Finite simple groups and localization

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Abstract

The purpose of this paper is to explore the concept of localization, which comes from homotopy theory, in the context of finite simple groups. We give an easy criterion for a finite simple group to be a localization of some simple subgroup and we apply it in various cases. Iterating this process allows us to connect many simple groups by a sequence of localizations. We prove that all sporadic simple groups (except possibly the Monster) and several groups of Lie type are connected to alternating groups. The question remains open whether or not there are several connected components within the family of finite simple groups. In some cases, we also consider automorphism groups and universal covering groups and we show that a localization of a finite simple group may not be simple.

Introduction

The concept of localization plays an important role in homotopy theory. The introduction by Bousfield of homotopical localization functors in [2] and more recently its popularization by Farjoun in [7] has led to the study of localization functors in other categories. Special attention has been set on the category of groups Gr, as the effect of a homotopical localization on the fundamental group is often best described by a localization functor $\mathrm{L} : Gr \to Gr$.

A *localization functor* is a pair (L, η) consisting of a functor $\mathrm{L} : Gr \to Gr$ together with a natural transformation $\eta : \mathrm{Id} \to \mathrm{L}$, such that L is idempotent, meaning that the two morphisms $\eta_{\mathrm{L}G}, \mathrm{L}(\eta_G) : \mathrm{L}G \to \mathrm{LL}G$ coincide and are isomorphisms. A group homomorphism $\varphi : H \to G$ is called in turn a *localization* if there exists a localization

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functor (L, η) such that $G = \mathrm{L}H$ and $\varphi = \eta_H : H \to \mathrm{L}H$ (but we note that the functor L is not uniquely determined by φ). In this situation, we often say that G is a localization of H. A very simple characterization of localizations can be given without mentioning localization functors: A group homomorphism $\varphi : H \to G$ is a localization if and only if φ induces a bijection

$$\varphi^* : \mathrm{Hom}(G, G) \cong \mathrm{Hom}(H, G) \tag{0.1}$$

as mentionned in [3, Lemma 2.1]. In the last decade several authors (Casacuberta, Farjoun, Libman, Rodríguez) have directed their efforts towards deciding which algebraic properties are preserved under localization. An exhaustive survey about this problem is nicely exposed in [3] by Casacuberta. For example, any localization of an abelian group is again abelian. Similarly, nilpotent groups of class at most 2 are preserved, but the question remains open for arbitrary nilpotent groups. Finiteness is not preserved, as shown by the example $A_n \to SO(n-1)$ (this is the main result in [16]). In fact, it has been shown in [8] that any non-abelian finite simple group has arbitrarily large localizations (under certain set-theoretical assumptions). In particular it is not easy to determine all possible localizations of a given object. Thus we restrict ourselves to the study of finite groups and wonder if it would be possible to understand the finite localizations of a given finite simple group. This paper is a first step in this direction.

Libman [17] observed recently that the inclusion of alternating groups $A_n \hookrightarrow A_{n+1}$ is a localization if $n \geq 7$. His motivation was to find a localization where new torsion elements appear (e.g. $A_{10} \hookrightarrow A_{11}$ is such a localization since A_{11} contains elements of order 11). In these examples, the groups are simple, which simplifies considerably the verification of formula (0.1). It suffices to check if $\mathrm{Aut}(G) \cong \mathrm{Hom}(H, G) - \{0\}$.

This paper is devoted to the study of the behaviour of injective localizations with respect to simplicity. We first give a criterion for an inclusion of a simple group in a finite simple group to be a localization. We then find several infinite families of such localizations, for example $L_2(p) \hookrightarrow A_{p+1}$ for any prime $p \geq 13$ (cf. Proposition 2.3). Here $L_2(p) = PSL_2(p)$ is the projective special linear group. It is striking to notice that the three conditions that appear in our criterion for an inclusion of simple groups $H \hookrightarrow G$ to be a localization already appeared in the literature. For example the main theorem of [15] states exactly that $J_3 \hookrightarrow E_6(4)$ is a localization (see Section 3). Similarly the main theorem in [21] states that $Sz(32) \hookrightarrow E_8(5)$ is a localization. Hence the language of localization theory can be useful to shortly reformulate some rather technical properties.

By Libman's result, the alternating groups A_n, for $n \geq 7$, are all connected by a sequence of localizations. We show that $A_5 \hookrightarrow A_6$ is also a localization. A more curious way allows us to connect A_6 to A_7 by a zigzag of localizations:

$$A_6 \hookrightarrow T \hookrightarrow Ru \hookleftarrow L_2(13) \hookrightarrow A_{14} \hookleftarrow \cdots \hookleftarrow A_7$$

where T is the Tits group, and Ru the Rudvalis group. This yields to the concept of rigid component of a simple group. The idea is that among all inclusions $H \hookrightarrow G$, those that are localizations deserve our attention because of the "rigidity condition" imposed by (0.1): Any automorphism of G is completely determined by its restriction to H. So, we say that two groups H and G lie in the same *rigid component* if H and G can be connected by a zigzag of inclusions which are all localizations.

Many finite simple groups can be connected to the alternating groups. Here is our main result:

Theorem *The following finite simple groups all lie in the same rigid component:*

(i) *All alternating groups A_n ($n \geq 5$).*

(ii) *The Chevalley groups $L_2(q)$ where q is a prime power ≥ 5.*

(iii) *The Chevalley groups $U_3(q)$ where q is a prime power, $q \neq 5$.*

(iv) *The Chevalley groups $G_2(p)$ where p is an odd prime such that $(p+1, 3) = 1$.*

(v) *All sporadic simple groups, except possibly the Monster.*

(vi) *The Chevalley groups $L_3(3)$, $L_3(5)$, $L_3(11)$, $L_4(3)$, $U_4(2)$, $U_4(3)$, $U_5(2)$, $U_6(2)$, $S_4(4)$, $S_6(2)$, $S_8(2)$, $D_4(2)$, ${}^2D_4(2)$, ${}^2D_5(2)$, ${}^3D_4(2)$, $D_4(3)$, $G_2(2)'$, $G_2(4)$, $G_2(5)$, $G_2(11)$, $E_6(4)$, $F_4(2)$, and $T = {}^2F_4(2)'$.*

The proof is an application of the localization criteria which are given in Sections 1 and 2, but requires a careful checking in the ATLAS [4], or in the more complete papers about maximal subgroups of finite simple groups (e.g. [12], [19], [24]). We do not know if the Monster can be connected to the alternating group, see Remark 6.7.

