptional divisor of the blow up $\pi$ and by $E'$ the exceptional divisor of the blow up $\pi'$, the blow up of the quotient space. In order to calculate the discrepancy of the blow up $\pi'$, we let $K_{\mathrm{Bl}_L\,\mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} = (\pi')^* K_{\mathbb{A}^3/\mathcal{F}} + aE'$, where $a$ denotes the discrepancy. Since the blow up $\pi$ describes the blow up of a line in $\mathbb{A}^3$, its discrepancy is equal to one and by the adjunction formula we have

$$
\begin{aligned}
K_{\mathrm{Bl}_L\,\mathbb{A}^3} &= \pi^* K_{\mathbb{A}^3} + E \\
&\cong \pi^*(\alpha^* K_{\mathbb{A}^3/\mathcal{F}} - (1-p)c_1(\mathcal{F})) + E, \\
&\quad \text{by adjunction, where } c_1(\mathcal{F}) = 0 \\
&= \pi^*\alpha^* K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(\pi')^* K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(K_{\mathrm{Bl}_L\,\mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} - aE') + E \\
&= \beta^*(K_{\mathrm{Bl}_L\,\mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}}) - a\beta^*(E') + E \\
&\cong (K_{\mathrm{Bl}_L\,\mathbb{A}^3} + (1-p)c_1(\mathcal{F}_{\mathrm{Bl}})) - a\beta^*(E') + E, \text{ by adjunction} \\
&= K_{\mathrm{Bl}_L\,\mathbb{A}^3} - E - a\beta^*(E') + E.
\end{aligned}
$$

The fact that $c_1(\mathcal{F}) = 0$ follows from the fact that $\mathcal{F} \cong \mathscr{O}_{\mathbb{A}^3}$, which holds due to the fact that every line bundle on $\mathbb{A}^3$ is trivial. Furthermore, the last equality holds because in order to obtain the foliation $\mathcal{F}_{\mathrm{Bl}}$ from $\mathcal{F}$, we divide by $\overline{x}$. This means that we divide by one time the exceptional divisor $E$, and so $c_1(\mathcal{F}_{\mathrm{Bl}}) = 1 \cdot E$.

It follows that

$$K_{\mathrm{Bl}_L \, \mathbb{A}^3} = K_{\mathrm{Bl}_L \, \mathbb{A}^3} - a\beta^*(E'),$$

and hence the discrepancy $a$ is equal to zero.

Considering the other chart $U_y$, we recall that the blow up is defined by

$$\begin{aligned}
\mathbb{F}_2[x, y, t] &\rightarrow \mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] \\
x &\mapsto \overline{x}\,\overline{y} \\
y &\mapsto \overline{y} \\
t &\mapsto \overline{t}
\end{aligned}$$

and the derivations transform as follows

$$x\frac{\partial}{\partial x} = \overline{x}\frac{\partial}{\partial \overline{x}}, \quad y\frac{\partial}{\partial y} = \overline{x}\frac{\partial}{\partial \overline{x}} + \overline{y}\frac{\partial}{\partial \overline{y}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.$$

Hence the foliation $\mathcal{F}$ becomes

$$\begin{aligned}
\mathcal{F} &= x^3\frac{\partial}{\partial x} + (xy^2 t + y^2)\frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \\
&= x^2 x\frac{\partial}{\partial x} + (xyt + y)y\frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \\
&= \overline{x}^2\overline{y}^2\overline{x}\frac{\partial}{\partial \overline{x}} + (\overline{x}\,\overline{y}^2\overline{t} + \overline{y})\left(\overline{x}\frac{\partial}{\partial \overline{x}} + \overline{y}\frac{\partial}{\partial \overline{y}}\right) + \overline{x}\,\overline{y}\frac{\partial}{\partial \overline{t}} \\
&= \left(\overline{x}^3\overline{y}^2 + \overline{x}^2\overline{y}^2\overline{t} + \overline{x}\,\overline{y}\right)\frac{\partial}{\partial \overline{x}} + (\overline{x}\,\overline{y}^3\overline{t} + \overline{y}^2)\frac{\partial}{\partial \overline{y}} + \overline{x}\,\overline{y}\frac{\partial}{\partial \overline{t}}
\end{aligned}$$

Dividing by $\overline{y}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_y} = \left(\overline{x}^3\overline{y} + \overline{x}^2\overline{y}\overline{t} + \overline{x}\right)\frac{\partial}{\partial \overline{x}} + (\overline{x}\,\overline{y}^2\overline{t} + \overline{y})\frac{\partial}{\partial \overline{y}} + \overline{x}\frac{\partial}{\partial \overline{t}}$$

Due to a similar argument as on the chart $U_x$, the discrepancy of the blow up on the quotient spaces is zero.

We get a singular line again for $\overline{x} = \overline{y} = 0$. Hence we blow up once more to get rid of this singular line.

On the chart $U_y$, the foliation is defined as

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_y} = \left(\overline{x}^3\overline{y} + \overline{x}^2\overline{y}\overline{t} + \overline{x}\right)\frac{\partial}{\partial \overline{x}} + (\overline{x}\,\overline{y}^2\overline{t} + \overline{y})\frac{\partial}{\partial \overline{y}} + \overline{x}\frac{\partial}{\partial \overline{t}}.$$

Similar to the first blow up, we blow up $\mathrm{Spec}\,\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}]$ along the line $L$ defined by the ideal $(\overline{x}, \overline{y})$. We denote the blow up by $\psi : \mathrm{Bl}_L\,\mathbb{A}^3_{\overline{x}, \overline{y}, \overline{t}} \rightarrow \mathbb{A}^3_{\overline{x}, \overline{y}, \overline{t}}$. Let $W_x$ and $W_y$ be the two affine charts covering $\mathrm{Bl}_L\,\mathbb{A}^3_{\overline{x}, \overline{y}, \overline{t}}$, which are both isomorphic to $\mathrm{Spec}\,\mathbb{F}_2[\hat{x}, \hat{y}, \hat{t}]$.

The blow up $\psi$ restricted to these charts is

$$\begin{aligned}
\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] &\rightarrow \mathbb{F}_2[\hat{x}, \hat{y}, \hat{t}] \\
\overline{x} &\mapsto \hat{x} \\
\overline{y} &\mapsto \hat{x}\hat{y} \\
\overline{t} &\mapsto \hat{t}
\end{aligned}$$

on the chart $W_x$, and

$$\begin{aligned}
\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] &\rightarrow \mathbb{F}_2[\hat{x}, \hat{y}, \hat{t}] \\
\overline{x} &\mapsto \hat{x}\hat{y} \\
\overline{y} &\mapsto \hat{y} \\
\overline{t} &\mapsto \hat{t}
\end{aligned}$$

on the chart $W_y$.

On the chart $W_x$, with the blow up defined as above, the derivations transform as follows

$$\overline{x}\frac{\partial}{\partial\overline{x}} = \hat{x}\frac{\partial}{\partial\hat{x}} + \hat{y}\frac{\partial}{\partial\hat{y}}, \quad \overline{y}\frac{\partial}{\partial\overline{y}} = \hat{y}\frac{\partial}{\partial\hat{y}}, \quad \frac{\partial}{\partial\overline{t}} = \frac{\partial}{\partial\hat{t}}.$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y}$ becomes

$$
\begin{aligned}
\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y} &= \left(\overline{x}^3\overline{y} + \overline{x}^2\overline{y}\overline{t} + \overline{x}\right)\frac{\partial}{\partial\overline{x}} + \left(\overline{x}\,\overline{y}^2\overline{t} + \overline{y}\right)\frac{\partial}{\partial\overline{y}} + \overline{x}\frac{\partial}{\partial\overline{t}} \\
&= \left(\overline{x}^2\overline{y} + \overline{x}\,\overline{y}\overline{t} + 1\right)\overline{x}\frac{\partial}{\partial\overline{x}} + (\overline{x}\,\overline{y}\,\overline{t} + 1)\overline{y}\frac{\partial}{\partial\overline{y}} + \overline{x}\frac{\partial}{\partial\overline{t}} \\
&= (\hat{x}^3\hat{y} + \hat{x}^2\hat{y}\hat{t} + 1)\left(\hat{x}\frac{\partial}{\partial\hat{x}} + \hat{y}\frac{\partial}{\partial\hat{y}}\right) + (\hat{x}^2\hat{y}\hat{t} + 1)\hat{y}\frac{\partial}{\partial\hat{y}} + \hat{x}\frac{\partial}{\partial\hat{t}} \\
&= (\hat{x}^4\hat{y} + \hat{x}^3\hat{y}\hat{t} + \hat{x})\frac{\partial}{\partial\hat{x}} + (\hat{x}^3\hat{y}^2 + \hat{x}^2\hat{y}^2\hat{t} + \hat{y} + \hat{x}^2\hat{y}^2\hat{t} + \hat{y})\frac{\partial}{\partial\hat{y}} + \hat{x}\frac{\partial}{\partial\hat{t}} \\
&= (\hat{x}^4\hat{y} + \hat{x}^3\hat{y}\hat{t} + \hat{x})\frac{\partial}{\partial\hat{x}} + \hat{x}^3\hat{y}^2\frac{\partial}{\partial\hat{y}} + \hat{x}\frac{\partial}{\partial\hat{t}}.
\end{aligned}
$$

Dividing by $\hat{x}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y,W_x} = (\hat{x}^3\hat{y} + \hat{x}^2\hat{y}\hat{t} + 1)\frac{\partial}{\partial\hat{x}} + \hat{x}^2\hat{y}^2\frac{\partial}{\partial\hat{y}} + 1\frac{\partial}{\partial\hat{t}}.$$

Hence the foliation becomes regular on this chart after the second blow up. The discrepancy of this blow up is equal to zero, due to a similar argument as for the first blow up. It remains to check the second affine chart.

On the chart $W_y$, the blow up is defined by

$$
\begin{aligned}
\mathbb{F}_2[\overline{x}, \overline{y}, \overline{t}] &\rightarrow \mathbb{F}_2[\hat{x}, \hat{y}, \hat{t}] \\
\overline{x} &\mapsto \hat{x}\hat{y} \\
\overline{y} &\mapsto \hat{y} \\
\overline{t} &\mapsto \hat{t}
\end{aligned}
$$

and the derivations transform as follows

$$\overline{x}\frac{\partial}{\partial\overline{x}} = \hat{x}\frac{\partial}{\partial\hat{x}}, \quad \overline{y}\frac{\partial}{\partial\overline{y}} = \hat{x}\frac{\partial}{\partial\hat{x}} + \hat{y}\frac{\partial}{\partial\hat{y}}, \quad \frac{\partial}{\partial\overline{t}} = \frac{\partial}{\partial\hat{t}}.$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y}$ becomes

$$
\begin{aligned}
\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y} &= \left(\overline{x}^3\overline{y} + \overline{x}^2\overline{y}\overline{t} + \overline{x}\right)\frac{\partial}{\partial\overline{x}} + \left(\overline{x}\,\overline{y}^2\overline{t} + \overline{y}\right)\frac{\partial}{\partial\overline{y}} + \overline{x}\frac{\partial}{\partial\overline{t}} \\
&= \left(\overline{x}^2\overline{y} + \overline{x}\,\overline{y}\overline{t} + 1\right)\overline{x}\frac{\partial}{\partial\overline{x}} + (\overline{x}\,\overline{y}\,\overline{t} + 1)\overline{y}\frac{\partial}{\partial\overline{y}} + \overline{x}\frac{\partial}{\partial\overline{t}} \\
&= (\hat{x}^2\hat{y}^3 + \hat{x}\hat{y}^2\hat{t} + 1)\hat{x}\frac{\partial}{\partial\hat{x}} + (\hat{x}\hat{y}^2\hat{t} + 1)\left(\hat{x}\frac{\partial}{\partial\hat{x}} + \hat{y}\frac{\partial}{\partial\hat{y}}\right) + \hat{x}\hat{y}\frac{\partial}{\partial\hat{t}} \\
&= (\hat{x}^3\hat{y}^3 + \hat{x}^2\hat{y}^2\hat{t} + \hat{x} + \hat{x}^2\hat{y}^2\hat{t} + \hat{x})\frac{\partial}{\partial\hat{x}} + (\hat{x}\hat{y}^3\hat{t} + \hat{y})\frac{\partial}{\partial\hat{y}} + \hat{x}\hat{y}\frac{\partial}{\partial\hat{t}} \\
&= (\hat{x}^3\hat{y}^3)\frac{\partial}{\partial\hat{x}} + (\hat{x}\hat{y}^3\hat{t} + \hat{y})\frac{\partial}{\partial\hat{y}} + \hat{x}\hat{y}\frac{\partial}{\partial\hat{t}}.
\end{aligned}
$$

Dividing by $\hat{y}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3,U_y,W_y} = (\hat{x}^3\hat{y}^2)\frac{\partial}{\partial\hat{x}} + (\hat{x}\hat{y}^2\hat{t} + 1)\frac{\partial}{\partial\hat{y}} + \hat{x}\frac{\partial}{\partial\hat{t}}.$$

The foliation is regular, from which we conclude that $X$ has an $A_2$ surface singularity. Hence $X$ is normal, but not regular. $\qquad\square$

Similarly, we analyze the foliation on the other affine charts.

**Proposition 3.5.2.** *On the affine charts $U_{x',y}$ and $U_{x',y'}$, the foliation is regular, and on the chart $U_{x,y'}$, there is an $A_2$ surface singularity, after passing to the function field of the base. This singularity is the same singularity as the one on the chart $U_{x,y}$.*

*Proof.* On the charts $U_{x',y}$ and $U_{x',y'}$, the foliation is defined via

$$v(x,y,t) = v\left(\frac{1}{x'},y,t\right) = \left(\frac{1}{x'}\right)\frac{\partial}{\partial x'} + \left(\left(\frac{1}{x'}\right)y^2t + \left(\frac{1}{x'}\right)t + y^2 + 1\right)\frac{\partial}{\partial y} + \left(\frac{1}{x'}\right)\frac{\partial}{\partial t},$$

and

$$v(x,y,t) = v\left(\frac{1}{x'},\frac{1}{y'},t\right) = \left(\frac{1}{x'}\right)\frac{\partial}{\partial x'} + \left(\left(\frac{1}{x'}\right)t + \left(\frac{1}{x'}\right)(y')^2t + 1 + (y')^2\right)\frac{\partial}{\partial y} + \left(\frac{1}{x'}\right)\frac{\partial}{\partial t},$$

respectively. Due to the coefficient $\frac{1}{x'}$ of $\frac{\partial}{\partial x'}$ which can not be zero, these two foliations are regular.

On the remaining chart $U_{x,y'}$, the foliation is defined via

$$v(x,y,t) = v\left(x,\frac{1}{y'},t\right) = x^3\frac{\partial}{\partial x} + \left(xt + x(y')^2t + 1 + (y')^2\right)\frac{\partial}{\partial y'} + x\frac{\partial}{\partial t}.$$

This foliation has a singular point at $x = 0$ and $(y')^2 + 1 = 0$. But this singularity is the same singularity as the one on the chart $U_{x,y}$, as $(y')^2 + 1 = 0 \Leftrightarrow y^2 + 1 = 0$. Hence this is the $A_2$ surface singularity we have already studied. $\qquad\square$

**Proposition 3.5.3.** *The surface $X$ is Gorenstein and hence the Property 3 in Properties 3.1.2 is satisfied.*

*Proof.* This follows from Proposition 3.5.2. The only occurring singularities are $A_2$ surface singularities, which are Gorenstein. $\qquad\square$

Furthermore, the following proposition states that $X$ is not geometrically normal.

**Proposition 3.5.4.** *The surface $X$ is not geometrically normal.*

*Proof.* We assume that $X$, which is equal to $X_{K(T)}$ by construction, is geometrically normal. By assumption, $X_{\overline{k}}$ is normal. According to [PW17, Proposition 2.1], over a perfect field, the properties of the geometric generic fiber of a morphism $\mathfrak{X} \to T$ are equal to the properties of a general fiber of the morphism. These properties include normality, regularity and reducedness.

In this case, the geometric generic fiber of $\mathfrak{X} \to T$ is $(\mathfrak{X} \times_T \operatorname{Spec} K(T)) \times_{K(T)} \overline{K(T)} = X_{\overline{k}}$, which is normal by assumption. It follows that a general fiber of the morphism is normal as well. This means that there is a nonempty open subset $W \subseteq T$, such that the scheme theoretic fiber over every closed point of $W$ is normal. Let $w \in W$ be a closed point. Then $\mathfrak{X}_w = \mathfrak{X} \times_T \operatorname{Spec} k(w)$ is normal. But the same holds for the Frobenius base change. Consider the fiber of the morphism $\mathfrak{X} \times_T T^1$ over the point $w$, where $\operatorname{Fr}_T : T^1 \to T$ denotes the Frobenius morphism. The fiber is equal to $(\mathfrak{X} \times_T T^1) \times_{T^1} \operatorname{Spec} k(w) = \mathfrak{X} \times_T \operatorname{Spec} k(w) = \mathfrak{X}_w$, which is normal. It follows that by shrinking $T$, we may assume that $\mathfrak{X} \times_T T^1$ is normal. Hence $\mathfrak{X} \times_T T^1$ agrees with its normalization $\mathfrak{Y}$.

On one hand, by the formula for the relative canonical divisor, we have $K_{\mathfrak{Y}/\mathfrak{X}} \cong -(p-1)\mathfrak{C}$, and so $K_{\mathfrak{Y}} + (p-1)\mathfrak{C} \cong \phi^*K_{\mathfrak{X}}$, where $\phi : \mathfrak{Y} \to \mathfrak{X}$.

On the other hand, we have the following Cartesian diagram, illustrated below.

$$
\begin{array}{ccc}
\mathfrak{X} & \xleftarrow{\ \phi\ } & \mathfrak{X} \times_T T^1 \cong \mathfrak{Y} \\
\downarrow & & \downarrow \\
T & \xleftarrow{\ \operatorname{Fr}_T\ } & T^1
\end{array}
$$

Figure 3.3: Cartesian diagram

Since $\mathfrak{Y}$ agrees with the fiber product, we can apply [Har66, Proposition 8.8.6], which states that $\omega_{\mathfrak{Y}/T^1} \cong \phi^* \omega_{\mathfrak{X}/T}$, from which it follows that $K_{\mathfrak{Y}} \cong \phi^* K_{\mathfrak{X}}$.

Comparing these two formulas, it follows that $\mathfrak{C} = 0$, which is a contradiction. Hence $X$ is not geometrically normal. $\qquad \square$

*Remark* 3.5.5. We note that the proof of the proposition above does not rely on the actual example constructed, but on the general analysis discussed in Chapter 2. Hence we will refer to this proposition to prove the fact that the following examples, constructed with the same general outline, are not geometrically normal.

Lastly, the following proposition states that $X$ is geometrically reduced.

**Proposition 3.5.6.** *The surface $X$ is geometrically reduced.*

*Proof.* As in the proof of Proposition 3.5.4, we consider the morphism $\mathfrak{X} \to T$, which is defined over a perfect field. Hence the properties of its geometric generic fiber are equal to the properties of a general fiber. The geometric generic fiber of $\mathfrak{X} \to T$ is equal to $X_{\overline{k}}$. Hence we show its reducedness by showing the reducedness of a general fiber of the morphism $\mathfrak{X} \to T$. This means that we want to show that there is a non-empty open set $W \subseteq T$ such that the scheme theoretic fiber over every closed point of $W$ is reduced. Reducedness is preserved by the Frobenius base change. We therefore show reduceness of a general fiber of the morphism $\mathfrak{X} \to T$ by showing reducedness of a general fiber of the morphism $\mathfrak{X} \times_T T^1 \to T^1$, where $\mathrm{Fr}_T : T^1 \to T$ denotes the Frobenius morphism.

Let $W \subseteq T$ be a non-empty open subset. Denote by $\mathfrak{X}_W$ the subscheme of $\mathfrak{X}$ restricted to $W$, and denote by $\mathfrak{X}_{W^1} = \mathfrak{X}_W \times_W W^1$ its Frobenius base change, where $\mathrm{Fr}_W : W^1 \to W$ denotes the Frobenius morphism. Since $\mathfrak{X}$ is a normal scheme, it has property $(S_2)$. In particular, it also has property $(S_1)$. From this, it follows that $\mathfrak{X}_W$ also has property $(S_1)$. By [Kun69], since $W$ is regular, the Frobienius morphism $\mathrm{Fr}_W$ is flat. From this, using [Pat13, Lemma 4.2], it follows that $\mathfrak{X}_{W^1}$ also has property $(S_1)$. This means that all embedded points are generic points. Hence if $\mathfrak{X}_{W^1}$ is non-reduced, it is non-reduced everywhere.

The morphism $\tau : \mathfrak{Y} \to \mathfrak{X}_W$ is of degree $p$ by construction. It factors through the scheme $\mathfrak{X}_{W^1}$, where we denote the morphisms by $\alpha : \mathfrak{Y} \to \mathfrak{X}_{W^1}$ and $\beta : \mathfrak{X}_{W^1} \to \mathfrak{X}_W$. Let $\eta$ denote the generic point of $\mathfrak{X}_W$. On the level of rings, the morphism $\tau$ is a field extension of degree $p$,

$$(\mathscr{O}_{\mathfrak{X}_W})_\eta \to \left( \beta_* \mathscr{O}_{\mathfrak{X}_{W^1}} \right)_\eta \to (\tau_* \mathscr{O}_{\mathfrak{Y}})_\eta .$$

With $(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}})_\eta$ being an Artinian local ring, where we denote by $\mathfrak{m}$ its maximal ideal, the morphism $(\mathscr{O}_{\mathfrak{X}_W})_\eta \to (\tau_* \mathscr{O}_{\mathfrak{Y}})_\eta$ factors through $(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}})_\eta / \mathfrak{m}$. On the level of rings, taking quotient by the nilradical is equal to reduction, and hence $(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}})_\eta / \mathfrak{m} = ((\mathscr{O}_{\mathfrak{X}_{W^1}})_{red})_\eta$. This intermediate field of the degree $p$ field extension is either of degree 1 or of degree $p$.

Assume first that the degree of the extension is $\left[ (\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}})_\eta / \mathfrak{m} : (\mathscr{O}_{\mathfrak{X}})_\eta \right] = 1$. Hence the morphism $(\mathfrak{X}_{W^1})_{red} \to \mathfrak{X}_W$ is birational. Since this morphism is finite, and additionally $\mathfrak{X}$ is normal, it follows that $(\mathfrak{X}_{W^1})_{red} \to \mathfrak{X}_W$ is an isomorphism.

We consider the following diagram, illustrated below.

$$\mathfrak{Y} \longrightarrow (\mathfrak{X}_{W^1})_{red} \overset{\cong}{\longrightarrow} \mathfrak{X}_W$$

Figure 3.4: Notation

From $(\mathfrak{X}_{W^1})_{red} \to \mathfrak{X}_W$ being an isomorphism it follows that $\phi^* \mathscr{O}_{W^1} \subseteq \mathscr{O}_{\mathfrak{X}_W}$ needs to be preserved by derivation. This means that the foliation $\mathcal{F}$ that defines $\mathfrak{X}$ by taking quotients needs to be zero on $\rho^* \mathscr{O}_{W^1}$. But as one can check using the definition of $\mathcal{F}$ on the chart $U_{x,y}$, this

does not hold, as the coefficient in front of $\frac{\partial}{\partial t}$ is non-zero. Hence the degree of the extension $\left[ \left(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}}\right)_\eta \big/ \mathfrak{m} : \left(\mathscr{O}_{\mathfrak{X}}\right)_\eta \right] \neq 1$.

Hence the degree of the field extension is $\left[ \left(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}}\right)_\eta \big/ \mathfrak{m} : \left(\mathscr{O}_{\mathfrak{X}}\right)_\eta \right] = p$. As $\dim_{\left(\mathscr{O}_{\mathfrak{X}_W}\right)_\eta} \left(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}}\right)_\eta = p$ we also obtain $\mathfrak{m} = 0$. Since both $\left(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}}\right)_\eta$ and $\left(\beta_* \mathscr{O}_{\mathfrak{X}_{W^1}}\right)_\eta \big/ \mathfrak{m}$ are of degree $p$ over the field $\left(\mathscr{O}_{\mathfrak{X}_W}\right)_\eta$, they are isomorphic. It follows that $\mathfrak{X}_{W^1}$ is reduced and the morphism $\mathfrak{Y} \to \mathfrak{X}_{W^1}$ is birational. This means that there is an open subset $U \subseteq \mathfrak{Y}$, such that the fibers of $\mathfrak{Y} \to W^1\big|_U$ are equal to the fibers of $\mathfrak{X}_{W^1} \to W^1\big|_U$. But since the fibers of the morphism $\mathfrak{Y} \to W^1$ are reduced, so are the fibers of $\mathfrak{X}_{W^1} \to W^1\big|_U$. From this, the claim follows.

$\square$

*Remark* 3.5.7. An other way to prove geometric reducedness is to use the main theorem in [Sch10]. The theorem states that if a proper normal $k$-scheme $X$ with $k = H^0(X, \mathscr{O}_X)$ is geometrically non-reduced, then the geometric generic embedding dimension of $X$ is smaller then the degree of inseparability of $k$. To apply the theorem to our example, we remark that the degree of inseparability of $k$ is 1. Hence the scheme $X$ is geometrically non-reduced only if its geometric generic embedding dimension is equal to 0, if $k = H^0(X, \mathscr{O}_X)$. This translates to the pushforward condition of the structure sheaf, as stated in the proof above.

*Remark* 3.5.8. We note again that the proposition above is largely independent of the actual foliation constructed. It holds under the general analysis of Chapter 2, with the additional requirement that $\rho^* \mathscr{O}_{W^1}$ is non-zero. In the present case, this is satisfied as the polynomial $h \in \mathbb{F}_2[x, y, t]$ is chosen to be $h(x, y, t) = x$. With this choice, the coefficient in front of $\frac{\partial}{\partial t}$ in the description of $\mathcal{F}$ on the chart $U_{x,y}$ is equal to $x$, and hence it is non-zero. We refer to this remark to prove the geometrical reducedness in the case where the additional requirement holds.

This proves the main result, Theorem 1.0.3.

# Chapter 4

# $\mathbb{P}^2$ Example

The goal of this chapter is to prove the second main theorem.

**Main Theorem** (Theorem 1.0.4)**.** *Let $(p, Y, C) = (2, \mathbb{P}^2, L)$. There exists a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface has one singular point, which is a Gorenstein $A_3$-singularity.*

In the previous chapter, we constructed an example of a surface $X$ which is normal, projective, del Pezzo and Gorenstein, but not geometrically normal. This has been constructed in the third case of Theorem 1.0.1, where the triple consisting of $p, Y$ and $C$ is $(2, \mathbb{P}^1 \times \mathbb{P}^1, F)$. In this chapter, we will apply the same techniques as in the previous chapter to obtain a surface with the same properties for the triple $(2, \mathbb{P}^2, L)$. The construction of this surface is found in Section 4.4, where we construct three foliations that are all candidates for surfaces of the required form. One of these foliations we study in depth, proving that its singularities are Du Val.

Chapter 2 describes the general idea of transforming the original question of finding a surface into a question about finding a foliation that satisfies certain properties. We use this process for the present case of finding a surface $X$ for the triple $(2, \mathbb{P}^2, L)$ as well.

As in the construction of the previous example in Chapter 3, we make certain choices that reduce the generality of the examples we are able to construct. However, making those choices throughout the construction of our example is necessary. By setting certain parameters we make the setup more workable. Furthermore, as we are only interested in constructing one example, setting parameters for the sake of having a better understanding of the setup does not pose any real loss, given that we choose parameters in a sensible way that is not too restrictive. If one was interested in giving a characterization of all surfaces that satisfy Theorem 1.0.1, this would not be the right approach. One could try to loosen these restrictions, but this is not being discussed in the present work. Similar to Chapter 3, we fix the following setup, stated in Assumption 4.1.1.

## 4.1 Setup

*Assumption* 4.1.1. We use the notations and constructions of Chapter 2. In what follows, we consider the triple $(p, Y, C) = (2, \mathbb{P}^2_{\overline{k}}, L)$ from Theorem 1.0.1, where $L$ is a line. Additionally, for the construction of an explicit example of a corresponding surface $X$, the base $T$ of $\mathfrak{X}$ is chosen to be $T = \mathbb{A}^1_{\mathbb{F}_2}$. This way, $k = \mathbb{F}_2(x)$.

It holds that the divisor $C$ on $Y$ is equal to $\left( \sum_{i=1}^{e} (\mathfrak{C}^i)_{\mathfrak{Y}^e} \right) \times_{T^e} \operatorname{Spec} \overline{k}$. But by Assumption 4.1.1, $C = L$ is a line, which can not be expressed as a sum of more than one summands. Hence we see that $e = 1$. Only one Frobenius base change is applied to obtain the scheme $\mathfrak{Y}$ as the normalization of $\mathfrak{X} \times_T T^1$. The divisor $C$ is equal to $C = \mathfrak{C} \times_T \operatorname{Spec} \overline{k}$, where $\mathfrak{C}$ is a divisor on $\mathfrak{Y}$.

Via the correspondence of foliations and purely inseparable morphisms of height one over perfect fields, Proposition 2.0.6, we can characterize the scheme $\mathfrak{X}$ by characterizing the foliation $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ for which $\mathfrak{X} = \mathfrak{Y}/\mathcal{F}$ holds.

The goal of this chapter is to construct a surface $X$ under Assumption 4.1.1.For this construction, the desired properties of the surface $X$ should be translated into properties that $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ needs to satisfy.

As we have stated in Section 2.1, we have chosen $\mathfrak{Y}$ to be the fiber product of $T$ and $\mathbb{P}^2_{\mathbb{F}_2}$ over Spec $\mathbb{F}_2$. With this choice, the correspondence between $\mathfrak{Y}$ and $Y$ is satisfied, with $\mathfrak{Y} \times_T \operatorname{Spec} \overline{k} = Y$.

*Properties* 4.1.2. The restrictions that ensure that the surface obtained by this construction is of the required form are the following, possibly allowing shrinking $T$.

*Property 1* The subsheaf $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ is a foliation. This property is necessary to obtain $\mathfrak{X}$ via the correspondence in Proposition 2.0.6.

*Property 2* $(\mathfrak{Y}, \mathfrak{C}) = (\mathbb{P}^2 \times U, L \times U)$, for $U \subseteq T$ open. The choice of $\mathfrak{Y}$ we have made above states that $\mathfrak{Y}$ is of this form.

*Property 3* The surface $X$ is Gorenstein. Equivalently, $\mathfrak{X}$ is Gorenstein over the generic point of $T$.

*Property 4* Lastly, the anticanonical divisor of $X$ is ample. Equivalently, the anticanonical divisor of $\mathfrak{X}$ over $T$ is ample. This means that $K_{\mathfrak{Y}} + (p-1)\mathfrak{C}$ is ample over some open subset $U \subseteq T$. This is automatic if $(p, Y, C)$ is of the chosen type.

Let $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ be the foliation on $\mathfrak{Y}$ such that $\mathfrak{X} = \mathfrak{Y}/\mathcal{F}$. Locally on an affine chart $U_{(x_0, x_1)}$ of $\mathfrak{Y}$, where

$$U_{(x_0,x_1)} := \mathbb{A}^2_{(x_0,x_1)} \times T \subseteq \mathbb{P}^2_{(x_0,x_1,x_2)} \times T = \mathfrak{Y},$$

the foliation $\mathcal{F}$ is of the form

$$\mathcal{F}_{U_{(x_0,x_1)}} := \mathcal{F}\big|_{U_{(x_0,x_1)}} = \mathscr{O}_{U_{(x_0,x_1)}} \cdot v,$$

where $v \in \operatorname{Der}_{\mathbb{F}_2}(\mathscr{O}_{U_{(x_0,x_1)}}, \mathscr{O}_{U_{(x_0,x_1)}}) = \mathbb{F}_2[a_1, a_2, t]\left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial t}\right)$, and $a_i$ denote the coordinates of $U_{(x_0,x_1)}$ and $t$ denotes the coordinate of $T$. Hence $v$ is of the form

$$v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t},$$

for $f, g, h \in \mathbb{F}_2[a_1, a_2, t]$.

In order for the foliation $\mathcal{F}$ to correspond to a threefold $\mathfrak{X}$, and hence to a surface $X$ of the required form, the Properties 4.1.2 need to be satisfied. Firstly, we note that the study of the restrictions posed by Property 1, the fact that $\mathcal{F}_{U_{(x_0,x_1)}}$ is indeed a foliation has already been done in Chapter 3. It results in Proposition 3.2.2, which states that $\mathcal{F}$ defines a foliation on the chart $U_{(x_0,x_1)}$ if and only if the polynomials $f, g$ and $h$ satisfy the following line of equations

$$gh(ff_{a_1} + gf_{a_2} + hf_t) = fh(fg_{a_1} + gg_{a_2} + hg_t) = fg(fh_{a_1} + gh_{a_2} + hh_t).$$

As in the previous chapter, we use Proposition 2.1.1 to restrict the form of the polynomial $h$. From this proposition, it follows that the support of the divisor $\mathfrak{C}$ is equal to the vanishing locus of the polynomial $h$.

**Important Consequence.** *By assumption, the divisor $C$ is a line $L$ inside $\mathbb{P}^2_{\overline{k}}$. We may choose coordinates so that $L$ is the line defined by $\{a_1 = 0\}$. Hence the polynomial $h \in \mathbb{F}_2[a_1, a_2, t]$ is $h(a_1, a_2, t) = a_1$, according to Proposition 2.1.1.*

With this, the line of equations which guarantee that the sheaf $\mathcal{F}$ defined by $f, g$ and $h$ is indeed a foliation translates to

$$ga_1(ff_{a_1} + gf_{a_2} + a_1 f_t) = fa_1(fg_{a_1} + gg_{a_2} + a_1 g_t) = fgf,$$

which is equal to (3.2.2). The divisibility conditions that need to be satisfied have already been studied in the previous chapter, leading to Claim 3.3.2.

To pose further restrictions on the polynomials $f$ and $g$, we use the fact that the anticanonical divisor $-K_X$ of the desired surface $X$ is ample, as stated in Property 4 of Properties 4.1.2. The divisors $C$ that appear in Theorem 1.0.1 are chosen in such a way that this condition is satisfied.

Since $\mathcal{F}^* \cong \mathscr{O}_{\mathfrak{Y}}(\mathfrak{C})$, the foliation $\mathcal{F} \cong \mathscr{O}_{\mathfrak{Y}}(-\mathfrak{C})$. Furthermore, the divisor $C$ is defined to be a line $L$ in $\mathbb{P}^2$. We have chosen coordinates such that the line $L$ is defined by $L = \{a_1 = 0\}$, and hence

$$\mathcal{F} \cong \mathscr{O}_{\mathfrak{Y}}(-\mathfrak{C}) \cong \mathscr{O}_{\mathfrak{Y}}(-1).$$

Knowing the degree of the foliation, we can further restrict the exponents of the variables $a_1$ and $a_2$ in $f$ and $g$.

In the remainder of this chapter, up until the explicit constructions in Section 4.4, we implement as many restrictions on the polynomials $f$ and $g$ as possible. We do so by studying the Properties 4.1.2 and their implications.

## 4.2 Restrictions

*Remark* 4.2.1. In the remainder of this chapter, we use the notations defined in Section 4.1. In particular, the polynomials $f, g$ and $h$ are as defined above.

