Fixed-order Controller Design for State Space Polytopic Systems by Convex Optimization [1]

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Abstract: In this paper, a new method for fixed-order controller design of systems with polytopic uncertainty in their state space representation is proposed. The approach uses the strictly positive realness (SPRness) of some transfer functions, as a tool to decouple the controller parameters and the Lyapunov matrices and represent the stability conditions and the performance criteria by a set of linear matrix inequalities. The quality of this convex approximation depends on the choice of a central state matrix. It is shown that this central matrix can be computed from a set of initial fixed-order controllers computed for each vertex of the polytope. The stability of the closed-loop polytopic system is guaranteed by a linear parameter dependent Lyapunov matrix. The results are extended to fixed-order H_∞ controller design for SISO systems.

1. INTRODUCTION

Most of the standard controller design methods usually lead to high-order controllers which have the same order as that of the generalized plant (i.e. the plant plus some frequency weighting functions) [Zhou (1998)]. The implementation of such controllers will result in high cost, difficult commissioning, poor reliability, fragility, numerical error and potential problems in maintenance [Gu et al. (2005)]. Therefore, they narrow the scope of use in practical applications.

Low-order controllers are always welcomed by control engineers. There has been a considerable interest in the design of low-, fixed-order controllers. Plant or controller reduction techniques do not always guarantee that the closed-loop performance is preserved. Therefore, a challenging problem is to design directly a low-, fixed-order controller for a system. The origin of difficulty in the development of efficient methods for designing fixed-order controllers is that it is a non-convex problem which are known to be NP-complete. Some researchers have been tried to solve the non-convex problem and find the local optimum. This problem has been formulated as Bilinear Matrix Inequalities (BMIs) in Safonov et al. (1994) and a non-convex matrix rank condition in Iwasaki and Skelton (1994), Scherer et al. (1997). A non-smooth H_∞ optimization approach has been also proposed by Apkarian and Noll (2006) for design of fixed-structure controllers.

Some researchers have been focused on the problem of full-order controller design for the systems with polytopic uncertainty. In Kanev et al. (2004), a locally optimal full-order output feedback controller for polytopic systems has been proposed by the use of local BMI optimization. This approach, which has iterative framework, starts from an initial controller and performs local optimization over a suitably defined non-convex function at each iteration. In Geromel et al. (2007), sufficient conditions for full-order robust output feedback controller in terms of LMIs with common Lyapunov matrices have been presented.

The problem of fixed-order controller design becomes more complicated for systems affected by polytopic uncertainty. Recently, new methods for fixed-order controller design of polytopic systems have been proposed in the polynomial framework for SISO systems. In Henrion et al. (2003), a convex parameterization of fixed-order stabilizing controllers for systems with polytopic uncertainty has been proposed. The same method is also utilized for fixed-order H_∞ controller design [Yang et al. (2007)]. The approach is based on the positivity of polynomials and depends on the choice of a so-called central polynomial. In Khatibi et al. (2008), the effect of the chosen central polynomial on the closed-loop poles is investigated.

A convex set of all stabilizing controllers for SISO polytopic systems is presented in Karimi et al. (2007) based on the Strict Positive Realness of the transfer functions. The results of this paper are extended to H_∞ controller design in H. Khatibi and A. Karimi (2010). For the case of fixed-order controller design, this approach leads to an inner approximation of the non-convex set of all stabilizing controllers presented by a set of LMIs originated from the Kalman-Yakoubovic-Popov (KYP) lemma. The quality of this approach for low-order controller design is related to the choice of some basis functions which is closely related to the choice of the central polynomial in Henrion et al. (2003).

In this paper, the problem of fixed-order controller design for polytopic systems is presented in the state space framework. It is clear that a polytopic state space representation is more general than a polytopic system in the coeffi-

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cients of the transfer function parameters. Moreover, the extension to MIMO systems is more straightforward for state space representation than the polynomial approach. It should be mentioned that the existing fixed-order controller design for polytopic systems considers only SISO systems with rational transfer function representation. The first contribution of this paper is to present a convex set of fixed-order stabilizing controllers for MIMO systems with polytopic uncertainty. The second contribution is to present a convex set of fixed-order stabilizing controllers with H_∞ bound on every weighted closed-loop transfer function for SISO polytopic systems. The main idea is to find an inner convex approximation of the non-convex set of all stabilizing, or H_∞, controllers around a desired central closed-loop state matrix.