We finally exhibit an example due to Viruel showing that if $H \hookrightarrow G$ is a localization with H simple then G need not be simple. There is a localization map from the Mathieu group M_{11} to the double cover of the Mathieu group M_{12}. This answers negatively a question posed by Libman in [17] and also by Casacuberta in [3] about the preservation of simplicity. In our context this also implies that the rigid component of a simple group may contain a non-simple group.

It is still an open problem to know how many rigid components of finite simple groups there are, even though our main theorem seems to suggest that there is only one. We note that the similar question for non-injective localizations has a trivial answer (see Section 1).

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1 A localization criterion

In Theorem 1.4 below we list necessary and sufficient conditions for an inclusion $H \hookrightarrow G$ between two non-abelian finite simple groups to be a localization. These conditions are easier to deal with if the groups H and G satisfy some extra assumptions, as we show in the corollaries after the theorem. The proof is a variation of that of Corollary 4 in [8].

We note here that we only deal with injective group homomorphisms because non-injective localizations abound. For example, for any two finite groups G_1 and G_2 of coprime orders, $G_1 \times G_2 \to G_1$ and $G_1 \times G_2 \to G_2$ are localizations. So the analogous concept of rigid component defined using non-injective localizations has no interest, since obviously any two finite groups are in the same component.

If the inclusion $i : H \hookrightarrow G$ is a localization, then so is the inclusion $H' \hookrightarrow G$ for any subgroup H' of G which is isomorphic to H. This shows that the choice of the subgroup H among isomorphic subgroups does not matter.

Let $c : G \to \mathrm{Aut}(G)$ be the natural injection of G defined as $c(g) = c_g$, where $c_g : G \to G$ denotes the inner automorphism given by $x \mapsto gxg^{-1}$. We shall always identify in this way a simple group G with a subgroup of $\mathrm{Aut}(G)$, without writing the map c. However, we use c explicitly in the following two easy results.

Lemma 1.1 *Let G be a non-abelian simple group. Then the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "c"'] & G \arrow[d, "c"] \\ \mathrm{Aut}(G) \arrow[r, "c_\alpha"'] & \mathrm{Aut}(G) \end{tikzcd}$$

for any automorphism $\alpha \in \mathrm{Aut}(G)$.

PROOF. This is a trivial check. $\square$

Lemma 1.2 *Let H be a non-abelian simple subgroup of a finite simple group G. Suppose that the inclusion $i : H \hookrightarrow G$ extends to an inclusion $i : \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$, i.e., the following diagram commutes*

$$\begin{tikzcd}
H \arrow[r, "i"] \arrow[d, "c"'] & G \arrow[d, "c"] \\
\mathrm{Aut}(H) \arrow[r, "i"'] & \mathrm{Aut}(G)
\end{tikzcd}$$

Then every automorphism $\alpha : H \to H$ extends to an automorphism $i(\alpha) : G \to G$.

PROOF. We have to show that the following square commutes:

$$\begin{tikzcd}
H \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & H \arrow[d, hook, "i"] \\
G \arrow[r, "i(\alpha)"'] & G
\end{tikzcd}$$

To do so we consider this square as the left-hand face of the cubical diagram

$$\begin{tikzcd}
H \arrow[rr, "c"] \arrow[dr, "\alpha"] \arrow[dd, hook] & & \mathrm{Aut}(H) \arrow[dr, "c_\alpha"] \arrow[dd, hook] & \\
& H \arrow[rr, "c"] \arrow[dd, hook] & & \mathrm{Aut}(H) \arrow[dd, hook] \\
G \arrow[rr, "c"] \arrow[dr, "i(\alpha)"'] & & \mathrm{Aut}(G) \arrow[dr, "c_{i(\alpha)}"] & \\
& G \arrow[rr, "c"'] & & \mathrm{Aut}(G)
\end{tikzcd}$$

The top and bottom squares commute by Lemma 1.1. The front and back squares are the same and commute by assumption. The right-hand square commutes as well because i is a homomorphism. This forces the left-hand square to commute and we are done. $\square$

Remark 1.3 As shown by the preceding lemma, it is stronger to require that the inclusion $i : H \hookrightarrow G$ extends to an inclusion $i : \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$ than to require that every automorphism of H extends to an automorphism of G. In general we have an exact sequence

$$1 \to C_{\mathrm{Aut}(G)}(H) \longrightarrow N_{\mathrm{Aut}(G)}(H) \longrightarrow \mathrm{Aut}(H)$$

so the second condition is equivalent to the fact that this is a short exact sequence. However, in the presence of the condition $C_{\mathrm{Aut}(G)}(H) = 1$, which plays a central role in this paper, we find that $N_{\mathrm{Aut}(G)}(H) \cong \mathrm{Aut}(H)$. Thus any automorphism of H extends to a unique automorphism of G, and this defines a homomorphism $i : \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$ extending the inclusion $H \hookrightarrow G$. Therefore, if the condition $C_{\mathrm{Aut}(G)}(H) = 1$ holds, we have a converse of the above lemma and both conditions are equivalent. We will use the first in the statements of the following results, even though it is the stronger one. It is indeed easier to check in the applications.

Theorem 1.4 *Let H be a non-abelian simple subgroup of a finite simple group G and let $i : H \hookrightarrow G$ be the inclusion. Then i is a localization if and only if the following three conditions are satisfied:*

1. *The inclusion $i : H \hookrightarrow G$ extends to an inclusion $i : \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$.*
2. *Any subgroup of G which is isomorphic to H is conjugate to H in $\mathrm{Aut}(G)$.*
3. *The centralizer $C_{\mathrm{Aut}(G)}(H) = 1$.*

Proof. If i is a localization, all three conditions have to be satisfied. By Lemma 1.2, condition (1) claims that the composite of an automorphism of H with i can be extended to an automorphism of G (see also Remark 1.3). Condition (2) claims that the inclusion of a subgroup of G isomorphic to H can be extended, while condition (3) says that there exists a unique extension for i, namely the identity.