**Claim 4.2.2.** *Let the polynomials $f, g$ and $h$ be as defined in Section 4.1. The exponents of the variables $a_0$ and $a_1$ are restricted in the following sense:*

- *In each monomial appearing in the polynomial $f$, the sum of the exponent of $a_0$ and of $a_1$ is at most 3. The same holds for the polynomial $g$.*

- *Furthermore, if the sum of the exponent of $a_0$ and $a_1$ is 3 in $f$ or in $g$, then there are additional restrictions, stated in Claim 4.2.3.*

*Proof.* In order to prove this, we need to exhibit the form of the foliation $\mathcal{F}$ on the other affine charts that cover $\mathbb{P}^2$. Denote the variables of $\mathbb{P}^2$ by $x_0, x_1$ and $x_2$. On the affine charts $U_{(x_0, x_1)}$, where $x_2 = 1$, the foliation $\mathcal{F}$ is defined by $\mathcal{F} = v \cdot \mathscr{O}_{U_{(x_0, x_1)}}$, where

$$v(a_0, a_1, t) = f(a_0, a_1, t)\frac{\partial}{\partial a_1} + g(a_0, a_1, t)\frac{\partial}{\partial a_2} + h(a_0, a_1, t)\frac{\partial}{\partial t}.$$

The foliation is regular on this chart. Since the foliation is not equal to $\mathscr{O}_{\mathfrak{Y}}$, but equal to $\mathscr{O}_{\mathfrak{Y}}(-1)$, there needs to be a pole of order 1 on one of the other affine charts. This pole needs to appear on the line $x_2 = 0$. This line is the only possibility for the pole to appear, due to the regularity of the foliation on the chart $U_{(x_0, x_1)}$.

We now examine the foliation $\mathcal{F}$ on the chart $U_{(x_1, x_2)}$, where $x_0 = 1$. The morphisms from $\mathbb{P}^2$ to $U_{(x_0, x_1)}$ and to $U_{(x_1, x_2)}$ are defined by

$$\varphi_2 : \begin{cases} \mathbb{P}^2 & \to & U_{(x_0, x_1)} \\ [x_0 : x_1 : x_2] & \mapsto & \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) \end{cases}$$

and

$$\varphi_0 : \begin{cases} \mathbb{P}^2 & \to & U_{(x_1, x_2)} \\ [x_0 : x_1 : x_2] & \mapsto & \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \end{cases}$$

Denote the variables of $U_{(x_0, x_1)}$ by $a_i$, and the variables of $U_{(x_1, x_2)}$ by $b_i$. Then the morphisms between these two charts are

$$\begin{array}{ccc} U_{(x_0, x_1)} & \leftrightarrow & U_{(x_1, x_2)} \\ (a_1, a_2) & \mapsto & \left(\frac{a_2}{a_1}, \frac{1}{a_1}\right) \\ \left(\frac{1}{b_2}, \frac{b_1}{b_2}\right) & \leftarrow & (b_1, b_2). \end{array}$$

The line $x_2 = 0$ on the chart $U_{(x_1, x_2)}$ is equal to $b_2 = 0$.

In order to express the foliation on the chart $U_{(x_1, x_2)}$, we express the partial derivatives $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ using $\frac{\partial}{\partial b_1}$ and $\frac{\partial}{\partial b_2}$. It holds that

$$a_1 = \frac{1}{b_2}, \quad a_2 = \frac{b_1}{b_2}.$$

With this, we have

$$\mathrm{d}a_1 = \mathrm{d}\left(\frac{1}{b_2}\right) = \frac{1}{b_2^2}\mathrm{d}b_2$$

and

$$\mathrm{d}a_2 = \mathrm{d}\left(\frac{b_1}{b_2}\right) = \frac{1}{b_2^2}(\mathrm{d}b_1 \cdot b_2 + \mathrm{d}b_2 \cdot b_1).$$

By definition, $\frac{\partial}{\partial b_1}$ and $\frac{\partial}{\partial b_2}$ are the completions of the following homomorphisms $\Omega_{\mathbb{F}_2[b_1,b_2]} \to \mathbb{F}_2[b_1,b_2]$ :

$$\frac{\partial}{\partial b_1} : \begin{cases} \mathrm{d}b_1 & \mapsto & 1 \\ \mathrm{d}b_2 & \mapsto & 0 \end{cases} \qquad \frac{\partial}{\partial b_2} : \begin{cases} \mathrm{d}b_1 & \mapsto & 0 \\ \mathrm{d}b_2 & \mapsto & 1 \end{cases}$$

Using the properties above, we have

$$\frac{\partial}{\partial b_1}(\mathrm{d}a_1) = \frac{\partial}{\partial b_1}\left(\frac{1}{b_2^2}\mathrm{d}b_2\right) = 0$$

and

$$\frac{\partial}{\partial b_1}(\mathrm{d}a_2) = \frac{\partial}{\partial b_1}\left(\frac{1}{b_2^2}(\mathrm{d}b_1 \cdot b_2 + \mathrm{d}b_2 \cdot b_1)\right) = \frac{1}{b_2^2}b_2 = \frac{1}{b_2} = a_1.$$

Hence,

$$\frac{\partial}{\partial b_1} = a_1\frac{\partial}{\partial a_2}. \tag{4.2.1}$$

On the other hand, we have

$$\frac{\partial}{\partial b_2}(\mathrm{d}a_1) = \frac{\partial}{\partial b_2}\left(\frac{1}{b_2^2}\mathrm{d}b_2\right) = \frac{1}{b_2^2} = a_1^2$$

and

$$\frac{\partial}{\partial b_2}(\mathrm{d}a_2) = \frac{\partial}{\partial b_2}\left(\frac{1}{b_2^2}(\mathrm{d}b_1 \cdot b_2 + \mathrm{d}b_2 \cdot b_1)\right) = \frac{1}{b_2^2}b_1 = \frac{1}{b_2}\cdot\frac{b_1}{b_2} = a_1 a_2.$$

Hence,

$$\frac{\partial}{\partial b_2} = a_1^2\frac{\partial}{\partial a_1} + a_1 a_2\frac{\partial}{\partial a_2}. \tag{4.2.2}$$

From (4.2.1) it follows that the partial derivative $\frac{\partial}{\partial a_2}$ can be expressed as

$$\frac{\partial}{\partial a_2} = \frac{1}{a_1}\frac{\partial}{\partial b_1} = b_2\frac{\partial}{\partial b_1}. \tag{4.2.3}$$

From (4.2.2), using (4.2.1) it follows that

$$\frac{\partial}{\partial b_2} = a_1^2\frac{\partial}{\partial a_1} + a_2\underbrace{a_1\frac{\partial}{\partial a_2}}_{=\frac{\partial}{\partial b_1}} = a_1^2\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial b_1},$$

and so

$$a_1^2\frac{\partial}{\partial a_1} = \frac{\partial}{\partial b_2} + a_2\frac{\partial}{\partial b_1}.$$

Hence the partial derivative $\frac{\partial}{\partial a_1}$ can be expressed as

$$\frac{\partial}{\partial a_1} = \frac{1}{a_1^2}\left(\frac{\partial}{\partial b_2} + a_2\frac{\partial}{\partial b_1}\right) = b_2^2\left(\frac{\partial}{\partial b_2} + \frac{b_1}{b_2}\frac{\partial}{\partial b_1}\right). \tag{4.2.4}$$

Using these descriptions of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$, we can express the foliation $\mathcal{F}$ on the chart $U_{(x_1,x_2)}$. On the chart $U_{(x_0,x_1)}$, the foliation is defined by $\mathcal{F} = v \cdot \mathscr{O}_{U_{(x_0,x_1)}}$ with $v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t}$. We

obtain on the chart $U_{x_1,x_2}$, with (4.2.3) and (4.2.4),

$$v(a_1, a_2, t) = v\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)$$

$$= f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)\frac{\partial}{\partial a_1} + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)\frac{\partial}{\partial a_2} + h\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)\frac{\partial}{\partial t}$$

$$= f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2^2\left(\frac{\partial}{\partial b_2} + \frac{b_1}{b_2}\frac{\partial}{\partial b_1}\right) + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2\frac{\partial}{\partial b_1} + h\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)\frac{\partial}{\partial t}$$

$$= \left(f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2\right)\frac{\partial}{\partial b_1} + f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2^2\frac{\partial}{\partial b_2} + \frac{1}{b_2}\frac{\partial}{\partial t}.$$

In the last equation, we use the fact that we were able to choose the polynomial $h$ to be of the form $h(a_1, a_2, t) = a_1$. The coefficient $\frac{1}{b_2}$ in front of $\frac{\partial}{\partial t}$ ensures that indeed there is a pole of order one in $b_2$, which means that the pole is on the line $\{x_2 = 0\}$. Additionally, the coefficients in front of $\frac{\partial}{\partial b_1}$ and $\frac{\partial}{\partial b_2}$ need be chosen in such a way that poles of order at most one in $b_2$ can appear. The coefficient in front of $\frac{\partial}{\partial b_2}$ is of the form

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2^2.$$

In order to ensure that poles in the variable $b_2$ are at most of order one, the sum of the exponent of the first and the second variable in every monomial appearing in $f$ is at most 3. The coefficient appearing in front of $\frac{\partial}{\partial b_1}$ is of the form

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2.$$

In order to ensure that poles in the variable $b_2$ are at most of order one, this gives two restrictions:

- The sum of the exponent of the first and second variable of each monomial appearing in $g$ is at most 3.

- Additionally, for monomials in $f$ or $g$, for which the sum of the exponent of the first and second variable is 3, there are further restrictions discussed in Claim 4.2.3, which ensure that the expressions $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_1 b_2$ and $g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)b_2$ cancel each other out.

By the restriction on the exponents in $f$, coming from $\frac{\partial}{\partial b_2}$, higher exponents in $g$ can not appear, since they would need to be canceled out by a counterpart of the same degree in $f$. This concludes the proof of the claim. $\qquad\square$

Summing up, the monomials that may appear in $f(a_1, a_2, t)$ and $g(a_1, a_2, t) \in \mathbb{F}_2[a_1, a_2, t]$ are

$$a_1^3 t^i, a_1^2 a_2 t^i, a_1 a_2^2 t^i, a_2^3 t^i, a_1^2 t^i, a_1 a_2 t^i, a_2^2 t^i, a_1 t^i, a_2 t^i, t^i$$

for $i \in \mathbb{N}$. Due to the fact that $f$ is divisible by $a_1$ according to Claim 3.3.2, the monomials $a_2^3 t^i, a_2^2 t^i, a_2 t^i$ and $t^i$ in $f$ can be eliminated. This results in the monomials

$$a_1^3 t^i, a_1^2 a_2 t^i, a_1 a_2^2 t^i, a_1^2 t^i, a_1 a_2 t^i, a_1 t^i$$

that may appear in $f(a_1, a_2, t)$. Denoting by $f'(a_1, a_2, t)$ the polynomial such that $f = a_1 \cdot f'$, as in Claim 3.3.2, the following monomials may appear in $f'$:

$$a_1^2 t^i, a_1 a_2 t^i, a_2^2 t^i, a_1 t^i, a_2 t^i, t^i.$$

We now study the additional restriction on monomials in $f$ and $g$, for which the sum of the exponent of the first and second variable is 3.

**Claim 4.2.3.** *The monomial $a_1^3 t^i$ can not appear in $g$, and the monomial $a_2^3 t^i$ can not appear in $f$. Furthermore, the monomials for which the sum of the exponent of the first and second variable is equal to 3 appear in $f$ and $g$ under restrictions. For simplicity, these restrictions are stated as follows, where we assume that $f$ and $g$ are monomials. The statement extends canonically to $f$ and $g$ being sums of monomials.*

- $g(a_1, a_2, t) = a_1^2 a_2 t^i \Leftrightarrow f(a_1, a_2, t) = a_1^3 t^i$

- $g(a_1, a_2, t) = a_1 a_2^2 t^i \Leftrightarrow f(a_1, a_2, t) = a_1^2 a_2 t^i$

- $g(a_1, a_2, t) = a_2^3 t^i \Leftrightarrow f(a_1, a_2, t) = a_1 a_2^2 t^i$

*Proof.* The proof of Claim 4.2.2 states that in the expression

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

poles of order one may appear in $b_2$. If $g$ is a monomial for which the sum of the exponent of the first and second variable is 3, then the expression

$$g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

gives a pole of order 2 in $b_2$. Hence this pole needs to be canceled out by the same expression coming from

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2,$$

and vice versa.

We now individually look at the four cases that may appear for the polynomial $g$.

Suppose that $g(a_1, a_2, t) = a_1^3$. Then $g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2 = \left(\frac{1}{b_2}\right)^3 \cdot b_2 = \left(\frac{1}{b_2}\right)^2$. In order for the expression

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

to be zero, $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2$ needs to be equal to $\left(\frac{1}{b_2}\right)^2$. From this, it follows that

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 = \left(\frac{1}{b_2}\right)^2$$

$$\Rightarrow f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) = \frac{1}{b_1 b_2^3} = \frac{1}{b_1} \cdot \left(\frac{1}{b_2}\right)^3.$$

But this is not an expression that can be obtained by the polynomial $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right)$, and hence the monomial $a_1^3$ can not appear in $g$. Analogous to this, one can show that the monomial $a_2^3$ can not appear in $f$.

Next, suppose that $g(a_1, a_2, t) = a_1^2 a_2$. Then $g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2 = \left(\frac{1}{b_2}\right)^2 \cdot \left(\frac{b_1}{b_2}\right) \cdot b_2 = \frac{b_1}{b_2^2}$. In order for the expression

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

to be zero, $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2$ needs to be equal to $\frac{b_1}{b_2^2}$. From this, it follows that

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 = \frac{b_1}{b_2^2}$$

$$\Rightarrow f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) = \frac{1}{b_2^3} = \left(\frac{1}{b_2}\right)^3.$$

Hence if $g(a_1, a_2, t) = a_1^2 a_2$, then $f$ needs to be of the form $f(a_1, a_2, t) = a_1^3$.

Next, suppose that $g(a_1, a_2, t) = a_1 a_2^2$. Then $g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2 = \left(\frac{1}{b_2}\right) \cdot \left(\frac{b_1}{b_2}\right)^2 \cdot b_2 = \frac{b_1^2}{b_2^2}$. In order for the expression

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

to be zero, $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2$ needs to be equal to $\frac{b_1^2}{b_2^2}$. From this, it follows that

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 = \frac{b_1^2}{b_2^2}$$

$$\Rightarrow f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) = \frac{b_1}{b_2^3} = \left(\frac{1}{b_2}\right)^2 \cdot \left(\frac{b_1}{b_2}\right).$$

Hence if $g(a_1, a_2, t) = a_1 a_2^2$, then $f$ needs to be of the form $f(a_1, a_2, t) = a_1^2 a_2$.

Lastly, suppose that $g(a_1, a_2, t) = a_2^3$. Then $g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2 = \left(\frac{b_1}{b_2}\right)^3 \cdot b_2 = \frac{b_1^3}{b_2^2}$. In order for the expression

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2$$

to be zero, $f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2$ needs to be equal to $\frac{b_1^3}{b_2^2}$. From this, it follows that

$$f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 = \frac{b_1^3}{b_2^2}$$

$$\Rightarrow f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) = \frac{b_1^2}{b_2^3} = \left(\frac{1}{b_2}\right) \cdot \left(\frac{b_1}{b_2}\right)^2.$$

Hence if $g(a_1, a_2, t) = a_2^3$, then $f$ needs to be of the form $f(a_1, a_2, t) = a_1 a_2^2$. $\square$

In the following section, we use the claim above to find explicit polynomials $f$ and $g$. We do so by explicitly setting certain parameters. This results in a loss of generality of the possible examples found, but ultimately, it leads to the construction of three foliations, one of which we prove to satisfy all requirements of Theorem 1.0.1.

## 4.3 Explicit Construcition

Summing up, using the fact that $f$ is divisible by $a_1$ and the restrictions of Claim 4.2.3, the following monomials may appear in $f$ and $g$.

*In $f$* : $a_1^3 t^i, a_1^2 a_2 t^i, a_1 a_2^2 t^i, a_1^2 t^i, a_1 a_2 t^i, a_1 t^i$, and hence

 *In $f'$* : $a_1^2 t^i, a_1 a_2 t^i, a_2^2 t^i, a_1 t^i, a_2 t^i, t^i$.

*In $g$* : $a_1^2 a_2 t^i, a_1 a_2^2 t^i, a_2^3 t^i, a_1^2 t^i, a_1 a_2 t^i, a_2^2 t^i, a_1 t^i, a_2 t^i, t^i$.

Furthermore, the list below shows that if $g$ contains a specific monomial, then $f$ needs to contain a corresponding monomial, as proved in Claim 4.2.3, and vice versa.

  ○ $g(a_1, a_2, t) = a_1^2 a_2 t^i \Leftrightarrow f(a_1, a_2, t) = a_1^3 t^i \Leftrightarrow f'(a_1, a_2, t) = a_1^2 t^i$

  ○ $g(a_1, a_2, t) = a_1 a_2^2 t^i \Leftrightarrow f(a_1, a_2, t) = a_1^2 a_2 t^i \Leftrightarrow f'(a_1, a_2, t) = a_1 a_2 t^i$

  ○ $g(a_1, a_2, t) = a_2^3 t^i \Leftrightarrow f(a_1, a_2, t) = a_1 a_2^2 t^i \Leftrightarrow f'(a_1, a_2, t) = a_2^2 t^i$

We now assume that $f'$ and $g$ are of the following form, where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}_2[t]$.

$$f'(a_1, a_2, t) = \alpha_1 a_1^2 + \alpha_2 a_1 a_2 + \alpha_3 a_2^2 + \alpha_4 a_1 + \alpha_5 a_2 + \alpha_6$$

$$g(a_1, a_2, t) = \gamma_1 a_1^2 a_2 + \gamma_2 a_1 a_2^2 + \gamma_3 a_2^3 + \beta_1 a_1^2 + \beta_2 a_1 a_2 + \beta_3 a_2^2 + \beta_4 a_1 + \beta_5 a_2 + \beta_6.$$

The conditions (3.3.2a) and (3.3.2b) from Claim 3.3.2 are necessary conditions for $f$ and $g$ to define a foliation on the chart $U_{(x_0,x_1)}$. The condition (3.3.2a) states that

$$f' \big| g f'_{a_2} + a_1 f'_t.$$

As we are interested in the construction of an explicit example, we are not necessarily obliged to operate with the largest generality possible. We may find it useful to set specific parameters to simplify our calculations. We do so by setting $f'_{a_2} = 0$. As this choice restricts the examples we may find, it would be possible that this prevents us from finding any examples at all. However, we set this as a working condition, with the intention of revisiting in case we would not be able to construct any examples with this choice. However, this was not necessary. As we will see in the remainder of this chapter, this construction does lead to explicit examples. With this choice, it follows from (3.3.2a) that

$$f' \big| a_1 f'_t.$$

But this is only possible if $f'_t = 0$, since the derivation with respect to $t$ decreases the exponent of the variable $t$ in each monomial of $f'$. Since the exponent of the variable $t$ is higher in $f'$, the only possibility for $f' \big| a_1 f'_t$ to hold is if $f'_t = 0$. This means that the variable $t$ appears to exponents divisible by 2 only. On the other hand, since we assumed $f'_{a_2} = 0$, we get

$$f'_{a_2} = \alpha_2 a_1 + \alpha_5 = 0,$$

from which $\alpha_2 = \alpha_5 = 0$ follows. This restricts the form of $f'$ to be

$$f'(a_1, a_2, t) = \alpha_1 a_1^2 + \alpha_3 a_2^2 + \alpha_4 a_1 + \alpha_6.$$

Additionally, since $f'_t = 0$, it holds that $\alpha_{i,t} = 0$ for all $i$, where $\alpha_{i,t}$ denotes the derivation of $\alpha_i$ with respect to $t$.

By Claim 4.2.3, the degree 3 monomials in $g$ depend on the monomials of degree 3 in $f$. This means that $\gamma_j = \alpha_j$ for $j = 1, 2, 3$, and it reduces the form of $g$ to

$$g(a_1, a_2, t) = \alpha_1 a_1^2 a_2 + \alpha_3 a_2^3 + \beta_1 a_1^2 + \beta_2 a_1 a_2 + \beta_3 a_2^2 + \beta_4 a_1 + \beta_5 a_2 + \beta_6.$$

In order to further restrict the possible forms of $f$ and $g$, we consider the initial condition (3.2.2). These conditions ensure that we obtain a foliation defined by $f$ and $g$. The conditions state that

$$g a_1 (f f_{a_1} + g f_{a_2} + a_1 f_t) = f a_1 (f g_{a_1} + g g_{a_2} + a_1 g_t) = f f g$$
$$\Leftrightarrow g a_1 (a_1 (f')^2 + a_1^2 f' f'_{a_1} + g a_1 f\!\!\!/_{a_2} + a_1^2 f\!\!\!/_t) = a_1^2 f' (a_1 f' g_{a_1} + g g_{a_2} + a_1 g_t) = a_1^2 (f')^2 g.$$

We look at the two equations separately. The first equation states that

$$a_1^2 (f')^2 g = g a_1 (a_1 (f')^2 + a_1^2 f' f'_{a_1}) \Leftrightarrow (f')^2 = (f')^2 + a_1 f' f'_{a_1} \Leftrightarrow a_1 f' f'_{a_1} = 0. \qquad (4.3.1)$$

Using the assumption that $f' \neq 0$, it follows that $f'_{a_1} = 0$.

The second equation states that

$$a_1^2 (f')^2 g = a_1^2 f' (a_1 f' g_{a_1} + g g_{a_2} + a_1 g_t) \Leftrightarrow f' g = a_1 f' g_{a_1} + g g_{a_2} + a_1 g_t. \qquad (4.3.2)$$

From (4.3.1) it follows that $f'_{a_1} = 0$. This means that

$$f'_{a_1} = \alpha_4 = 0.$$

We now study the restrictions posed by (4.3.2). The polynomials $f'$ and $g$ are of the form

$$f'(a_1, a_2, t) = \alpha_1 a_1^2 + \alpha_3 a_2^2 + \alpha_6,$$
$$g(a_1, a_2, t) = \alpha_1 a_1^2 a_2 + \alpha_3 a_2^3 + \beta_1 a_1^2 + \beta_2 a_1 a_2 + \beta_3 a_2^2 + \beta_4 a_1 + \beta_5 a_2 + \beta_6.$$

The partial derivatives of $g$ are

$\circ$ $g_{a_1} = \beta_2 a_2 + \beta_4,$

○ $g_{a_2} = \alpha_1 a_1^2 + \alpha_3 a_2^2 + \beta_2 a_1 + \beta_5$,

○ $g_t = \alpha_{1,t} a_1^2 a_2 + \alpha_{3,t} a_2^3 + \beta_{1,t} a_1^2 + \beta_{2,t} a_1 a_2 + \beta_{3,t} a_2^2 + \beta_{4,t} a_1 + \beta_{5,t} a_2 + \beta_{6,t}$.

The equation (4.3.2) states that

$$f'g = a_1 f' g_{a_1} + g g_{a_2} + a_1 g_t.$$

We compare the right hand side of this equation to the left hand side. For the right hand side, we first compute the three terms individually.

$$
\begin{aligned}
a_1 f' g_{a_1} =\ & a_1(\alpha_1 a_1^2 + \alpha_3 a_2^2 + \alpha_6)(\beta_2 a_2 + \beta_4) \\
=\ & a_1(\alpha_1 \beta_2 a_1^2 a_2 + \alpha_1 \beta_4 a_1^2 + \alpha_3 \beta_2 a_2^3 + \alpha_3 \beta_4 a_2^2 + \alpha_6 \beta_2 a_2 + \alpha_6 \beta_4) \\
=\ & \alpha_1 \beta_2 a_1^3 a_2 + \alpha_1 \beta_4 a_1^3 + \alpha_3 \beta_2 a_1 a_2^3 + \alpha_3 \beta_4 a_1 a_2^2 + \alpha_6 \beta_2 a_1 a_2 + \alpha_6 \beta_4 a_1 \\
=\ & \alpha_1 \beta_2 a_1^3 a_2 + \alpha_3 \beta_2 a_1 a_2^3 \\
& + \alpha_1 \beta_4 a_1^3 + \alpha_3 \beta_4 a_1 a_2^2 \\
& + \alpha_6 \beta_2 a_1 a_2 \\
& + \alpha_6 \beta_4 a_1.
\end{aligned}
$$

$$
\begin{aligned}
g g_{a_2} =\ & (\alpha_1 a_1^2 + \alpha_3 a_2^2 + \beta_2 a_1 + \beta_5)(\alpha_1 a_1^2 a_2 + \alpha_3 a_2^3 + \beta_1 a_1^2 + \beta_2 a_1 a_2 + \beta_3 a_2^2 + \beta_4 a_1 + \beta_5 a_2 + \beta_6) \\
=\ & (\alpha_1)^2 a_1^4 a_2 + \alpha_1 \alpha_3 a_1^2 a_2^3 + \alpha_1 \beta_1 a_1^4 + \alpha_1 \beta_2 a_1^3 a_2 + \alpha_1 \beta_3 a_1^2 a_2^2 + \alpha_1 \beta_4 a_1^3 + \alpha_1 \beta_5 a_1^2 a_2 \\
& + \alpha_1 \beta_6 a_1^2 + \alpha_3 \alpha_1 a_1^2 a_2^3 + (\alpha_3)^2 a_2^5 + \alpha_3 \beta_1 a_1^2 a_2^2 + \alpha_3 \beta_2 a_1 a_2^3 + \alpha_3 \beta_3 a_2^4 + \alpha_3 \beta_4 a_1 a_2^2 \\
& + \alpha_3 \beta_5 a_2^3 + \alpha_3 \beta_6 a_2^2 + \beta_2 \alpha_1 a_1^3 a_2 + \beta_2 \alpha_3 a_1 a_2^3 + \beta_2 \beta_1 a_1^3 + (\beta_2)^2 a_1^2 a_2 + \beta_2 \beta_3 a_1 a_2^2 \\
& + \beta_2 \beta_4 a_1^2 + \beta_2 \beta_5 a_1 a_2 + \beta_2 \beta_6 a_1 + \beta_5 \alpha_1 a_1^2 a_2 + \beta_5 \alpha_3 a_2^3 + \beta_5 \beta_1 a_1^2 + \beta_5 \beta_2 a_1 a_2 \\
& + \beta_5 \beta_3 a_2^2 + \beta_5 \beta_4 a_1 + (\beta_5)^2 a_2 + \beta_5 \beta_6 \\
=\ & (\alpha_1)^2 a_1^4 a_2 + (2\alpha_1 \alpha_3) a_1^2 a_2^3 + (\alpha_3)^2 a_2^5 \\
& + \alpha_1 \beta_1 a_1^4 + (2\alpha_1 \beta_2) a_1^3 a_2 + (\alpha_1 \beta_3 + \alpha_3 \beta_1) a_1^2 a_2^2 + (2\alpha_3 \beta_2) a_1 a_2^3 + \alpha_3 \beta_3 a_2^4 \\
& + (\alpha_1 \beta_4 + \beta_2 \beta_1) a_1^3 + (2\alpha_1 \beta_5 + (\beta_2)^2) a_1^2 a_2 + (\alpha_3 \beta_4 + \beta_2 \beta_3) a_1 a_2^2 + (2\alpha_3 \beta_5) a_2^3 \\
& + (\alpha_1 \beta_6 + \beta_2 \beta_4 + \beta_5 \beta_1) a_1^2 + (2\beta_2 \beta_5) a_1 a_2 + (\alpha_3 \beta_6 + \beta_5 \beta_3) a_2^2 \\
& + (\beta_2 \beta_6 + \beta_5 \beta_4) a_1 + (\beta_5)^2 a_2 \\
& + \beta_5 \beta_6.
\end{aligned}
$$

$$
\begin{aligned}
a_1 g_t =\ & a_1(\alpha_{1,t} a_1^2 a_2 + \alpha_{3,t} a_2^3 + \beta_{1,t} a_1^2 + \beta_{2,t} a_1 a_2 + \beta_{3,t} a_2^2 + \beta_{4,t} a_1 + \beta_{5,t} a_2 + \beta_{6,t}) \\
=\ & \alpha_{1,t} a_1^3 a_2 + \alpha_{3,t} a_1 a_2^3 + \beta_{1,t} a_1^3 + \beta_{2,t} a_1^2 a_2 + \beta_{3,t} a_1 a_2^2 + \beta_{4,t} a_1^2 + \beta_{5,t} a_1 a_2 + \beta_{6,t} a_1 \\
=\ & \alpha_{1,t} a_1^3 a_2 + \alpha_{3,t} a_1 a_2^3 \\
& + \beta_{1,t} a_1^3 + \beta_{2,t} a_1^2 a_2 + \beta_{3,t} a_1 a_2^2 \\
& + \beta_{4,t} a_1^2 + \beta_{5,t} a_1 a_2 \\
& + \beta_{6,t} a_1.
\end{aligned}
$$

Summing up these terms, we get for the right hand side of equation (4.3.2)

$$
\begin{aligned}
& a_1 f' g_{a_1} + g g_{a_2} + a_1 g_t \\
&= (\alpha_1)^2 a_1^4 a_2 + (\alpha_3)^2 a_2^5 \\
&\quad + \alpha_1 \beta_1 a_1^4 + (\alpha_1 \beta_2 + \alpha_{1,t}) a_1^3 a_2 + (\alpha_1 \beta_3 + \alpha_3 \beta_1) a_1^2 a_2^2 + (\alpha_3 \beta_2 + \alpha_{3,t}) a_1 a_2^3 + \alpha_3 \beta_3 a_2^4 \\
&\quad + (\beta_2 \beta_1 + \beta_{1,t}) a_1^3 + ((\beta_2)^2 + \beta_{2,t}) a_1^2 a_2 + (\beta_2 \beta_3 + \beta_{3,t}) a_1 a_2^2 \\
&\quad + (\alpha_1 \beta_6 + \beta_2 \beta_4 + \beta_5 \beta_1 + \beta_{4,t}) a_1^2 + (\alpha_6 \beta_2 + \beta_{5,t}) a_1 a_2 + (\alpha_3 \beta_6 + \beta_5 \beta_3) a_2^2
\end{aligned}
$$

$$+(\alpha_6\beta_4 + \beta_2\beta_6 + \beta_5\beta_4 + \beta_{6,t})a_1 + (\beta_5)^2 a_2$$
$$+\beta_5\beta_6.$$

The left hand side is of the following form.

$$
\begin{aligned}
f'g &= (\alpha_1 a_1^2 + \alpha_3 a_2^2 + \alpha_6)(\alpha_1 a_1^2 a_2 + \alpha_3 a_2^3 + \beta_1 a_1^2 + \beta_2 a_1 a_2 + \beta_3 a_2^2 + \beta_4 a_1 + \beta_5 a_2 + \beta_6) \\
&= (\alpha_1)^2 a_1^4 a_2 + \alpha_1\alpha_3 a_1^2 a_2^3 + \alpha_1\beta_1 a_1^4 + \alpha_1\beta_2 a_1^3 a_2 + \alpha_1\beta_3 a_1^2 a_2^2 + \alpha_1\beta_4 a_1^3 + \alpha_1\beta_5 a_1^2 a_2 \\
&\quad + \alpha_1\beta_6 a_1^2 + \alpha_3\alpha_1 a_1^2 a_2^3 + (\alpha_3)^2 a_2^5 + \alpha_3\beta_1 a_1^2 a_2^2 + \alpha_3\beta_2 a_1 a_2^3 + \alpha_3\beta_3 a_2^4 + \alpha_3\beta_4 a_1 a_2^2 \\
&\quad + \alpha_3\beta_5 a_2^3 + \alpha_3\beta_6 a_2^2 + \alpha_6\alpha_1 a_1^2 a_2 + \alpha_6\alpha_3 a_2^3 + \alpha_6\beta_1 a_1^2 + \alpha_6\beta_2 a_1 a_2 + \alpha_6\beta_3 a_2^2 + \alpha_6\beta_4 a_1 \\
&\quad + \alpha_6\beta_5 a_2 + \alpha_6\beta_6 \\
&= (\alpha_1)^2 a_1^4 a_2 + (2\alpha_1\alpha_3)a_1^2 a_2^3 + (\alpha_3)^2 a_2^5 \\
&\quad + \alpha_1\beta_1 a_1^4 + \alpha_1\beta_2 a_1^3 a_2 + (\alpha_1\beta_3 + \alpha_3\beta_1)a_1^2 a_2^2 + \alpha_3\beta_2 a_1 a_2^3 + \alpha_3\beta_3 a_2^4 \\
&\quad + \alpha_1\beta_4 a_1^3 + (\alpha_6\alpha_1 + \alpha_1\beta_5)a_1^2 a_2 + \alpha_3\beta_4 a_1 a_2^2 + (\alpha_3\beta_5 + \alpha_6\alpha_3)a_2^3 \\
&\quad + (\alpha_6\beta_1 + \alpha_1\beta_6)a_1^2 + \alpha_6\beta_2 a_1 a_2 + (\alpha_3\beta_6 + \alpha_6\beta_3)a_2^2 \\
&\quad + \alpha_6\beta_4 a_1 + \alpha_6\beta_5 a_2 \\
&\quad + \alpha_6\beta_6.
\end{aligned}
$$

We now compare the right hand side and the left hand side in order to obtain restrictions on the polynomials $\alpha_i, \beta_i \in \mathbb{F}_2[t]$. This comparison yields

for $a_1^4 a_2$ : $\quad (\alpha_1)^2 = (\alpha_1)^2$

for $a_2^5$ : $\quad (\alpha_3)^2 = (\alpha_3)^2$

for $a_1^4$ : $\quad \alpha_1\beta_1 = \alpha_1\beta_1$

for $a_1^3 a_2$ : $\quad \alpha_1\beta_2 = \alpha_1\beta_2 + \alpha_{1,t}$

for $a_1^2 a_2^2$ : $\quad \alpha_1\beta_3 + \alpha_3\beta_1 = \alpha_1\beta_3 + \alpha_3\beta_1$

for $a_1 a_2^3$ : $\quad \alpha_3\beta_2 = \alpha_3\beta_2 + \alpha_{3,t}$

for $a_2^4$ : $\quad \alpha_3\beta_3 = \alpha_3\beta_3$

for $a_1^3$ : $\quad \alpha_1\beta_4 = \beta_2\beta_1 + \beta_{1,t}$

for $a_1^2 a_2$ : $\quad \alpha_6\alpha_1 + \alpha_1\beta_5 = (\beta_2)^2 + \beta_{2,t}$

for $a_1 a_2^2$ : $\quad \alpha_3\beta_4 = \beta_2\beta_3 + \beta_{3,t}$

for $a_2^3$ : $\quad \alpha_3\beta_5 + \alpha_6\alpha_3 = 0$

for $a_1^2$ : $\quad \alpha_6\beta_1 + \alpha_1\beta_6 = \alpha_1\beta_6 + \beta_2\beta_4 + \beta_5\beta_1 + \beta_{4,t}$

for $a_1 a_2$ : $\quad \alpha_6\beta_2 = \alpha_6\beta_2 + \beta_{5,t}$

for $a_2^2$ : $\quad \alpha_3\beta_6 + \alpha_6\beta_3 = \alpha_3\beta_6 + \beta_5\beta_3$

for $a_1$ : $\quad \alpha_6\beta_4 = \alpha_6\beta_4 + \beta_2\beta_6 + \beta_5\beta_4 + \beta_{6,t}$

for $a_2$ : $\quad \alpha_6\beta_5 = (\beta_5)^2$

for $1$ : $\quad \alpha_6\beta_6 = \beta_5\beta_6$

Some of these equations do not pose actual restrictions, as the left and right hand side already agree. We list below the ones that are not automatically satisfied, and their implications. These are

for $a_1^3 a_2$ : $\quad \alpha_{1,t} = 0$ satisfied since $f'_t = 0$

for $a_1 a_2^3$ : $\quad \alpha_{3,t} = 0$ satisfied

for $a_1^3$ : $\quad \alpha_1\beta_4 = \beta_1\beta_2 + \beta_{1,t}$

for $a_1^2 a_2$ : $\quad \alpha_1\alpha_6 + \alpha_1\beta_5 = (\beta_2)^2 + \beta_{2,t}$

for $a_1 a_2^2$ : $\qquad \alpha_3 \beta_4 = \beta_2 \beta_3 + \beta_{3,t}$

for $a_2^3$ : $\qquad \alpha_3 \beta_5 + \alpha_6 \alpha_3 = 0 \Rightarrow$ either $\alpha_3 = 0$ or $\alpha_6 = \beta_5$

for $a_1^2$ : $\qquad \alpha_6 \beta_1 = \beta_2 \beta_4 + \beta_1 \beta_5 + \beta_{4,t}$

for $a_1 a_2$ : $\qquad \beta_{5,t} = 0$

for $a_2^2$ : $\qquad \alpha_6 \beta_3 = \beta_5 \beta_3 \Rightarrow$ either $\beta_3 = 0$ or $\alpha_6 = \beta_5$

for $a_1$ : $\qquad \beta_2 \beta_6 + \beta_4 \beta_5 + \beta_{6,t} = 0$

for $a_2$ : $\qquad \alpha_6 \beta_5 = (\beta_5)^2 \Rightarrow$ either $\beta_5 = 0$ or $\alpha_6 = \beta_5$

for $1$ : $\qquad \alpha_6 \beta_6 = \beta_5 \beta_6 \Rightarrow$ either $\beta_6 = 0$ or $\alpha_6 = \beta_5$

In order to construct an explicit example, we are free to set certain parameters. We first set $\alpha_6 = \beta_5$, as suggested by the restrictions coming from $a_2^3, a_2^2, a_2$ and $1$. With this choice, the equation coming from $a_1^2 a_2$ is

$$\alpha_1 \alpha_6 + \alpha_1 \underbrace{\beta_5}_{=\alpha_6} = (\beta_2)^2 + \beta_{2,t} \Rightarrow (\beta_2)^2 = \beta_{2,t}.$$

This is true only if $\beta_2 = 0$. If $\beta_2 \neq 0$, then the degree of $\beta_2^2$ is strictly bigger than the degree of $\beta_{2,t}$.