The organization of the paper is as follows: The problem formulation, the basic idea, the concept of central state matrix and simulation results are presented in Section 2. In Section 3, the convex set of fixed-order H_∞ controllers for SISO systems together with a simulation example are given. Finally, Section 4 presents some concluding remarks.

2. FIXED-ORDER STABILIZING CONTROLLERS

2.1 Problem formulation

Consider a linear time-invariant multi-input multi-output polytopic system represented by the following state space realization:

$$\begin{aligned} \dot{x}_g(t) &= A_g x_g(t) + B_g u(t) \\ y(t) &= C_g x_g(t) \end{aligned} \tag{1}$$

where $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, and $y \in \mathbb{R}^{n_o}$ are the state, the input and the output of the system, respectively. The model is supposed to be strictly proper as it is a characteristic of the real physical systems. For the technical reasons, it is assumed that the matrices A_g and C_g have polytopic uncertainty as follows:

$$A_g(\lambda) = \sum_{i=1}^{q} \lambda_i A_i \qquad C_g(\lambda) = \sum_{i=1}^{q} \lambda_i C_i \tag{2}$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{q} \lambda_i = 1$, and $(A_i, B_g, C_i, 0)$ is the state space realization of each vertex of the polytope. Note that if the matrix B_g has polytopic uncertainty and C_g is fixed, similar results can be obtained.

The first objective is to find a convex set of fixed-order stabilizing output feedback controllers for the polytopic system. The controller is represented by:

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \tag{3}$$

where $A_k \in \mathbb{R}^{m\times m}$ and B_k, C_k, and D_k are of appropriate dimensions. Then, the state matrix of the closed-loop system A_c is given by:

$$A_c(\lambda) = \begin{bmatrix} A_g(\lambda) - B_g D_k C_g(\lambda) & B_g C_k \\ -B_k C_g(\lambda) & A_k \end{bmatrix} \tag{4}$$

This matrix is called stable if all its eigenvalues have strictly negative real part.

2.2 Basic idea

In Henrion et al. (2003), the main idea for synthesis of a fixed order controller for a SISO polytopic system with a rational transfer function representation is given as follows. Suppose that $c_i(s)$ for $i = 1, \ldots, q$ is the characteristic polynomials of the closed-loop system at i-th vertex, then the polytopic system is stable if $c_i(s)/d(s)$ for $i = 1, \ldots, q$ is an SPR transfer function where $d(s)$ is a given stable polynomial called the central polynomial. The choice of the central polynomial is very crucial and affects the control performance as well as the conservatism of the approach. In this paper, the same idea is used to find a convex set of fixed-order controllers for systems with polytopic uncertainty in their state space representation. The main idea is presented in the following lemma, definition and theorem.

Lemma 1. The following statements are equivalent:

(1) $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & I \end{array}\right]$ is SPR.

(2) $H^{-1}(s) = \left[\begin{array}{c|c} A - BC & B \\ \hline -C & I \end{array}\right]$ is SPR.

Proof: According to the KYP lemma, the statement (1) is equivalent to the existence of $P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -2I \end{bmatrix} < 0 \tag{5}$$

which leads to the following inequality:

$$A^T P + PA + \frac{1}{2}(PB - C^T)(B^T P - C) < 0 \tag{6}$$

This inequality can be rearranged to

$$(A - BC)^T P + P(A - BC) + \frac{1}{2}(PB + C^T)(B^T P + C) < 0 \tag{7}$$

which is equivalent to

$$\begin{bmatrix} (A - BC)^T P + P(A - BC) & PB + C^T \\ B^T P + C & -2I \end{bmatrix} < 0 \tag{8}$$

Therefore, the statement (2) follows. □

Remark 1: Note that A and $A - BC$ are both stable with a common Lyapunov matrix P.

Definition 1. Two matrices M and A in $\mathbb{R}^{n\times n}$ are called *SPR-pair* matrices if :

$$H(s) = \left[\begin{array}{c|c} M & I \\ \hline M - A & I \end{array}\right] \tag{9}$$

is SPR.