Assume now that all three conditions are satisfied. For any given homomorphism $\varphi : H \to G$, we have to find a unique homomorphism $\Phi : G \to G$ such that $\Phi \circ i = \varphi$. The trivial homomorphism $G \to G$ obviously extends the trivial homomorphism from H to G. It is unique since H is in the kernel of Φ, which must be equal to G by simplicity. Hence, we can suppose that φ is not trivial. Since H is simple we have that $\varphi(H) \leq G$ and $H \cong \varphi(H)$.

By (2) there is an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $c_\alpha(\varphi(H)) = H$, or equivalently by Lemma 1.1, $\alpha(\varphi(H)) = H$. Therefore the composite map

$$H \xrightarrow{\varphi} \varphi(H) \xrightarrow{\alpha|_{\varphi(H)}} H$$

is some automorphism β of H. By condition (1) this automorphism of H extends to an automorphism $i(\beta) : G \to G$. That is, the following square commutes:

$$\begin{tikzcd} H \arrow[r, "\beta"] \arrow[d, hook, "i"'] & H \arrow[d, hook, "i"] \\ G \arrow[r, "i(\beta)"'] & G \end{tikzcd}$$

The homomorphism $\Phi = \alpha^{-1} i(\beta)$ extends φ as desired. We prove now it is unique. Suppose that $\Phi' : G \to G$ is a homomorphism such that $\Phi' \circ i = \varphi$. Then, since G is simple, $\Phi' \in \mathrm{Aut}(G)$. The composite $\Phi^{-1}\Phi'$ is an element in the centralizer $C_{\mathrm{Aut}(G)}(H)$, which is trivial by (3). This finishes the proof of the theorem. $\square$

Remark 1.5 As already mentionned in Remark 1.3, conditions (1) and (3) imply that $\mathrm{Aut}(H) = N_{\mathrm{Aut}(G)}(H)$. By condition (2), the cardinal of the orbit of H under the conjugation action of $\mathrm{Aut}(G)$ is equal to the number k of conjugacy classes of H in G multiplied by the cardinal of the orbit of H under the conjugation action of G. That is,

$$\frac{\mid \mathrm{Aut}(G) \mid}{\mid N_{\mathrm{Aut}(G)}(H) \mid} = k \cdot \frac{\mid G \mid}{\mid N_G(H) \mid},$$

Condition (3) is thus equivalent to the following one, which is sometimes easier to verify:

3'. The number of conjugacy classes of subgroups of G isomorphic to H is equal to

$$\frac{\mid \mathrm{Out}(G) \mid}{\mid \mathrm{Out}(H) \mid} \cdot \frac{\mid N_G(H) \mid}{\mid H \mid}.$$

We obtain immediately the following corollaries. Using the terminology in [20, p.158], recall that a group is *complete* if it has no outer-automorphism and trivial centre. The first corollary describes the situation when the groups involved are complete.

Corollary 1.6 *Let H be a non-abelian simple subgroup of a finite simple group G and let $i : H \hookrightarrow G$ be the inclusion. Assume that H and G are complete groups. Then i is a localization if and only if the following two conditions are satisfied:*

1. *Any subgroup of G which is isomorphic to H is conjugate to H.*

2. *$C_G(H) = 1$.* $\square$

The condition $C_G(H) = 1$ is here equivalent to $N_G(H) = H$. This is often easier to check. It is in particular always the case when H is a maximal subgroup of G. This leads us to the next corollary.

Corollary 1.7 *Let H be a non-abelian simple subgroup of a finite simple group G and let $i : H \hookrightarrow G$ be the inclusion. Assume that H is a maximal subgroup of G. Then i is a localization if and only if the following three conditions are satisfied:*

1. *The inclusion $i : H \hookrightarrow G$ extends to an inclusion $i : \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$.*
2. *Any subgroup of G which is isomorphic to H is conjugate to H in $\mathrm{Aut}(G)$.*
3. *The number of conjugacy classes of H in G is equal to $\dfrac{|\mathrm{Out}(G)|}{|\mathrm{Out}(H)|}$.*

Proof. Since H is a maximal subgroup of G, $N_G(H) = H$. The corollary is now a direct consequence of Theorem 1.4 taking into account Remark 1.5 about the number of conjugacy classes of subgroups of G isomorphic to H. $\square$

2 Localization in alternating groups

We describe in this section a method for finding localizations of finite simple groups in alternating groups. Let H be a simple group and K a subgroup of index n. The (left) action of H on the cosets of K in H defines a *permutation representation* $H \to S_n$ as in [20, Theorem 3.14, p.53]. The *degree* of the representation is the number n of cosets. As H is simple, this homomorphism is actually an inclusion $H \hookrightarrow A_n$. Recall that $\mathrm{Aut}(A_n) = S_n$ if $n \geq 7$.

Theorem 2.1 *Let H be a non-abelian finite simple group and K a maximal subgroup of index $n \geq 7$. Suppose that the following two conditions hold:*

1. *The order of K is maximal (among all maximal subgroups).*
2. *Any subgroup of H of index n is conjugate to K.*

Then the permutation representation $H \hookrightarrow A_n$ is a localization.

Proof. We show that the conditions of Theorem 1.4 are satisfied, starting with condition (1). Since K is maximal, it is self-normalizing and therefore the action of H on the cosets of K is isomorphic to the conjugation action of H on the set of conjugates of K. By our second assumption, this set is left invariant under $\mathrm{Aut}(H)$. Thus the action of H extends to $\mathrm{Aut}(H)$ and this yields the desired extension $\mathrm{Aut}(H) \to S_n = \mathrm{Aut}(A_n)$.

To check condition (2) of Theorem 1.4, let H' be a subgroup of A_n which is isomorphic to H and denote by $\alpha : H \to H'$ an isomorphism. Let J be the stabilizer of a point in $\{1, \dots, n\}$ under the action of H'. Since the orbit of this point has cardinality $\leq n$, the index of J is at most n, hence equal to n by our first assumption. Thus H' acts transitively. So H has a second transitive action via α and the action of H'. For this action, the stabilizer of a point is a subgroup of H of index n, hence conjugate to K by assumption. So K is also the stabilizer of a point for this second action and this shows that this action of H is isomorphic to the permutation action of H on the cosets of K, that is, to the first action. It follows that the permutation representation $H \xrightarrow{\alpha} H' \hookrightarrow A_n$ is conjugate in S_n to $H \hookrightarrow A_n$.