With $\beta_2 = 0$, and $\beta_5 = \alpha_6$, the restriction coming from $a_1^2$ simplifies to

$$\alpha_6 \beta_1 = \cancel{\beta_2} \beta_4 + \beta_1 \underbrace{\beta_5}_{=\alpha_6} + \beta_{4,t} \Rightarrow \beta_{4,t} = 0.$$

We sum up the remaining restrictions in the following lemma.

**Lemma 4.3.1.** *Let $f'$ and $g \in \mathbb{F}_2[a_1, a_2, t]$ be defined as follows*

$$f'(a_1, a_2, t) = \alpha_1 a_1^2 + \alpha_3 a_2^2 + \alpha_6$$

*with $\alpha_{i,t} = 0$ for $i = 1, 3, 6$ and*

$$g(a_1, a_2, t) = \alpha_1 a_1^2 a_2 + \alpha_3 a_2^3 + \beta_1 a_1^2 + \beta_3 a_2^2 + \beta_4 a_1 + \alpha_6 a_2 + \beta_6$$

*with $\beta_{4,t} = 0$. Additionally, the polynomials $\alpha_i, \beta_i \in \mathbb{F}_2[t]$ satisfy the equations*

$$\alpha_1 \beta_4 = \beta_{1,t} \tag{4.3.3a}$$
$$\alpha_3 \beta_4 = \beta_{3,t} \tag{4.3.3b}$$
$$\alpha_6 \beta_4 = \beta_{6,t} \tag{4.3.3c}$$

*Then, the polynomials $f'$ and $g$ defining $\mathcal{F}$ via $\mathcal{F}|_{U_{(x_0,x_1)}} = \mathcal{O}_{U_{(x_0,x_1)}} \cdot v$, with $v(a_1, a_2, t) = f(a_1, a_2, t) \frac{\partial}{\partial a_1} + g(a_1, a_2, t) \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial t}$ define a foliation on this chart.*

This concludes the study of Property 1 in Properties 4.1.2, which ensures that $\mathcal{F}$ is a foliation. Before we construct explicit examples with the use of the lemma above, we first exhibit the form of the foliation on the third affine chart $U_{(x_0,x_2)}$. On this affine chart, the variable $x_1$ is equal to $1$. The morphisms from $\mathbb{P}^2$ to $U_{(x_0,x_1)}$ and to $U_{(x_0,x_2)}$ are defined by

$$\varphi_2 : \begin{cases} \mathbb{P}^2 & \to & U_{(x_0,x_1)} \\ [x_0 : x_1 : x_2] & \mapsto & \left( \frac{x_0}{x_2}, \frac{x_1}{x_2} \right) \end{cases}$$

and

$$\varphi_1 : \begin{cases} \mathbb{P}^2 & \to & U_{(x_0,x_2)} \\ [x_0 : x_1 : x_2] & \mapsto & \left( \frac{x_0}{x_1}, \frac{x_2}{x_1} \right) \end{cases}$$

The variables of $U_{(x_0,x_1)}$ are denoted by $a_i$. The variables of $U_{(x_0,x_2)}$ we denote by $c_i$. Then the morphisms between these two charts are

$$\begin{array}{ccc} U_{(x_0,x_1)} & \leftrightarrow & U_{(x_0,x_2)} \\ (a_1, a_2) & \mapsto & \left( \frac{a_1}{a_2}, \frac{1}{a_2} \right) \\ \left( \frac{c_1}{c_2}, \frac{1}{c_2} \right) & \leftarrow & (c_1, c_2). \end{array}$$

The line $x_2 = 0$ on the chart $U_{(x_0,x_2)}$ is equal to $c_2 = 0$.

In order to express the foliation on the chart $U_{(x_0,x_2)}$, we express the partial derivatives $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ using $\frac{\partial}{\partial c_1}$ and $\frac{\partial}{\partial c_2}$. It holds that

$$a_1 = \frac{c_1}{c_2}, \quad a_2 = \frac{1}{c_2}.$$

With this, we have

$$da_1 = d\left(\frac{c_1}{c_2}\right) = \frac{1}{c_2^2}(c_1 dc_2 + c_2 dc_1)$$

and

$$da_2 = d\left(\frac{1}{c_2}\right) = \frac{1}{c_2^2} dc_2.$$

By definition, $\frac{\partial}{\partial c_1}$ and $\frac{\partial}{\partial c_2}$ are the completions of the following homomorphisms $\Omega_{\mathbb{F}_2[c_1,c_2]} \to \mathbb{F}_2[c_1,c_2]$ :

$$\frac{\partial}{\partial c_1} : \begin{cases} dc_1 & \mapsto & 1 \\ dc_2 & \mapsto & 0 \end{cases} \qquad \frac{\partial}{\partial c_2} : \begin{cases} dc_1 & \mapsto & 0 \\ dc_2 & \mapsto & 1 \end{cases}$$

Using the properties above, we have

$$\frac{\partial}{\partial c_1}(da_1) = \frac{\partial}{\partial c_1}\left(\frac{1}{c_2^2}(c_1 dc_2 + c_2 dc_1)\right) = \frac{1}{c_2} = a_2,$$

and

$$\frac{\partial}{\partial c_1}(da_2) = \frac{\partial}{\partial c_1}\left(\frac{1}{c_2^2} dc_2\right) = 0.$$

Hence,

$$\frac{\partial}{\partial c_1} = a_2 \frac{\partial}{\partial a_1}. \tag{4.3.4}$$

On the other hand, we have

$$\frac{\partial}{\partial c_2}(da_1) = \frac{\partial}{\partial c_2}\left(\frac{1}{c_2^2}(c_1 dc_2 + c_2 dc_1)\right) = \frac{c_1}{c_2^2} = \frac{c_1}{c_2} \cdot \frac{1}{c_2} = a_1 a_2,$$

and

$$\frac{\partial}{\partial c_2}(da_2) = \frac{\partial}{\partial c_2}\left(\frac{1}{c_2^2} dc_2\right) = \frac{1}{c_2^2} = a_2^2.$$

Hence,

$$\frac{\partial}{\partial c_2} = a_1 a_2 \frac{\partial}{\partial a_1} + a_2^2 \frac{\partial}{\partial a_2}. \tag{4.3.5}$$

From (4.3.4) it follows that the partial derivative $\frac{\partial}{\partial a_1}$ can be expressed as

$$\frac{\partial}{\partial a_1} = \frac{1}{a_2}\frac{\partial}{\partial c_1} = c_2 \frac{\partial}{\partial c_1}. \tag{4.3.6}$$

From (4.3.5), using (4.3.4) it follows that

$$\frac{\partial}{\partial c_2} = a_1 \underbrace{a_2 \frac{\partial}{\partial a_1}}_{=\frac{\partial}{\partial c_1}} + a_2^2 \frac{\partial}{\partial a_2} = a_1 \frac{\partial}{\partial c_1} + a_2^2 \frac{\partial}{\partial a_2},$$

and so

$$a_2^2 \frac{\partial}{\partial a_2} = a_1 \frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_2}.$$

Hence the partial derivative $\frac{\partial}{\partial a_2}$ can be expressed as

$$\frac{\partial}{\partial a_2} = \frac{1}{a_2^2}\left(a_1 \frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_2}\right) = c_2^2\left(\frac{c_1}{c_2}\frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_2}\right). \tag{4.3.7}$$

Using these descriptions of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$, we can express the foliation $\mathcal{F}$ on the chart $U_{(x_0,x_2)}$. On the chart $U_{(x_0,x_1)}$, the foliation is defined by $\mathcal{F} = v \cdot \mathscr{O}_{U_{(x_0,x_1)}}$ with $v = f(a_1,a_2,t)\frac{\partial}{\partial a_1} + g(a_1,a_2,t)\frac{\partial}{\partial a_2} + h(a_1,a_2,t)\frac{\partial}{\partial t}$. We obtain on the chart $U_{(x_0,x_2)}$, with (4.3.6) and (4.3.7),

$$
\begin{aligned}
v(a_1,a_2,t) &= v\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right) \\
&= f\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)\frac{\partial}{\partial a_1} + g\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)\frac{\partial}{\partial a_2} + h\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)\frac{\partial}{\partial t} \\
&= f\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)c_2\frac{\partial}{\partial c_1} + g\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)c_2^2\left(\frac{c_1}{c_2}\frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_2}\right) + h\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)\frac{\partial}{\partial t} \\
&= \left(f\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)c_2 + g\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)c_1c_2\right)\frac{\partial}{\partial c_1} + g\left(\frac{c_1}{c_2},\frac{1}{c_2},t\right)c_2^2\frac{\partial}{\partial c_2} + \frac{c_1}{c_2}\frac{\partial}{\partial t}.
\end{aligned}
$$

In the last equation, we use the fact that $h(a_1,a_2,t) = a_1$.

## 4.4 Examples of Foliations

In this section, we construct three foliations, using Lemma 4.3.1. The first two examples, in Section 4.4.1, we do not study in depth. The third example, constructed in Section 4.4.2, satisfies all requirements of Theorem 1.0.1. This is proved in Section 4.5.

### 4.4.1 Two Foliations without Study of Singularities

**Example 4.4.1.** We create an explicit example using Lemma 4.3.1. We set $\alpha_1 = \alpha_3 = 0$. The equations (4.3.3a) to (4.3.3c) are satisfied if $\beta_{1,t} = \beta_{3,t} = 0$ and $\alpha_6\beta_4 = \beta_{6,t}$. In this case, the polynomials $f'$ and $g$ are of the form

$$
f'(a_1,a_2,t) = \alpha_6
$$

with $\alpha_{6,t} = 0$ and

$$
g(a_1,a_2,t) = \beta_1 a_1^2 + \beta_3 a_2^2 + \beta_4 a_1 + \alpha_6 a_2 + \beta_6
$$

with $\beta_{1,t} = \beta_{3,t} = \beta_{4,t} = 0$ and $\alpha_6\beta_4 = \beta_{6,t}$.

With the choice of $\alpha_6 = 1, \beta_4 = 1$ and $\beta_6 = t$ and the remaining $\beta_i = 0$, the equations (4.3.3a) to (4.3.3c) are satisfied. We obtain

$$
f' = 1 \Rightarrow f = a_1 f' = a_1
$$

and

$$
g = a_1 + a_2 + t.
$$

Additionally, we verify that the equations (3.2.2) are satisfied. These equations are

$$
ga_1(ff_{a_1} + gf_{a_2} + a_1f_t) = fa_1(fg_{a_1} + gg_{a_2} + a_1g_t) = ffg.
$$

One easily verifies that with our choice of $f$ and $g$ these equations are satisfied, since all three expressions are equal to $a_1^3 + a_1^2 a_2 + a_1^2 t$. The foliation we obtain this way on the chart $U_{(x_0,x_1)}$ is defined by

$$
v(a_1,a_2,t) = a_1\frac{\partial}{\partial a_1} + (a_1 + a_2 + t)\frac{\partial}{\partial a_2} + a_1\frac{\partial}{\partial t}.
$$

There is a singularity for $a_1 = 0$ and $a_2 + t = 0$.

We now exhibit this foliation on the other two affine charts. On the chart $U_{(x_1,x_2)}$, the foliation is defined by

$$
\begin{aligned}
v(a_1, a_2, t) &= \left( f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2 \right) \frac{\partial}{\partial b_1} + f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2^2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= \left( \frac{1}{b_2} b_1 b_2 + \left(\frac{1}{b_2} + \frac{b_1}{b_2} + t\right) b_2 \right) \frac{\partial}{\partial b_1} + \frac{1}{b_2} b_2^2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= (b_1 + 1 + b_1 + b_2 t) \frac{\partial}{\partial b_1} + b_2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= (b_2 t + 1) \frac{\partial}{\partial b_1} + b_2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t}.
\end{aligned}
$$

On this chart, the foliation is of degree $\mathscr{O}(-1)$ due to the coefficient $\frac{1}{b_2}$ in front of $\frac{\partial}{\partial t}$, and furthermore it is regular, since after multiplying with $b_2$, we obtain the coefficient 1 in front of $\frac{\partial}{\partial t}$.

Lastly, on the chart $U_{(x_0,x_2)}$, the foliation is defined by

$$
\begin{aligned}
v(a_1, a_2, t) &= \left( f\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right) c_2 + g\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right) c_1 c_2 \right) \frac{\partial}{\partial c_1} + g\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right) c_2^2 \frac{\partial}{\partial c_2} + \frac{c_1}{c_2} \frac{\partial}{\partial t} \\
&= \left( \frac{c_1}{c_2} \cdot c_2 + \left(\frac{c_1}{c_2} + \frac{1}{c_2} + t\right) c_1 c_2 \right) \frac{\partial}{\partial c_1} + \left( \frac{c_1}{c_2} + \frac{1}{c_2} + t\right) c_2^2 \frac{\partial}{\partial c_2} + \frac{c_1}{c_2} \frac{\partial}{\partial t} \\
&= \left( c_1 + c_1^2 + c_1 + c_1 c_2 t \right) \frac{\partial}{\partial c_1} + \left( c_1 c_2 + c_2 + c_2^2 t \right) \frac{\partial}{\partial c_2} + \frac{c_1}{c_2} \frac{\partial}{\partial t} \\
&= \left( c_1^2 + c_1 c_2 t \right) \frac{\partial}{\partial c_1} + \left( c_1 c_2 + c_2 + c_2^2 t \right) \frac{\partial}{\partial c_2} + \frac{c_1}{c_2} \frac{\partial}{\partial t}.
\end{aligned}
$$

After multiplying with $c_2$, we get

$$
c_2 \cdot v = \left( c_1^2 c_2 + c_1 c_2^2 t \right) \frac{\partial}{\partial c_1} + \left( c_1 c_2^2 + c_2^2 + c_2^3 t \right) \frac{\partial}{\partial c_2} + c_1 \frac{\partial}{\partial t},
$$

which locally on the chart $U_{(x_0,x_2)}$ defines the foliation. For $c_1 = 0$ and $c_2^2 + c_2^3 t = 0$ there is a singularity.

**Example 4.4.2.** We construct an other example by using Lemma 4.3.1 and setting $\alpha_6 = t^2, \beta_6 = 1$ and letting the other coefficients be zero. The lemma is satisfied with this choice. We obtain the polynomials

$$
f = a_1 t^2
$$

$$
g = a_2 t^2 + 1.
$$

One can easily verify that these polynomials satisfy the equations (3.2.2), which confirms that the polynomials define a foliation on the chart $U_{(x_0,x_1)}$. On the three charts, the foliation is defined by:
On $U_{x_0,x_1}$ :

$$
v(a_1, a_2, t) = a_1 t^2 \frac{\partial}{\partial a_1} + (a_2 t^2 + t) \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial t}.
$$

On $U_{x_1,x_2}$ :

$$
v = b_2 \frac{\partial}{\partial b_1} + b_2 t^2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t}.
$$

On $U_{x_0,x_2}$ :

$$
v = c_1 c_2 \frac{\partial}{\partial c_1} + (c_2 t^2 + c_2^2) \frac{\partial}{\partial c_2} + \frac{c_1}{c_2} \frac{\partial}{\partial t}.
$$

There are singularities both on the chart $U_{(x_0,x_1)}$ and on the chart $U_{(x_0,x_2)}$.

### 4.4.2 Foliation with Study of Singularity

**Example 4.4.3.** Using Lemma 4.3.1, we construct a foliation by setting the parameters $\alpha_1(t) = 1, \beta_6(t) = 1$, and the remaining $\alpha_i(t)$ and $\beta_i(t)$ zero. With this choice, the requirements of the lemma are satisfied, meaning that $\alpha_{i,t} = 0$ for all $i$, and $\beta_{4,t} = 0$. Furthermore, the equations (4.3.3a) to (4.3.3c) are satisfied.

The polynomials $f(a_1, a_2, t)$ and $g(a_1, a_2, t)$ we obtain with these choices are

$$f' = a_1^2 \Rightarrow f = a_1^3,$$
$$g = a_1^2 a_2 + 1.$$

The partial derivatives of the two polynomials are

$$f_{a_1} = a_1^2, \quad f_{a_2} = 0, \quad f_t = 0$$
$$g_{a_1} = 0, \quad g_{a_2} = a_1^2, \quad g_t = 0.$$

With the derivatives as above, it is easy to check that the equations (3.2.2) are satisfied, which ensure that the polynomials $f$ and $g$ define a foliation on the affine chart $U_{(x_0, x_1)}$. These equations are

$$g a_1 (f f_{a_1} + g f_{a_2} + a_1 f_t) = f a_1 (f g_{a_1} + g g_{a_2} + a_1 g_t) = f f g.$$

On the affine chart $U_{(x_0, x_1)}$, the foliation is defined as $\mathcal{F} = \mathscr{O}_{U_{(x_0, x_1)}} \cdot v$ with

$$v(a_1, a_2, t) = a_1^3 \frac{\partial}{\partial a_1} + (a_1^2 a_2 + 1) \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial t}.$$

On this chart, the foliation is regular, since it is not possible for all coefficients in front of $\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}$ and $\frac{\partial}{\partial t}$ to be zero simultaneously.

We recall that on the chart $U_{(x_1, x_2)}$ with coordinates $b_i$, the foliation is defined by

$$
\begin{aligned}
v(a_1, a_2, t) &= v\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) \\
&= \left(f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_1 b_2 + g\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2\right) \frac{\partial}{\partial b_1} + f\left(\frac{1}{b_2}, \frac{b_1}{b_2}, t\right) b_2^2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= \left(\left(\frac{1}{b_2}\right)^3 b_1 b_2 + \left(\left(\frac{1}{b_2}\right)^2 \left(\frac{b_1}{b_2}\right) + 1\right) b_2\right) \frac{\partial}{\partial b_1} + \left(\frac{1}{b_2}\right)^3 b_2^2 \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= \left(\frac{b_1}{b_2^2} + \frac{b_1}{b_2^2} + b_2\right) \frac{\partial}{\partial b_1} + \frac{1}{b_2} \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t} \\
&= b_2 \frac{\partial}{\partial b_1} + \frac{1}{b_2} \frac{\partial}{\partial b_2} + \frac{1}{b_2} \frac{\partial}{\partial t}.
\end{aligned}
$$

The pole of order one in the variable $b_2$ verifies that the foliation is of degree $\mathscr{O}_{\mathfrak{Y}}(-1)$. This ensures that the surface $X$ constructed using this foliation is indeed anti-ample, and hence verifying Property 4 in Properties 4.1.2. After multiplying with $b_2$, we obtain

$$b_2 \cdot v = b_2^2 \frac{\partial}{\partial b_1} + 1 \frac{\partial}{\partial b_2} + 1 \frac{\partial}{\partial t},$$

which defines the foliation locally on the chart $U_{(x_1, x_2)}$. Hence the foliation is regular on this affine chart as well.

Lastly, on the affine chart $U_{(x_0,x_2)}$, with variables $c_i$, the foliation is defined by

$$
\begin{aligned}
v(a_1, a_2, t) &= v\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right) \\
&= \left(f\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right)c_2 + g\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right)c_1 c_2\right)\frac{\partial}{\partial c_1} + g\left(\frac{c_1}{c_2}, \frac{1}{c_2}, t\right)c_2^2\frac{\partial}{\partial c_2} + \frac{c_1}{c_2}\frac{\partial}{\partial t} \\
&= \left(\left(\frac{c_1}{c_2}\right)^3 c_2 + \left(\left(\frac{c_1}{c_2}\right)^2\frac{1}{c_2} + 1\right)c_1 c_2\right)\frac{\partial}{\partial c_1} + g\left(\left(\frac{c_1}{c_2}\right)^2\frac{1}{c_2} + 1\right)c_2^2\frac{\partial}{\partial c_2} + \frac{c_1}{c_2}\frac{\partial}{\partial t} \\
&= \left(\frac{c_1^3}{c_2^2} + \frac{c_1^3}{c_2^2} + c_1 c_2\right)\frac{\partial}{\partial c_1} + \left(\frac{c_1^2}{c_2} + c_2^2\right)\frac{\partial}{\partial c_2} + \frac{c_1}{c_2}\frac{\partial}{\partial t} \\
&= c_1 c_2\frac{\partial}{\partial c_1} + \left(\frac{c_1^2}{c_2} + c_2^2\right)\frac{\partial}{\partial c_2} + \frac{c_1}{c_2}\frac{\partial}{\partial t}.
\end{aligned}
$$

After multiplying with $c_2$, we obtain

$$
c_2 \cdot v = c_1 c_2^2\frac{\partial}{\partial c_1} + \left(c_1^2 + c_2^3\right)\frac{\partial}{\partial c_2} + c_1\frac{\partial}{\partial t},
$$

which defines the foliation locally on the chart $U_{(x_0,x_2)}$.

For $c_1 = c_2 = 0$, and $t \in T$, there is a singular point.

**Proposition 4.4.4.** *On the two charts $U_{x_0,x_1}$ and $U_{x_1,x_2}$, the foliation is regular. On the chart $U_{(x_0,x_2)}$ there is one singular point, which is an $A_3$ surface singularity after passing to the function field of the base.*

*Proof.* As we have seen by the description of the foliation on the first two charts, it is regular. On the third chart, we have remarked that for $c_1 = c_2 = 0$, there is a singular point.

We schematically illustrate below the charts and three blow ups we have to perform. At the root of the two arrows we have the chart which is being blown up. Above it, connected with the two-branched arrow are the two charts of the blow up. This illustration helps to keep track of the notation of both charts and coordinates. We remark that we use the same notation for the coordinates of the two open charts of a blow up. This is indicated in the diagram below as well.
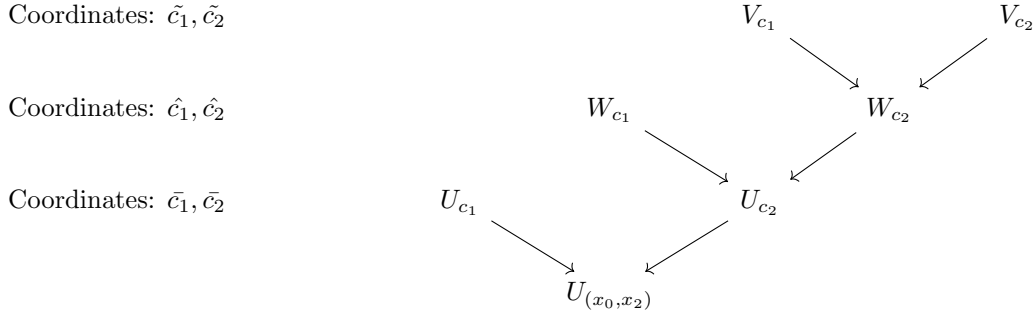


Figure 4.1: Schematic illustration of the blow ups

We let $\mathbb{A}^3_{c_1,c_2,t} := \operatorname{Spec}\mathbb{F}_2[c_1, c_2, t]$, and $L := \{(0, 0, t)\,|\,t \in T\}$ be the line along which we blow up, with defining ideal $I(L) = (c_1, c_2)$. Consider the map

$$
\begin{aligned}
\mathbb{A}^3_{c_1,c_2,t} \setminus L &\to \mathbb{A}^3 \times \mathbb{P}^1 \\
(c_1, c_2, t) &\mapsto ((c_1, c_2, t), [c_1 : c_2]).
\end{aligned}
$$

The blow up is defined to be the closure in $\mathbb{A}^3 \times \mathbb{P}^1$ of the image of the above map. Hence $\operatorname{Bl}_L\mathbb{A}^3_{c_1,c_2,t} = \overline{\{((c_1, c_2, t), [x : y]) \in \mathbb{A}^3 \times \mathbb{P}^1\,|\,c_1 y = c_2 x\}}$. We denote the blow up by $\pi : \operatorname{Bl}_L\mathbb{A}^3_{c_1,c_2,t} \to \mathbb{A}^3_{c_1,c_2,t}$. As $\pi$ is an isomorphism when restricted to the preimage of $\mathbb{A}^3_{c_1,c_2,t} \setminus L$, it holds that

$\pi^*\mathcal{F}\big|_{\mathbb{A}^3\setminus L}$ is a foliation on this preimage and hence extends uniquely to a foliation $\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3}$ on $\mathrm{Bl}_L\mathbb{A}^3$ by saturatedness.

$\mathrm{Bl}_L\mathbb{A}^3$ can be covered by two affine charts $U_{c_1}$ and $U_{c_2}$, which are given by

$$U_{c_1} = \{((c_1, c_2, t), [1 : y]) \in \mathbb{A}^3 \times \mathbb{P}^1 \big| c_1 y = c_2\},$$

$$U_{c_2} = \{((c_1, c_2, t), [x : 1]) \in \mathbb{A}^3 \times \mathbb{P}^1 \big| c_2 x = c_1\}.$$

Both charts are isomorphic to $\mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}]$ via the following isomorphisms

$$\begin{array}{rcl} \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] & \to & U_{c_1} \\ (\overline{c_1}, \overline{c_2}, \overline{t}) & \mapsto & ((\overline{c_1}, \overline{c_1 c_2}, \overline{t}), [1 : \overline{c_2}]) \end{array}$$

and

$$\begin{array}{rcl} \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] & \to & U_{c_2} \\ (\overline{c_1}, \overline{c_2}, \overline{t}) & \mapsto & ((\overline{c_1 c_2}, \overline{c_2}, \overline{t}), [\overline{c_1} : 1]). \end{array}$$

The map $\pi$ restricted to these charts is

$$\begin{array}{rcl} \mathbb{F}_2[c_1, c_2, t] & \to & \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] \\ c_1 & \mapsto & \overline{c_1} \\ c_2 & \mapsto & \overline{c_1 c_2} \\ t & \mapsto & \overline{t} \end{array}$$

on the chart $U_{c_1}$, and

$$\begin{array}{rcl} \mathbb{F}_2[c_1, c_2, t] & \to & \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] \\ c_1 & \mapsto & \overline{c_1 c_2} \\ c_2 & \mapsto & \overline{c_2} \\ t & \mapsto & \overline{t} \end{array}$$

on the chart $U_{c_2}$.

On the chart $U_{c_1}$, with the blow up defined as above, the derivations transform as follows

$$c_1 \frac{\partial}{\partial c_1} = \overline{c_1} \frac{\partial}{\partial \overline{c_1}} + \overline{c_2} \frac{\partial}{\partial \overline{c_2}}, \quad c_2 \frac{\partial}{\partial c_2} = \overline{c_2} \frac{\partial}{\partial \overline{c_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.$$

Hence the foliation $\mathcal{F}$ becomes

$$\begin{aligned} \mathcal{F} &= c_1 c_2^2 \frac{\partial}{\partial c_1} + (c_1^2 + c_2^3) \frac{\partial}{\partial c_2} + c_1 \frac{\partial}{\partial t} \\ &= c_2^2 \left(c_1 \frac{\partial}{\partial c_1}\right) + \left(\frac{c_1^2}{c_2} + c_2^2\right) \left(c_2 \frac{\partial}{\partial c_2}\right) + c_1 \frac{\partial}{\partial t} \\ &= \overline{c_1}^2 \overline{c_2}^2 \left(\overline{c_1} \frac{\partial}{\partial \overline{c_1}} + \overline{c_2} \frac{\partial}{\partial \overline{c_2}}\right) + \left(\frac{\overline{c_1}^2}{\overline{c_1 c_2}} + \overline{c_1}^2 \overline{c_2}^2\right) \overline{c_2} \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}} \\ &= \overline{c_1}^3 \overline{c_2}^2 \frac{\partial}{\partial \overline{c_1}} + \overline{c_1} \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}} \end{aligned}$$

Dividing by $\overline{c_1}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L\,\mathbb{A}^3, U_{c_1}} = \overline{c_1}^2 \overline{c_2}^2 \frac{\partial}{\partial \overline{c_1}} + 1 \frac{\partial}{\partial \overline{c_2}} + 1 \frac{\partial}{\partial \overline{t}}$$

on the chart $U_{c_1}$.

Hence after blowing up, the foliation becomes regular. It remains to compute the discrepancy of this blow up. Consider the following diagram,

$$\begin{array}{ccc} \mathrm{Bl}_L \mathbb{A}^3 & \xrightarrow{\ \beta\ } & \mathrm{Bl}_L \mathbb{A}^3 / \mathcal{F}_{\mathrm{Bl}} \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{A}^3 & \xrightarrow[\ \alpha\ ]{} & \mathbb{A}^3 / \mathcal{F} \end{array}$$

Figure 4.2: Notation for the blow up

where we denote by $E$ the exceptional divisor of the blow up $\pi$ and by $E'$ the exceptional divisor of the blow up $\pi'$, the blow up of the quotient space. In order to calculate the discrepancy of the blow up $\pi'$, we let $K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} = (\pi')^* K_{\mathbb{A}^3/\mathcal{F}} + aE'$, where $a$ denotes the discrepancy. Since the blow up $\pi$ describes the blow up of a line in $\mathbb{A}^3$, its discrepancy is equal to one and by the adjunction formula we have

$$
\begin{aligned}
K_{\mathrm{Bl}_L \mathbb{A}^3} &= \pi^* K_{\mathbb{A}^3} + E \\
&\cong \pi^*(\alpha^* K_{\mathbb{A}^3/\mathcal{F}} - (1-p)c_1(\mathcal{F})) + E, \\
&\quad \text{by adjunction, where } c_1(\mathcal{F}) = 0, \\
&= \pi^* \alpha^* K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(\pi')^* K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} - aE') + E \\
&= \beta^*(K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}}) - a\beta^*(E') + E \\
&\cong (K_{\mathrm{Bl}_L \mathbb{A}^3} + (1-p)c_1(\mathcal{F}_{\mathrm{Bl}})) - a\beta^*(E') + E, \text{ by adjunction} \\
&= K_{\mathrm{Bl}_L \mathbb{A}^3} - E - a\beta^*(E') + E.
\end{aligned}
$$

The fact that $c_1(\mathcal{F}) = 0$ follows from the fact that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^3}$, which holds due to the fact that every line bundle on $\mathbb{A}^3$ is trivial. The last equality holds because in order to obtain the foliation $\mathcal{F}_{\mathrm{Bl}}$ from $\mathcal{F}$, we divide by $\overline{c_1}$. This means that we divide by one time the exceptional divisor $E$, and so $c_1(\mathcal{F}_{\mathrm{Bl}}) = 1 \cdot E$.

It follows that
$$
K_{\mathrm{Bl}_L \mathbb{A}^3} = K_{\mathrm{Bl}_L \mathbb{A}^3} - a\beta^*(E'),
$$
and hence the discrepancy $a$ is equal to zero.

Considering the other chart $U_{c_2}$, we recall that the blow up is defined by

$$
\begin{aligned}
\mathbb{F}_2[c_1, c_2, t] &\rightarrow \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] \\
c_1 &\mapsto \overline{c_1 c_2} \\
c_2 &\mapsto \overline{c_2} \\
t &\mapsto \overline{t}
\end{aligned}
$$

and the derivations transform as follows

$$
c_1 \frac{\partial}{\partial c_1} = \overline{c_1} \frac{\partial}{\partial \overline{c_1}}, \quad c_2 \frac{\partial}{\partial c_2} = \overline{c_1} \frac{\partial}{\partial \overline{c_1}} + \overline{c_2} \frac{\partial}{\partial \overline{c_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.
$$

Hence the foliation $\mathcal{F}$ becomes

$$
\begin{aligned}
\mathcal{F} &= c_1 c_2^2 \frac{\partial}{\partial c_1} + (c_1^2 + c_2^3) \frac{\partial}{\partial c_2} + c_1 \frac{\partial}{\partial t} \\
&= c_2^2 \left( c_1 \frac{\partial}{\partial c_1} \right) + \left( \frac{c_1^2}{c_2} + c_2^2 \right) \left( c_2 \frac{\partial}{\partial c_2} \right) + c_1 \frac{\partial}{\partial t} \\
&= \overline{c_2}^2 \overline{c_1} \frac{\partial}{\partial \overline{c_1}} + \left( \frac{\overline{c_1}^2 \overline{c_2}^2}{\overline{c_2}} + \overline{c_2}^2 \right) \left( \overline{c_1} \frac{\partial}{\partial \overline{c_1}} + \overline{c_2} \frac{\partial}{\partial \overline{c_2}} \right) + \overline{c_1 c_2} \frac{\partial}{\partial \overline{t}} \\
&= \overline{c_1}^3 \overline{c_2} \frac{\partial}{\partial \overline{c_1}} + (\overline{c_1}^2 \overline{c_2}^2 + \overline{c_2}^3) \frac{\partial}{\partial \overline{c_2}} + \overline{c_1 c_2} \frac{\partial}{\partial \overline{t}}
\end{aligned}
$$

Dividing by $\overline{c_2}$, we get

$$
\mathcal{F}_{\mathrm{Bl}_L \mathbb{A}^3, U_{c_2}} = \overline{c_1}^3 \frac{\partial}{\partial \overline{c_1}} + (\overline{c_1}^2 \overline{c_2} + \overline{c_2}^2) \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}}
$$

Due to a similar argument as on the chart $U_{c_1}$, the discrepancy of the blow up on the quotient spaces is zero.

We get a singularity again for $\overline{c_1} = \overline{c_2} = 0$. Hence we blow up once more in an attempt to get rid of this singularity.

On the chart $U_{c_2}$, the foliation is defined as

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}} = \overline{c_1}^3 \frac{\partial}{\partial \overline{c_1}} + (\overline{c_1}^2 \overline{c_2} + \overline{c_2}^2) \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}}.$$

Similar to the first blow up, we blow up $\mathrm{Spec} \, \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}]$ along the line $L$ defined by the ideal $(\overline{c_1}, \overline{c_2})$. We denote the blow up by $\psi : \mathrm{Bl}_L \, \mathbb{A}^3_{\overline{c_1}, \overline{c_2}, \overline{t}} \to \mathbb{A}^3_{\overline{c_1}, \overline{c_2}, \overline{t}}$. Let $W_{c_1}$ and $W_{c_2}$ be the two affine charts covering $\mathrm{Bl}_L \, \mathbb{A}^3_{\overline{c_1}, \overline{c_2}, \overline{t}}$, which are both isomorphic to $\mathrm{Spec} \, \mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}]$.

The blow up $\psi$ restricted to these charts is

$$\begin{aligned} \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] &\to \mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}] \\ \overline{c_1} &\mapsto \hat{c_1} \\ \overline{c_2} &\mapsto \hat{c_1}\hat{c_2} \\ \overline{t} &\mapsto \hat{t} \end{aligned}$$

on the chart $W_{c_1}$, and

$$\begin{aligned} \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] &\to \mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}] \\ \overline{c_1} &\mapsto \hat{c_1}\hat{c_2} \\ \overline{c_2} &\mapsto \hat{c_2} \\ \overline{t} &\mapsto \hat{t} \end{aligned}$$

on the chart $W_{c_2}$.