By applying Lemma 1, it is evident that if M and A are SPR-pair, then A and M are also SPR-pair and they are both stable with a common Lyapunov matrix. As a result, the following LMIs are equivalent:

$$\begin{bmatrix} M^T P + PM & P - M^T + A^T \\ P - M + A & -2I \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} A^T P + PA & P - A^T + M^T \\ P - A + M & -2I \end{bmatrix} < 0 \tag{11}$$

Theorem 1. The fixed-order controller defined in (3) stabilizes the polytopic system in (1) and (2) if a given stable matrix M, makes an SPR-pair with A_c^i for $i = 1, \ldots, q$,

where A_c^i is the closed-loop state matrix of the i-th vertex defined by:

$$A_c^i = \begin{bmatrix} A_i - B_g D_k C_i & B_g C_k \\ -B_k C_i & A_k \end{bmatrix} \tag{12}$$

Thus, a convex set of stabilizing controllers can be given using the KYP lemma by the following set of LMIs:

$$\begin{bmatrix} M^T P_i + P_i M & P_i - M^T + (A_c^i)^T \\ P_i - M + A_c^i & -2I \end{bmatrix} < 0 \tag{13}$$

for $i = 1, \ldots, q$. The variables are the controller parameters (A_k, B_k, C_k, D_k) and q symmetric matrices P_i for $i = 1, \ldots, q$.

Proof : Based on Lemma 1 and the equivalence of (10) and (11), the LMIs in (13) is equivalent to the following inequalities:

$$\begin{bmatrix} (A_c^i)^T P_i + P_i A_c^i & P_i - (A_c^i)^T + M^T \\ P_i - A_c^i + M & -2I \end{bmatrix} < 0 \tag{14}$$

for $i = 1, \ldots, q$. which ensures the stability of A_c^i. By convex combination of (13) for all vertices, we get:

$$\begin{bmatrix} M^T P(\lambda) + P(\lambda) M & P(\lambda) - M^T + A_c^T(\lambda) \\ P(\lambda) - M + A_c(\lambda) & -2I \end{bmatrix} < 0 \tag{15}$$

that shows that M and $A_c(\lambda)$ are SPR-pair. Therefore, using again the equivalence of (10) and (11), we can conclude that (13) is equivalent to:

$$\begin{bmatrix} A_c^T(\lambda) P(\lambda) + P(\lambda) A_c(\lambda) & P(\lambda) - A_c^T(\lambda) + M^T \\ P(\lambda) - A_c(\lambda) + M & -2I \end{bmatrix} < 0 \tag{16}$$

Thus, the closed-loop state matrix of the polytopic system $A_c(\lambda) = \sum_{i=1}^{q} \lambda_i A_c^i$ is stable with a linearly dependent Lyapunov matrix $P(\lambda) = \sum_{i=1}^{q} \lambda_i P_i$. □

The convex set of fixed-order stabilizing controller presented in this theorem is an inner convex approximation of the non-convex set of all fixed-order stabilizing controllers for the polytopic system. The quality of this approximation depends on the choice M, the central state matrix, which will be discussed in the next subsection.

2.3 Choice of the central state matrix

In the polynomial approaches to fixed-order controller design for polytopic systems, the central polynomial is interpreted as the desired closed-loop characteristic polynomial. In a similar way, the central state matrix M can be seen as the desired closed-loop state matrix. Suppose that $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{n+m}]$ is the vector of desired closed-loop eigenvalues, therefore a choice for the central matrix is $M = diag(\alpha)$. It should be mentioned that in addition to the eigenvalues, desired eigenvectors of the closed-loop state matrix can also be assigned. The eigenstructure assignment is used as a closed-loop specification in aerospace engineering [Andry et al. (1983)] and fault detection [Patton and Chen (2000)]. Suppose that the desired eigenvectors are given in $V = [V_1, V_2, \ldots, V_{m+n}]$ that corresponds to $m+n$ distinct eigenvalue in vector α, then a choice for the central matrix is $M = V diag(\alpha) V^{-1}$.

In the case that a desired state matrix cannot be defined or the problem becomes infeasible, an alternative is to use a set of initial stabilizing controllers designed for each vertex. Nowadays, there are several fixed-order controller design methods to deal with systems without parametric uncertainty. Therefore, it is reasonable to suppose that a set of fixed-order controllers that satisfy the control performance for each vertex of the polytopic system is available. These controllers may be designed by balanced controller order reduction of a full-order controller [Zhou (1998)], or by convex relaxation of a rank constraint in the classical full-order controller design [Grigoriadis and Skelton (1996)] or by non smooth H_∞ optimization [Apkarian and Noll (2006)] or finally by a fixed-order linearly parameterized controller based on the spectral models of each vertex [Karimi and Galdos (2010); Galdos et al. (2010)].