Finally, since H is a transitive subgroup of S_n with maximal stabilizer, the centralizer $C_{S_n}(H)$ is trivial by [6, Theorem 4.2A (vi)] and thus condition (3) of Theorem 1.4 is satisfied. □

Among the twenty-six sporadic simple groups, twenty have a subgroup which satisfies the conditions of Theorem 2.1.

Corollary 2.2 *The following inclusions are localizations:*
$M_{11} \hookrightarrow A_{11}$, $M_{22} \hookrightarrow A_{22}$, $M_{23} \hookrightarrow A_{23}$, $M_{24} \hookrightarrow A_{24}$, $J_1 \hookrightarrow A_{266}$, $J_2 \hookrightarrow A_{100}$, $J_3 \hookrightarrow A_{6156}$, $J_4 \hookrightarrow A_{173067389}$, $HS \hookrightarrow A_{100}$, $McL \hookrightarrow A_{275}$, $Co_1 \hookrightarrow A_{98280}$, $Co_2 \hookrightarrow A_{2300}$, $Co_3 \hookrightarrow A_{276}$, $Suz \hookrightarrow A_{1782}$, $He \hookrightarrow A_{2058}$, $Ru \hookrightarrow A_{4060}$, $Fi_{22} \hookrightarrow A_{3510}$, $Fi_{23} \hookrightarrow A_{31671}$, $HN \hookrightarrow A_{1140000}$, $Ly \hookrightarrow A_{8835156}$.

Proof. In each case, it suffices to check in the ATLAS [4] that the conditions of Theorem 2.1 are satisfied. It is necessary to check the complete list of maximal subgroups in [13] for the Fischer group Fi_{23} and [14] for the Janko group J_4. □

We obtain now two infinite families of localizations. The classical projective special linear groups $L_2(q) = PSL_2(q)$ of type $A_1(q)$, as well as the projective special unitary groups $U_3(q) = PSU_3(q)$ of type ${}^2A_2(q)$, are almost all connected to an alternating group by a localization. Recall that the notation $L_2(q)$ is used only for the simple projective special linear groups, that is if the prime power $q \geq 4$. Similarly the notation $U_3(q)$ is used for $q > 2$.

Proposition 2.3 *(i) The permutation representation $L_2(q) \hookrightarrow A_{q+1}$ induced by the action of $SL_2(q)$ on the projective line is a localization for any prime power $q \notin \{4, 5, 7, 9, 11\}$.*

(ii) The permutation representation $U_3(q) \hookrightarrow A_{q^3+1}$ induced by the action of $SU_3(q)$ on the set of isotropic points in the projective plane is a localization for any prime power $q \neq 5$.

Proof. We prove both statements at the same time. The group $L_2(q)$ acts on the projective line, whereas $U_3(q)$ acts on the set of isotropic points in the projective plane. In both cases, let B be the stabilizer of a point for this action (Borel subgroup). Let us also denote by G either $L_2(q)$ or $U_3(q)$, where q is a prime power as specified above, and r is $q+1$, or q^3+1 respectively. Then B is a subgroup of G of index r by [11, Satz II-8.2] and [11, Satz II-10.12].

By [11, Satz II-8.28], which is an old theorem of Galois when q is a prime, $L_2(q)$ has no non-trivial permutation representation of degree less than r if $q \notin \{4, 5, 7, 9, 11\}$. The same holds for $U_3(q)$ by [5, Table 1] if $q \neq 5$. Thus B satisfies condition (1) of Theorem 2.1.

It remains to show that condition (2) is also satisfied. The subgroup B is the normalizer of a Sylow p-subgroup U, and $B = UT$, where T is a complement of U in B. If N denotes the normalizer of T in G, we know that $G = UNU$ (Bruhat decomposition). We are now ready to prove that any subgroup of G of index r is conjugate to B. Let H be such a subgroup. It contains a Sylow p-subgroup, and we can thus assume it actually contains U. Since G is generated by U and N, the subgroup H is generated by U and $N \cap H$. Assume H contains an element $x \in N - T$. The class of x in the Weyl group $N/T \cong C_2$ is a generator and we have $G = < U, xUx^{-1} >$ (see for example [10, Theorem 2.3.8 (e)]). But both U and its conjugate xUx^{-1} are contained in H. This is impossible because $H \neq G$, so $N \cap H = T \cap H$. It follows that H is contained in $< U, T > = B$. But H and B have the same order and therefore $H = B$. □

Remark 2.4 This proof does not work for the action of $L_{n+1}(q)$ on the n-dimensional projective space if $n \geq 2$, because there is a second action of the same degree, namely the action on the set of all hyperplanes in $(\mathbb{F}_q)^{n+1}$. Thus there is another conjugacy class of subgroups of the same index, so condition 1 does not hold.

3 Proof of the main theorem

In order to prove our main theorem, we have to check that any group of the list is connected to an alternating group by a zigzag of localizations. When no specific proof is indicated for an inclusion to be a localization, it means that all the necessary information for checking conditions (1)-(3) of Theorem 1.4 is available in the ATLAS [4].

(i) Alternating groups.

The inclusions $A_n \hookrightarrow A_{n+1}$, for $n \geq 7$, studied by Libman in [17, Example 3.4] are localizations by Corollary 1.7, with $\mathrm{Out}(A_n) \cong C_2 \cong \mathrm{Out}(A_{n+1})$. The inclusion $A_5 \hookrightarrow A_6$ is a localization as well, since $\mathrm{Out}(A_6) = (C_2)^2$, $\mathrm{Out}(A_5) = C_2$, and there are indeed two conjugacy classes of subgroups of A_6 isomorphic to A_5 with fusion in $\mathrm{Aut}(A_6)$. The inclusion $A_6 \hookrightarrow A_7$ is not a localization, but we can connect these two groups via a zigzag of localizations, for example as follows:

$$A_6 \hookrightarrow T \hookrightarrow Ru \hookleftarrow L_2(13) \hookrightarrow A_{14}$$

where T denotes the Tits group, Ru the Rudvalis group and the last arrow is a localization by Proposition 2.3.