On the chart $W_{c_1}$, with the blow up defined as above, the derivations transform as follows

$$\overline{c_1} \frac{\partial}{\partial \overline{c_1}} = \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + \hat{c_2} \frac{\partial}{\partial \hat{c_2}}, \quad \overline{c_2} \frac{\partial}{\partial \overline{c_2}} = \hat{c_2} \frac{\partial}{\partial \hat{c_2}}, \quad \frac{\partial}{\partial \overline{t}} = \frac{\partial}{\partial \hat{t}}.$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}}$ becomes

$$\begin{aligned} \mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}} &= \overline{c_1}^3 \frac{\partial}{\partial \overline{c_1}} + (\overline{c_1}^2 \overline{c_2} + \overline{c_2}^2) \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}} \\ &= \overline{c_1}^2 \left( \overline{c_1} \frac{\partial}{\partial \overline{c_1}} \right) + (\overline{c_1}^2 + \overline{c_2}) \left( \overline{c_2} \frac{\partial}{\partial \overline{c_2}} \right) + \overline{c_1} \frac{\partial}{\partial \overline{t}} \\ &= \hat{c_1}^2 \left( \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + \hat{c_2} \frac{\partial}{\partial \hat{c_2}} \right) + (\hat{c_1}^2 + \hat{c_1}\hat{c_2}) \left( \hat{c_2} \frac{\partial}{\partial \hat{c_2}} \right) + \hat{c_1} \frac{\partial}{\partial \hat{t}} \\ &= \hat{c_1}^3 \frac{\partial}{\partial \hat{c_1}} + \hat{c_1}\hat{c_2}^2 \frac{\partial}{\partial \hat{c_2}} + \hat{c_1} \frac{\partial}{\partial \hat{t}}. \end{aligned}$$

Dividing by $\hat{c_1}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_1}} = \hat{c_1}^2 \frac{\partial}{\partial \hat{c_1}} + \hat{c_2}^2 \frac{\partial}{\partial \hat{c_2}} + 1 \frac{\partial}{\partial \hat{t}}.$$

Hence the foliation becomes regular on this chart after the second blow up, due to the coefficient 1 in front of $\frac{\partial}{\partial \hat{t}}$. The discrepancy of this blow up is equal to zero, due to a similar argument as for the first blow up. It remains to check the second affine chart.

On the chart $W_{c_2}$, the blow up is defined by

$$\begin{aligned} \mathbb{F}_2[\overline{c_1}, \overline{c_2}, \overline{t}] &\to \mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}] \\ \overline{c_1} &\mapsto \hat{c_1}\hat{c_2} \\ \overline{c_2} &\mapsto \hat{c_2} \\ \overline{t} &\mapsto \hat{t} \end{aligned}$$

and the derivations transform as follows

$$\overline{c_1} \frac{\partial}{\partial \overline{c_1}} = \hat{c_1} \frac{\partial}{\partial \hat{c_1}}, \quad \overline{c_2} \frac{\partial}{\partial \overline{c_2}} = \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + \hat{c_2} \frac{\partial}{\partial \hat{c_2}}, \quad \frac{\partial}{\partial \overline{t}} = \frac{\partial}{\partial \hat{t}}.$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}}$ becomes

$$
\begin{aligned}
\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}} &= \overline{c_1}^3 \frac{\partial}{\partial \overline{c_1}} + (\overline{c_1}^2 \overline{c_2} + \overline{c_2}^2) \frac{\partial}{\partial \overline{c_2}} + \overline{c_1} \frac{\partial}{\partial \overline{t}} \\
&= \overline{c_1}^2 \left( \overline{c_1} \frac{\partial}{\partial \overline{c_1}} \right) + (\overline{c_1}^2 + \overline{c_2}) \left( \overline{c_2} \frac{\partial}{\partial \overline{c_2}} \right) + \overline{c_1} \frac{\partial}{\partial \overline{t}} \\
&= \hat{c_1}^2 \hat{c_2}^2 \left( \hat{c_1} \frac{\partial}{\partial \hat{c_1}} \right) + (\hat{c_1}^2 \hat{c_2}^2 + \hat{c_2}) \left( \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + \hat{c_2} \frac{\partial}{\partial \hat{c_2}} \right) + \hat{c_1} \hat{c_2} \frac{\partial}{\partial \hat{t}} \\
&= \hat{c_1} \hat{c_2} \frac{\partial}{\partial \hat{c_1}} + (\hat{c_1}^2 \hat{c_2}^3 + \hat{c_2}^2) \frac{\partial}{\partial \hat{c_2}} + \hat{c_1} \hat{c_2} \frac{\partial}{\partial \hat{t}}.
\end{aligned}
$$

Dividing by $\hat{c_2}$, we get

$$
\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}} = \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + (\hat{c_1}^2 \hat{c_2}^2 + \hat{c_2}) \frac{\partial}{\partial \hat{c_2}} + \hat{c_1} \frac{\partial}{\partial \hat{t}}.
$$

Due to a similar calculation as before, the discrepancy of this blow up is zero. Again, there is a singularity for $\hat{c_1} = \hat{c_2} = 0$. In an attempt to get rid of this singularity we blow up a third time.

On the chart $W_{c_2}$, the foliation is defined as

$$
\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}} = \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + (\hat{c_1}^2 \hat{c_2}^2 + \hat{c_2}) \frac{\partial}{\partial \hat{c_2}} + \hat{c_1} \frac{\partial}{\partial \hat{t}}.
$$

Similar to the first two blow up, we blow up $\mathrm{Spec}\,\mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}]$ along the line $L$ defined by the ideal $(\hat{c_1}, \hat{c_2})$. We denote the blow up by $\phi : \mathrm{Bl}_L \, \mathbb{A}^3_{\hat{c_1}, \hat{c_2}, \hat{t}} \to \mathbb{A}^3_{\hat{c_1}, \hat{c_2}, \hat{t}}$. Let $V_{c_1}$ and $V_{c_2}$ be the two affine charts covering $\mathrm{Bl}_L \, \mathbb{A}^3_{\hat{c_1}, \hat{c_2}, \hat{t}}$, which are both isomorphic to $\mathrm{Spec}\,\mathbb{F}_2[\tilde{c_1}, \tilde{c_2}, \tilde{t}]$.

The blow up $\phi$ restricted to these charts is

$$
\begin{aligned}
\mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}] &\to \mathbb{F}_2[\tilde{c_1}, \tilde{c_2}, \tilde{t}] \\
\hat{c_1} &\mapsto \tilde{c_1} \\
\hat{c_2} &\mapsto \tilde{c_1} \tilde{c_2} \\
\hat{t} &\mapsto \tilde{t}
\end{aligned}
$$

on the chart $V_{c_1}$, and

$$
\begin{aligned}
\mathbb{F}_2[\hat{c_1}, \hat{c_2}, \hat{t}] &\to \mathbb{F}_2[\tilde{c_1}, \tilde{c_2}, \tilde{t}] \\
\hat{c_1} &\mapsto \tilde{c_1} \tilde{c_2} \\
\hat{c_2} &\mapsto \tilde{c_2} \\
\hat{t} &\mapsto \tilde{t}
\end{aligned}
$$

on the chart $V_{c_2}$.

On the chart $V_{c_1}$, with the blow up defined as above, the derivations transform as follows

$$
\hat{c_1} \frac{\partial}{\partial c_1} = \tilde{c_1} \frac{\partial}{\partial \tilde{c_1}} + \tilde{c_2} \frac{\partial}{\partial \tilde{c_2}}, \quad \hat{c_2} \frac{\partial}{\partial \hat{c_2}} = \tilde{c_2} \frac{\partial}{\partial \tilde{c_2}}, \quad \frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial \tilde{t}}.
$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}}$ becomes

$$
\begin{aligned}
\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}} &= \hat{c_1} \frac{\partial}{\partial \hat{c_1}} + (\hat{c_1}^2 \hat{c_2}^2 + \hat{c_2}) \frac{\partial}{\partial \hat{c_2}} + \hat{c_1} \frac{\partial}{\partial \hat{t}} \\
&= \left( \hat{c_1} \frac{\partial}{\partial \hat{c_1}} \right) + (\hat{c_1}^2 \hat{c_2} + 1) \left( \hat{c_2} \frac{\partial}{\partial \hat{c_2}} \right) + \hat{c_1} \frac{\partial}{\partial \hat{t}} \\
&= \left( \tilde{c_1} \frac{\partial}{\partial \tilde{c_1}} + \tilde{c_2} \frac{\partial}{\partial \tilde{c_2}} \right) + (\tilde{c_1}^2 \tilde{c_1} \tilde{c_2} + 1) \left( \tilde{c_2} \frac{\partial}{\partial \tilde{c_2}} \right) + \tilde{c_1} \frac{\partial}{\partial \tilde{t}} \\
&= \tilde{c_1} \frac{\partial}{\partial \tilde{c_1}} + \tilde{c_1}^3 \tilde{c_2}^2 \frac{\partial}{\partial \tilde{c_2}} + \tilde{c_1} \frac{\partial}{\partial \tilde{t}}
\end{aligned}
$$

Dividing by $\tilde{c}_1$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_1}, V_{c_1}} = 1 \frac{\partial}{\partial \tilde{c}_1} + \tilde{c}_1{}^2 \tilde{c}_2{}^2 \frac{\partial}{\partial \tilde{c}_2} + 1 \frac{\partial}{\partial \tilde{t}}.$$

Hence the foliation becomes regular on this chart after the third blow up, due to the coefficient 1 in front of $\frac{\partial}{\partial \tilde{t}}$. The discrepancy of this blow up is equal to zero, due to a similar argument as for the first two blow ups. It remains to check the second affine chart.

On the chart $V_{c_2}$, the blow up is defined by

$$\begin{array}{rcl}
\mathbb{F}_2[\hat{c}_1, \hat{c}_2, \hat{t}] & \rightarrow & \mathbb{F}_2[\tilde{c}_1, \tilde{c}_2, \tilde{t}] \\
\hat{c}_1 & \mapsto & \tilde{c}_1 \tilde{c}_2 \\
\hat{c}_2 & \mapsto & \tilde{c}_2 \\
\hat{t} & \mapsto & \tilde{t}
\end{array}$$

and the derivations transform as follows

$$\hat{c}_1 \frac{\partial}{\partial \hat{c}_1} = \tilde{c}_1 \frac{\partial}{\partial \tilde{c}_1}, \quad \hat{c}_2 \frac{\partial}{\partial \hat{c}_2} = \tilde{c}_1 \frac{\partial}{\partial \tilde{c}_1} + \tilde{c}_2 \frac{\partial}{\partial \tilde{c}_2}, \quad \frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial \tilde{t}}.$$

Hence the foliation $\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}}$ becomes

$$\begin{aligned}
\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}} &= \hat{c}_1 \frac{\partial}{\partial \hat{c}_1} + (\hat{c}_1{}^2 \hat{c}_2{}^2 + \hat{c}_2) \frac{\partial}{\partial \hat{c}_2} + \hat{c}_1 \frac{\partial}{\partial \hat{t}} \\
&= \left( \hat{c}_1 \frac{\partial}{\partial \hat{c}_1} \right) + (\hat{c}_1{}^2 \hat{c}_2 + 1) \left( \hat{c}_2 \frac{\partial}{\partial \hat{c}_2} \right) + \hat{c}_1 \frac{\partial}{\partial \hat{t}} \\
&= \tilde{c}_1 \frac{\partial}{\partial \tilde{c}_1} + \left( \tilde{c}_1{}^2 \tilde{c}_2{}^2 \tilde{c}_2 + 1 \right) \left( \tilde{c}_1 \frac{\partial}{\partial \tilde{c}_1} + \tilde{c}_2 \frac{\partial}{\partial \tilde{c}_2} \right) + \tilde{c}_1 \tilde{c}_2 \frac{\partial}{\partial \tilde{t}} \\
&= \tilde{c}_1{}^3 \tilde{c}_2{}^3 \frac{\partial}{\partial \tilde{c}_1} + (\tilde{c}_1{}^2 \tilde{c}_2{}^4 + \tilde{c}_2) \frac{\partial}{\partial \tilde{c}_2} + \tilde{c}_1 \tilde{c}_2 \frac{\partial}{\partial \tilde{t}}.
\end{aligned}$$

Dividing by $\tilde{c}_2$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \, \mathbb{A}^3, U_{c_2}, W_{c_2}, V_{c_2}} = \tilde{c}_1{}^3 \tilde{c}_2{}^2 \frac{\partial}{\partial \tilde{c}_1} + (\tilde{c}_1{}^2 \tilde{c}_2{}^3 + 1) \frac{\partial}{\partial \tilde{c}_2} + \tilde{c}_1 \frac{\partial}{\partial \tilde{t}}.$$

There is no singularity on this chart anymore, since if $\tilde{c}_1 = 0$, then the coefficient $(\tilde{c}_1{}^2 \tilde{c}_2{}^3 + 1)$ in front of $\frac{\partial}{\partial \tilde{c}_2}$ is not equal to zero. $\qquad \square$

## 4.5   Properties of the Resulting Surface

The surface we obtain in Example 4.4.3 satisfies all requirements of Theorem 1.0.1. As we will prove in this section, the surface is normal and Gorenstein, with an $A_3$ surface singularity. Furthermore, the surface is geometrically reduced, but not geometrically normal. This proves the main result Theorem 1.0.4.

**Proposition 4.5.1.** *The surface $X$ is Gorenstein. This verifies that Property 3 in Properties 4.1.2 is satisfied.*

*Proof.* The only occurring singularity is an $A_3$-surface singularity, which is Gorenstein. $\qquad \square$

**Proposition 4.5.2.** *The surface $X$ is not geometrically normal.*

*Proof.* As stated in Remark 3.5.5, the proof of Proposition 3.5.4 relies only on the general construction of Chapter 2. The results of Chapter 4 are based on this general construction, which proves that $X$ is not geometrically normal. $\qquad \square$

**Proposition 4.5.3.** *The surface $X$ is geometrically reduced.*

*Proof.* The Remark 3.5.8 states that the proof of Proposition 3.5.6 relies on the construction of Chapter 2, with one additional assumption. This assumption states that on one of the affine charts, the coefficient in front of $\frac{\partial}{\partial t}$ in the description of $\mathcal{F}$ is non-zero. On the chart $U_{(x_0,x_1)}$, with variables $a_i$, the foliation is described as $\mathcal{F}\big|_{U_{(x_0,x_1)}} = \mathscr{O}_{U_{(x_0,x_1)}} \cdot v$, with $v$ of the form

$$v(a_1, a_2, t) = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}.$$

We chose the polynomial $h \in \mathbb{F}_2[a_1, a_2, t]$ to be of the form $h = a_1$. This choice guarantees that the coefficient in front of $\frac{\partial}{\partial t}$ is non-zero. The remark is hence applicable, and the surface $X$ is geometrically reduced.                                                                              $\square$

# Chapter 5

# Hirzebruch Surface Example

The goal of this chapter is to prove the third and fourth main theorem.

**Main Theorem** (Theorem 1.0.5). *Let $(p, Y, C) = (2, H_2, D)$. There exists a regular, geometrically reduced, projective surface $X$ that satisfies the properties of Theorem 1.0.1.*

**Main Theorem** (Theorem 1.0.6). *Let $(p, Y, C) = (2, H_3, D)$. There exists a normal, geometrically reduced, but non-regular projective surface $X$ that satisfies the properties of Theorem 1.0.1. The surface has two singular points.*

As in the two previous chapters, we are constructing a specific surface $X$ for one of the triples $(p, Y, C)$ of Theorem 1.0.1. In this chapter, we are considering the case of a Hirzebruch surface, with $(p, Y, C) = (2, H_d, D)$ for $d \geq 1$, where $H_d$ denotes the Hirzebruch surface of degree $d$ and $D$ its exceptional section. The approach we use in order to construct this surface is the same as in the previous chapter. We use the general analysis described in Chapter 2 to transform the setup into a question of finding a foliation that satisfies certain properties, in accordance with the surface we are constructing. For a Hirzebruch surface of degree 2 and of degree 3, we do find a corresponding surface. The construction of the first surface is found in Section 5.4.1.3, with the proof of its properties in Section 5.4.1.4. The construction of the second surface is found in Section 5.5.2, with the proof of its properties in Section 5.5.2.1. The key to the construction of a corresponding surface is to take into consideration the degree of the Hirzebruch surface. It is essential to distinguish the case where either the characteristic of the field divides the degree, or not. These two studies are separated into two sections, Section 5.4 and Section 5.5.

Firstly, in order to define a foliation on the open charts that cover a Hirzebruch surface, we need to study these open charts, and the maps between them. The following section will contain this study.

## 5.1 Open Charts Covering a Hirzebruch Surface

Consider the Hirzebruch surface of degree $d$, denoted by $H_d = \mathbb{P}_{\mathbb{P}^1_{x,y}}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d))$. In this section, we describe the four affine open charts that cover $H_d$, along with the maps between them. The first two charts that are affine over the projective base are denoted by $G_1$ and $G_2$, and illustrated as the green rectangles in Figure 5.1. They are obtained by localizing $H_d = \mathbb{P}_{\mathbb{P}^1_{x,y}}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d)) = \mathrm{Proj}_{\mathbb{P}^1_{x,y}} \mathrm{Sym}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d))$. Localizing at $\mathscr{O}_{\mathbb{P}^1}(d)$ and taking the degree zero contributions with respect to the grading of Proj gives the first affine chart, $G_1$. By localizing

$$\mathrm{Sym}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d)) = \mathscr{O}_{\mathbb{P}^1} \oplus (\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d)) \oplus (\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1}(2d)) \oplus \dots$$

at $\mathscr{O}_{\mathbb{P}^1}(d)$, we obtain

$$\mathscr{O}_{\mathbb{P}^1} \oplus (\mathscr{O}_{\mathbb{P}^1}(-d) \oplus \mathscr{O}_{\mathbb{P}^1}) \oplus (\mathscr{O}_{\mathbb{P}^1}(-2d) \oplus \mathscr{O}_{\mathbb{P}^1}(-d) \oplus \mathscr{O}_{\mathbb{P}^1}) \oplus \dots.$$

Hence the chart $G_1$ is

$$G_1 = \mathrm{Spec}_{\mathbb{P}^1_{x,y}} \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-d) \oplus \mathscr{O}_{\mathbb{P}^1}(-2d) \oplus \dots$$
$$= \mathrm{Spec}_{\mathbb{P}^1_{x,y}} (\mathrm{Sym}\, \mathscr{O}_{\mathbb{P}^1}(-d)).$$

The second chart $G_2$ is obtained by localizing $\mathrm{Sym}\,(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d))$ at $\mathscr{O}_{\mathbb{P}^1}$, and taking the degree zero contributions. With this definition, the chart $G_2$ is

$$G_2 = \mathrm{Spec}_{\mathbb{P}^1_{x,y}} \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1}(2d) \oplus \dots$$
$$= \mathrm{Spec}_{\mathbb{P}^1_{x,y}} (\mathrm{Sym}\, \mathscr{O}_{\mathbb{P}^1}(d)).$$
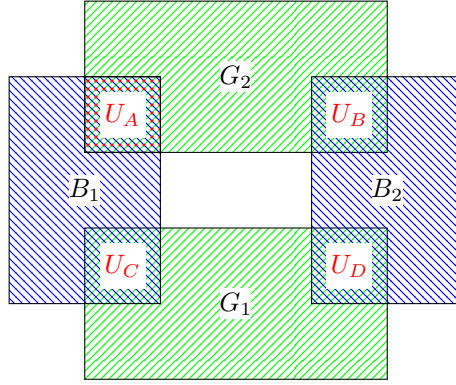


Figure 5.1: Affine charts covering a Hirzebruch surface

In order to determine what the sections on these open charts look like, we note that for any line bundle $\mathcal{L}$, the sections of $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{L}$ are defined by the maps $\mathcal{L} \to \mathscr{O}_{\mathbb{P}^1}$. Using this remark, it follows that $G_2$ does not have any non-obvious sections. If it had a non-obvious sections, there would need to be a map $\mathscr{O}_{\mathbb{P}^1}(d) \to \mathscr{O}_{\mathbb{P}^1}$, which is not possible for $d > 0$. Hence $G_2$ has the exceptional section. On the other hand, the chart $G_1$ has many sections. They are defined by the maps $\mathscr{O}_{\mathbb{P}^1}(-d) \to \mathscr{O}_{\mathbb{P}^1}$, and since it holds that $\mathrm{Hom}(\mathscr{O}_{\mathbb{P}^1}(-d), \mathscr{O}_{\mathbb{P}^1}) \cong H^0(\mathscr{O}_{\mathbb{P}^1}(d))$, it follows that $G_1$ has the general sections.

The other two charts, $B_1$ and $B_2$, which are projective over the affine base are illustrated as blue rectangles in Figure 5.1. They are defined by

$$B_1 = \mathscr{O}_{\mathbb{P}^1}(d)\big|_{D(x)} \cong \mathscr{O}_{\mathbb{A}^1_{\frac{y}{x}}}(d) \quad \text{and} \quad B_2 = \mathscr{O}_{\mathbb{P}^1}(d)\big|_{D(y)} \cong \mathscr{O}_{\mathbb{A}^1_{\frac{x}{y}}}(d).$$

The maps between the two charts $B_1$ and $B_2$ are obtained by multiplying the generator by $(\frac{y}{x})^d$, respectively $(\frac{x}{y})^d$. This is illustrated below.

$$B_1 = \mathscr{O}_{\mathbb{A}^1_{\frac{y}{x}}}(d) \quad \cong \quad \mathscr{O}_{\mathbb{P}^1}(d)\big|_{D(x)} \qquad \mathscr{O}_{\mathbb{P}^1}(d)\big|_{D(y)} \quad \cong \quad \mathscr{O}_{\mathbb{A}^1_{\frac{x}{y}}}(d) = B_2$$

$$1 \longmapsto x^d \underset{\cdot(\frac{x}{y})^d}{\overset{\cdot(\frac{y}{x})^d}{\rightleftarrows}} y^d \longleftarrow\!\shortmid 1$$

The four intersections $G_i \cap B_j$, and the morphisms between them are obtained by first localizing in order to obtain $G_i$, and then localizing further in order to obtain said intersections. We denote by $u$ and $v$ the variables of $\mathscr{O}_{\mathbb{P}^1}$ and $\mathscr{O}_{\mathbb{P}^1}(d)$ respectively. We recall that $G_1$ is constructed by localizing $\mathrm{Sym}(\mathscr{O}_{\mathbb{P}^1} u \oplus \mathscr{O}_{\mathbb{P}^1}(d)v)$ at the variable $v$, which results in the generator $\mathscr{O}_{\mathbb{P}^1}(-d)\frac{u}{v}$. Analogously, $G_2$ is constructed by localizing $\mathrm{Sym}(\mathscr{O}_{\mathbb{P}^1} u \oplus \mathscr{O}_{\mathbb{P}^1}(d)v)$ at $u$, which results in the generator $\mathscr{O}_{\mathbb{P}^1}(d)\frac{v}{u}$. The Figure 5.2 illustrates these localizations.

$$\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\,u \oplus \mathcal{O}_{\mathbb{P}^1}(d)v)$$

$$G_1 = \mathrm{Sym}\left(\mathcal{O}_{\mathbb{P}^1}(-d)\tfrac{u}{v}\right) \qquad\qquad \mathrm{Sym}\left(\mathcal{O}_{\mathbb{P}^1}(d)\tfrac{v}{u}\right) = G_2$$

$$G_1 \cap G_2 = \mathrm{Sym}\left(\mathcal{O}_{\mathbb{P}^1}(-d)\tfrac{u}{v} \oplus \mathcal{O}_{\mathbb{P}^1}(d)\tfrac{v}{u}\right)$$

Figure 5.2: Localization to the charts $G_i$.

It holds that

$$\mathrm{Proj}_{\mathbb{P}^1_{x,y}}\left(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1_k}u \oplus \mathcal{O}_{\mathbb{P}^1_k}(d)v)\big|_{D(x)}\right) = \mathrm{Proj}\left(k\left[\tfrac{y}{x}\right][u, x^d v]\right),$$

where on the right hand side, the degree of $\frac{y}{x}$ is zero, and the degree of $u$ and $x^d v$ is one. In order to glue from the restriction of $\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}u \oplus \mathcal{O}_{\mathbb{P}^1}(d)v)$ to either $D(x)$ or $D(y)$, we first need to fix an isomorphism of these two restrictions to their intersection.

$$k\left[\tfrac{y}{x}\right][u, x^d v] \longrightarrow \underbrace{k\left[\tfrac{y}{x}, \tfrac{x}{y}\right][u, x^d v] \cong k\left[\tfrac{x}{y}, \tfrac{y}{x}\right][u, y^d v]}_{=\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}u \oplus \mathcal{O}_{\mathbb{P}^1}(d)v)\big|_{D(xy)}} \longleftarrow k\left[\tfrac{x}{y}\right][u, y^d v]$$

The isomorphism we fix is the following

$$
\begin{aligned}
k\left[\tfrac{y}{x}, \tfrac{x}{y}\right][u, x^d v] &\rightarrow k\left[\tfrac{x}{y}, \tfrac{y}{x}\right][u, y^d v] \\
u &\mapsto u \\
x^d v &\mapsto y^d v \left(= \left(\tfrac{y}{x}\right)^d x^d v\right)
\end{aligned}
$$

Having fixed this isomorphism, we are now able to localize the charts $G_i$ further in order to obtain $G_i \cap B_j$. First, localizing $G_2$ we get $G_2 \cap B_1$ and $G_2 \cap B_2$, and the following morphism between these two intersections:

$$G_2 \cap B_1 = \mathrm{Spec}\, k\underbrace{\left[\tfrac{y}{x}\right]}_{=:a_1}\underbrace{\left[x^d \tfrac{v}{u}\right]}_{=:a_2} \rightarrow \mathrm{Spec}\, k\underbrace{\left[\tfrac{x}{y}\right]}_{=:b_1}\underbrace{\left[y^d \tfrac{v}{u}\right]}_{=:b_2} = G_2 \cap B_2.$$

Similarly, localizing $G_1$ we get $G_1 \cap B_1$ and $G_1 \cap B_2$, and the following morphism between these intersections:

$$G_1 \cap B_1 = \mathrm{Spec}\, k\underbrace{\left[\tfrac{y}{x}\right]}_{=:c_1}\underbrace{\left[x^{-d}\tfrac{u}{v}\right]}_{=:c_2} \rightarrow \mathrm{Spec}\, k\underbrace{\left[\tfrac{x}{y}\right]}_{=:d_1}\underbrace{\left[y^{-d}\tfrac{u}{v}\right]}_{=:d_2} = G_1 \cap B_2.$$

Summing up, we have the four intersections $U_A := G_2 \cap B_1, U_B := G_2 \cap B_2, U_C := G_1 \cap B_1$ and $U_D := G_1 \cap B_2$. Expressing the coordinates of all intersections in the coordinates $a_1$ and $a_2$, belonging to the chart $U_A$, we obtain

*for $U_B$*: $b_1 = a_1^{-1}$, $\quad b_2 = y^d \tfrac{v}{u} = \left(\tfrac{y^d}{x^d}\right)\left(x^d \tfrac{v}{u}\right) = a_1^d a_2$

*for $U_C$*: $c_1 = a_1$, $\quad c_2 = x^{-d}\tfrac{u}{v} = \left(\tfrac{u}{x^d v}\right) = \left(\tfrac{x^d v}{u}\right)^{-1} = a_2^{-1}$

*for $U_D$*: $d_1 = a_1^{-1}$, $\quad d_2 = y^{-d}\tfrac{u}{v} = \left(\tfrac{u}{y^d v}\right) = \left(\tfrac{x^d}{y^d}\right)\left(\tfrac{u}{x^d v}\right) = \left(\tfrac{y}{x}\right)^{-d}\left(\tfrac{x^d v}{u}\right)^{-1} = a_1^{-d} a_2^{-1},$

where we denote the coordinates of $U_B$ by $b_1, b_2$, the coordinates of $U_C$ by $c_1, c_2$ and the coordinates of the chart $U_D$ by $d_1, d_2$.

**Lemma 5.1.1.** *The exceptional section on the chart $G_2$ is defined by $a_2 = 0$ on $U_A$, and by $b_2 = 0$ on $U_B$. A general section on the chart $G_1$ is defined by a polynomial of degree at most $d$ in the variable $c_2$ on $U_C$ and in the variable $d_2$ in $U_D$.*

*Proof.* For the proof of the statement about the exceptional section, we study the morphism $G_2 \cap B_1 \to G_2 \cap B_2$, defined by

$$G_2 \cap B_1 = \operatorname{Spec} k \underbrace{\left[\frac{y}{x}\right]}_{=:a_1} \underbrace{\left[x^d \frac{v}{u}\right]}_{=:a_2} \quad \to \quad \operatorname{Spec} k \underbrace{\left[\frac{x}{y}\right]}_{=:b_1} \underbrace{\left[y^d \frac{v}{u}\right]}_{=:b_2} = G_2 \cap B_2.$$

We set

$$(a_1, a_2) = \left(\frac{y}{x}, 0\right).$$

Changing from the variables $(a_1, a_2) = \left(\frac{y}{x}, 0\right)$, to the variables $b_i$, we obtain $b_1 = a_1^{-1} = \frac{x}{y}$ and $b_2 = a_1^d a_2 = 0$. The exceptional section on the chart $U_B$ is hence defined by

$$(b_1, b_2) = \left(\frac{x}{y}, 0\right).$$

On the chart $G_1$, we have the general sections. Consider the morphism $G_1 \cap B_1 \to G_1 \cap B_2$, defined by

$$G_1 \cap B_1 = \operatorname{Spec} k \underbrace{\left[\frac{y}{x}\right]}_{=:c_1} \underbrace{\left[x^{-d} \frac{u}{v}\right]}_{=:c_2} \quad \to \quad \operatorname{Spec} k \underbrace{\left[\frac{x}{y}\right]}_{=:d_1} \underbrace{\left[y^{-d} \frac{u}{v}\right]}_{=:d_2} = G_1 \cap B_2.$$

Let $\varphi$ be a polynomial of degree at most $d$. We set

$$(c_1, c_2) = \left(\frac{y}{x}, \varphi\left(\frac{y}{x}\right)\right).$$

Changing from the variables $c_i$ to the variables $d_i$, we obtain

$$d_1 = c_1^{-1} = \frac{x}{y}, \quad d_2 = c_1^{-d} c_2 = \left(\frac{y}{x}\right)^{-d} \varphi\left(\frac{y}{x}\right).$$

Now suppose the polynomial $\varphi$ is of the form $\varphi(s) = \sum_{i=0}^d \gamma_i s^i$. Then $d_2$ is of the form

$$\begin{aligned}
d_2 &= \left(\frac{y}{x}\right)^{-d} \varphi\left(\frac{y}{x}\right) = \left(\frac{x}{y}\right)^d \sum_{i=0}^d \gamma_i \left(\frac{y}{x}\right)^i \\
&= d_1^d \sum_{i=0}^d \gamma_i \left(\frac{x}{y}\right)^{-i} = \sum_{i=0}^d \gamma_i d_1^{d-i} \\
&= \sum_{j=0}^d \gamma_{d-j} d_1^j,
\end{aligned}$$

which is a polynomial of degree at most $d$ in the variable $d_1$. The general section on the chart $U_D$ is hence defined by

$$(d_1, d_2) = \left(\frac{x}{y}, \sum_{j=0}^d \gamma_{d-j} \left(\frac{x}{y}\right)^j\right).$$

$\square$

## 5.2 Properties for the Construction

As in the construction of examples described in the previous two chapters, we will have to make certain choices which will on one hand restrict the generality of the examples we may find, but which are necessary working conditions. Since we are only interested in finding one explicit example of a corresponding surface, and not a general description of all examples, setting certain parameters does not pose any real constraint. As previously, we use the setup described in Chapter 2. Furthermore, we work with the following assumptions.

*Assumption* 5.2.1. We use the notations and constructions of Chapter 2. In this chapter, we consider the triple $(p, Y, C) = (2, H_d, D)$, for $d \geq 1$, from Theorem 1.0.1. The exceptional section of the Hirzebruch surface is denoted by $D$. For the construction of an explicit example, we have additionally chosen the base $T$ of $\mathfrak{X}$ to be $T = \mathbb{A}^1_{\mathbb{F}_2}$. Furthermore, only one Frobenius base change is necessary to obtain the scheme $\mathfrak{Y}$ as the normalization of $\mathfrak{X} \times_T T^1$. Hence the divisor $C$ is equal to $C = \mathfrak{C} \times_T \operatorname{Spec} \overline{k}$.

Again, as stated in Section 2.1 we have chosen $\mathfrak{Y}$ to be the fiber product of $T$ and $H_d$ over $\operatorname{Spec} \mathbb{F}_2$. With this choice, the correspondence between $\mathfrak{Y}$ and $Y$ is satisfied, meaning that $\mathfrak{Y} \times_T \operatorname{Spec} \overline{k} = Y$.

Under these assumptions, we construct a surface $X$ via the construction of a scheme $\mathfrak{X}$, which is defined by the quotient of $\mathfrak{Y}$ with a foliation $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$. As in the previous chapters, the properties the surface $X$ needs to satisfy are translated into properties of the foliation $\mathcal{F}$. These properties are equivalent to Properties 4.1.2, adapted to the Hirzebruch case.

*Properties* 5.2.2. The restrictions that ensure that the surface obtained by this construction is of the required form are the following, possibly allowing shrinking $T$.

*Property 1* The subsheaf $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ is a foliation. This property is necessary for the construction of the surface via the correspondence in Proposition 2.0.6.

*Property 2* $(\mathfrak{Y}, \mathfrak{C}) = (H_d \times U, D \times U)$, for $U \subseteq T$ open. The choice of $\mathfrak{Y}$ we have made above states that $\mathfrak{Y}$ is of this form.

*Property 3* The surface $X$ is Gorenstein. Equivalently, $\mathfrak{X}$ is Gorenstein over the generic point of $T$.

*Property 4* Lastly, the anticanonical divisor of $X$ is ample. Equivalently, the anticanonical divisor of $\mathfrak{X}$ over $T$ is ample. This means that $K_{\mathfrak{Y}} + (p-1)\mathfrak{C}$ is ample over some open subset $U \subseteq T$. This is automatic if $(p, Y, C)$ is of the chosen type.

We now let $\mathcal{F} \subseteq \mathcal{T}_{\mathfrak{Y}}$ be a foliation on $\mathfrak{Y}$ such that $\mathfrak{X} = \mathfrak{Y}/\mathcal{F}$.

The affine charts covering the Hirzebruch surface $H_d$ have been studied in Section 5.1. Locally on the first affine chart $U_A \times T$ of $\mathfrak{Y}$, with

$$U_A \times T = \operatorname{Spec} \mathbb{F}_2[a_1, a_2] \times T \subseteq H_{d, \mathbb{F}_2} \times T = \mathfrak{Y},$$

the foliation is of the form

$$\mathcal{F}_{U_A} := \mathcal{F}\big|_{U_A} = \mathscr{O}_{U_A} \cdot v,$$

where $v \in \operatorname{Der}_{\mathbb{F}_2}(\mathscr{O}_{U_A}, \mathscr{O}_{U_A}) = \mathbb{F}_2[a_1, a_2, t]\left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial t}\right)$, there $t$ denotes the coordinate of $T$. It follows that $v$ is of the form

$$v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t}$$

for some polynomials $f, g, h \in \mathbb{F}_2[a_1, a_2, t]$. As the restrictions that are posed on the polynomials $f, g$ and $h$ in order for $\mathcal{F}\big|_{U_A}$ to be a foliation have already been studied in Chapter 3, we obtain the same restrictions as stated in Proposition 3.2.2, which results in the equations

$$gh(ff_{a_1} + gf_{a_2} + hf_t) = fh(fg_{a_1} + gg_{a_2} + hg_t) = fg(fh_{a_1} + gh_{a_2} + hh_t)$$

that need to be satisfied by $f, g$ and $h$. Furthermore, Proposition 2.1.1 implies that the support of the divisor $\mathfrak{C}$ is equal to the vanishing locus of the polynomial $h$.

**Important Consequence.** *By Assumption 5.2.1, the divisor $C$ is defined to be the exceptional section of the Hirzebruch surface. On $U_A$, the exceptional section is defined by $a_2 = 0$, according to Lemma 5.1.1. It hence follows that $h(a_1, a_2, t) = a_2$.*

This means that the equations which ensure that $\mathcal{F}_{U_A}$ is indeed a foliation simplify to

$$ga_2(ff_{a_1} + gf_{a_2} + a_2 f_t) = fa_2(fg_{a_1} + gg_{a_2} + a_2 g_t) = fgg. \tag{5.2.1}$$

In order to restrict the form of the remaining polynomials $f$ and $g$ further, we exhibit how the foliation behaves on the three other charts.

## 5.3   Foliation on all Charts

The change of coordinates between the chart $U_A$ and the other charts $U_B, U_C$ and $U_D$ has been elaborated in Section 5.1. For each chart, we first express the partial derivatives $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ in terms of the partial derivatives of that respective chart. Then, using this description, we express how the foliation behaves on each of the charts.

*Remark* 5.3.1. For the remainder of this chapter, we use the notation introduced in Section 5.2 and in Section 5.1.

**Lemma 5.3.2.** *The foliation $\mathcal{F}$ is defined as $\mathcal{F}\big|_{U_A} = \mathscr{O}_{U_A} \cdot v$ on the chart $U_A$, with*

$$v(a_1, a_2, t) = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial t}.$$

*Restricted to the charts $U_B, U_C$ and $U_D$, the foliation is of the following form. For the chart $U_B$, we obtain $\mathcal{F}\big|_{U_B} = \mathscr{O}_{U_B} \cdot v_{U_B}$, with*

$$v_{U_B} = f\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^2 \frac{\partial}{\partial b_1} + \left(f\left(\frac{1}{b_1}, b_1^d b_2, t\right)db_1 b_2 + g\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^{-d}\right)\frac{\partial}{\partial b_2} + b_1^d b_2 \frac{\partial}{\partial t}.$$

*For the chart $U_C$, we obtain $\mathcal{F}\big|_{U_C} = \mathscr{O}_{U_C} \cdot v_{U_C}$, where*

$$v_{U_C} = f\left(c_1, \frac{1}{c_2}, t\right)\frac{\partial}{\partial c_1} + g\left(c_1, \frac{1}{c_2}, t\right)c_2^2 \frac{\partial}{\partial c_2} + \frac{1}{c_2}\frac{\partial}{\partial t}.$$

*Lastly, on the chart $U_D$, the restriction of the foliation is $\mathcal{F}\big|_{U_D} = \mathscr{O}_{U_D} \cdot v_{U_D}$, with*

$$v_{U_D} = f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right)d_1^2 \frac{\partial}{\partial d_1} + \left(f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right)dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right)d_1^{-d}d_2^2\right)\frac{\partial}{\partial d_2} + d_1^d \frac{1}{d_2}\frac{\partial}{\partial t}.$$

*Proof.* We apply this process first to the chart $U_B$. The variables $a_1$ and $a_2$ in terms of the variables of the chart $U_B$ are

$$a_1 = b_1^{-1}, \quad a_2 = b_1^d b_2.$$

With this, we obtain

$$da_1 = d\left(\frac{1}{b_1}\right) = \frac{1}{b_1^2}db_1$$

and

$$da_2 = d\left(b_1^d b_2\right) = db_1^{d-1}b_2(db_1) + b_1^d(db_2).$$

By definition, $\frac{\partial}{\partial b_1}$ and $\frac{\partial}{\partial b_2}$ are the completions of the following homomorphisms $\Omega_{\mathbb{F}_2[b_1, b_2]} \rightarrow \mathbb{F}_2[b_1, b_2]$:

$$\frac{\partial}{\partial b_1} : \left\{ \begin{array}{ccc} db_1 & \mapsto & 1 \\ db_2 & \mapsto & 0 \end{array} \right. \qquad \frac{\partial}{\partial b_2} : \left\{ \begin{array}{ccc} db_1 & \mapsto & 0 \\ db_2 & \mapsto & 1 \end{array} \right. .$$

Using these properties, we have

$$\frac{\partial}{\partial b_1}(\mathrm{d}a_1) = \frac{\partial}{\partial b_1}\left(\frac{1}{b_1^2}\mathrm{d}b_1\right) = b_1^{-2} = a_1^2$$

and

$$\frac{\partial}{\partial b_1}(\mathrm{d}a_2) = \frac{\partial}{\partial b_1}(\mathrm{d}b_1^{d-1}b_2(\mathrm{d}b_1) + b_1^d(\mathrm{d}b_2)) = \mathrm{d}b_1^{d-1}b_2 = \mathrm{d}b_1^{-1}b_1^d b_2 = \mathrm{d}a_1 a_2.$$

Hence

$$\frac{\partial}{\partial b_1} = a_1^2\frac{\partial}{\partial a_1} + \mathrm{d}a_1 a_2\frac{\partial}{\partial a_2} \tag{5.3.1}$$

On the other hand, we have

$$\frac{\partial}{\partial b_2}(\mathrm{d}a_1) = \frac{\partial}{\partial b_2}\left(\frac{1}{b_1^2}\mathrm{d}b_1\right) = 0$$

and

$$\frac{\partial}{\partial b_2}(\mathrm{d}a_2) = \frac{\partial}{\partial b_2}(\mathrm{d}b_1^{d-1}b_2(\mathrm{d}b_1) + b_1^d(\mathrm{d}b_2)) = b_1^d = a_1^{-d}.$$

Hence

$$\frac{\partial}{\partial b_2} = a_1^{-d}\frac{\partial}{\partial a_2}. \tag{5.3.2}$$

From (5.3.2), it follows that $\frac{\partial}{\partial a_2}$ can be expressed as

$$\frac{\partial}{\partial a_2} = a_1^d\frac{\partial}{\partial b_2} = b_1^{-d}\frac{\partial}{\partial b_2}. \tag{5.3.3}$$

Replacing $\frac{\partial}{\partial a_2}$ in (5.3.1), it follows that

$$\frac{\partial}{\partial b_1} = a_1^2\frac{\partial}{\partial a_1} + \mathrm{d}a_1 a_2 \underbrace{\frac{\partial}{\partial a_2}}_{=b_1^{-d}\frac{\partial}{\partial b_2}} = a_1^2\frac{\partial}{\partial a_1} + \mathrm{d}a_1 a_2 b_1^{-d}\frac{\partial}{\partial b_2}.$$

With this, it holds that

$$a_1^2\frac{\partial}{\partial a_1} = \frac{\partial}{\partial b_1} + \mathrm{d}a_1 a_2 b_1^{-d}\frac{\partial}{\partial b_2},$$

and hence the partial derivative $\frac{\partial}{\partial a_1}$ can be expressed as

$$\frac{\partial}{\partial a_1} = a_1^{-2}\frac{\partial}{\partial b_1} + \mathrm{d}a_1^{-1}a_2 b_1^{-d} = b_1^2\frac{\partial}{\partial b_1} + \mathrm{d}b_1 b_2\frac{\partial}{\partial b_2}. \tag{5.3.4}$$

With these descriptions of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ we can express the foliation $\mathcal{F}$ on the chart $U_B$. On the chart $U_A$, the foliation is defined as $\mathcal{F}\big|_{U_A} = v \cdot \mathscr{O}_{U_A}$ with $v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t}$. On the chart $U_B$, we obtain with (5.3.3) and (5.3.4) that

$$v(a_1, a_2, t) = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}$$

$$= f(b_1^{-1}, b_1^d b_2, t)\left(b_1^2\frac{\partial}{\partial b_1} + \mathrm{d}b_1 b_2\frac{\partial}{\partial b_2}\right) + g(b_1^{-1}, b_1^d b_2, t)\left(b_1^{-d}\frac{\partial}{\partial b_2}\right) + h(b_1^{-1}, b_1^d b_2, t)\frac{\partial}{\partial t}$$

$$= f\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^2\frac{\partial}{\partial b_1} + \left(f\left(\frac{1}{b_1}, b_1^d b_2, t\right)\mathrm{d}b_1 b_2 + g\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^{-d}\right)\frac{\partial}{\partial b_2}$$

$$+ h\left(\frac{1}{b_1}, b_1^d b_2, t\right)\frac{\partial}{\partial t}$$

$$=: v_{U_B}(b_1, b_2, t).$$

We repeat this process with the chart $U_C$. It holds that

$$a_1 = c_1, \quad a_2 = c_2^{-1}.$$

With this, we obtain

$$\mathrm{d}a_1 = \mathrm{d}c_1$$

and

$$\mathrm{d}a_2 = \mathrm{d}\left(\frac{1}{c_2}\right) = \frac{1}{c_2^2}\mathrm{d}c_2.$$

By definition, $\frac{\partial}{\partial c_1}$ and $\frac{\partial}{\partial c_2}$ are the completions of the following homomorphisms $\Omega_{\mathbb{F}_2[c_1,c_2]} \to \mathbb{F}_2[c_1,c_2]$ :

$$\frac{\partial}{\partial c_1} : \begin{cases} \mathrm{d}c_1 & \mapsto & 1 \\ \mathrm{d}c_2 & \mapsto & 0 \end{cases} \qquad \frac{\partial}{\partial c_2} : \begin{cases} \mathrm{d}c_1 & \mapsto & 0 \\ \mathrm{d}c_2 & \mapsto & 1 \end{cases}.$$

Using these properties, we have

$$\frac{\partial}{\partial c_1}(\mathrm{d}a_1) = \frac{\partial}{\partial c_1}(\mathrm{d}c_1) = 1$$

and

$$\frac{\partial}{\partial c_1}(\mathrm{d}a_2) = \frac{\partial}{\partial c_1}\left(\frac{1}{c_2^2}\mathrm{d}c_2\right) = 0.$$

Hence

$$\frac{\partial}{\partial c_1} = \frac{\partial}{\partial a_1}. \tag{5.3.5}$$

On the other hand, we have

$$\frac{\partial}{\partial c_2}(\mathrm{d}a_1) = \frac{\partial}{\partial c_2}(\mathrm{d}c_1) = 0$$

and

$$\frac{\partial}{\partial c_2}(\mathrm{d}a_2) = \frac{\partial}{\partial c_2}\left(\frac{1}{c_2^2}\mathrm{d}c_2\right) = c_2^{-2} = a_2^2.$$

Hence

$$\frac{\partial}{\partial c_2} = a_2^2 \frac{\partial}{\partial a_2}. \tag{5.3.6}$$

From (5.3.6) it follows that

$$\frac{\partial}{\partial a_2} = a_2^{-2}\frac{\partial}{\partial c_2} = c_2^2 \frac{\partial}{\partial c_2}.$$

With these descriptions of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ we can express the foliation $\mathcal{F}$ on the chart $U_C$. On the chart $U_A$, the foliation is defined as $\mathcal{F}\big|_{U_A} = v \cdot \mathscr{O}_{U_A}$ with $v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t}$. On the chart $U_C$, we obtain with (5.3.5) and (5.3.6) that

$$\begin{aligned}
v(a_1, a_2, t) &= f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t} \\
&= f\left(c_1, \frac{1}{c_2}, t\right)\frac{\partial}{\partial c_1} + g\left(c_1, \frac{1}{c_2}, t\right)c_2^2\frac{\partial}{\partial c_2} + h\left(c_1, \frac{1}{c_2}, t\right)\frac{\partial}{\partial t} \\
&=: v_{U_C}(c_1, c_2, t).
\end{aligned}$$

Lastly, we repeat this process for the chart $U_D$. It holds that

$$a_1 = d_1^{-1}, \quad a_2 = d_1^d d_2^{-1}.$$

With this, we obtain

$$\mathrm{d}a_1 = \mathrm{d}\left(\frac{1}{d_1}\right) = \frac{1}{d_1^2}\mathrm{d}d_1$$

and

$$\mathrm{d}a_2 = \mathrm{d}\left(d_1^d d_2^{-1}\right) = d d_1^{d-1} d_2^{-1}(\mathrm{d}d_1) + d_1^d d_2^{-2}(\mathrm{d}d_2).$$

By definition, $\frac{\partial}{\partial d_1}$ and $\frac{\partial}{\partial d_2}$ are the completions of the following homomorphisms $\Omega_{\mathbb{F}_2[d_1,d_2]} \to \mathbb{F}_2[d_1,d_2]$ :

$$\frac{\partial}{\partial d_1} : \begin{cases} \mathrm{d}d_1 & \mapsto & 1 \\ \mathrm{d}d_2 & \mapsto & 0 \end{cases} \qquad \frac{\partial}{\partial d_2} : \begin{cases} \mathrm{d}d_1 & \mapsto & 0 \\ \mathrm{d}d_2 & \mapsto & 1 \end{cases}.$$

Using these properties, we have

$$\frac{\partial}{\partial d_1}(\mathrm{d}a_1) = \frac{\partial}{\partial d_1}\left(d_1^{-2}\mathrm{d}d_1\right) = d_1^{-2}$$

and

$$\frac{\partial}{\partial d_1}(\mathrm{d}a_2) = \frac{\partial}{\partial d_1}(dd_1^{d-1}d_2^{-1}(\mathrm{d}d_1) + d_1^d d_2^{-2}(\mathrm{d}d_2)) = dd_1^{d-1}d_2^{-1}.$$

Hence

$$\frac{\partial}{\partial d_1} = d_1^{-2}\frac{\partial}{\partial a_1} + dd_1^{d-1}d_2^{-1}\frac{\partial}{\partial a_2}. \tag{5.3.7}$$

On the other hand, we have

$$\frac{\partial}{\partial d_2}(\mathrm{d}a_1) = \frac{\partial}{\partial d_2}\left(d_1^{-2}\mathrm{d}d_1\right) = 0$$

and

$$\frac{\partial}{\partial d_2}(\mathrm{d}a_2) = \frac{\partial}{\partial d_2}(dd_1^{d-1}d_2^{-1}(\mathrm{d}d_1) + d_1^d d_2^{-2}(\mathrm{d}d_2)) = d_1^d d_2^{-2}.$$

Hence

$$\frac{\partial}{\partial d_2} = d_1^d d_2^{-2}\frac{\partial}{\partial a_2}. \tag{5.3.8}$$

From (5.3.8), it follows that $\frac{\partial}{\partial a_2}$ can be expressed as

$$\frac{\partial}{\partial a_2} = d_1^{-d} d_2^2 \frac{\partial}{\partial d_2}. \tag{5.3.9}$$

Replacing $\frac{\partial}{\partial a_2}$ in (5.3.7), it follows that

$$\frac{\partial}{\partial d_1} = d_1^{-2}\frac{\partial}{\partial a_1} + dd_1^{d-1}d_2^{-1}d_1^{-d}d_2^2\frac{\partial}{\partial d_2} = d_1^{-2}\frac{\partial}{\partial a_1} + dd_1^{-1}d_2\frac{\partial}{\partial d_2}.$$

With this, it holds that

$$d_1^{-2}\frac{\partial}{\partial a_1} = \frac{\partial}{\partial d_1} + dd_1^{-1}d_2\frac{\partial}{\partial d_2},$$

and hence the partial derivative $\frac{\partial}{\partial a_1}$ can be expressed as

$$\frac{\partial}{\partial a_1} = d_1^2\frac{\partial}{\partial d_1} + dd_1 d_2\frac{\partial}{\partial d_2}. \tag{5.3.10}$$

With these descriptions of $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$ we can express the foliation $\mathcal{F}$ on the chart $U_D$. On the chart $U_A$, the foliation is defined as $\mathcal{F}|_{U_A} = v \cdot \mathscr{O}_{U_A}$ with $v = f\frac{\partial}{\partial a_1} + g\frac{\partial}{\partial a_2} + h\frac{\partial}{\partial t}$. On the chart $U_D$, we obtain with (5.3.9) and (5.3.10) that

$$\begin{aligned}
v(a_1, a_2, t) &= f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}\\
&= f\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)\left(d_1^2\frac{\partial}{\partial d_1} + dd_1 d_2\frac{\partial}{\partial d_2}\right) + g\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)\left(d_1^{-d}d_2^2\frac{\partial}{\partial d_2}\right)\\
&\quad + h\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)\frac{\partial}{\partial t}\\
&= f\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)d_1^2\frac{\partial}{\partial d_1} + \left(f\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)d_1^{-d}d_2^2\right)\frac{\partial}{\partial d_2}\\
&\quad + h\left(\frac{1}{d_1}, d_1^d\frac{1}{d_2}, t\right)\frac{\partial}{\partial t}\\
&=: v_{U_D}(d_1, d_2, t).
\end{aligned}$$

$\square$

   With this description of the foliation on all four charts, we are able to further restrict the form of the polynomials $f$ and $g$ defining the foliation. For this, we use the fact that as stated as Property 4 in Properties 5.2.2, the anticanonical divisor $-K_X$ of the surface $X$ is ample. The divisors in Theorem 1.0.1 are constructed in such a way that this is satisfied. It holds that $\mathcal{F} \cong \mathcal{O}_{\mathfrak{Y}}(-\mathfrak{C})$. Using the fact that the divisor $C$ is equal to the exceptional section, which is defined by $\{a_2 = 0\}$ on the chart $U_A$, it follows that $\mathcal{F} \cong \mathcal{O}_{\mathfrak{Y}}(-1)$. On the chart $U_A$, the foliation is regular by definition. But since $\mathcal{F} \cong \mathcal{O}_{\mathfrak{Y}}(-1)$, there must be a pole of order one on one of the other charts. This pole can not occur on the chart $U_B$, which also contains the exceptional section. Instead, it appears along $\{c_2 = 0\}$ or $\{d_2 = 0\}$. Along the fibers of the Hirzebruch surface, defined by $\{a_1 = 0\}, \{b_1 = 0\}, \{c_1 = 0\}$ and $\{d_1 = 0\}$ no poles may occur. To sum up the study of Property 3, there needs to be a pole of of order one along either $\{c_2 = 0\}$ or $\{d_2 = 0\}$, or along both lines, and no poles elsewhere.

   We use this knowledge about the potential poles by studying each of the charts $U_B, U_C$ and $U_D$ separately. For each chart, we list how the restrictions on the possible poles restrict the form of the polynomials $f$ and $g$. In the remainder of this section, we use this to implement as many restrictions as possible on the form of the polynomials $f$ and $g$.

**Lemma 5.3.3.** *The exponent of the second variable in the polynomial $f$ is at most one. The exponent of the second variable in the polynomial $g$ is at most three. Additional restrictions on both polynomials occur. These restrictions depend on whether the characteristic divides the degree of the Hirzebruch surface or not.*

*Remark* 5.3.4. The proof of this lemma covers the restrictions that occur independently of whether the characteristic divides the degree of the Hirzebruch surface or not. It is mentioned in the proof that we need to study both cases separately in order to be able to restrict the polynomials enough to find explicit examples of foliations.

*Proof.* Firstly, we study the chart $U_C$. The foliation $\mathcal{F}$ restricted to $U_C$ is defined by $\mathcal{F}\big|_{U_C} = \mathcal{O}_{U_C} \cdot v_{U_C}$, where

$$v_{U_C} = f\left(c_1, \frac{1}{c_2}, t\right)\frac{\partial}{\partial c_1} + g\left(c_1, \frac{1}{c_2}, t\right)c_2^2\frac{\partial}{\partial c_2} + \frac{1}{c_2}\frac{\partial}{\partial t}.$$

Since poles of order 1 may occur along $c_2 = 0$, we obtain the following two restrictions:

   ∘ In the polynomial $f$, the exponent of the second variable is at most one.

   ∘ In the polynomial $g$, the exponent of the second variable is at most three.

Furthermore, we note that due to the coefficient in front of $\frac{\partial}{\partial t}$, there is a pole of order one along $\{c_2 = 0\}$, which ensures that $\mathcal{F} \cong \mathcal{O}_{\mathfrak{Y}}(-1)$ holds. Further poles along $\{c_2 = 0\}$ or $\{d_2 = 0\}$ are hence optional.

   Secondly, we study the chart $U_B$. The foliation $\mathcal{F}$ restricted to $U_B$ is defined by $\mathcal{F}\big|_{U_B} = \mathcal{O}_{U_B} \cdot v_{U_B}$, with

$$v_{U_B} = f\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^2\frac{\partial}{\partial b_1} + \left(f\left(\frac{1}{b_1}, b_1^d b_2, t\right)db_1 b_2 + g\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^{-d}\right)\frac{\partial}{\partial b_2} + b_1^d b_2\frac{\partial}{\partial t}$$

Since no poles occur along $b_1 = 0$, we obtain the following restrictions. First, we consider the polynomial $f$. Without loss of generality, we assume that $f$ is a monomial of the form $f(a_1, a_2, t) = a_1^i a_2^j$ for some $i, j \in \mathbb{N}$. The result we obtain can be extended to $f$ being a sum of such monomials, as well as to $f$ containing the variable $t$ to any exponent. The coefficient in front of $\frac{\partial}{\partial b_1}$ is

$$f\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^2 = b_1^{-i}(b_1^d b_2)^j b_1^2 = b_1^{jd-i+2}b_2^j.$$

The fact that no pole occurs in the variable $b_2$ means that $j \geq 0$, which is satisfied. Furthermore, the fact that no pole occurs in the variable $b_1$ means that $jd - i + 2 \geq 0 \Rightarrow jd \geq i - 2 \Rightarrow j \geq \frac{i-2}{d}$.

   The coefficient in front of $\frac{\partial}{\partial b_2}$ is

$$\left(f\left(\frac{1}{b_1}, b_1^d b_2, t\right)db_1 b_2 + g\left(\frac{1}{b_1}, b_1^d b_2, t\right)b_1^{-d}\right).$$

Again, we may assume that the polynomial $g$ is of the form $g(a_1, a_2, t) = a_1^k a_2^l$ for some $k, l \in \mathbb{N}$. With this, the coefficient in front of $\frac{\partial}{\partial b_2}$ is equal to

$$\left( f\left( \frac{1}{b_1}, b_1^d b_2, t \right) db_1 b_2 + g\left( \frac{1}{b_1}, b_1^d b_2, t \right) b_1^{-d} \right) = b_1^{-i} (b_1^d b_2)^j db_1 b_2 + b_1^{-k} (b_1^d b_2)^l b_1^{-d} \quad (5.3.11)$$

$$= db_1^{jd-i+1} b_2^{j+1} + b_1^{dl-k-d} b_2^l. \quad (5.3.12)$$

We can not make any further claims of what relations exist between $i, j, k$ and $l$ without separating the two possible cases of either $d$ being divisible by the characteristic $p = 2$, or $d$ not being divisible by $p$. In the first case, the first term of the sum is canceled. In the second case, the first term is not canceled, but potential poles of higher orders could appear, if they cancel each other out. These two cases are studied separately in Section 5.4 and Section 5.5.

Lastly, we study the chart $U_D$. The foliation $\mathcal{F}$ restricted to $U_D$ is defined by $\mathcal{F}\big|_{U_D} = \mathscr{O}_{U_D} \cdot v_{U_D}$, with

$$v_{U_D} = f\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) d_1^2 \frac{\partial}{\partial d_1} + \left( f\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) dd_1 d_2 + g\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) d_1^{-d} d_2^2 \right) \frac{\partial}{\partial d_2} + d_1^d \frac{1}{d_2} \frac{\partial}{\partial t}.$$

With the same notation as on the chart $U_B$, the coefficient in front of $\frac{\partial}{\partial d_1}$ is

$$f\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) d_1^2 = d_1^{-i} (d_1^d d_2^{-1})^j d_1^2 = d_1^{dj+2-i} d_2^{-j}.$$

Since no poles occur along $d_1$, it needs to hold that $dj + 2 - i \geq 0 \Rightarrow dj \geq i - 2 \Rightarrow j \geq \frac{i-2}{d}$. Furthermore, poles of order one may occur along $d_2$, which leads to and $-j \geq -1 \Rightarrow j \leq 1$.

The coefficient in front of $\frac{\partial}{\partial d_2}$ is

$$\left( f\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) dd_1 d_2 + g\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) d_1^{-d} d_2^2 \right) = d_1^{-i} (d_1^d d_2^{-1})^j dd_1 d_2 + (d_1^{-1})^k (d_1^d d_2^{-1})^l d_1^{-d} d_2^2$$

$$= dd_1^{dj+1-i} d_2^{1-j} + d_1^{dl-d-k} d_2^{2-l}.$$

As remarked for the chart $U_B$, no further claims can be made in this general case. At this point, it is necessary to separate our study into two different cases. $\qquad \square$

**Important Remark.** *As mentioned in the proof above, in order to find additional restrictions on the polynomials $f$ and $g$ to Lemma 5.3.3, we need to study two cases separately. The two cases where $p$ divides the degree of the Hirzebruch surface, $d$, and where $p$ does not divide $d$ lead to different restrictions. The difference of these two cases can for example be illustrated by equation (5.3.12) in the proof above. If $p \mid d$, then the first summand is zero, which is not the case if $p \nmid d$. Hence these two possibilities lead to different relations between the integers $i, j, k$ and $l$ which describe the polynomials $f$ and $g$.*

We now separate these two cases. The first case we treat is the case for $(p, H_d, D)$, where the characteristic $p = 2$ divides the degree $d$ of the Hirzebruch surface. The following section, Section 5.4 covers this case. The case where $p$ does not divide the degree $d$ is covered in Section 5.5.

## 5.4 Characteristic Divides Degree of the Hirzebruch Surface

We assume that the characteristic $p$, which is equal to 2, divides the degree of the Hirzebruch surface, $2 \big| d$. Summing up the conditions on the three affine charts $U_B, U_C$ and $U_D$, we obtain the following lemma.

**Lemma 5.4.1.** *The polynomials $f$ and $g$, which define the foliation $\mathcal{F}$ are sums of the monomials $f(a_1, a_2, t) = a_1^i a_2^j$ and $g(a_1, a_2, t) = a_1^k a_2^l$, where $i, j, k, l \in \mathbb{N}$ satisfy*

$$j \leq 1 \tag{5.4.1a}$$

$$l \leq 3 \tag{5.4.1b}$$

$$j \geq \frac{i-2}{d} \tag{5.4.1c}$$

$$l \geq 1 + \frac{k}{d}. \tag{5.4.1d}$$

*Additionally, in each monomial, the variable $t$ may appear to any exponent.*

The proof of this lemma uses the restrictions we obtain by studying all affine charts, and the poles that may appear on each chart.

*Proof.* We continue with the study we have started above, in Lemma 5.3.3. Due to that lemma, the natural numbers $i, j, k, l$ need to satisfy the following restrictions imposed

*by $U_C$ :*    ∘ $j \leq 1$

∘ $l \leq 3$

*by $U_B$ :* The coefficient in front of $\frac{\partial}{\partial b_1}$ has already been studied above, and leads to the restriction $j \geq \frac{i-2}{d}$. Using the fact that $2|d$, the coefficient in front of $\frac{\partial}{\partial b_2}$ simplifies to

$$\left( f\left( \frac{1}{b_1}, b_1^d b_2, t \right) db_1 b_2 + g\left( \frac{1}{b_1}, b_1^d b_2, t \right) b_1^{-d} \right) = d b_1^{jd-i+1} b_2^{j+1} + b_1^{dl-k-d} b_2^l.$$

Since no poles along $b_1$ may occur, we get $dl - k - d \geq 0 \Rightarrow dl \geq k + d \Rightarrow l \geq \frac{k+d}{d} = 1 + \frac{k}{d}$. Summing up, we get

∘ $j \geq \frac{i-2}{d}$

∘ $l \geq 1 + \frac{k}{d}$

*by $U_D$ :* Studying the coefficient in front of $\frac{\partial}{\partial d_1}$ has lead to the restrictions $j \geq \frac{i-2}{d}$ and $j \leq 1$. Using the fact that $2|d$, the coefficient in front of $\frac{\partial}{\partial d_2}$ simplifies to

$$\left( f\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) dd_1 d_2 + g\left( \frac{1}{d_1}, d_1^d \frac{1}{d_2}, t \right) d_1^{-d} d_2^2 \right) = d d_1^{jd+1-i} d_2^{1-j} + d_1^{dl-d-k} d_2^{2-l}.$$

With the fact no poles along $d_1$ may occur, and poles of order 1 along $d_2$ may occur, we get

*for $d_1$ :* $dl - d - k \geq 0 \Rightarrow dl \geq d + k \Rightarrow l \geq \frac{d+k}{d} = 1 + \frac{k}{d}$

*for $d_2$ :* $2 - l \geq -1 \Rightarrow 3 \geq l$.

Summing up the restrictions posed by the chart $U_D$, we get

∘ $j \geq \frac{i-2}{d}$

∘ $j \leq 1$

∘ $l \geq 1 + \frac{k}{d}$

∘ $l \leq 3$.

Summing up the requirements on all charts, the claim follows.    □

We can explicitly characterize all cases that can occur, by combining the restriction (5.4.1a) with (5.4.1c), and (5.4.1b) with (5.4.1d). This results in the following lemma.

**Lemma 5.4.2.** *If $2 \mid d$, then the polynomials $f$ and $g$ are sums of the following monomials. For $f$, these monomials are*

$$1, \ a_1, \ a_1^2, \ a_1^i a_2,$$

*with $i \leq d + 2$. For $g$, the monomials are*

$$a_2, \ a_1^{k_1} a_2^2, \ a_1^{k_2} a_2^3$$

*with $k_1 \leq d$ and $k_2 \leq 2d$. Additionally, the variable $t$ can appear to any exponent in all of these monomials.*

*Proof.* The restriction (5.4.1a) states that $j = 0$ or $j = 1$. With the restriction (5.4.1c), the following cases exist.

○ If $j = 0 \overset{(5.4.1c)}{\Rightarrow} \frac{i-2}{d} \leq 0$. But since $d \geq 0$ it follows from this that $i - 2 \leq 0$, and so with $i \geq 0$ this holds for $i \in \{0, 1, 2\}$.

○ If $j = 1 \overset{(5.4.1c)}{\Rightarrow} \frac{i-2}{d} \leq 1$. Since $d \geq 0$, it follows that $i - 2 \leq d \Rightarrow i \leq d + 2$.

The restriction (5.4.1b) states that $l = 0, 1, 2$ or $l = 3$. With the restriction (5.4.1d), the following cases exist.

○ If $l = 0 \overset{(5.4.1d)}{\Rightarrow} 1 + \frac{k}{d} \leq 0$. But since both $k$ and $d \in \mathbb{N}$, this case can not occur.

○ If $l = 1 \overset{(5.4.1d)}{\Rightarrow} 1 + \frac{k}{d} \leq 1$. From this it follows that $\frac{k}{d} \leq 0$. But since both are natural numbers, this holds only if $\frac{k}{d} = 0$, and hence if $k = 0$.

○ If $l = 2 \overset{(5.4.1d)}{\Rightarrow} 1 + \frac{k}{d} \leq 2$. From this it follows that $\frac{k}{d} \leq 1$. This holds if $k \leq d$.

○ If $l = 3 \overset{(5.4.1d)}{\Rightarrow} 1 + \frac{k}{d} \leq 3$. From this it follows that $\frac{k}{d} \leq 2$, which holds if $k \leq 2d$.

<div align="right">□</div>

With the restrictions posed by the Lemma 5.4.2, we now construct an explicit example for $Y$ being a Hirzebruch surface of degree 2.

## 5.4.1 Construction of an Example in the Case $d = 2$

If $d = 2$, then the polynomials $f$ and $g$ that define the foliation $\mathcal{F}$ consist of the following monomials, according to Lemma 5.4.2. For $f$, these monomials are

$$1, a_1, a_1^2, a_2, a_1 a_2, a_1^2 a_2, a_1^3 a_2, a_1^4 a_2,$$

with the variable $t$ appearing to any exponent. For $g$, these monomials, again with the variable $t$ appearing to any exponent are

$$a_2, a_2^2, a_1 a_2^2, a_1^2 a_2^2, a_2^3, a_1 a_2^3, a_1^2 a_2^3, a_1^3 a_2^3, a_1^4 a_2^3.$$

Hence we may write these two polynomials in the following way, with $\alpha_i, \beta_i \in \mathbb{F}_2[t]$.

$$f(a_1, a_2, t) = \alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2,$$

$$g(a_1, a_2, t) = \beta_1 a_2 + \beta_2 a_2^2 + \beta_3 a_1 a_2^2 + \beta_4 a_1^2 a_2^2 + \beta_5 a_2^3 + \beta_6 a_1 a_2^3 + \beta_7 a_1^2 a_2^3 + \beta_8 a_1^3 a_2^3 + \beta_9 a_1^4 a_2^3.$$

In the subsection below, we study the restrictions on $f$ and $g$ posed by the equations (5.2.1), which ensure that the polynomials define a foliation. Throughout the subsection, we set certain parameters to take specific values. We do so in order to eventually be able to construct explicit foliations. These foliations are constructed in Section 5.4.1.3.

#### 5.4.1.1   Restrictions

The restrictions that ensure that the polynomials $f$ and $g$ define a foliation are (5.2.1). They state that the following equations need to be satisfied

$$ga_2(ff_{a_1} + gf_{a_2} + a_2f_t) = fa_2(fg_{a_1} + gg_{a_2} + a_2g_t) = fgg.$$

In order to relate these restrictions to the polynomials $\alpha_i$ and $\beta_i$, we first study what divisibility condition (5.2.1) implies.