(ii) Chevalley groups $L_2(q)$.

By Proposition 2.3, all but five linear groups $L_2(q)$ are connected to an alternating group. The group $L_2(4) \cong L_2(5)$ is A_5, and $L_2(9) \cong A_6$, which are connected to all alternating groups by the argument above. We connect $L_2(7)$ to A_{28} via a chain of two localizations

$$L_2(7) \hookrightarrow U_3(3) = G_2(2)' \hookrightarrow A_{28}$$

where we use Theorem 2.1 for the second map. Similarly, we connect $L_2(11)$ to A_{22} via the Mathieu group M_{22}:

$$L_2(11) \hookrightarrow M_{22} \hookrightarrow A_{22}.$$

(iii) Chevalley groups $U_3(q)$.

For $q \neq 5$, we have seen in Proposition 2.3 (ii) that $U_3(q) \hookrightarrow A_{q^3+1}$ is a localization. Recall that $U_3(2)$ is not simple. We do not know if $U_3(5)$ is connected to the alternating groups.

(iv) Chevalley groups $G_2(p)$.

When p is an odd prime such that $(p+1,3)=1$, we will see in Proposition 4.2 that $U_3(p) \hookrightarrow G_2(p)$ is a localization. We can conclude by (iii), since 5 is not a prime in the considered family.

(v) Sporadic simple groups.

By Corollary 2.2, we already know that twenty sporadic simple groups are connected with some alternating group. We now show how to connect all the other sporadic groups, except the Monster for which we do not know what happens (see Remark 6.7).

For the Mathieu group M_{12}, we note that the inclusion $M_{11} \hookrightarrow M_{12}$ is a localization because there are two conjugacy classes of inclusions of M_{11} in M_{12} (of index 11) with fusion in $\mathrm{Aut}(M_{12})$ (cf. [4, p.33]). We conclude by Corollary 1.7.

The list of all maximal subgroups of Fi_{24}' is given in [19] and one sees that $He \hookrightarrow Fi_{24}'$ is a localization.

Looking at the complete list of maximal subgroups of the Baby Monster B in [24], we see that $Fi_{23} \hookrightarrow B$ is a localization, as well as $Th \hookrightarrow B$, $HN \hookrightarrow B$, and $L_2(11) \hookrightarrow B$ (see Proposition 4.1 in [24]). This connects Thompson's group Th and the Baby Monster (as well as the Harada-Norton group HN) to the Fischer groups and also to the Chevalley groups $L_2(q)$.

Finally we consider the O'Nan group $O'N$. By [23, Proposition 3.9] we see that $M_{11} \hookrightarrow O'N$ is a localization.

(vi) Other Chevalley groups.

Suzuki's construction of the sporadic group Suz provides a sequence of graphs whose groups of automorphisms are successively $\mathrm{Aut}(L_2(7))$, $\mathrm{Aut}(G_2(2)')$, $\mathrm{Aut}(J_2)$, $\mathrm{Aut}(G_2(4))$ and $\mathrm{Aut}(Suz)$ (see [9, p.108-9]). Each one of these five groups is an extension of C_2 by the appropriate finite simple group. All arrows in the sequence

$$L_2(7) \hookrightarrow G_2(2)' \hookrightarrow J_2 \hookrightarrow G_2(4) \hookrightarrow Suz \tag{3.1}$$

are thus localizations by Corollary 1.7 because they are actually inclusions of the largest maximal subgroup (cf. [4]). This connects the groups $G_2(2)'$ and $G_2(4)$ to alternating groups since we already know that Suz is connected to A_{1782} by Corollary 2.2. Alternatively, note that $G_2(4) \hookleftarrow L_2(13) \hookrightarrow A_{14}$ are localizations, using Proposition 2.3 for the second one.

The Suzuki group provides some more examples of localizations: $L_3(3) \hookrightarrow Suz$ by [22, Section 6.6], and $U_5(2) \hookrightarrow Suz$ by [22, Section 6.1]. We also have localizations

$$A_9 \hookrightarrow D_4(2) = O_8^+(2) \hookrightarrow F_4(2) \hookleftarrow {}^3D_4(2)$$

which connect these Chevalley groups (see Proposition 4.3 for the last arrow). We are able to connect three symplectic groups since $A_8 \hookrightarrow S_6(2)$ and $S_4(4) \hookrightarrow He$ are localizations, as well as $S_8(2) \hookrightarrow A_{120}$ by Theorem 2.1. This allows us in turn to connect more Chevalley groups as $U_4(2) \hookrightarrow S_6(2)$, and $O_8^- = {}^2D_4(2) \hookrightarrow S_8(2)$ are all localizations.

Each of the following localizations involves a linear group and connects some new group to the component of the alternating groups:
$L_2(11) \hookrightarrow U_5(2)$, $L_3(3) \hookrightarrow T$, $L_2(7) \hookrightarrow L_3(11)$, and $L_4(3) \hookrightarrow F_4(2)$.

The localization $U_3(3) \hookrightarrow G_2(5)$ connects $G_2(5)$ and thus $L_3(5)$ by Proposition 4.1 below. Likewise, since we just showed above that $L_3(11)$ belongs to the same rigid component, then so does $G_2(11)$.

Next $M_{22} \hookrightarrow U_6(2)$ and $A_{12} \hookrightarrow O_{10}^- = {}^2D_5(2)$ are also localizations.

In the last three localizations, connecting the groups $U_4(3)$, $E_6(4)$, and $D_4(3)$, the orders of the outer-automorphism groups is larger than 2. Nevertheless, Theorem 1.4 applies easily. There is a localization $A_7 \hookrightarrow U_4(3)$. There are four conjugacy classes of subgroups of $U_4(3)$ isomorphic to A_7, all of them being maximal. We have $\mathrm{Out}(U_4(3)) \cong D_8$ acting transitively on those classes and S_7 is contained in $\mathrm{Aut}(U_4(3))$ (see [4, p.52]).

We have also a localization $J_3 \hookrightarrow E_6(4)$. Here $\mathrm{Out}(E_6(4)) \cong D_{12}$ and there are exactly six conjugacy classes of J_3 in $E_6(4)$ which are permuted transitively by D_{12}. This is exactly the statement of the main theorem of [15].