**Lemma 5.4.3.** *The equations in (5.2.1) imply that the following divisibility conditions need to hold for the polynomials $f$ and $g$:*

$$f \mid (g'f_{a_2} + f_t) \tag{5.4.2a}$$
$$g' \mid (ff_{a_1} + a_2f_t) \tag{5.4.2b}$$
$$g' \mid (fg'_{a_1} + a_2g'_t). \tag{5.4.2c}$$

*Proof.* The equality
$$a_2(ff_{a_1} + gf_{a_2} + a_2f_t) = fg$$
implies

$$f \mid a_2(gf_{a_2} + a_2f_t) \tag{5.4.3a}$$
$$g \mid a_2(ff_{a_1} + a_2f_t) \tag{5.4.3b}$$
$$a_2 \mid fg. \tag{5.4.3c}$$

The equality
$$a_2(fg_{a_1} + gg_{a_2} + a_2g_t) = g^2$$
implies

$$g \mid a_2(fg_{a_1} + a_2g_t) \tag{5.4.4a}$$
$$a_2 \mid g^2 \Rightarrow a_2 \mid g. \tag{5.4.4b}$$

Lastly, the equality
$$g(ff_{a_1} + gf_{a_2} + a_2f_t) = f(fg_{a_1} + gg_{a_2} + a_2g_t)$$
implies

$$g \mid f(fg_{a_1} + a_2g_t) \tag{5.4.5a}$$
$$f \mid g(gf_{a_2} + a_2f_t). \tag{5.4.5b}$$

Since by (5.4.4b), $a_2 \mid g$ we may write $g = a_2g'$ for some polynomial $g' \in \mathbb{F}_2[a_1, a_2, t]$. Furthermore, since we assume that no common divisor of $f, g$ and $h$ exists, with $h$ being defined as $h = a_2$, it follows that $f \nmid a_2$ and $a_2 \nmid f$. The partial derivatives of $g$ are

$$g_{a_1} = \frac{\partial}{\partial a_1}(a_2g') = a_2g'_{a_1}, \qquad g_{a_2} = \frac{\partial}{\partial a_2}(a_2g') = g' + a_2g'_{a_2}, \qquad g_t = \frac{\partial}{\partial t}(a_2g') = a_2g'_t.$$

With this description of $g$, the conditions above transform into

$$f \mid a_2(a_2g'f_{a_2} + a_2f_t) \Rightarrow f \mid a_2^2(g'f_{a_2} + f_t) \Rightarrow f \mid (g'f_{a_2} + f_t) \tag{5.4.6a}$$
$$a_2g' \mid a_2(ff_{a_1} + a_2f_t) \Rightarrow g' \mid (ff_{a_1} + a_2f_t) \tag{5.4.6b}$$
$$a_2 \mid fa_2g' \text{ is satisfied} \tag{5.4.6c}$$
$$a_2g' \mid a_2(fa_2g'_{a_1} + a_2^2g'_t) \Rightarrow g' \mid a_2(fg'_{a_1} + a_2g'_t) \tag{5.4.6d}$$
$$a_2g' \mid f(fa_2g'_{a_1} + a_2^2g'_t) \Rightarrow g' \mid f(fg'_{a_1} + a_2g'_t) \tag{5.4.6e}$$
$$f \mid a_2g'(a_2g'f_{a_2} + a_2f_t) \Rightarrow f \mid a_2^2g'(g'f_{a_2} + f_t) \tag{5.4.6f}$$

The conditions (5.4.6d) and (5.4.6e) imply that $g' \mid fg'_{a_1} + a_2g'_t$. Additionally, using that $f \nmid a_2$, the condition (5.4.6f) agrees with (5.4.6a). Summing up, we hence obtain the restrictions 5.4.2a to 5.4.2c.                                                                                                                                                                    $\square$

From now on, up until Lemma 5.3.3, we incorporate these three condition 5.4.2a to 5.4.2c from Lemma 5.4.3 into the description of the polynomials $f$ and $g$ based on Lemma 5.4.2. Based on Lemma 5.4.2, we expressed the polynomials $f$ and $g$ in the following way, with $\alpha_i, \beta_i \in \mathbb{F}_2[t]$.

$$f(a_1, a_2, t) = \alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2,$$

$$g(a_1, a_2, t) = \beta_1 a_2 + \beta_2 a_2^2 + \beta_3 a_1 a_2^2 + \beta_4 a_1^2 a_2^2 + \beta_5 a_2^3 + \beta_6 a_1 a_2^3 + \beta_7 a_1^2 a_2^3 + \beta_8 a_1^3 a_2^3 + \beta_9 a_1^4 a_2^3.$$

The partial derivatives of the polynomial $f$ are

$$f_{a_1} = \alpha_2 + \alpha_5 a_2 + \alpha_7 a_1^2 a_2 \qquad\qquad f_{a_2} = \alpha_4 + \alpha_5 a_1 + \alpha_6 a_1^2 + \alpha_7 a_1^3 + \alpha_8 a_1^4.$$

The polynomial $g'$ is of the form

$$g'(a_1, a_2, t) = \beta_1 + \beta_2 a_2 + \beta_3 a_1 a_2 + \beta_4 a_1^2 a_2 + \beta_5 a_2^2 + \beta_6 a_1 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_8 a_1^3 a_2^2 + \beta_9 a_1^4 a_2^2.$$

Its partial derivatives are

$$g'_{a_1} = \beta_3 a_2 + \beta_6 a_2^2 + \beta_8 a_1^2 a_2^2 \qquad\qquad g'_{a_2} = \beta_2 + \beta_3 a_1 + \beta_4 a_1^2.$$

We note that the divisibility condition 5.4.2c states that $g' \mid fg'_{a_1} + a_2 g'_t$. In order to find two explicit polynomials $f$ and $g$ such that this condition holds, we may assume that $g'_{a_1} = 0$. In doing so, we limit the generality of possible examples that we find. On the other hand, it is necessary at this point to implement some constraints that allow us to simplify the situation. As we will see later, with this constraint we able to produce examples of the desired form. The condition then states that $g' \mid a_2 g'_t$. But since $g'_t$ is of lower degree than $g'$ in the variable $t$, this can only hold if $g'_t = 0$. The assumption $g'_{a_1} = 0$ implies that $g'_t = 0$ as well. Furthermore, this implies that $\beta_3 = \beta_6 = \beta_8 = 0$, and that the variable $t$ appears in $g'$ only to exponents that are divisible by 2.

Therefore, the polynomial $g'$ is of the form

$$g'(a_1, a_2, t) = \beta_1 + \beta_2 a_2 + \beta_4 a_1^2 a_2 + \beta_5 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_9 a_1^4 a_2^2$$

with partial derivatives

$$g'_{a_1} = 0 \qquad\qquad g'_{a_2} = \beta_2 + \beta_4 a_1^2 \qquad\qquad g'_t = 0.$$

The equation $g'_t = 0$ implies that in each polynomial $\beta_i$, the variable $t$ appears to an exponent divisible by 2. Going back to the original equation (5.2.1) which ensures that the polynomials $f$ and $g$ indeed define a foliation, we obtain

$$g a_2 (f f_{a_1} + g f_{a_2} + a_2 f_t) = f a_2 (f g_{a_1} + g g_{a_2} + a_2 g_t) = f g g$$

$$\Leftrightarrow a_2^2 g'(f f_{a_1} + a_2 g' f_{a_2} + a_2 f_t) = a_2 f(f a_2 \cancel{g'_{a_1}} + a_2 g'(g' + a_2 g'_{a_2}) + a_2^2 \cancel{g'_t}) = a_2^2 f(g')^2$$

$$\Leftrightarrow a_2^2 g'(f f_{a_1} + a_2 g' f_{a_2} + a_2 f_t) = a_2^2 f g'(g' + a_2 g'_{a_2}) = a_2^2 f(g')^2$$

$$\Leftrightarrow f f_{a_1} + a_2 g' f_{a_2} + a_2 f_t = f(g' + a_2 g'_{a_2}) = f g'$$

This gives us the following two equations that $f$ and $g$ need to satisfy.

$$f g' = f g' + a_2 f g'_{a_2} \Rightarrow a_2 f g'_{a_2} = 0 \tag{5.4.7a}$$

$$f g' = f f_{a_1} + a_2 g' f_{a_2} + a_2 f_t \tag{5.4.7b}$$

The first equation 5.4.7a implies that $a_2 f g'_{a_2} = 0$. But since by assumption, $f \neq 0$, it follows that $g'_{a_2} = 0$. With this, it holds that $\beta_2 = \beta_4 = 0$ in the explicit description of $g$. Hence $g'$ is of the form

$$g'(a_1, a_2, t) = \beta_1 + \beta_5 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_9 a_1^4 a_2^2.$$

The second equation 5.4.7b states that $f g' = f f_{a_1} + a_2 g' f_{a_2} + a_2 f_t$. With the description of $f$ and $g$ as follows

$$f(a_1, a_2, t) = \alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2$$

$$g'(a_1, a_2, t) = \beta_1 + \beta_5 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_9 a_1^4 a_2^2,$$

we calculate the left and right hand side of equation 5.4.7b separately. We then compare these two sides, in order to get restrictions on the polynomials $\alpha_i$ and $\beta_i \in \mathbb{F}_2[t]$.

The left hand side of 5.4.7b is

$$
\begin{aligned}
fg' \quad =& \ (\beta_1 + \beta_5 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_9 a_1^4 a_2^2) \\
& \ (\alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2) \\
=& \ \alpha_1\beta_1 + \alpha_2\beta_1 a_1 + \alpha_3\beta_1 a_1^2 + \alpha_4\beta_1 a_2 + \alpha_5\beta_1 a_1 a_2 + \alpha_6\beta_1 a_1^2 a_2 + \alpha_7\beta_1 a_1^3 a_2 + \alpha_8\beta_1 a_1^4 a_2 \\
& +\alpha_1\beta_5 a_2^2 + \alpha_2\beta_5 a_1 a_2^2 + \alpha_3\beta_5 a_1^2 a_2^2 + \alpha_4\beta_5 a_2^3 + \alpha_5\beta_5 a_1 a_2^3 + \alpha_6\beta_5 a_1^2 a_2^3 + \alpha_7\beta_5 a_1^3 a_2^3 \\
& +\alpha_8\beta_5 a_1^4 a_2^3 + \alpha_1\beta_7 a_1^2 a_2^2 + \alpha_2\beta_7 a_1^3 a_2^2 + \alpha_3\beta_7 a_1^4 a_2^2 + \alpha_4\beta_7 a_1^2 a_2^3 + \alpha_5\beta_7 a_1^3 a_2^3 + \alpha_6\beta_7 a_1^4 a_2^3 \\
& +\alpha_7\beta_7 a_1^5 a_2^3 + \alpha_8\beta_7 a_1^6 a_2^3 + \alpha_1\beta_9 a_1^4 a_2^2 + \alpha_2\beta_9 a_1^5 a_2^2 + \alpha_3\beta_9 a_1^6 a_2^2 + \alpha_4\beta_9 a_1^4 a_2^3 + \alpha_5\beta_9 a_1^5 a_2^3 \\
& +\alpha_6\beta_9 a_1^6 a_2^3 + \alpha_7\beta_9 a_1^7 a_2^3 + \alpha_8\beta_9 a_1^8 a_2^3 \\
=& \ (\alpha_1\beta_1) \\
& +(\alpha_2\beta_1)a_1 + (\alpha_4\beta_1)a_2 \\
& +(\alpha_3\beta_1)a_1^2 + (\alpha_5\beta_1)a_1 a_2 + (\alpha_1\beta_5)a_2^2 \\
& +(\alpha_6\beta_1)a_1^2 a_2 + (\alpha_2\beta_5)a_1 a_2^2 + (\alpha_4\beta_5)a_2^3 \\
& +(\alpha_7\beta_1)a_1^3 a_2 + (\alpha_3\beta_5 + \alpha_1\beta_7)a_1^2 a_2^2 + (\alpha_5\beta_5)a_1 a_2^3 \\
& +(\alpha_8\beta_1)a_1^4 a_2 + (\alpha_2\beta_7)a_1^3 a_2^2 + (\alpha_6\beta_5 + \alpha_4\beta_7)a_1^2 a_2^3 \\
& +(\alpha_1\beta_9 + \alpha_3\beta_7)a_1^4 a_2^2 + (\alpha_5\beta_7 + \alpha_7\beta_5)a_1^3 a_2^3 \\
& +(\alpha_2\beta_9)a_1^5 a_2^2 + (\alpha_8\beta_5 + \alpha_6\beta_7 + \alpha_4\beta_9)a_1^4 a_2^3 \\
& +(\alpha_3\beta_9)a_1^6 a_2^2 + (\alpha_7\beta_7 + \alpha_5\beta_9)a_1^5 a_2^3 \\
& +(\alpha_8\beta_7 + \alpha_6\beta_9)a_1^6 a_2^3 \\
& +(\alpha_7\beta_9)a_1^7 a_2^3 \\
& +(\alpha_8\beta_9)a_1^8 a_2^3
\end{aligned}
$$

The right hand side of 5.4.7b consists of three summands. We first calculate each one individually.

$$
\begin{aligned}
ff_{a_1} \quad =& \ (\alpha_2 + \alpha_5 a_2 + \alpha_7 a_1^2 a_2) \\
& \ (\alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2) \\
=& \ \alpha_1\alpha_2 + \alpha_2^2 a_1 + \alpha_2\alpha_3 a_1^2 + \alpha_2\alpha_4 a_2 + \alpha_2\alpha_5 a_1 a_2 + \alpha_2\alpha_6 a_1^2 a_2 + \alpha_2\alpha_7 a_1^3 a_2 + \alpha_2\alpha_8 a_1^4 a_2 \\
& +\alpha_1\alpha_5 a_2 + \alpha_2\alpha_5 a_1 a_2 + \alpha_3\alpha_5 a_1^2 a_2 + \alpha_4\alpha_5 a_2^2 + \alpha_5^2 a_1 a_2^2 + \alpha_5\alpha_6 a_1^2 a_2^2 + \alpha_5\alpha_7 a_1^3 a_2^2 \\
& +\alpha_5\alpha_8 a_1^4 a_2^2 + \alpha_1\alpha_7 a_1^2 a_2 + \alpha_2\alpha_7 a_1^3 a_2 + \alpha_3\alpha_7 a_1^4 a_2 + \alpha_4\alpha_7 a_1^2 a_2^2 + \alpha_5\alpha_7 a_1^3 a_2^2 + \alpha_6\alpha_7 a_1^4 a_2^2 \\
& +\alpha_7^2 a_1^5 a_2^2 + \alpha_7\alpha_8 a_1^6 a_2^2 \\
=& \ (\alpha_1\alpha_2) \\
& +(\alpha_2^2)a_1 + (\alpha_2\alpha_4 + \alpha_1\alpha_5)a_2 \\
& +(\alpha_2\alpha_3)a_1^2 + \underline{(\alpha_2\alpha_5 + \overline{\alpha_2\alpha_5}})a_1 a_2 + (\alpha_4\alpha_5)a_2^2 \\
& +(\alpha_2\alpha_6 + \alpha_3\alpha_5 + \alpha_1\alpha_7)a_1^2 a_2 + (\alpha_5^2)a_1 a_2^2 \\
& +\underline{(\alpha_2\alpha_7 + \overline{\alpha_2\alpha_7}})a_1^3 a_2 + (\alpha_5\alpha_6 + \alpha_4\alpha_7)a_1^2 a_2^2 \\
& +(\alpha_2\alpha_8 + \alpha_3\alpha_7)a_1^4 a_2 + \underline{(\alpha_5\alpha_7 + \overline{\alpha_5\alpha_7}})a_1^3 a_2^2 \\
& +(\alpha_5\alpha_8 + \alpha_6\alpha_7)a_1^4 a_2^2 \\
& +(\alpha_7^2)a_1^5 a_2^2 \\
& +(\alpha_7\alpha_8)a_1^6 a_2^2. \\
a_2 g' f_{a_2} =& \ (\beta_1 a_2 + \beta_5 a_2^3 + \beta_7 a_1^2 a_2^3 + \beta_9 a_1^4 a_2^3)(\alpha_4 + \alpha_5 a_1 + \alpha_6 a_1^2 + \alpha_7 a_1^3 + \alpha_8 a_1^4) \\
=& \ \alpha_4\beta_1 a_2 + \alpha_5\beta_1 a_1 a_2 + \alpha_6\beta_1 a_1^2 a_2 + \alpha_7\beta_1 a_1^3 a_2 + \alpha_8\beta_1 a_1^4 a_2 \\
& +\alpha_4\beta_5 a_2^3 + \alpha_5\beta_5 a_1 a_2^3 + \alpha_6\beta_5 a_1^2 a_2^3 + \alpha_7\beta_5 a_1^3 a_2^3 + \alpha_8\beta_5 a_1^4 a_2^3 \\
& +\alpha_4\beta_7 a_1^2 a_2^3 + \alpha_5\beta_7 a_1^3 a_2^3 + \alpha_6\beta_7 a_1^4 a_2^3 + \alpha_7\beta_7 a_1^5 a_2^3 + \alpha_8\beta_7 a_1^6 a_2^3 \\
& +\alpha_4\beta_9 a_1^4 a_2^3 + \alpha_5\beta_9 a_1^5 a_2^3 + \alpha_6\beta_9 a_1^6 a_2^3 + \alpha_7\beta_9 a_1^7 a_2^3 + \alpha_8\beta_9 a_1^8 a_2^3 \\
=& \ (\alpha_4\beta_1)a_2 \\
& +(\alpha_5\beta_1)a_1 a_2
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha_6\beta_1)a_1^2a_2 + (\alpha_4\beta_5)a_2^3 \\
&+(\alpha_7\beta_1)a_1^3a_2 + (\alpha_5\beta_5)a_1a_2^3 \\
&+(\alpha_8\beta_1)a_1^4a_2 + (\alpha_6\beta_5 + \alpha_4\beta_7)a_1^2a_2^3 \\
&+(\alpha_7\beta_5 + \alpha_5\beta_7)a_1^3a_2^3 \\
&+(\alpha_8\beta_5 + \alpha_6\beta_7 + \alpha_4\beta_9)a_1^4a_2^3 \\
&+(\alpha_7\beta_7 + \alpha_5\beta_9)a_1^5a_2^3 \\
&+(\alpha_8\beta_7 + \alpha_6\beta_9)a_1^6a_2^3 \\
&+(\alpha_7\beta_9)a_1^7a_2^3 \\
&+(\alpha_8\beta_9)a_1^8a_2^3.
\end{aligned}
$$

$$
a_2 f_t = \alpha_{1,t}a_2 + \alpha_{2,t}a_1a_2 + \alpha_{3,t}a_1^2a_2 + \alpha_{4,t}a_2^2 + \alpha_{5,t}a_1a_2^2 + \alpha_{6,t}a_1^2a_2^2 + \alpha_{7,t}a_1^3a_2^2 + \alpha_{8,t}a_1^4a_2^2,
$$

where $\alpha_{i,t}$ denotes the derivative of $\alpha_i$ with respect to the variable $t$.

Summing up, we obtain for the right hand side of 5.4.7b

$$
\begin{aligned}
ff_{a_1} &+ a_2 g' f_{a_2} + a_2 f_t \\
&= (\alpha_1\alpha_2) \\
&+(\alpha_2^2)a_1 + (\alpha_2\alpha_4 + \alpha_1\alpha_5 + \alpha_4\beta_1 + \alpha_{1,t})a_2 \\
&+(\alpha_2\alpha_3)a_1^2 + (\alpha_5\beta_1 + \alpha_{2,t})a_1a_2 + (\alpha_4\alpha_5 + \alpha_{4,t})a_2^2 \\
&+(\alpha_2\alpha_6 + \alpha_3\alpha_5 + \alpha_1\alpha_7 + \alpha_6\beta_1 + \alpha_{3,t})a_1^2a_2 + (\alpha_5^2 + \alpha_{5,t})a_1a_2^2 + (\alpha_4\beta_5)a_2^3 \\
&+(\alpha_7\beta_1)a_1^3a_2 + (\alpha_5\alpha_6 + \alpha_4\alpha_7 + \alpha_{6,t})a_1^2a_2^2 + (\alpha_5\beta_5)a_1a_2^3 \\
&+(\alpha_2\alpha_8 + \alpha_3\alpha_7 + \alpha_8\beta_1)a_1^4a_2 + (\alpha_{7,t})a_1^3a_2^2 + (\alpha_6\beta_5 + \alpha_4\beta_7)a_1^2a_2^3 \\
&+(\alpha_5\alpha_8 + \alpha_6\alpha_7 + \alpha_{8,t})a_1^4a_2^2 + (\alpha_7\beta_5 + \alpha_5\beta_7)a_1^3a_2^3 \\
&+(\alpha_7^2)a_1^5a_2^2 + (\alpha_8\beta_5 + \alpha_6\beta_7 + \alpha_4\beta_9)a_1^4a_2^3 \\
&+(\alpha_7\alpha_8)a_1^6a_2^2 + (\alpha_7\beta_7 + \alpha_5\beta_9)a_1^5a_2^3 \\
&+(\alpha_8\beta_7 + \alpha_6\beta_9)a_1^6a_2^3 \\
&+(\alpha_7\beta_9)a_1^7a_2^3 \\
&+(\alpha_8\beta_9)a_1^8a_2^3
\end{aligned}
$$

We now compare the two sides, indicating in each step which one of the summands $a_1^i a_2^j$ we are comparing. This comparison yields

$$
\begin{array}{ll}
\text{for } 1: & \alpha_1\alpha_2 = \alpha_1\beta_1 \Rightarrow \alpha_2 = \beta_1 \text{ if } \alpha_1 \neq 0 \\
\text{for } a_1: & \alpha_2^2 = \alpha_2\beta_1 \Rightarrow \alpha_2 = \beta_1 \text{ if } \alpha_2 \neq 0 \\
\text{for } a_2: & \alpha_2\alpha_4 + \alpha_1\alpha_5 + \cancel{\alpha_4\beta_1} + \alpha_{1,t} = \cancel{\alpha_4\beta_1} \Rightarrow \alpha_2\alpha_4 + \alpha_1\alpha_5 + \alpha_{1,t} = 0 \\
\text{for } a_1^2: & \alpha_2\alpha_3 = \alpha_3\beta_1 \Rightarrow \alpha_2 = \beta_1 \text{ if } \alpha_3 \neq 0 \\
\text{for } a_1a_2: & \cancel{\alpha_5\beta_1} + \alpha_{2,t} = \cancel{\alpha_5\beta_1} \Rightarrow \alpha_{2,t} = 0 \\
\text{for } a_2^2: & \alpha_4\alpha_5 + \alpha_{4,t} = \alpha_1\beta_5 \\
\text{for } a_1^2a_2: & \alpha_2\alpha_6 + \alpha_3\alpha_5 + \alpha_1\alpha_7 + \cancel{\alpha_6\beta_1} + \alpha_{3,t} = \cancel{\alpha_6\beta_1} \Rightarrow \alpha_2\alpha_6 + \alpha_3\alpha_5 + \alpha_1\alpha_7 + \alpha_{3,t} = 0 \\
\text{for } a_1a_2^2: & \alpha_5^2 + \alpha_{5,t} = \alpha_2\beta_5 \\
\text{for } a_2^3: & \alpha_4\beta_5 = \alpha_4\beta_5 \\
\text{for } a_1^3a_2: & \alpha_7\beta_1 = \alpha_7\beta_1 \\
\text{for } a_1^2a_2^2: & \alpha_5\alpha_6 + \alpha_4\alpha_7 + \alpha_{6,t} = \alpha_3\beta_5 + \alpha_1\beta_7 \\
\text{for } a_1a_2^3: & \alpha_5\beta_5 = \alpha_5\beta_5 \\
\text{for } a_1^4a_2: & \alpha_2\alpha_8 + \alpha_3\alpha_7 + \cancel{\alpha_8\beta_1} = \cancel{\alpha_8\beta_1} \Rightarrow \alpha_2\alpha_8 + \alpha_3\alpha_7 = 0 \\
\text{for } a_1^3a_2^2: & \alpha_{7,t} = \alpha_2\beta_7
\end{array}
$$

for $a_1^2 a_2^3$ : $\quad \alpha_6 \beta_5 + \alpha_4 \beta_7 = \alpha_6 \beta_5 + \alpha_4 \beta_7$

for $a_1^4 a_2^2$ : $\quad \alpha_5 \alpha_8 + \alpha_6 \alpha_7 + \alpha_{8,t} = \alpha_1 \beta_9 + \alpha_3 \beta_7$

for $a_1^3 a_2^3$ : $\quad \alpha_7 \beta_5 + \alpha_5 \beta_7 = \alpha_5 \beta_7 + \alpha_7 \beta_5$

for $a_1^5 a_2^2$ : $\quad \alpha_7^2 = \alpha_2 \beta_9$

for $a_1^4 a_2^3$ : $\quad \alpha_8 \beta_5 + \alpha_6 \beta_7 + \alpha_4 \beta_9 = \alpha_8 \beta_5 + \alpha_6 \beta_7 + \alpha_4 \beta_9$

for $a_1^6 a_2^2$ : $\quad \alpha_7 \alpha_8 = \alpha_3 \beta_9$

for $a_1^5 a_2^3$ : $\quad \alpha_7 \beta_7 + \alpha_5 \beta_9 = \alpha_7 \beta_7 + \alpha_5 \beta_9$

for $a_1^6 a_2^3$ : $\quad \alpha_8 \beta_7 + \alpha_6 \beta_9 = \alpha_8 \beta_7 + \alpha_6 \beta_9$

for $a_1^7 a_2^3$ : $\quad \alpha_7 \beta_9 = \alpha_7 \beta_9$

for $a_1^8 a_2^3$ : $\quad \alpha_8 \beta_9 = \alpha_8 \beta_9$

### 5.4.1.2 Explicit Construction

Some of these equations above are redundant, as the left and right hand side already agree. We use the equations stated above, and set some of the parameters to take specific values. The following lemma states one instance in which the polynomials $f$ and $g$ defined through the parameters $\alpha_i$ and $\beta_i$ define a foliations. The lemma does not cover all instances in which this can be true, due to the parameters we have set to take specific values. It only states that in this one specific instance, we do obtain a foliation.

**Lemma 5.4.4.** *Let the polynomials $f$ and $g$ be defined as follows, with $\alpha_i, \beta_i \in \mathbb{F}_2[t]$ :*

$$f(a_1, a_2, t) = \alpha_1 + \alpha_2 a_1 + \alpha_3 a_1^2 + \alpha_4 a_2 + \alpha_5 a_1 a_2 + \alpha_6 a_1^2 a_2 + \alpha_7 a_1^3 a_2 + \alpha_8 a_1^4 a_2$$
$$g'(a_1, a_2, t) = \beta_1 + \beta_5 a_2^2 + \beta_7 a_1^2 a_2^2 + \beta_9 a_1^4 a_2^2.$$

*If the following restrictions are satisfied, then $f$ and $g$ define a foliation.*

$$\alpha_2 = \beta_1 = 0 \tag{5.4.8a}$$
$$\alpha_1 = 1 \tag{5.4.8b}$$
$$\alpha_{4,t} = \beta_5 \tag{5.4.8c}$$
$$\alpha_{3,t} = 0 \tag{5.4.8d}$$
$$\alpha_5 = 0 \tag{5.4.8e}$$
$$\alpha_{6,t} = \alpha_3 \beta_5 + \beta_7 \tag{5.4.8f}$$
$$\alpha_{8,t} = \beta_9 + \alpha_3 \beta_7 \tag{5.4.8g}$$
$$\alpha_7 = 0 \tag{5.4.8h}$$
$$\alpha_3 \beta_9 = 0. \tag{5.4.8i}$$

*Proof.* The equations coming from the comparison of the left hand side and the right hands side above are listed below. As is suggested by three of the equations, coming from $1, a_1$ and $a_1^2$, we assume that $\alpha_2 = \beta_1$. With this, we obtain the following restrictions

(i) $\alpha_2 = \beta_1$

(ii) $\alpha_2 \alpha_4 + \alpha_1 \alpha_5 + \alpha_{1,t} = 0$

(iii) $\alpha_{2,t} = 0$

(iv) $\alpha_4 \alpha_5 + \alpha_{4,t} = \alpha_1 \beta_5$

(v) $\alpha_2 \alpha_6 + \alpha_3 \alpha_5 + \alpha_1 \alpha_7 + \alpha_{3,t} = 0$

(vi) $\alpha_5^2 + \alpha_{5,t} = \alpha_2 \beta_5$

(vii) $\alpha_5 \alpha_6 + \alpha_4 \alpha_7 + \alpha_{6,t} = \alpha_3 \beta_5 + \alpha_1 \beta_7$

(viii) $\alpha_2 \alpha_8 + \alpha_3 \alpha_7 = 0$

(ix) $\alpha_{7,t} = \alpha_2 \beta_7$

(x) $\alpha_5 \alpha_8 + \alpha_6 \alpha_7 + \alpha_{8,t} = \alpha_1 \beta_9 + \alpha_3 \beta_7$

(xi) $\alpha_7^2 = \alpha_2 \beta_9$

(xii) $\alpha_7 \alpha_8 = \alpha_3 \beta_9.$

Note that since $g'_t = 0$, the exponent of the variable $t$ is divisible by 2 in each polynomial $\beta_i$.

In order to find one or more explicit examples, we may fix some of these polynomials $\alpha_i$ and $\beta_i$ to simplify the restrictions above. We set $\alpha_2 = 0$ and as we will see below, we do indeed find explicit examples with this choice. The restrictions simplify to

(i) $\alpha_2 = \beta_1 = 0$

(ii) $\alpha_1 \alpha_5 + \alpha_{1,t} = 0$

(iii) $\alpha_{2,t} = 0$ is satisfied

(iv) $\alpha_4 \alpha_5 + \alpha_{4,t} = \alpha_1 \beta_5$

(v) $\alpha_3 \alpha_5 + \alpha_1 \alpha_7 + \alpha_{3,t} = 0$

(vi) $\alpha_5^2 + \alpha_{5,t} = 0$

(vii) $\alpha_5 \alpha_6 + \alpha_4 \alpha_7 + \alpha_{6,t} = \alpha_3 \beta_5 + \alpha_1 \beta_7$

(viii) $\alpha_3 \alpha_7 = 0$

(ix) $\alpha_{7,t} = 0$

(x) $\alpha_5 \alpha_8 + \alpha_6 \alpha_7 + \alpha_{8,t} = \alpha_1 \beta_9 + \alpha_3 \beta_7$

(xi) $\alpha_7^2 = 0$

(xii) $\alpha_7 \alpha_8 = \alpha_3 \beta_9$.

Equation (vi) states that $\alpha_5^2 = \alpha_{5,t}$. This can only hold for $\alpha_5 = 0$. If we assume that $\alpha_5$ is non-zero, then the degree of $\alpha_5^2$ is strictly bigger than the degree of $\alpha_{5,t}$. Additionally, from (xi) it follows that $\alpha_7 = 0$. The remaining restrictions are

(i) $\alpha_2 = \beta_1 = 0$

(ii) $\alpha_{1,t} = 0$

(iv) $\alpha_{4,t} = \alpha_1 \beta_5$

(v) $\alpha_{3,t} = 0$

(vi) $\alpha_5 = 0$

(vii) $\alpha_{6,t} = \alpha_3 \beta_5 + \alpha_1 \beta_7$

(x) $\alpha_{8,t} = \alpha_1 \beta_9 + \alpha_3 \beta_7$

(xi) $\alpha_7 = 0$

(xii) $\alpha_3 \beta_9 = 0$.

To simplify these restrictions further, we set $\alpha_1 = 1$. With this choice, (ii) is satisfied. For the remaining restrictions we obtain

(i) $\alpha_2 = \beta_1 = 0$

(iv) $\alpha_{4,t} = \beta_5$

(v) $\alpha_{3,t} = 0$

(vi) $\alpha_5 = 0$

(vii) $\alpha_{6,t} = \alpha_3 \beta_5 + \beta_7$

(x) $\alpha_{8,t} = \beta_9 + \alpha_3 \beta_7$

(xi) $\alpha_7 = 0$

(xii) $\alpha_3 \beta_9 = 0$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Throughout the process above, we have set some of the polynomials $\alpha_i$ and $\beta_i$ to be take specific values. This process was not as straight forward as the lemma might suggest. In fact, it took several attempts to set values that would eventually work, and give back examples of foliations.

### 5.4.1.3 Two Examples of Foliations

This subsection contains two foliations. The second foliation we study in depth, and prove that it satisfies all the requirements of Theorem 1.0.1.

**Example 5.4.5.** We now construct an explicit example such that all the requirements of Lemma 5.4.4 are satisfied. Note again that in all $\beta_i$, the exponents of the variable $t$ are multiples of 2. We set $\alpha_1 = 1, \beta_5 = t^2$ and $\beta_7 = 1$. With the choice $\beta_5 = t^2$, the equation 5.4.8c in Lemma 5.4.4 implies that $\alpha_{4,t} = t^2$. This is satisfied for the choice $\alpha_4 = t^3$. The choice $\beta_7 = 1$ implies with equation 5.4.8f that $\alpha_3 \beta_5 + 1 = \alpha_{6,t} \Rightarrow \alpha_{6,t} = 1$. This is satisfied for the choice $\alpha_6 = t$. The remaining polynomials are all chosen to be zero. With these choices, the polynomials $f$ and $g'$ are

$$f(a_1, a_2, t) = 1 + t^3 a_2 + t a_1^2 a_2,$$
$$g'(a_1, a_2, t) = t^2 a_2^2 + a_1^2 a_2^2.$$

A short computation verifies that these two polynomials indeed satisfy (5.2.1), which ensures that the polynomials define a foliation.

On the affine chart $U_A$, the resulting foliation is of the form

$$\mathcal{F}|_{U_A} = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}$$

$$= (1 + t^3 a_2 + t a_1^2 a_2)\frac{\partial}{\partial a_1} + (t^2 a_2^3 + a_1^2 a_2^3)\frac{\partial}{\partial a_2} + a_2\frac{\partial}{\partial t}.$$

According to Lemma 5.3.2, on the affine chart $U_C$, the foliation $\mathcal{F}$ restricted to $U_C$ is

$$\mathcal{F}|_{U_C} = f(c_1, c_2^{-1}, t)\frac{\partial}{\partial c_1} + g(c_1, c_2^{-1}, t)c_2^2\frac{\partial}{\partial c_2} + h(c_1, c_2^{-1}, t)\frac{\partial}{\partial t}$$

$$= (1 + t^3 c_2^{-1} + t c_1^2 c_2^{-1})\frac{\partial}{\partial c_1} + (t^2 c_2^{-3} + c_1^2 c_2^{-3})c_2^2\frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + t^3 c_2^{-1} + t c_1^2 c_2^{-1})\frac{\partial}{\partial c_1} + (t^2 c_2^{-1} + c_1^2 c_2^{-1}) + c_2^{-1}\frac{\partial}{\partial t}.$$

On the affine chart $U_B$, the foliation $\mathcal{F}$ restricted to $U_B$ is

$$\mathcal{F}|_{U_B} = f\left(b_1^{-1}, b_1^2 b_2, t\right) b_1^2 \frac{\partial}{\partial b_1} + \left(\cancel{f\left(b_1^{-1}, b_1^2 b_2, t\right) 2 b_1 b_2} + g\left(b_1^{-1}, b_1^2 b_2, t\right) b_1^{-2}\right)\frac{\partial}{\partial b_2}$$

$$+ h\left(b_1^{-1}, b_1^2 b_2, t\right)\frac{\partial}{\partial t}$$

$$= (1 + t^3 b_1^2 b_2 + t b_1^{-2} b_1^2 b_2) b_1^2 \frac{\partial}{\partial b_1} + (t^2 b_1^6 b_2^3 + b_1^{-2} b_1^6 b_2^3) b_1^{-2}\frac{\partial}{\partial b_2} + b_1^2 b_2\frac{\partial}{\partial t}$$

$$= (b_1^2 + t^3 b_1^4 b_2 + t b_1^2 b_2)\frac{\partial}{\partial b_1} + (t^2 b_1^4 b_2^3 + b_1^2 b_2^3)\frac{\partial}{\partial b_2} + b_1^2 b_2\frac{\partial}{\partial t}.$$

On the affine chart $U_D$, the foliation $\mathcal{F}$ restricted to $U_D$ is

$$\mathcal{F}|_{U_D} = f\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right) d_1^2 \frac{\partial}{\partial d_1} + \left(\cancel{f\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right) 2 d_1 d_2} + g\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right) d_1^{-2} d_2^2\right)\frac{\partial}{\partial d_2}$$

$$+ h\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right)\frac{\partial}{\partial t}$$

$$= (1 + t^3 d_1^2 d_2^{-1} + t d_1^{-2} d_1^2 d_2^{-1}) d_1^2 \frac{\partial}{\partial d_1} + (t^2 d_1^6 d_2^{-3} + d_1^{-2} d_1^6 d_2^{-3}) d_1^{-2} d_2^2\frac{\partial}{\partial d_2} + d_1^2 d_2^{-1}\frac{\partial}{\partial t}$$

$$= (d_1^2 + t^3 d_1^4 d_2^{-1} + t d_1^2 d_2^{-1})\frac{\partial}{\partial d_1} + (t^2 d_1^4 d_2^{-1} + d_1^2 d_2^{-1})\frac{\partial}{\partial d_2} + d_1^2 d_2^{-1}\frac{\partial}{\partial t}.$$

An other example is constructed below. It is the foliation that defines the surface of the required form, as we will show in the following subsection.

**Example 5.4.6.** We again construct an explicit example such that all the requirements of Lemma 5.4.4 are satisfied. Note that in all polynomials $\beta_i \in \mathbb{F}_2[t]$, the exponents of the variable $t$ are multiples of 2. By our assumption, we choose $\alpha_1 = 1$. Additionally, we set $\beta_5 = 1$. The equation 5.4.8c in Lemma 5.4.4 states that $\alpha_{4,t} = \beta_5 = 1$. This is satisfied for the choice $\alpha_4 = t$. All other polynomials $\alpha_i$ and $\beta_i$ are chosen to be zero. With this choices, the polynomials $f$ and $g$ are

$$f(a_1, a_2, t) = 1 + a_2 t,$$
$$g'(a_1, a_2, t) = a_2^2 \Rightarrow g(a_1, a_2, t) = a_2^3.$$

A quick computation verifies that these two polynomials indeed satisfy (5.2.1), which ensures that the polynomials define a foliation on the chart $U_A$.

On the chart $U_A$, the resulting foliation is of the form

$$\mathcal{F}|_{U_A} = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}$$

$$= (1 + t a_2)\frac{\partial}{\partial a_1} + a_2^3\frac{\partial}{\partial a_2} + a_2\frac{\partial}{\partial t}.$$

According to Lemma 5.3.2, the foliation transforms to the other charts in the following manner. On the chart $U_C$, the foliation $\mathcal{F}$ restricted to $U_C$ is defined as $\mathcal{F}|_{U_C} = \mathcal{O}_{U_C} \cdot v_{U_C}$, with

$$v_{U_C}(c_1, c_2, t) = f(c_1, c_2^{-1}, t)\frac{\partial}{\partial c_1} + g(c_1, c_2^{-1}, t)c_2^2\frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + tc_2^{-1})\frac{\partial}{\partial c_1} + c_2^{-3}c_2^2\frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + tc_2^{-1})\frac{\partial}{\partial c_1} + c_2^{-1}\frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

In order for the foliation on the chart $U_C$ to be saturated, we need to multiply $v_{U_C}$ with $c_2$. In doing so, we obtain

$$v_{U_C} \cdot c_2 = (c_2 + t)\frac{\partial}{\partial c_1} + 1\frac{\partial}{\partial c_2} + 1\frac{\partial}{\partial t},$$

which locally defines the foliation on $U_C$.

Due to the coefficient 1 in front of $\frac{\partial}{\partial c_2}$ and $\frac{\partial}{\partial t}$, this is regular. The class of the foliation $\mathcal{F}$ changes in the following manner: it decreases by $\{c_2 = 0\}$. By the definition of the charts $U_A, U_B, U_C$ and $U_D$, the line $\{c_2 = 0\}$ is a general section of the Hirzebruch surface $Y$, which we denote by $G$. From this, it follows that the class of $\mathcal{F}$ changes to $\mathcal{F} \cong \mathcal{O}_Y(-G)$.

On the affine chart $U_B$, the foliation $\mathcal{F}$ restricted to $U_B$ is defined as $\mathcal{F}|_{U_B} = \mathcal{O}_{U_B} \cdot v_{U_B}$, with

$$v_{U_B}(b_1, b_2, t) = f\left(b_1^{-1}, b_1^2 b_2, t\right)b_1^2\frac{\partial}{\partial b_1} + \left(f\left(b_1^{-1}, b_1^2 b_2, t\right)2b_1 b_2 + g\left(b_1^{-1}, b_1^2 b_2, t\right)b_1^{-2}\right)\frac{\partial}{\partial b_2} + b_1^2 b_2\frac{\partial}{\partial t}$$

$$= (1 + tb_1^2 b_2)b_1^2\frac{\partial}{\partial b_1} + b_1^6 b_2^3 b_1^{-2}\frac{\partial}{\partial b_2} + b_1^2 b_2\frac{\partial}{\partial t}$$

$$= (b_1^2 + tb_1^4 b_2)\frac{\partial}{\partial b_1} + b_1^4 b_2^3\frac{\partial}{\partial b_2} + b_1^2 b_2\frac{\partial}{\partial t}.$$

In order for this foliation to be saturated, we need to divide $v_{U_B}$ by $b_1^2$. We obtain

$$v_{U_B} \cdot \frac{1}{b_1^2} = (1 + tb_1^2 b_2)\frac{\partial}{\partial b_1} + b_1^2 b_2^3\frac{\partial}{\partial b_2} + b_2\frac{\partial}{\partial t},$$

which defines the foliation locally on $U_B$.

This is regular, since not all coefficients in front of $\frac{\partial}{\partial b_1}$, $\frac{\partial}{\partial b_2}$ and $\frac{\partial}{\partial t}$ can simultaneously be zero. If the coefficient $b_2$ in front of $\frac{\partial}{\partial t}$ is zero, then $(1 + tb_1^2 b_2) \neq 0$. The division by $b_1^2$ changes the class of $\mathcal{F}$ in the following manner: it becomes bigger by two times $\{b_1 = 0\}$. By the definition of the charts $U_A, U_B, U_C$ and $U_D$, the line $\{b_1 = 0\}$ is a fiber, and hence after this division, $\mathcal{F} \cong \mathcal{O}_Y(2F)$, where $F$ denotes a fiber of the Hirzebruch surface.

On the affine chart $U_D$, the foliation $\mathcal{F}$ restricted to $U_D$ is defined as $\mathcal{F}|_{U_D} = \mathcal{O}_{U_D} \cdot v_{U_D}$, where

$$v_{U_D}(d_1, d_2, t) = f\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right)d_1^2\frac{\partial}{\partial d_1} + \left(f\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right)2d_1 d_2 + g\left(d_1^{-1}, d_1^2 d_2^{-1}, t\right)d_1^{-2}d_2^2\right)\frac{\partial}{\partial d_2}$$

$$+ d_1^2 d_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + td_1^2 d_2^{-1})d_1^2\frac{\partial}{\partial d_1} + d_1^6 d_2^{-3}d_1^{-2}d_2^2\frac{\partial}{\partial d_2} + d_1^2 d_2^{-1}\frac{\partial}{\partial t}$$

$$= (d_1^2 + td_1^4 d_2^{-1})\frac{\partial}{\partial d_1} + d_1^4 d_2^{-1}\frac{\partial}{\partial d_2} + d_1^2 d_2^{-1}\frac{\partial}{\partial t}.$$

In order for this to be saturated, we need to divide the foliation $\mathcal{F}|_{U_D}$ by $d_1^2$. We obtain

$$v_{U_D} \cdot \frac{1}{d_1^2} = (1 + td_1^4 d_2^{-1})\frac{\partial}{\partial d_1} + d_1^2 d_2^{-1}\frac{\partial}{\partial d_2} + d_2^{-1}\frac{\partial}{\partial t}.$$

By the definition of the charts $U_A, U_B, U_C$ and $U_D$, it holds that $b_1 = d_1$, and hence the change of the class of $\mathcal{F}$ has already been covered above by the study of $\mathcal{F}$ restricted to $U_B$. Furthermore, we

need to multiply by $d_2$ in order to obtain

$$v_{U_D} \cdot \frac{1}{d_1^2} \cdot d_2 = (d_2 + t d_1^4) \frac{\partial}{\partial d_1} + d_1^2 \frac{\partial}{\partial d_2} + 1 \frac{\partial}{\partial t},$$

which defines the foliation locally on $U_D$.

Due to the coefficient 1 in front of $\frac{\partial}{\partial t}$, this is regular. With this multiplication, class of $\mathcal{F}$ becomes smaller by $\{d_2 = 0\}$. Expressing the variable $d_2$ in the variables $c_1, c_2$, we get that $d_2 = c_1^{-d} d_2$. Hence comparing $c_2$ and $d_2$, we see that for $c_1 \neq 0$, the line $\{d_2 = 0\}$ coincides with the line $\{c_2 = 0\}$. Hence the change of the class $\mathcal{F}$ due to this multiplication has been covered by the changes discussed for the chart $U_C$ already.

In summary, $\mathcal{F} \cong \mathscr{O}_Y(2F - G)$, with $F$ and $G$ fiber and general section of the Hirzebruch surface.

### 5.4.1.4   Properties of the Resulting Surface

The surface we have constructed in Example 5.4.6 satisfies all the requirements of Theorem 1.0.1 in the case where $(p, Y, C) = (2, H_2, D)$. As we will prove in this subsection, the surface $X$ is regular, with $-K_X$ ample. Furthermore, $X$ is geometrically reduced, but not geometrically normal.

**Proposition 5.4.7.** *The surface $X$ is regular.*

*Proof.* As we have seen in the construction of the example, the foliation $\mathcal{F}$ is regular on all charts. $\qquad\square$

**Proposition 5.4.8.** *The anticanonical divisor $-K_X$ is ample.*

*Proof.* As we have seen in the study of $\mathcal{F}$ on all charts $U_A, U_B, U_C$ and $U_D$, it holds that $\mathcal{F} \cong \mathscr{O}_Y(2F - G)$, with $F$ and $G$ fiber and general section of the Hirzebruch surface. The exceptional section $D$ is equal to $D = -(2F - G) = G - 2F$. We have calculated the canonical divisor of $Y$ in Example 1.1.6, where we saw that $K_Y + D = -4F - D$. With $D = G - 2F$, we obtain

$$K_Y = -4F - 2D = -4F - 2(G - 2F) = -4F - 2G + 4F$$
$$= -2G.$$

We calculate the canonical divisor of the surface $X$, which is defined as the quotient of $Y/\mathcal{F}$, by using [PW17, Proposition 2.10]. This proposition states that

$$K_{Y/X} \cong (p-1)(\det \mathcal{F}).$$

If we denote the morphism from $Y$ to $X = Y/\mathcal{F}$ by $\pi$, then it follows that

$$\pi^*(K_X) \cong K_Y - (p-1)(\det \mathcal{F}) = K_Y - (\det \mathcal{F}).$$

With $K_Y = -2G$ and $\det \mathcal{F} = 2F - G$, it follows that

$$\pi^*(K_X) = -2G - (2F - G) = -G - 2F.$$

The proof follows from the claim below. $\qquad\square$

**Claim 5.4.9.** *It holds that $G + 2F$ is ample, and hence $-G - 2F$ is anti-ample.*

*Proof.* The general section $G$ is nef, since there are no fixed components in its linear system, and hence it moves. The fiber $F$ is nef, since it is by definition the pullback of one point on the base. Since the ample cone is the interior of the nef cone, it follows from both $F$ and $G$ being nef that $G + 2F$ is ample. $\qquad\square$

**Proposition 5.4.10.** *The surface $X$ is not geometrically normal.*

*Proof.* We refer to Remark 3.5.5, which states that Proposition 3.5.4 is independent of the explicit foliation, if the foliation is constructed according to the setup of Chapter 2. $\qquad\square$

**Proposition 5.4.11.** *The surface $X$ is geometrically reduced.*

*Proof.* We refer to Remark 3.5.8. This remark states that the proof of the geometric reducedness relies on the setup of Chapter 2, with an additional assumption. The assumption states that the polynomial $h(a_1, a_2, t) \in \mathbb{F}_2[a_1, a_2, t]$ is nonzero. This holds here, which concludes the proof. $\qquad\square$

## 5.5 Degree not Divisible by Characteristic

We assume that $2 \nmid d$. As in the previous section, we base our further study on Lemma 5.3.3. In the subsection below, we implement restrictions on the polynomials $f$ and $g$ based on that lemma.

### 5.5.1 Restrictions

We let $f(a_1, a_2, t) = a_1^i a_2^j$ and $g(a_1, a_2, t) = a_1^k a_2^l$. The natural numbers $i, j, k, l$ need to satisfy the following restrictions imposed

by $U_C$: (a) $j \leq 1$

(b) $l \leq 3$

by $U_B$: The restriction given by the coefficient in front of $\frac{\partial}{\partial b_1}$ is $j \geq \frac{i-2}{d}$. The coefficient in front of $\frac{\partial}{\partial b_2}$ is

$$\left( f\left(\frac{1}{b_1}, b_1^d b_2, t\right) db_1 b_2 + g\left(\frac{1}{b_1}, b_1^d b_2, t\right) b_1^{-d} \right) = b_1^{-i}(b_1^d b_2)^j db_1 b_2 + b_1^{-k}(b_1^d b_2)^l b_1^{-d}$$
$$= \underbrace{d}_{=1} b_1^{dj+1-i} b_2^{j+1} + b_1^{dl-k-d} b_2^l$$
$$= b_1^{dj+1-i} b_2^{j+1} + b_1^{dl-k-d} b_2^l$$

On this chart, no poles can occur along $b_1 = 0$. To ensure this, there are two possibilities. Either both of the summands satisfy this fact, or else the two summands may have poles of higher degree, but then these poles need to appear in both summands, and be canceled out by each other in characteristic 2. These two options are discussed below.

*No cancellation* If no cancellation occurs, then both exponents $dj+1-i \geq 0$ and $dl-k-d \geq 0$. From this, we obtain
   ○ $dj + 1 - i \geq 0 \Rightarrow dj \geq i - 1 \Rightarrow j \geq \frac{i-1}{d}$
   ○ $dl - k - d \geq 0 \Rightarrow dl \geq k + d \Rightarrow l \geq \frac{k+d}{d} = 1 + \frac{k}{d}$.

*With cancellation* If poles of higher order appear along $b_1 = 0$ in one of the summands, then the other summand needs to be of the exact same form, so that they cancel each other out in characteristic 2. This means that the expressions $b_1^{dj+1-i} b_2^{j+1}$ and $b_1^{dl-k-d} b_2^l$ need to be equal if $dj + 1 - i < 0$ of $dl - k - d < 0$. Hence
   ○ $j + 1 = l$ and
   ○ $dj + 1 - i = dl - k - d \Rightarrow d(j - l + 1) = i - k - 1$.

Using the fact that $j + 1 = l$, the second equation transforms into
$$d(j - (j + 1) + 1) = i - k - 1 \Rightarrow i - k - 1 = 0 \Rightarrow i = k + 1.$$

by $U_D$: The restriction given by the coefficient in front of $\frac{\partial}{\partial d_1}$ are $j \geq \frac{i-2}{d}$ and $j \leq 1$. The coefficient in front of $\frac{\partial}{\partial d_2}$ is

$$\left( f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) d_1^{-d} d_2^2 \right) = d_1^{-i}(d_1^d d_2^{-1})^j dd_1 d_2 + (d_1^{-1})^k (d_1^d d_2^{-1})^l d_1^{-d} d_2^2$$
$$= dd_1^{jd+1-i} d_2^{1-j} + d_1^{dl-d-k} d_2^{2-l}$$
$$= d_1^{jd+1-i} d_2^{1-j} + d_1^{dl-d-k} d_2^{2-l}.$$

On this chart, no poles can occur along $d_1 = 0$, but poles of order one may occur along $d_2 = 0$. To ensure this, there are two possibilities. Either both of the summands satisfy this fact, or else the two summands may have poles of higher degree, but then these poles need to appear in both summands, and be canceled out by each other in characteristic 2. These two options are discussed below.

*No cancellation* If no cancellation occurs, then

- $jd + 1 - i \geq 0 \Rightarrow jd \geq i - 1 \Rightarrow j \geq \frac{i-1}{d}$
- $1 - j \geq -1 \Rightarrow j \leq 2$
- $dl - d - k \geq 0 \Rightarrow dl \geq d + k \Rightarrow l \geq 1 + \frac{k}{d}$
- $2 - l \geq -1 \Rightarrow l \leq 3$.

*With cancellation* If poles of higher order appear along $d_1 = 0$ or $d_2 = 0$ in one of the summands, then the other summand needs to be of the exact same form, so that they cancel each other out in characteristic 2. This means that the expressions $d_1^{jd+1-i} d_2^{1-j}$ and $d_1^{dl-d-k} d_2^{2-l}$ need to be equal under certain circumstances. These circumstances are if either $jd + 1 - i < 0$, and therefore $dl - d - k < 0$ as well, or if $1 - j < -1$ and therefore $2 - l < -1$ as well. Then

- $1 - j = 2 - l \Rightarrow l = j + 1$ and
- $jd + 1 - i = dl - d - k \Rightarrow d(j - l + 1) = i - 1 - k$.

Using the fact that $j + 1 = l$, the second equation transforms into

$$d(j - (j + 1) + 1) = i - k - 1 \Rightarrow i - k - 1 = 0 \Rightarrow i = k + 1.$$

Summing up these requirements, we obtain polynomials $f$ and $g$, with $f(a_1, a_2, t) = a_1^i a_2^j$ and $g(a_1, a_2, t) = a_1^k a_2^l$, such that $i, j, k, n \in \mathbb{N}$ satisfy the following restrictions, which are listed according to the chart they are coming form.

*Chart $U_C$* On the chart $U_C$, we obtain the following two restrictions.

$$
\begin{array}{ll}
\text{C.1} & j \leq 1 \\
\text{C.2} & l \leq 3
\end{array}
$$

*Chart $U_B$* On the chart $U_B$, there are two options. Either cancellation occurs, or not. Without cancellation, we have the restrictions below, denoted by Bnc, for "No Cancellation", and with cancellation, we denote the restrictions by Bwc, for "With Cancellation".

| | | | | |
|---|---|---|---|---|
| Bnc.1 | $j \geq \frac{i-2}{d}$ | | Bwc.1 | $j \geq \frac{i-2}{d}$ |
| Bnc.2 | $j \geq \frac{i-1}{d}$ | | Bwc.2 | $l = j + 1$ |
| Bnc.3 | $l \geq 1 + \frac{k}{d}$ | | Bwc.3 | $i = k + 1$. |

*Chart $U_D$* On the chart $U_D$, there are two options. Either cancellation occurs, or not. Without cancellation, we have the restrictions below, denoted by Dnc, for "No Cancellation", and with cancellation, we denote the restrictions by Dwc, for "With Cancellation".

| | | | | |
|---|---|---|---|---|
| Dnc.1 | $j \geq \frac{i-2}{d}$ | | Dwc.1 | $j \geq \frac{i-2}{d}$ |
| Dnc.2 | $j \leq 1$ | | Dwc.2 | $j \leq 1$ |
| Dnc.3 | $j \geq \frac{i-1}{d}$ | | Dwc.3 | $l = j + 1$ |
| Dnc.4 | $j \leq 2$ | | Dwc.4 | $i = k + 1$. |
| Dnc.5 | $l \geq 1 + \frac{k}{d}$ | | | |
| Dnc.6 | $l \leq 3$. | | | |

This leads to an explicit characterization of the possible monomials that may occur. The characterization is explained in the following lemma.

**Lemma 5.5.1.** *If $2 \nmid d$, then the polynomials $f$ and $g$ are sums of the following monomials. For $f$, these monomials are*

$$1, a_1, a_1^i a_2$$

*with $i \leq d + 1$. For $g$, these monomials are*

$$a_2, a_1^{k_1} a_2^2, a_1^{k_2} a_2^3,$$

with $k_1 \leq d$ and $k_2 \leq 2d$. Additionally, the variable $t$ can appear to any exponent in all of these monomials. Furthermore, there are additional monomials that may appear in $f$ and $g$. If $a_1^2 t^s$ appears in $f$, then $a_1 a_2 t^s$ needs to appear in $g$. If $a_1^{d+2} a_2 t^s$ appears in $f$, then $a_1^{d+1} a_2^2 t^s$ needs to appear in $g$.

*Proof.* We need to consider four cases separately. On both charts $U_B$ and $U_D$, cancellation can occur. So the four cases consist of both of these options in both charts. Firstly, we consider the case when no cancellation occurs in either chart. Then we need to consider the restrictions denoted by C, Bnc and Dnc. Summed up, these are the following

$$j \leq 1 \tag{5.5.1a}$$
$$l \leq 3 \tag{5.5.1b}$$
$$j \geq \frac{i-2}{d} \tag{5.5.1c}$$
$$j \geq \frac{i-1}{d} \tag{5.5.1d}$$
$$l \geq 1 + \frac{k}{d} \tag{5.5.1e}$$
$$j \leq 2 \tag{5.5.1f}$$

The restrictions (5.5.1c) and (5.5.1f) are implied by (5.5.1d) and (5.5.1a) respectively. With the restrictions (5.5.1a) and (5.5.1b) we are able to go through the cases that may occur explicitly. From (5.5.1a) we obtain the following.

- Either $j = 0 \stackrel{(5.5.1d)}{\Rightarrow} \frac{i-1}{d} \leq 0$. Since $d \geq 1$, it follows that $i - 1 \leq 0$. Since $i \in \mathbb{N}$, this implies that either $i = 0$ or $i = 1$.

- Or $j = 1 \stackrel{(5.5.1d)}{\Rightarrow} \frac{i-1}{d} \leq 1 \Rightarrow i - 1 \leq d \Rightarrow i \leq d + 1$.

From (5.5.1b) we obtain the following.

- Either $l = 0 \stackrel{(5.5.1e)}{\Rightarrow} 1 + \frac{k}{d} \leq 0 \Rightarrow \frac{k}{d} \leq -1$. Using the fact that $k, d \in \mathbb{N}$, this case can not occur.

- Or $l = 1 \stackrel{(5.5.1e)}{\Rightarrow} 1 + \frac{k}{d} \leq 1 \Rightarrow \frac{k}{d} = 0$. This implies that $k = 0$.

- Or $l = 2 \stackrel{(5.5.1e)}{\Rightarrow} 1 + \frac{k}{d} \leq 2 \Rightarrow \frac{k}{d} \leq 1 \Rightarrow k \leq d$.

- Or $l = 3 \stackrel{(5.5.1e)}{\Rightarrow} 1 + \frac{k}{d} \leq 3 \Rightarrow \frac{k}{d} \leq 2 \Rightarrow k \leq 2d$.

Summing up, in this case the polynomials $f$ and $g$ are made up of the monomials

$$1, a_1, a_1^i a_2$$

for $f$, with $i \leq d + 1$, and

$$a_2, a_1^{k_1} a_2^2, a_1^{k_2} a_2^3$$

for $g$, with $k_1 \leq d$ and $k_2 \leq 2d$.

Secondly, we consider the case when cancellation occurs in $U_D$, but not in $U_B$. For this, we need to consider the restrictions denoted by C, Bnc and Dwc. These are

$$j \leq 1 \tag{5.5.2a}$$
$$l \leq 3 \tag{5.5.2b}$$
$$j \geq \frac{i-2}{d} \tag{5.5.2c}$$
$$j \geq \frac{i-1}{d} \tag{5.5.2d}$$
$$l \geq 1 + \frac{k}{d} \tag{5.5.2e}$$
$$l = j + 1 \tag{5.5.2f}$$
$$i = k + 1 \tag{5.5.2g}$$

Again, (5.5.2c) is implied by (5.5.2d). Using the description of $i$ and $l$ in (5.5.2f) and (5.5.2g), we obtain

$$j \leq 1 \tag{5.5.3a}$$

$$j + 1 \leq 3 \Rightarrow j \leq 2 \tag{5.5.3b}$$

$$j \geq \frac{(k+1) - 1}{d} \Rightarrow j \geq \frac{k}{d} \tag{5.5.3c}$$

$$j + 1 \geq 1 + \frac{k}{d} \Rightarrow j \geq \frac{k}{d} \tag{5.5.3d}$$

$$l = j + 1 \tag{5.5.3e}$$

$$i = k + 1 \tag{5.5.3f}$$

We are able to explicitly go through the cases occurring with the restriction (5.5.3a). From (5.5.3a) we obtain the following.

○ Either $j = 0 \overset{(5.5.3c)}{\Rightarrow} \frac{k}{d} \leq 0$. This implies that $k = 0$. With (5.5.3e) and (5.5.3f) we get $l = 1$ and $i = 1$. This implies that if $f$ is of the form $a_1$, then $g$ needs to be of the form $a_2$. But if we exhibit the coefficient in front of $\frac{\partial}{\partial d_2}$ in this case, we get

$$\left( f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) d_1^{-d} d_2^2 \right) = dd_2 + d_2.$$

Hence there is no actual pole in either $d_1 = 0$ or $d_2 = 0$, and so this case is covered by the study above, where no cancellation occurs.

○ Or $j = 1 \overset{(5.5.3c)}{\Rightarrow} \frac{k}{d} \leq 1 \Rightarrow k \leq d$. With (5.5.3e) and (5.5.3f) we get $l = 2$ and $i = k + 1$. This implies that if $f$ is of the form $a_1^{k+1} a_2$, then $g$ needs to be of the form $a_1^k a_2^2$. But if we exhibit the coefficient in front of $\frac{\partial}{\partial d_2}$ in this case, we get

$$\left( f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) d_1^{-d} d_2^2 \right) = dd_1^{d-k} + d_1^{d-k}.$$

With the fact that $k \leq d$, we do not obtain actual poles in $d_1 = 0$. Hence this case is covered by the study above, where no cancellation occurs.

Thirdly, we consider the case when cancellation occurs in $U_B$, but not in $U_D$. For this, we need to consider the restrictions denoted by C, Bwc, Dnc. These are

$$j \leq 1 \tag{5.5.4a}$$

$$l \leq 3 \tag{5.5.4b}$$

$$j \geq \frac{i - 2}{d} \tag{5.5.4c}$$

$$l = j + 1 \tag{5.5.4d}$$

$$i = k + 1 \tag{5.5.4e}$$

$$j \geq \frac{i - 1}{d} \tag{5.5.4f}$$

$$j \leq 2 \tag{5.5.4g}$$

$$l \geq 1 + \frac{k}{d} \tag{5.5.4h}$$

Again, (5.5.4g) can be dismissed, and (5.5.4c) is implied by (5.5.4f). Replacing $i$ and $l$ by the

expressions obtained by (5.5.4d) and (5.5.4e) in the remaining conditions we obtain the following.

$$j \leq 1 \tag{5.5.5a}$$

$$j + 1 \leq 3 \Rightarrow j \leq 2 \tag{5.5.5b}$$

$$l = j + 1 \tag{5.5.5c}$$

$$i = k + 1 \tag{5.5.5d}$$

$$j \geq \frac{i-1}{d} = \frac{k}{d} \tag{5.5.5e}$$

$$j + 1 \geq 1 + \frac{k}{d} \Rightarrow j \geq \frac{k}{d} \tag{5.5.5f}$$

We explicitly go through all cases, according to 5.5.5a.

- Either $j = 0 \overset{(5.5.5e)}{\Rightarrow} \frac{k}{d} \leq 0 \Rightarrow k = 0$. Then with (5.5.5c) and (5.5.5d) we get $l = 1$ and $i = 1$. This implies that if $f$ is of the form $a_1$, then $g$ needs to be of the form $a_2$. But as in the second case, there is no actual pole in either $d_1 = 0$ or $d_2 = 0$, so this case if covered by the first case.

- Or $j = 1 \overset{(5.5.5e)}{\Rightarrow} \frac{k}{d} \leq 1 \Rightarrow k \leq d$. Then with (5.5.5c) and (5.5.5d) we get $l = 2$ and $i = k + 1$. This implies that if $f$ is of the form $a_1^{k+1} a_2$, then $g$ needs to be of the form $a_1^k a_2^2$. But as in the second case, there is no actual pole in either $d_1 = 0$ or $d_2 = 0$, so this case if covered by the first case.

Lastly, we consider the case when cancellation occurs in $U_B$ and in $U_D$. For this, we need to consider the restrictions posed by C, Bwc and Dwc. These are

$$j \leq 1 \tag{5.5.6a}$$

$$l \leq 3 \tag{5.5.6b}$$

$$j \geq \frac{i-2}{d} \tag{5.5.6c}$$

$$l = j + 1 \tag{5.5.6d}$$

$$i = k + 1 \tag{5.5.6e}$$

We replace $l$ and $i$ by their description in (5.5.6d) and (5.5.6e). The remaining restrictions are

$$j \leq 1 \tag{5.5.7a}$$

$$j + 1 \leq 3 \Rightarrow j \leq 2 \tag{5.5.7b}$$

$$j \geq \frac{k-1}{d} \tag{5.5.7c}$$

$$l = j + 1 \tag{5.5.7d}$$

$$i = k + 1 \tag{5.5.7e}$$

We go through all cases explicitly, according to (5.5.7a).

- Either $j = 0 \overset{(5.5.7c)}{\Rightarrow} \frac{k-1}{d} \leq 0 \Rightarrow k - 1 \leq 0$. Since $k \in \mathbb{N}$, this is satisfied for $k = 0$ or 1.

  ◇ In the case where $k = 0$, we get with (5.5.7d) and (5.5.7e) that $l = 1$ and $i = 1$. As we have seen previously, we obtain no actual poles this way, and hence this case is covered by the first case.

  ◇ In the case where $k = 1$, we get with (5.5.7d) and (5.5.7e) that $l = 1$ and $i = 2$. Hence if $f$ is of the form $a_1^2$, then $g$ needs to be of the form $a_1 a_2$. The coefficient in front of $\frac{\partial}{\partial d_2}$ is

$$f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) d_1^{-d} d_2^2 = d_1^{-1} d_2 + d_1^{-1} d_2.$$

  Hence we see that there is a pole of order 1 along the line $d_1 = 0$, which gets canceled by the expression above. This is a different case from the first case, where no cancellation occurs.

○ Or $j = 1 \overset{(5.5.7c)}{\Rightarrow} \frac{k-1}{d} \leq 1 \Rightarrow k - 1 \leq d \Rightarrow k \leq d + 1$. We get with (5.5.7d) and (5.5.7e) that $l = 2$ and $i = k + 1$. This means that if $f$ is of the form $a_1^{k+1} a_2$, then $g$ needs to be of the form $a_1^k d_2^2$. The coefficient in front of $\frac{\partial}{\partial d_2}$ is

$$f\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) dd_1 d_2 + g\left(\frac{1}{d_1}, d_1^d \frac{1}{d_2}, t\right) d_1^{-d} d_2^2 = d_1^{d-k} + d_1^{d-k}.$$

Hence with the restriction $k \leq d + 1$ we obtain a pole of order 1 maximally, when $k = d + 1$.

This sums up the study of all possible cases, and the claim follows.　　　　　　　□

The lemma above is the analogous to Lemma 5.4.2, in the case where $p$ does not divide the degree of the Hirzebruch surface.

### 5.5.2　Construction of an Example in the Case $d = 3$

Similar to the case of $d = 2$, we base the construction of an explicit example on Lemma 5.5.1. Writing the polynomials $f$ and $g$ in a form that agrees with the lemma, we then use the equations (5.2.1) that ensure that the polynomials $f$ and $g$ define a foliation to find as many constraints on the form of $f$ and $g$ as possible. During that process, we set certain parameters to take specific values, in order to simplify the situation. We will not go into detail about this process, as its analogous has already been studied in the case where $2 \mid d$. Instead, we state the example we find below, without going into detail about the calculations that were necessary to construct the example.

**Example 5.5.2.** Let $f(t) = 1 + a_2 t + a_1^2 a_2 t$ and $g(t) = a_2^3 + a_1^2 a_2^3$. With this choice, the equations (5.2.1) are satisfied, which ensures that $f$ and $g$ define a foliation on the chart $U_A$. On this chart, the foliation is of the form

$$\mathcal{F}\big|_{U_A} = f(a_1, a_2, t)\frac{\partial}{\partial a_1} + g(a_1, a_2, t)\frac{\partial}{\partial a_2} + h(a_1, a_2, t)\frac{\partial}{\partial t}$$

$$= (1 + a_2 t + a_1^2 a_2 t)\frac{\partial}{\partial a_1} + (a_2^3 + a_1^2 a_2^3)\frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial t}.$$

On this chart, the foliation is regular. On the other affine charts, the foliation transforms as follows according to Lemma 5.3.2. For the chart $U_C$, we get

$$v_{U_C}(c_1, c_2, t) = f(c_1, c_2^{-1}, t)\frac{\partial}{\partial c_1} + g(c_1, c_2^{-1}, t)c_2^2 \frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + c_2^{-1} t + c_1^2 c_2^{-1} t)\frac{\partial}{\partial c_1} + (c_2^{-3} + c_1^2 c_2^{-3})c_2^2 \frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}$$

$$= (1 + c_2^{-1} t + c_1^2 c_2^{-1} t)\frac{\partial}{\partial c_1} + (c_2^{-1} + c_1^2 c_2^{-1})\frac{\partial}{\partial c_2} + c_2^{-1}\frac{\partial}{\partial t}.$$

In order for the foliation on the chart $U_C$ to be saturated, we need to multiply $v_{U_C}$ with $c_2$. In doing so, we obtain

$$v_{U_C} \cdot c_2 = (c_2 + t + c_1^2 t)\frac{\partial}{\partial c_1} + (1 + c_1^2)\frac{\partial}{\partial c_2} + 1\frac{\partial}{\partial t},$$

which locally defines the foliation on $U_C$.

Due to the coefficient 1 in front of $\frac{\partial}{\partial t}$, this is regular. The class of the foliation $\mathcal{F}$ changes in the following manner: it decreases by $\{c_2 = 0\}$. By the definition of the charts $U_A, U_B, U_C$ and $U_D$, the line $\{c_2 = 0\}$ is a general section of the Hirzebruch surface $Y$, which we denote by $G$. From this, it follows that the class of $\mathcal{F}$ changes to $\mathcal{F} \cong \mathcal{O}_Y(-G)$.

On the affine chart $U_B$, the foliation $\mathcal{F}$ restricted to $U_B$ is defined as $\mathcal{F}|_{U_B} = \mathscr{O}_{U_B} \cdot v_{U_B}$, with

$$
\begin{aligned}
v_{U_B}(b_1, b_2, t) &= f\left(b_1^{-1}, b_1^3 b_2, t\right) b_1^2 \frac{\partial}{\partial b_1} + \left(f\left(b_1^{-1}, b_1^3 b_2, t\right) 3 b_1 b_2 + g\left(b_1^{-1}, b_1^3 b_2, t\right) b_1^{-3}\right) \frac{\partial}{\partial b_2} + b_1^3 b_2 \frac{\partial}{\partial t} \\
&= (1 + b_1^3 b_2 t + b_1^{-2} b_1^3 b_2 t) b_1^2 \frac{\partial}{\partial b_1} \\
&\quad + \left((1 + b_1^3 b_2 t + b_1^{-2} b_1^3 b_2 t) b_1 b_2 + (b_1^9 b_2^3 + b_1^{-2} b_1^9 b_2^3) b_1^{-3}\right) \frac{\partial}{\partial b_2} + b_1^3 b_2 \frac{\partial}{\partial t} \\
&= (b_1^2 + b_1^5 b_2 t + b_1^3 b_2 t) \frac{\partial}{\partial b_1} + \left(b_1 b_2 + b_1^4 b_2^2 t + b_1^2 b_2^2 t + b_1^6 b_2^3 + b_1^4 b_2^3\right) \frac{\partial}{\partial b_2} + b_1^3 b_2 \frac{\partial}{\partial t}.
\end{aligned}
$$

In order for this foliation to be saturated, we need to divide $v_{U_B}$ by $b_1$. We obtain

$$
v_{U_B} \cdot \frac{1}{b_1} = (b_1 + b_1^4 b_2 t + b_1^2 b_2 t) \frac{\partial}{\partial b_1} + \left(b_2 + b_1^3 b_2^2 t + b_1 b_2^2 t + b_1^5 b_2^3 + b_1^3 b_2^3\right) \frac{\partial}{\partial b_2} + b_1^2 b_2 \frac{\partial}{\partial t},
$$

which defines the foliation locally on $U_B$.

For $b_1 = b_2 = 0$, there is a singularity on this chart.

The division by $b_1$ changes the class of $\mathcal{F}$ in the following manner: it becomes bigger by one times $\{b_1 = 0\}$. By the definition of the charts $U_A, U_B, U_C$ and $U_D$, the line $\{b_1 = 0\}$ is a fiber, and hence after this division, $\mathcal{F} \cong \mathscr{O}_Y(F)$, where $F$ denotes a fiber of the Hirzebruch surface.

On the affine chart $U_D$, the foliation $\mathcal{F}$ restricted to $U_D$ is defined as $\mathcal{F}|_{U_D} = \mathscr{O}_{U_D} \cdot v_{U_D}$, where

$$
\begin{aligned}
v_{U_D}(d_1, d_2, t) &= f\left(d_1^{-1}, d_1^3 d_2^{-1}, t\right) d_1^2 \frac{\partial}{\partial d_1} + \left(f\left(d_1^{-1}, d_1^3 d_2^{-1}, t\right) 3 d_1 d_2 + g\left(d_1^{-1}, d_1^3 d_2^{-1}, t\right) d_1^{-3} d_2^2\right) \frac{\partial}{\partial d_2} \\
&\quad + d_1^3 d_2^{-1} \frac{\partial}{\partial t} \\
&= (1 + d_1^3 d_2^{-1} t + d_1^{-2} d_1^3 d_2^{-1} t) d_1^2 \frac{\partial}{\partial d_1} \\
&\quad + \left((1 + d_1^3 d_2^{-1} t + d_1^{-2} d_1^3 d_2^{-1} t) d_1 d_2 + (d_1^9 d_2^{-3} + d_1^{-2} d_1^9 d_2^{-3}) d_1^{-3} d_2^2\right) \frac{\partial}{\partial d_2} + d_1^3 d_2^{-1} \frac{\partial}{\partial t} \\
&= (d_1^2 + d_1^5 d_2^{-1} t + d_1^3 d_2^{-1} t) \frac{\partial}{\partial d_1} + \left(d_1 d_2 + d_1^4 t + d_1^2 t + d_1^6 d_2^{-1} + d_1^4 d_2^{-1}\right) \frac{\partial}{\partial d_2} + d_1^3 d_2^{-1} \frac{\partial}{\partial t}
\end{aligned}
$$

In order for this to be saturated, we need to divide the foliation $\mathcal{F}|_{U_D}$ by $d_1$. We obtain

$$
v_{U_D} \cdot \frac{1}{d_1} = (d_1 + d_1^4 d_2^{-1} t + d_1^2 d_2^{-1} t) \frac{\partial}{\partial d_1} + \left(d_2 + d_1^3 t + d_1 t + d_1^5 d_2^{-1} + d_1^3 d_2^{-1}\right) \frac{\partial}{\partial d_2} + d_1^2 d_2^{-1} \frac{\partial}{\partial t}.
$$

By the definition of the charts $U_A, U_B, U_C$ and $U_D$, it holds that $b_1 = d_1$, and hence the change of the class of $\mathcal{F}$ has already been covered above by the study of $\mathcal{F}$ restricted to $U_B$. Furthermore, we need to multiply by $d_2$ in order to obtain

$$
v_{U_D} \cdot \frac{1}{d_1} \cdot d_2 = (d_1 d_2 + d_1^4 t + d_1^2 t) \frac{\partial}{\partial d_1} + \left(d_2^2 + d_1^3 d_2 t + d_1 d_2 t + d_1^5 + d_1^3\right) \frac{\partial}{\partial d_2} + d_1^2 \frac{\partial}{\partial t},
$$

which defines the foliation locally on $U_D$.