Finally $D_4(2) \hookrightarrow D_4(3)$ is a localization. Here $\mathrm{Out}(D_4(2)) \cong S_3$ and $\mathrm{Out}(D_4(3)) \cong S_4$. There are four conjugacy classes of subgroups of $D_4(3)$ isomorphic to $D_4(2)$.

4 Other localizations

In this section, we give further examples of localizations between simple groups. We start with three infinite families of localizations. Except the second family, we do not know if the groups belong to the rigid component of alternating groups.

Proposition 4.1 *Let p be an odd prime with $(3, p-1) = 1$. Then there is a localization $L_3(p) \hookrightarrow G_2(p)$.*

Proof. We first treat the case $p = 3$. Then $\mathrm{Out}(L_3(3)) = C_2 = \mathrm{Out}(G_2(3))$. By [18, Table 1, p.300] we know there are two conjugacy classes of $\mathrm{Aut}(L_3(3))$ in $G_2(3)$, which are both maximal. They fuse in $\mathrm{Aut}(G_2(3))$ by a graph automorphism by [12, Proposition 2.2]. Also $N_{\mathrm{Aut}(G_2(3))}(L_3(3)) = \mathrm{Aut}(L_3(3))$ by the same proposition, so that condition (3) of Theorem 1.4 is satisfied. There are also only two copies of $L_3(3)$ in $G_2(3)$, as one can see in [12, Theorem A] that the only subgroups of type $L_3(3)$ in $G_2(3)$ are those contained in $\mathrm{Aut}(L_3(3))$.

The case $p \neq 3$ is simpler as there is only one conjugacy class of $\mathrm{Aut}(L_3(p))$ in $G_2(p)$ by [18, Table 1, p.300]. In this case $G_2(p)$ is complete. □

Proposition 4.2 *Let p be an odd prime with $(3, p+1) = 1$. Then there is a localization $U_3(p) \hookrightarrow G_2(p)$.*

Proof. The proof is similar to that of the preceding proposition. Apply also [12, Proposition 2.2]. □

Proposition 4.3 *Let p be any prime. Then there is a localization ${}^3D_4(p) \hookrightarrow F_4(p)$.*

Proof. We have $\mathrm{Out}(F_4(2)) = C_2$ while for an odd prime p, $F_4(p)$ is complete. On the other hand $\mathrm{Out}({}^3D_4(p)) = C_3$. By [18, Proposition 7.2] there are exactly $(2, p)$ conjugacy classes of ${}^3D_4(p)$ in $F_4(p)$, fused by an automorphism if $p = 2$. The inclusion ${}^3D_4(p) \hookrightarrow \mathrm{Aut}({}^3D_4(p)) \hookrightarrow F_4(p)$ given by [18, Table 1, p.300] is thus a localization by Theorem 1.4. □

For sporadic simple groups, we have seen various localizations in the proof of the main theorem. We give here further examples involving sporadic groups.

We start with the five Mathieu groups. Recall that the Mathieu groups M_{12} and M_{22} have C_2 as outer-automorphism groups, while the three other Mathieu groups are complete. The inclusions $M_{11} \hookrightarrow M_{23}$ and $M_{23} \hookrightarrow M_{24}$ are localizations by Corollary 1.6. We have already seen in Section 3 that the inclusion $M_{11} \hookrightarrow M_{12}$ is a localization. The inclusion $M_{12} \hookrightarrow M_{24}$ is also a localization. Indeed $\mathrm{Aut}(M_{12})$ is the stabilizer in M_{24} of a pair of dodecads, the stabilizer of a single dodecad is a copy of M_{12}. Up to conjugacy, these are the only subgroups of M_{24} isomorphic to M_{12} and thus the formula (3') in Remark 1.5 about the number of conjugacy classes of M_{12} in M_{24} is satisfied. Similarly $M_{22} \hookrightarrow M_{24}$ is also a localization, because $\mathrm{Aut}(M_{22})$ can be identified as the stabilizer of a duad in M_{24}

whereas M_{22} is the pointwise stabilizer. In short we have the following diagram, where all inclusions are localizations:

$$\begin{tikzcd} M_{11} \arrow[r, hook] \arrow[d, hook] & M_{23} \arrow[d, hook] & \\ M_{12} \arrow[r, hook] & M_{24} & M_{22} \arrow[l, hook'] \end{tikzcd} .$$

We consider next the sporadic groups linked to the Conway group Co_1. Inside Co_1 sits Co_2 as stabilizer of a certain vector OA of type 2 and Co_3 as stabilizer of another vector OB of type 3. These vectors are part of a triangle OAB and its stabilizer is the group HS, whereas its setwise stabilizer is $\mathrm{Aut}(HS) = HS.2$. The Conway groups are complete, the smaller ones are maximal simple subgroups of Co_1 and there is a unique conjugacy class of each of them in Co_1 as indicated in the ATLAS [4, p.180]. Hence $Co_2 \hookrightarrow Co_1$ and $Co_3 \hookrightarrow Co_1$ are localizations by Corollary 1.6. Likewise the inclusions $HS \hookrightarrow Co_2$ and $McL \hookrightarrow Co_3$ are also localizations: They factor through their group of automorphisms, since for example $\mathrm{Aut}(McL)$ is the setwise stabilizer of a triangle of type 223 in the Leech lattice, a vertex of which is stabilized by Co_3. Finally, $M_{22} \hookrightarrow HS$ is a localization for similar reasons, since any automorphism of M_{22} can be seen as an automorphism of the Higman-Sims graph (cf. [1, Theorem 8.7 p.273]). We get here the following diagram of localizations:

$$\begin{tikzcd} M_{22} \arrow[d, hook] & McL \arrow[r, hook] & Co_3 \arrow[d, hook] \\ HS \arrow[r, hook] & Co_2 \arrow[r, hook] & Co_1 \end{tikzcd}$$

Some other related localizations are $M_{23} \hookrightarrow Co_3$, $M_{23} \hookrightarrow Co_2$ and $M_{11} \hookrightarrow HS$.