For $d_1 = d_2 = 0$, there is a singularity.

With this multiplication, the class of $\mathcal{F}$ becomes smaller by $\{d_2 = 0\}$. Expressing the variable $d_2$ in the variables $c_1, c_2$, we get that $d_2 = c_1^{-d} d_2$. Hence comparing $c_2$ and $d_2$, we see that for $c_1 \neq 0$, the line $\{d_2 = 0\}$ coincides with the line $\{c_2 = 0\}$. Hence the change of the class $\mathcal{F}$ due to this multiplication has been covered by the changes discussed for the chart $U_C$ already.

In summary, $\mathcal{F} \cong \mathscr{O}_Y(F - G)$, with $F$ and $G$ fiber and general section of the Hirzebruch surface.

### 5.5.2.1 Properties of the Resulting Surface

The surface we have constructed in Example 5.5.2 satisfies all the requirements of Theorem 1.0.1 in the case where $(p, Y, C) = (2, H_3, D)$. As we will prove in this subsection, the surface $X$ has Gorenstein singularities, with $-K_X$ ample. Furthermore, $X$ is geometrically reduced, but not geometrically normal.

**Lemma 5.5.3.** *There are two singularities, one on the chart $U_B$ and one on the chart $U_D$. The singularity on the chart $U_B$ is an $A_1$ surface singularity, after passing to the function field of the base. The singularity on the chart $U_D$ is a singularity with discrepancy $-1$ that is resolved after one blow up.*

*Proof.* We first study the foliation on the chart $U_B$. There, it is defined as

$$\mathcal{F}\big|_{U_B} = (b_1 + b_1^4 b_2 t + b_1^2 b_2 t)\frac{\partial}{\partial b_1} + \left(b_2 + b_1^3 b_2^2 t + b_1 b_2^2 t + b_1^5 b_2^3 + b_1^3 b_2^3\right)\frac{\partial}{\partial b_2} + b_1^2 b_2 \frac{\partial}{\partial t},$$

with a singular point for $b_1 = b_2 = 0$. We let $\mathbb{A}^3_{b_1,b_2,t} := \operatorname{Spec}\mathbb{F}_2[b_1,b_2,t]$, and $L := \{(0,0,t)\big|t \in T\}$ be the line along which we blow up, with defining ideal $I(L) = (b_1, b_2)$. Consider the map

$$\begin{array}{rcl}
\mathbb{A}^3_{b_1,b_2,t} \setminus L & \to & \mathbb{A}^3 \times \mathbb{P}^1 \\
(b_1, b_2, t) & \mapsto & ((b_1, b_2, t), [b_1 : b_2]).
\end{array}$$

The blow up is defined to be the closure in $\mathbb{A}^3 \times \mathbb{P}^1$ of the image of the above map. Hence $\operatorname{Bl}_L \mathbb{A}^3_{b_1,b_2,t} = \overline{\{((b_1, b_2, t), [u : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 \big| b_1 v = u b_2\}}$. We denote the blow up by $\pi : \operatorname{Bl}_L \mathbb{A}^3_{b_1,b_2,t} \to \mathbb{A}^3_{b_1,b_2,t}$. As $\pi$ is an isomorphism when restricted to the preimage of $\mathbb{A}^3_{b_1,b_2,t} \setminus L$, it holds that $\pi^*\mathcal{F}\big|_{\mathbb{A}^3\setminus L}$ is a foliation on this preimage and hence extends uniquely to a foliation $\mathcal{F}_{\operatorname{Bl}_L \mathbb{A}^3}$ on $\operatorname{Bl}_L \mathbb{A}^3$ by saturatedness.

$\operatorname{Bl}_L \mathbb{A}^3$ can be covered by two affine charts $U_{b_1}$ and $U_{b_2}$, which are given by

$$U_{b_1} = \{((b_1, b_2, t), [1 : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 \big| b_1 v = b_2\},$$

$$U_{b_2} = \{((b_1, b_2, t), [u : 1]) \in \mathbb{A}^3 \times \mathbb{P}^1 \big| b_1 = u b_2\}.$$

Both charts are isomorphic to $\mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}]$ via the following isomorphisms

$$\begin{array}{rcl}
\mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}] & \to & U_{b_1} \\
(\overline{b_1}, \overline{b_2}, \overline{t}) & \mapsto & ((\overline{b_1}, \overline{b_1 b_2}, \overline{t}), [1 : \overline{b_2}])
\end{array}$$

and

$$\begin{array}{rcl}
\mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}] & \to & U_{b_2} \\
(\overline{b_1}, \overline{b_2}, \overline{t}) & \mapsto & ((\overline{b_1 b_2}, \overline{b_2}, \overline{t}), [\overline{b_1} : 1]).
\end{array}$$

The map $\pi$ restricted to these charts is

$$\begin{array}{rcl}
\mathbb{F}_2[b_1, b_2, t] & \to & \mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}] \\
b_1 & \mapsto & \overline{b_1} \\
b_2 & \mapsto & \overline{b_1 b_2} \\
t & \mapsto & \overline{t}
\end{array}$$

on the chart $U_{b_1}$, and

$$\begin{array}{rcl}
\mathbb{F}_2[b_1, b_2, t] & \to & \mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}] \\
b_1 & \mapsto & \overline{b_1 b_2} \\
b_2 & \mapsto & \overline{b_2} \\
t & \mapsto & \overline{t}
\end{array}$$

on the chart $U_{b_2}$.

On the chart $U_{b_1}$, with the blow up defined as above, the derivations transform as follows

$$b_1 \frac{\partial}{\partial b_1} = \overline{b_1}\frac{\partial}{\partial \overline{b_1}} + \overline{b_2}\frac{\partial}{\partial \overline{b_2}}, \quad b_2 \frac{\partial}{\partial b_2} = \overline{b_2}\frac{\partial}{\partial \overline{b_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.$$

Hence the foliation $\mathcal{F}$ is defined by

$$\mathcal{F}\big|_{U_B} = (b_1 + b_1^4 b_2 t + b_1^2 b_2 t)\frac{\partial}{\partial b_1} + \left(b_2 + b_1^3 b_2^2 t + b_1 b_2^2 t + b_1^5 b_2^3 + b_1^3 b_2^3\right)\frac{\partial}{\partial b_2} + b_1^2 b_2 \frac{\partial}{\partial t}$$

$$= (1 + b_1^3 b_2 t + b_1 b_2 t)\left(b_1 \frac{\partial}{\partial b_1}\right) + \left(1 + b_1^3 b_2 t + b_1 b_2 t + b_1^5 b_2^2 + b_1^3 b_2^2\right)\left(b_2 \frac{\partial}{\partial b_2}\right) + b_1^2 b_2 \frac{\partial}{\partial t}$$

$$= (1 + \overline{b_1}^4 \overline{b_2 t} + \overline{b_1}^2 \overline{b_2 t})\left(\overline{b_1}\frac{\partial}{\partial \overline{b_1}} + \overline{b_2}\frac{\partial}{\partial \overline{b_2}}\right)$$

$$+ (1 + \overline{b_1}^4 \overline{b_2 t} + \overline{b_1}^2 \overline{b_2 t} + \overline{b_1}^7 \overline{b_2}^2 + \overline{b_1}^5 \overline{b_2}^2)\left(\overline{b_2}\frac{\partial}{\partial \overline{b_2}}\right) + \overline{b_1}^3 \overline{b_2}\frac{\partial}{\partial \overline{t}}$$

$$= (\overline{b_1} + \overline{b_1}^5 \overline{b_2 t} + \overline{b_1}^3 \overline{b_2 t})\frac{\partial}{\partial \overline{b_1}} + (\overline{b_1}^7 \overline{b_2}^3 + \overline{b_1}^5 \overline{b_2}^3)\frac{\partial}{\partial \overline{b_2}} + \overline{b_1}^3 \overline{b_2}\frac{\partial}{\partial \overline{t}}$$

Dividing by $\overline{b_1}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \mathbb{A}^3, U_{b_1}} = (1 + \overline{b_1}^4 \overline{b_2 t} + \overline{b_1}^2 \overline{b_2 t})\frac{\partial}{\partial \overline{b_1}} + (\overline{b_1}^6 \overline{b_2}^3 + \overline{b_1}^4 \overline{b_2}^3)\frac{\partial}{\partial \overline{b_2}} + \overline{b_1}^2 \overline{b_2}\frac{\partial}{\partial \overline{t}}$$

on the chart $U_{b_1}$. Hence after blowing up, the foliation becomes regular. The discrepancy of this blow up is computed in the usual way, which results in the discrepancy zero.

Considering the other chart $U_{b_2}$, we recall that the blow up is defined by

$$\begin{array}{ccc} \mathbb{F}_2[b_1, b_2, t] & \to & \mathbb{F}_2[\overline{b_1}, \overline{b_2}, \overline{t}] \\ b_1 & \mapsto & \overline{b_1 b_2} \\ b_2 & \mapsto & \overline{b_2} \\ t & \mapsto & \overline{t} \end{array}$$

and the derivations transform as follows

$$b_1 \frac{\partial}{\partial b_1} = \overline{b_1}\frac{\partial}{\partial \overline{b_1}}, \quad b_2 \frac{\partial}{\partial b_2} = \overline{b_1}\frac{\partial}{\partial \overline{b_1}} + \overline{b_2}\frac{\partial}{\partial \overline{b_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.$$

Hence the foliation $\mathcal{F}$ becomes

$$\mathcal{F}\big|_{U_B} = (b_1 + b_1^4 b_2 t + b_1^2 b_2 t)\frac{\partial}{\partial b_1} + \left(b_2 + b_1^3 b_2^2 t + b_1 b_2^2 t + b_1^5 b_2^3 + b_1^3 b_2^3\right)\frac{\partial}{\partial b_2} + b_1^2 b_2 \frac{\partial}{\partial t}$$

$$= (1 + b_1^3 b_2 t + b_1 b_2 t)\left(b_1 \frac{\partial}{\partial b_1}\right) + \left(1 + b_1^3 b_2 t + b_1 b_2 t + b_1^5 b_2^2 + b_1^3 b_2^2\right)\left(b_2 \frac{\partial}{\partial b_2}\right) + b_1^2 b_2 \frac{\partial}{\partial t}$$

$$= (1 + \overline{b_1}^3 \overline{b_2}^4 \overline{t} + \overline{b_1 b_2}^2 \overline{t})\left(\overline{b_1}\frac{\partial}{\partial \overline{b_1}}\right) + (1 + \overline{b_1}^3 \overline{b_2}^4 \overline{t} + \overline{b_1 b_2}^2 \overline{t} + \overline{b_1}^5 \overline{b_2}^7 + \overline{b_1}^3 \overline{b_2}^5)\left(\overline{b_1}\frac{\partial}{\partial \overline{b_1}} + \overline{b_2}\frac{\partial}{\partial \overline{b_2}}\right)$$

$$+ \overline{b_1}^2 \overline{b_2}^3 \frac{\partial}{\partial \overline{t}}$$

$$= (\overline{b_1}^6 \overline{b_2}^7 + \overline{b_1}^4 \overline{b_2}^5)\frac{\partial}{\partial \overline{b_1}} + (\overline{b_2} + \overline{b_1}^3 \overline{b_2}^5 \overline{t} + \overline{b_1 b_2}^3 \overline{t} + \overline{b_1}^5 \overline{b_2}^8 + \overline{b_1}^3 \overline{b_2}^6)\frac{\partial}{\partial \overline{b_2}} + \overline{b_1}^2 \overline{b_2}^3 \frac{\partial}{\partial \overline{t}}$$

Dividing by $\overline{b_2}$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \mathbb{A}^3, U_{b_2}} = (\overline{b_1}^6 \overline{b_2}^6 + \overline{b_1}^4 \overline{b_2}^4)\frac{\partial}{\partial \overline{b_1}} + (1 + \overline{b_1}^3 \overline{b_2}^4 \overline{t} + \overline{b_1 b_2}^2 \overline{t} + \overline{b_1}^5 \overline{b_2}^7 + \overline{b_1}^3 \overline{b_2}^5)\frac{\partial}{\partial \overline{b_2}} + \overline{b_1}^2 \overline{b_2}^2 \frac{\partial}{\partial \overline{t}}$$

This is regular. Due to a similar argument as on the chart $U_{b_1}$, the discrepancy of the blow up on the quotient spaces is zero.

This concludes the study of the singularity on the chart $U_B$.

On the chart $U_D$, the foliation is defined as

$$\mathcal{F}\big|_{U_D} = (d_1 d_2 + d_1^4 t + d_1^2 t)\frac{\partial}{\partial d_1} + (d_2^2 + d_1^3 d_2 t + d_1 d_2 t + d_1^5 + d_1^3)\frac{\partial}{\partial d_2} + d_1^2 \frac{\partial}{\partial t},$$

with a singular point for $d_1 = d_2 = 0$. We let $\mathbb{A}^3_{d_1,d_2,t} := \operatorname{Spec} \mathbb{F}_2[d_1, d_2, t]$, and $L := \{(0, 0, t) | t \in T\}$ be the line along which we blow up, with defining ideal $I(L) = (d_1, d_2)$. Consider the map

$$
\begin{array}{ccc}
\mathbb{A}^3_{d_1,d_2,t} \setminus L & \to & \mathbb{A}^3 \times \mathbb{P}^1 \\
(d_1, d_2, t) & \mapsto & ((d_1, d_2, t), [d_1 : d_2]).
\end{array}
$$

The blow up is defined to be the closure in $\mathbb{A}^3 \times \mathbb{P}^1$ of the image of the above map. Hence $\operatorname{Bl}_L \mathbb{A}^3_{d_1,d_2,t} = \overline{\{((d_1, d_2, t), [u : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 | d_1 v = u d_2\}}$. We denote the blow up by $\pi : \operatorname{Bl}_L \mathbb{A}^3_{d_1,d_2,t} \to \mathbb{A}^3_{d_1,d_2,t}$. As $\pi$ is an isomorphism when restricted to the preimage of $\mathbb{A}^3_{d_1,d_2,t} \setminus L$, it holds that $\pi^* \mathcal{F}|_{\mathbb{A}^3 \setminus L}$ is a foliation on this preimage and hence extends uniquely to a foliation $\mathcal{F}_{\operatorname{Bl}_L \mathbb{A}^3}$ on $\operatorname{Bl}_L \mathbb{A}^3$ by saturatedness.

$\operatorname{Bl}_L \mathbb{A}^3$ can be covered by two affine charts $U_{d_1}$ and $U_{d_2}$, which are given by

$$
U_{d_1} = \{((d_1, d_2, t), [1 : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 | d_1 v = d_2\},
$$

$$
U_{d_2} = \{((d_1, d_2, t), [u : 1]) \in \mathbb{A}^3 \times \mathbb{P}^1 | d_1 = u d_2\}.
$$

Both charts are isomorphic to $\mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}]$ via the following isomorphisms

$$
\begin{array}{ccc}
\mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}] & \to & U_{d_1} \\
(\overline{d_1}, \overline{d_2}, \overline{t}) & \mapsto & ((\overline{d_1}, \overline{d_1 d_2}, \overline{t}), [1 : \overline{d_2}])
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}] & \to & U_{d_2} \\
(\overline{d_1}, \overline{d_2}, \overline{t}) & \mapsto & ((\overline{d_1 d_2}, \overline{d_2}, \overline{t}), [\overline{d_1} : 1]).
\end{array}
$$

The map $\pi$ restricted to these charts is

$$
\begin{array}{ccc}
\mathbb{F}_2[d_1, d_2, t] & \to & \mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}] \\
d_1 & \mapsto & \overline{d_1} \\
d_2 & \mapsto & \overline{d_1 d_2} \\
t & \mapsto & \overline{t}
\end{array}
$$

on the chart $U_{d_1}$, and

$$
\begin{array}{ccc}
\mathbb{F}_2[d_1, d_2, t] & \to & \mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}] \\
d_1 & \mapsto & \overline{d_1 d_2} \\
d_2 & \mapsto & \overline{d_2} \\
t & \mapsto & \overline{t}
\end{array}
$$

on the chart $U_{d_2}$.

On the chart $U_{d_1}$, with the blow up defined as above, the derivations transform as follows

$$
d_1 \frac{\partial}{\partial d_1} = \overline{d_1} \frac{\partial}{\partial \overline{d_1}} + \overline{d_2} \frac{\partial}{\partial \overline{d_2}}, \quad d_2 \frac{\partial}{\partial d_2} = \overline{d_2} \frac{\partial}{\partial \overline{d_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.
$$

Hence the foliation $\mathcal{F}$ is defined by

$$
\begin{aligned}
\mathcal{F}|_{U_D} &= (d_1 d_2 + d_1^4 t + d_1^2 t) \frac{\partial}{\partial d_1} + (d_2^2 + d_1^3 d_2 t + d_1 d_2 t + d_1^5 + d_1^3) \frac{\partial}{\partial d_2} + d_1^2 \frac{\partial}{\partial t} \\
&= (d_2 + d_1^3 t + d_1 t)\left(d_1 \frac{\partial}{\partial d_1}\right) + \left(d_2 + d_1^3 t + d_1 t + \frac{d_1^5}{d_2} + \frac{d_1^3}{d_2}\right)\left(d_2 \frac{\partial}{\partial d_2}\right) + d_1^2 \frac{\partial}{\partial t} \\
&= (\overline{d_1 d_2} + \overline{d_1}^3 \overline{t} + \overline{d_1 t})\left(\overline{d_1} \frac{\partial}{\partial \overline{d_1}} + \overline{d_2} \frac{\partial}{\partial \overline{d_2}}\right) + \left(\overline{d_1 d_2} + \overline{d_1}^3 \overline{t} + \overline{d_1 t} + \frac{\overline{d_1}^5}{\overline{d_1 d_2}} + \frac{\overline{d_1}^3}{\overline{d_1 d_2}}\right)\left(\overline{d_2} \frac{\partial}{\partial \overline{d_2}}\right) \\
&\quad + \overline{d_1}^2 \frac{\partial}{\partial \overline{t}} \\
&= (\overline{d_1}^2 \overline{d_2} + \overline{d_1}^4 \overline{t} + \overline{d_1}^2 \overline{t}) \frac{\partial}{\partial \overline{d_1}} + (\overline{d_1}^4 + \overline{d_1}^2) \frac{\partial}{\partial \overline{d_2}} + \overline{d_1}^2 \frac{\partial}{\partial \overline{t}}
\end{aligned}
$$

Dividing by $\overline{d_1}^2$, we get

$$\mathcal{F}_{\mathrm{Bl}_L \mathbb{A}^3, U_{d_1}} = (\overline{d_2} + \overline{d_1}^2\overline{t} + \overline{t})\frac{\partial}{\partial \overline{d_1}} + (\overline{d_1}^2 + 1)\frac{\partial}{\partial \overline{d_2}} + 1\frac{\partial}{\partial \overline{t}}$$

on the chart $U_{b_1}$. Hence after blowing up, the foliation becomes regular. It remains to compute the discrepancy of this blow up. Consider the following diagram,

$$
\begin{array}{ccc}
\mathrm{Bl}_L \mathbb{A}^3 & \xrightarrow{\ \beta\ } & \mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}} \\
\pi \downarrow & & \downarrow \pi' \\
\mathbb{A}^3 & \xrightarrow[\ \alpha\ ]{} & \mathbb{A}^3/\mathcal{F}
\end{array}
$$

Figure 5.3: Notation for the blow up

where we denote by $E$ the exceptional divisor of the blow up $\pi$ and by $E'$ the exceptional divisor of the blow up $\pi'$, the blow up of the quotient space. In order to calculate the discrepancy of the blow up $\pi'$, we let $K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} = (\pi')^*K_{\mathbb{A}^3/\mathcal{F}} + aE'$, where $a$ denotes the discrepancy. Since the blow up $\pi$ describes the blow up of a line in $\mathbb{A}^3$, its discrepancy is equal to one and by the adjunction formula we have

$$
\begin{aligned}
K_{\mathrm{Bl}_L \mathbb{A}^3} &= \pi^*K_{\mathbb{A}^3} + E \\
&\cong \pi^*(\alpha^*K_{\mathbb{A}^3/\mathcal{F}} - (1-p)c_1(\mathcal{F})) + E, \\
&\quad \text{by adjunction, where } c_1(\mathcal{F}) = 0 \\
&= \pi^*\alpha^*K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(\pi')^*K_{\mathbb{A}^3/\mathcal{F}} + E \\
&= \beta^*(K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}} - aE') + E \\
&= \beta^*(K_{\mathrm{Bl}_L \mathbb{A}^3/\mathcal{F}_{\mathrm{Bl}}}) - a\beta^*(E') + E \\
&\cong (K_{\mathrm{Bl}_L \mathbb{A}^3} + (1-p)c_1(\mathcal{F}_{\mathrm{Bl}})) - a\beta^*(E') + E, \text{ by adjunction} \\
&= K_{\mathrm{Bl}_L \mathbb{A}^3} - 2E - a\beta^*(E') + E.
\end{aligned}
$$

The fact that $c_1(\mathcal{F}) = 0$ follows from the fact that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^3}$, which holds due to the fact that every line bundle on $\mathbb{A}^3$ is trivial. Furthermore, the last equality holds because in order to obtain the foliation $\mathcal{F}_{\mathrm{Bl}}$ from $\mathcal{F}$, we divide by $\overline{d_1}^2$. This means that we divide by two times the exceptional divisor $E$, and so $c_1(\mathcal{F}_{\mathrm{Bl}}) = 2 \cdot E$.

It follows that

$$K_{\mathrm{Bl}_L \mathbb{A}^3} = K_{\mathrm{Bl}_L \mathbb{A}^3} - E - a\beta^*(E'),$$

and hence $-E - a\beta^*(E') = 0 \Leftrightarrow E = -a\beta^*(E')$, from which it follows that $a = -1$. The discrepancy is therefore $-1$.

Considering the other chart $U_{d_2}$, we recall that the blow up is defined by

$$
\begin{aligned}
\mathbb{F}_2[d_1, d_2, t] &\rightarrow \mathbb{F}_2[\overline{d_1}, \overline{d_2}, \overline{t}] \\
d_1 &\mapsto \overline{d_1 d_2} \\
d_2 &\mapsto \overline{d_2} \\
t &\mapsto \overline{t}
\end{aligned}
$$

and the derivations transform as follows

$$d_1\frac{\partial}{\partial d_1} = \overline{d_1}\frac{\partial}{\partial \overline{d_1}}, \quad d_2\frac{\partial}{\partial d_2} = \overline{d_1}\frac{\partial}{\partial \overline{d_1}} + \overline{d_2}\frac{\partial}{\partial \overline{d_2}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \overline{t}}.$$

Hence the foliation $\mathcal{F}$ becomes

$$
\begin{aligned}
\mathcal{F}\big|_{U_D} &= (d_1 d_2 + d_1^4 t + d_1^2 t)\frac{\partial}{\partial d_1} + (d_2^2 + d_1^3 d_2 t + d_1 d_2 t + d_1^5 + d_1^3)\frac{\partial}{\partial d_2} + d_1^2 \frac{\partial}{\partial t} \\
&= (d_2 + d_1^3 t + d_1 t)\left(d_1 \frac{\partial}{\partial d_1}\right) + \left(d_2 + d_1^3 t + d_1 t + \frac{d_1^5}{d_2} + \frac{d_1^3}{d_2}\right)\left(d_2 \frac{\partial}{\partial d_2}\right) + d_1^2 \frac{\partial}{\partial t} \\
&= (\overline{d_2} + \overline{d_1}^3 \overline{d_2}^3 \overline{t} + \overline{d_1 d_2 t})\left(\overline{d_1}\frac{\partial}{\partial \overline{d_1}}\right) \\
&\quad + \left(\overline{d_2} + \overline{d_1}^3\overline{d_2}^3\overline{t} + \overline{d_1 d_2 t} + \frac{\overline{d_1}^5 \overline{d_2}^5}{\overline{d_2}} + \frac{\overline{d_1}^3\overline{d_2}^3}{\overline{d_2}}\right)\left(\overline{d_1}\frac{\partial}{\partial\overline{d_1}} + \overline{d_2}\frac{\partial}{\partial\overline{d_2}}\right) + \overline{d_1}^2\overline{d_2}^2\frac{\partial}{\partial\overline{t}} \\
&= \left(\overline{d_1}^6\overline{d_2}^4 + \overline{d_1}^4\overline{d_2}^2\right)\frac{\partial}{\partial\overline{d_1}} + \left(\overline{d_2}^2 + \overline{d_1}^3\overline{d_2}^4\overline{t} + \overline{d_1 d_2}^2\overline{t} + \overline{d_1}^5\overline{d_2}^5 + \overline{d_1}^3\overline{d_2}^3\right)\frac{\partial}{\partial\overline{d_2}} + \overline{d_1}^2\overline{d_2}^2\frac{\partial}{\partial\overline{t}}
\end{aligned}
$$

Dividing by $\overline{d_2}^2$, we get

$$
\mathcal{F}_{\mathrm{Bl}_L \mathbb{A}^3, U_{b_2}} = \left(\overline{d_1}^6\overline{d_2}^2 + \overline{d_1}^4\right)\frac{\partial}{\partial\overline{d_1}} + \left(1 + \overline{d_1}^3\overline{d_2}^2\overline{t} + \overline{d_1 t} + \overline{d_1}^5\overline{d_2}^3 + \overline{d_1}^3\overline{d_2}\right)\frac{\partial}{\partial\overline{d_2}} + \overline{d_1}^2\frac{\partial}{\partial\overline{t}}
$$

This is regular. Due to a similar argument as on the chart $U_{d_1}$, the discrepancy of the blow up on the quotient space is $-1$.

This concludes the study of the singularity on the chart $U_D$. $\qquad\square$

*Remark* 5.5.4. One might be interested in whether the singularity of discrepancy $-1$ on the chart $U_D$ is elliptic or cuspidal. Using techniques not mentioned in the thesis, such as $W\mathcal{O}$-rationality, one can see that it is probably neither. Instead, the singularity is a Gorenstein log-canonical singularity that does not appear over algebraically closed fields. The only exceptional divisor of the minimal resolution is a regular genus one curve that is geometrically a cuspidal rational curve.

The fact that the singularity is Gorenstein is proved below.

**Proposition 5.5.5.** *All singularities are Gorenstein.*

*Proof.* There are two singularities, according to Lemma 5.5.3. One of the two singularities is an $A_1$ singularity, which is Gorenstein. The other singularity is of discrepancy $-1$, and is resolved after one blow up. The following claim proves that this is a Gorenstein singularity as well. $\qquad\square$

**Claim 5.5.6.** *A singularity $x$ on a normal surface $X$ of discrepancy $-1$, which is resolved after one blow up is Gorenstein.*

*Proof.* We denote by $\tilde{X}$ the resolution of the singularity, by $f$ the map $f : \tilde{X} \to X$, and by $E$ the exceptional divisor. In the sense of Cartier divisors, we have the following equality, $f^*K_X = K_{\tilde{X}} + E$, from which it follows that $f_*(K_{\tilde{X}} + E) = K_X$.

We prove the statement in two steps. In the first step, we show that it is enough to show that $K_{\tilde{X}} + E$ is $f$-base point free. In the second step, we prove that $K_{\tilde{X}} + E$ is indeed base point free.

Firstly, we assume that $K_{\tilde{X}} + E$ is base point free. Let $U \subseteq X$ open, with $x \in U$. Take $y \in E \subseteq f^{-1}U$ a point on the exceptional divisor. Then, there exists a section $s \in H^0(f^{-1}U, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}} + E))$ such that $s \otimes k(y) \neq 0$. With $K_{\tilde{X}} + E\big|_E \sim 0$, it follows that $s$ does not vanish at any point of $E$. Furthermore, it follows that $s$ does not vanish in a neighborhood of $E$, and hence $s$ is trivial along $E$. We conclude with $K_X$ being locally trivial, meaning that $K_X$ is Cartier.

Secondly, we show that $K_{\tilde{X}} + E$ is indeed base point free. We consider the adjunction sequence,

$$
0 \to \mathscr{O}_{\tilde{X}}(K_{\tilde{X}}) \to \mathscr{O}_{\tilde{X}}(K_{\tilde{X}} + E) \to \mathscr{O}_E(K_{\tilde{X}} + E\big|_E) \to 0.
$$

As $K_{\tilde{X}} + E = f^*K_X$, it follows that $K_{\tilde{X}} + E\big|_E = (f\big|_E)^*K_X \sim 0$. We now consider part of the long exact sequence induced by the short exact sequence above,

$$
H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}} + E)) \to H^0(E, \mathscr{O}_E) \to H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}})).
$$

According to [KK94, Corollary 2.2.5], relative Kawamata-Viehweg vanishing holds in dimension 2, and hence $H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}})) = 0$. From this, it follows that

$$H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}} + E)) \twoheadrightarrow H^0(E, \mathscr{O}_E).$$

This means that there is a section $s \in H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}(K_{\tilde{X}} + E))$ that maps to $1 \in H^0(E, \mathscr{O}_E)$. This section $s$ is the section we consider in the first step of the proof, using the property that $s$ is non-zero in a neighborhood of $E$. $\square$

**Proposition 5.5.7.** *The anticanonical divisor $-K_X$ is ample.*

*Proof.* As we have seen in the study of $\mathcal{F}$ on all charts $U_A, U_B, U_C$ and $U_D$, it holds that $\mathcal{F} \cong \mathscr{O}_Y(F - G)$, with $F$ and $G$ fiber and general section of the Hirzebruch surface. Using the proof of Proposition 5.4.8, we have that $K_Y + D = -4F - D$. With $D = G - F$, we obtain

$$\begin{aligned} K_Y &= -4F - 2D = -4F - 2(G - F) = -4F - 2G + 2F \\ &= -2F - 2G. \end{aligned}$$

If we denote the morphism from $Y$ to $X = Y/\mathcal{F}$ by $\pi$, then it follows that

$$\pi^*(K_X) \cong K_Y - (p-1)(\det \mathcal{F}) = K_Y - (\det \mathcal{F}),$$

as we have seen in the proof of Proposition 5.4.8.
With $K_Y = -2F - 2G$ and $\det \mathcal{F} = F - G$, it follows that

$$\pi^*(K_X) = -2F - 2G - (F - G) = -3F - G.$$

The proof follows from the claim below. $\square$

**Claim 5.5.8.** *It holds that $3F + G$ is ample, and hence $-3F - G$ is anti-ample.*

*Proof.* This proof is equivalent to the proof of Claim 5.4.9. $\square$

**Proposition 5.5.9.** *The surface $X$ is not geometrically normal.*

*Proof.* We refer to Remark 3.5.5, which states that Proposition 3.5.4 is independent of the explicit foliation, if the foliation is constructed according to the setup of Chapter 2. $\square$

**Proposition 5.5.10.** *The surface $X$ is geometrically reduced.*

*Proof.* We refer to Remark 3.5.8. This remark states that the proof of the geometric reducedness relies on the setup of Chapter 2, with an additional assumption. The assumption states that the polynomial $h(a_1, a_2, t) \in \mathbb{F}_2[a_1, a_2, t]$ is nonzero. This holds here, which concludes the proof. $\square$

# Bibliography

[BT20]   Fabio Bernasconi and Hiromu Tanaka, *On del pezzo fibrations in positive characteristic*, Journal of the Institute of Mathematics of Jussieu (2020), 1–43.

[FS20]   Andrea Fanelli and Stefan Schröer, *Del Pezzo surfaces and Mori fiber spaces in positive characteristic*, Trans. Amer. Math. Soc. **373** (2020), no. 3, 1775–1843. MR 4068282

[Har66]  Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966. MR 0222093

[Har77]  ———, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har80]  ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176. MR 597077

[Har94]  ———, *Generalized divisors on Gorenstein schemes*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), vol. 8, 1994, pp. 287–339. MR 1291023

[JW19]   Lena Ji and Joe Waldron, *Structure of geometrically Non-Reduced Varieties*, 2019, *https://arxiv.org/abs/1909.04014*. To appear in Transactions of the AMS.

[KK94]   János Kollár and Sándor Kovács, *Birational geometry of log surfaces*, Seminar on Moduli of Surfaces, 1994, *web.math.princeton.edu/ kollar/FromMyHomePage/BiratLogSurf.ps*.

[Kun69]  Ernst Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **91** (1969), 772–784. MR 252389

[Mad16]  Zachary Maddock, *Regular del Pezzo surfaces with irregularity*, J. Algebraic Geom. **25** (2016), no. 3, 401–429. MR 3493588

[Pat13]  Zsolt Patakfalvi, *Base change behavior of the relative canonical sheaf related to higher dimensional moduli*, Algebra Number Theory **7** (2013), no. 2, 353–378. MR 3123642

[PW17]   Zsolt Patakfalvi and Joe Waldron, *Singularities of General Fibers and the LMMP*, 2017, *https://arxiv.org/abs/1708.04268*, to appear in American Journal of Mathematics.

[SB97]   N. I. Shepherd-Barron, *Fano threefolds in positive characteristic*, Compositio Math. **105** (1997), no. 3, 237–265. MR 1440723

[Sch07]  Stefan Schröer, *Weak del Pezzo surfaces with irregularity*, Tohoku Math. J. (2) **59** (2007), no. 2, 293–322. MR 2347424

[Sch10]  ———, *On fibrations whose geometric fibers are nonreduced*, Nagoya Math. J. **200** (2010), 35–57. MR 2747877

[Tan18]  Hiromu Tanaka, *Behavior of canonical divisors under purely inseparable base changes*, J. Reine Angew. Math. **744** (2018), 237–264. MR 3871445

[Tan19]  ———, *Boundedness of regular del Pezzo surfaces over imperfect fields*, *https://arxiv.org/abs/1905.00143*, 2019.

[Tan20]  _____, *Invariants of algebraic varieties over imperfect fields*, *https://arxiv.org/abs/1903.10113*, 2020.

# Index

# Curriculum Vitae

## Personal information

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| name | Ursina SCHWEIZER |
| date of birth | 22.04.1993 |
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## Education

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|---:|---|
| | **École Polytechnique Fédérale de Lausanne** |
| 02.2017 - 07.2021 | PhD in Mathematics |
| | *Advisor: Zsolt Patakfalvi* |
| | *Thesis: Examples of normal, not geometrically normal, projective Gorenstein del Pezzo surfaces with at most Du Val singularities* |
| | |
| | **University of Basel** |
| 09.2014 - 12.2016 | MsC in Mathematics |
| | *Advisor: Jérémy Blanc* |
| | *Thesis:* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{Q}))$ |
| 09.2011 - 09.2014 | BSc in Mathematics |

## Teaching activities

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| | Teaching assistant for the following courses: |
| Spring 2021 | Anneaux et Corps |
| Spring 2020 | Géométrie |
| Fall 2019 | Analysis I (German) |
| Spring 2019 | Géométrie |
| Fall 2018 | Analysis I (German) |
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## Personal skills

| | |
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| languages | German *(native)*, English *(fluent)*, French *(intermediate)* |
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