We move now to the Fischer groups and Janko's group J_4. The inclusion $T \hookrightarrow Fi_{22}$ is a localization (both have C_2 as outer automorphism groups) as well as $M_{12} \hookrightarrow Fi_{22}$, and $A_{10} \hookrightarrow Fi_{22}$. Associated to the second Fischer group, we have a chain of localizations

$$A_{10} \hookrightarrow S_8(2) \hookrightarrow Fi_{23}.$$

By [13, Theorem 1] the inclusion $A_{12} \hookrightarrow Fi_{23}$ is also a localization. Finally $M_{11} \hookrightarrow J_4$ and $M_{23} \hookrightarrow J_4$ are localizations by Corollaries 6.3.2 and 6.3.4 in [14].

Let us now end this section with a list without proofs of a few inclusions we know to be localizations. We start with two examples of localizations of alternating groups:
$A_{12} \hookrightarrow HN$, and $A_7 \hookrightarrow Suz$ by [22, Section 4.4].

Finally we list a few localizations of Chevalley groups: $L_2(8) \hookrightarrow S_6(2)$, $L_2(13) \hookrightarrow G_2(3)$, $L_2(32) \hookrightarrow J_4$ (by [14, Proposition 5.3.1]), $U_3(3) \hookrightarrow S_6(2)$, ${}^3D_4(2) \hookrightarrow Th$, $G_2(5) \hookrightarrow Ly$, $E_6(2) \hookrightarrow E_7(2)$, and $E_6(3) \hookrightarrow E_7(3)$. The inclusion $E_6(q) \hookrightarrow E_7(q)$ is actually a localization if and only if $q = 2$ or $q = 3$ by [18, Table 1]. The main theorem in [21] states that $Sz(32) \hookrightarrow E_8(5)$ is a localization. There is one conjugacy class of $Sz(32)$, and $\mathrm{Out}(Sz(32)) \cong C_5$.

5 Localizations between automorphism groups

The purpose of this section is to show that a localization $H \hookrightarrow G$ can often be extended to a localization $\mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$. This generalizes the observation made by Libman (cf. [17, Example 3.4]) that the localization $A_n \hookrightarrow A_{n+1}$ extends to a localization $S_n \hookrightarrow S_{n+1}$ if $n \geq 7$. This result could be the starting point for determining the rigid component of the symmetric groups, but we will not go further in this direction.

Lemma 5.1 *Let G be a finite simple group with $\mathrm{Out}(G) \cong C_p$, where p is a prime. Then G is the only proper normal subgroup of $\mathrm{Aut}(G)$ and any non-trivial endomorphism of $\mathrm{Aut}(G)$ is either an isomorphism, or has G as kernel.* □

Lemma 5.2 *Let $H \hookrightarrow G$ be an inclusion of simple groups. Then any subgroup of $\mathrm{Aut}(G)$ isomorphic to H is contained in G.*

Proof. Let H' be a subgroup of $\mathrm{Aut}(G)$ isomorphic to H. The kernel of the projection $\mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G)$ contains H' because H' is simple, while $\mathrm{Out}(G)$ is solvable (this is the Schreier conjecture, whose proof depends on the classification of finite simple groups, see [10, Theorem 7.1.1]). □

Theorem 5.3 *Let $H \hookrightarrow G$ be a localization between two finite simple groups. Suppose that $\mathrm{Out}(H) \cong \mathrm{Out}(G) \cong C_p$, where p is a prime. Then $\mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(G)$ is also a localization.*

Proof. The idea is similar to the proof of Theorem 1.4. Let $\varphi : \mathrm{Aut}(H) \to \mathrm{Aut}(G)$ be any homomorphism. Let us assume that φ is not trivial. If it is an injection, the composite $\phi : H \hookrightarrow \mathrm{Aut}(H) \xrightarrow{\varphi} \mathrm{Aut}(G)$ actually lies in G by Lemma 5.2 and because $H \hookrightarrow G$ is a localization by Theorem 1.4, there is a unique automorphism α of G making the appropriate diagram commute. Conjugation by α on $\mathrm{Aut}(G)$ is the unique extension we need. Indeed in the following diagram all squares are commutative and so is the top triangle:

$$
\begin{tikzcd}
H \arrow[rr, "i"] \arrow[dr, "\phi"] \arrow[dd] & & G \arrow[dl, "\alpha"'] \arrow[dd] \\
& G \arrow[dd] & \\
\mathrm{Aut}(H) \arrow[rr, "i"] \arrow[dr, "\varphi"] & & \mathrm{Aut}(G) \arrow[dl, "c_\alpha"'] \\
& \mathrm{Aut}(G) &
\end{tikzcd}
$$

Conjugation by α of an automorphism of H is an automorphism of $\phi(H)$. Therefore $c_\alpha(\mathrm{Aut}(H)) \subset \mathrm{Aut}(\phi(H))$ and thus $c_\alpha(\mathrm{Aut}(H))$ and $\varphi(\mathrm{Aut}(H))$ coincide because they both are equal to $\mathrm{Aut}(\phi(H))$. The composite $\mathrm{Aut}(\phi(H)) \xrightarrow{c_{\alpha^{-1}}} \mathrm{Aut}(H) \xrightarrow{\phi} \mathrm{Aut}(\phi(H))$ is conjugation by some automorphism β of $\phi(H)$ since $\mathrm{Aut}(\phi(H))$ is complete by [20, Theorem 7.14]. We have thus shown that $\varphi = c_\beta \circ c_\alpha \circ i$. In particular ϕ is the restriction of α to H composed with β. But by construction $\phi = \alpha \mid_H$ and so β has to be trivial.

By Lemma 5.1, the only other case is when $\ker \varphi = H$. In that case C_p is a subgroup of $\mathrm{Out}(G)$. Thus it clearly extends to a unique endomorphism of $\mathrm{Aut}(G)$. $\square$

Many examples can be directly derived from previous examples, such as $S_n \hookrightarrow S_{n+1}$ and $SL_2(p) \hookrightarrow S_{p+1}$. Suzuki's chain of groups (3.1), as well as $M_{22} \hookrightarrow HS$, also extend to localizations of their automorphism groups.

The converse of the above theorem is false, as shown by the following example. There exists an inclusion $\mathrm{Aut}(L_3(2)) \hookrightarrow S_8$ which is actually a localization (Condition (0.1) can be checked for example with the help of MAGMA). However the induced morphism $L_3(2) \hookrightarrow A_8$ fails to be a localization: There are two conjugacy classes of subgroups of A_8 isomorphic to $L_3(2)$, which are not conjugate in S_8.

6 Further results

It was asked in [16] and also in [3] whether simple groups are preserved under localization, i.e. if $H \hookrightarrow G$ is a localization and H is simple, is G necessary simple? We next show that the answer is affirmative if H is maximal in G. However, without this assumption G need not be simple, as illustrated by Proposition 6.5, where we show that under certain conditions a localization $H \hookrightarrow G$ induces a localization $H \hookrightarrow \tilde{G}$ from H to the universal cover $\tilde{G}$ of G. This result was elaborated on an observation made by A. Viruel (cf. Example 6.6 below).

Proposition 6.1 *Let G be a finite group and let H be a maximal subgroup which is simple. If the inclusion $H \hookrightarrow G$ is a localization, then G is simple.*

Proof. Let N be a normal subgroup of G. As H is simple, $N \cap H$ is either equal to $\{1\}$ or H.

If $N \cap H = H$, as H is maximal, then either $N = G$ or $N = H$, and we show that the latter case is impossible. By maximality of H, the quotient G/H does not have any non-trivial proper subgroup, so $G/H \cong C_p$ for some prime p. Then G has a subgroup of order p and there is an endomorphism of G factoring through C_p, whose restriction to H is trivial. This contradicts the assumption that the inclusion $H \hookrightarrow G$ is a localization.

If $N \cap H = \{1\}$, then either $N = \{1\}$ or $NH = G$ as H is maximal. The second case cannot occur because it would imply that $G = N \rtimes H$, but $H \hookrightarrow N \rtimes H$ cannot be a localization since both the identity of G and the projection onto H extend the inclusion $H \hookrightarrow G$. □

We indicate now a generic situation where the localization of a simple group can be non-simple (it will actually be a double cover of a simple group). We first need to recall some basic facts. Let $Mult(G)$ be the Schur multiplier of a finite simple group G. It is well known that the universal cover $\tilde{G} \twoheadrightarrow G$ induces an exact sequence

$$1 \to S \to \mathrm{Aut}(\tilde{G}) \to \mathrm{Aut}(G) \to 1, \tag{6.2}$$

where S is the subgroup of automorphisms of $\tilde{G}$ which induce the identity on G. Thus, if $S = 1$, then $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(G)$.

Lemma 6.3 *Let G be a non-abelian finite simple group with $Mult(G) \cong C_2$. Then $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(G)$.*

Proof. We show that any automorphism α of $\tilde{G}$ which induces the identity on G is itself the identity. Such an automorphism induces an automorphism on $Mult(G) \cong C_2$. The only automorphism of C_2 is the identity, so we have to determine the set of automorphisms of $\tilde{G}$ inducing the identity on both G and $Mult(G)$. This set is in bijection with $\mathrm{Hom}(G, Mult(G))$, which is trivial since G is simple and $Mult(G)$ abelian. □

Proposition 6.4 *Let G be a finite simple group with Schur multiplier $Mult(G)$. Suppose that $S = 1$ in (6.2). Then, the universal cover $\tilde{G} \twoheadrightarrow G$ is a localization. In particular, if $Mult(G) \cong C_2$, we have that $\tilde{G} \to G$ is a localization.*

Proof. We have to show that $\tilde{G} \to G$ induces a bijection $\mathrm{Hom}(G, G) \cong \mathrm{Hom}(\tilde{G}, G)$ or equivalently, $\mathrm{Aut}(G) \cong \mathrm{Hom}(\tilde{G}, G) \setminus \{0\}$. This follows easily since the only non-trivial proper normal subgroups of $\tilde{G}$ are contained in its centre $Mult(G)$. Thus any non-trivial homomorphism $\tilde{G} \to G$ can be decomposed as the canonical projection $\tilde{G} \to G$ followed by an automorphism of G. □

Proposition 6.5 *Let $i : H \hookrightarrow G$ be an inclusion of two finite simple groups. Suppose that the Schur multipliers of H and G have orders 1 and 2 respectively and let $j : H = \tilde{H} \hookrightarrow \tilde{G}$ be the induced homomorphism. Then $i : H \hookrightarrow G$ is a localization if and only if $j : H \hookrightarrow \tilde{G}$ is a localization.*

Proof. Suppose that $i : H \hookrightarrow G$ is a localization and let $\varphi : H \to \tilde{G}$ be a non-trivial homomorphism. We have to show that this homomorphism extends to an automorphism of $\tilde{G}$. The composite $H \xrightarrow{\varphi} \tilde{G} \twoheadrightarrow G$ extends to a unique automorphism ψ of G, since i is a localization. Now, as $S = 1$ in (6.2) above, ψ can be lifted to a unique automorphism of $\tilde{G}$, which is the desired automorphism. The proof of the other implication is similar.
□

Example 6.6 This example was communicated to us by Antonio Viruel. The inclusion $M_{11} \hookrightarrow \tilde{M}_{12}$ of the Mathieu group M_{11} into the double cover of the Mathieu group M_{12} is a localization. This follows from the above proposition. Note that M_{11} is not maximal in $\tilde{M}_{12}$ (the maximal subgroup is $M_{11} \times C_2$), so this does not contradict Proposition 6.1. Since $Mult(A_n) \cong C_2$, we get many other examples of this type using Corollary 2.2. All sporadic groups appearing in this list which have trivial Schur multiplier (that is $M_{11}, M_{23}, M_{24}, J_1, J_4, Co_2, Co_3, He, Fi_{23}, HN$, and Ly) admit the double cover of an alternating group as localization. Here are some more examples of localizations which give rise to similar examples: $Co_2 \hookrightarrow Co_1$, $Co_3 \hookrightarrow Co_1$, $G_2(2)' \hookrightarrow J_2$, $Fi_{23} \hookrightarrow B$.

Remark 6.7 The latter example $Fi_{23} \hookrightarrow B$ produces a localization $Fi_{23} \hookrightarrow \tilde{B}$ and the double cover $\tilde{B}$ is a maximal subgroup of the Monster M. It would be nice to know if $\tilde{B} \hookrightarrow M$ is a localization, which would connect the Monster to the rigid component of the alternating groups.

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