Take $\bar{A}_c^i$ as the closed loop state matrix of each vertex with its corresponding controller, then a good candidate for the central state matrix will be a matrix which is SPR-pair with $\bar{A}_c^i$ for all $i = 1, \ldots, q$. Thus, the central state matrix M can be chosen as a feasible solution to the following LMIs:

$$\begin{bmatrix} (\bar{A}_c^i)^T P_i + P_i \bar{A}_c^i & P_i - (\bar{A}_c^i)^T + M^T \\ P_i - \bar{A}_c^i + M & -2I \end{bmatrix} < 0 \tag{17}$$

for $i = 1, \ldots, q$.

The results can be further improved if the resulted robust controller is used for computing $\bar{A}_c^i$ and then for updating the matrix M iteratively. This is illustrated in the following simulation examples.

2.4 Simulation examples

Example 1: Consider the following forth-order polytopic system. This example is borrowed from Wu (2001) and represents a mechanical system:

$$A_g(\rho_1, \rho_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \rho_1 - 200 & -200 & -1 & 0 \\ -400 & \rho_2 - 400 & 0 & -2 \end{bmatrix} \tag{18}$$

$$Bg = [\,0\ 0\ 1\ 0\,]^T$$

$$Cg = [\,0\ 1\ 0\ 0\,]$$

where $\rho_1 \in [0 \quad 9k]$ and $\rho_2 \in [0 \quad 25k]$. This is a polytopic system with 4 vertices ($A_1 = A_g(0,0), A_2 = A_g(9k, 0), A_3 = A_g(9k, 25k)$ and $A_g(0, 25k)$) and k is used to increase or decrease the size of the polytope. It should be mentioned that the open loop system is unstable except for $\rho_1 = \rho_2 = 0$. The proposed approach is used to compute a second-order stabilizing controller for the whole polytope. The central matrix is chosen using four simple PID controllers computed for each vertex using a controller design toolbox [Karimi (2012)]. These controllers are used to compute a feasible solution M of (14). Finally, a robust controller is computed using the results of Theorem 1.

Using this approach k could be increased up to 0.58 and for greater value a feasible solution could not be obtained. The initial PID controllers for $k = 0.58$ are given below:

$$\mathrm{PID}_1(s) = \frac{-11621.35(s + 0.407)(s + 8.65)}{s(s + 833.3)}$$

$$\mathrm{PID}_2(s) = \frac{-12786.91(s + 0.5226)(s + 13.5)}{s(s + 833.3)}$$

$$\mathrm{PID}_3(s) = \frac{-13088.56(s+0.7177)(s+19.63)}{s(s+833.3)}$$

$$\mathrm{PID}_4(s) = \frac{-12546.07(s+0.602)(s+15.85)}{s(s+833.3)}$$

and the resulting second-order robust controller is:

$$K_0(s) = \frac{-13038.35(s+1.064)(s+16.71)}{(s+0.7969)(s+883.9)} \tag{19}$$

Now, if this controller is used to find a new M, less conservative results can be obtained. In fact, k can be increased to $k = 1$. This controller is :

$$K_1(s) = \frac{-12284.83(s+46.73)(s+13.82)}{(s+34.2)(s+937.5)} \tag{20}$$

The results can be further improved by an iterative approach in which M is computed based on the controller in the last iteration. It should be mentioned that this approach is similar to solving a BMI using an iterative approach. However, the main difference is that the Lyapunov matrices P_i are always optimization variables and are not fixed in any iteration. In other words, from LMI in (14) M is computed and then from LMI in (13) the controller parameters are computed that use to compute $\bar{A}_c^i$ in (14) for the next iteration.

To solve the optimization problems in MATLAB, YALMIP [Löfberg (2004)] as the interface and SDPT3 [Toh et al. (1999)] as the solver are used.

3. CONVEX SET OF FIXED-ORDER H_∞ CONTROLLERS

In this section, the objective is to design a fixed-order stabilizing controller for the polytopic system which satisfies some H_∞ norm bounds on some weighted transfer functions of the closed-loop system. The results of this section are valid only for SISO systems (i. e. $n_i = n_o = 1$) for the reason that becomes clear in the sequel.

For simplicity of the presentation, the infinity norm of the sensitivity function $S = (1 + GK)^{-1}$ is considered but it can be applied to any other closed-loop transfer function as well. The objective is therefore to design a fixed-order controller for the polytopic system in (1) and (2) to achieve

$$\|WS(\lambda)\|_\infty < \gamma \tag{21}$$

where γ is given and W is a weighting transfer function with the realization $(A_w, B_w, C_w, 0)$. Then, the state space realization of $WS(\lambda)$ is as follows:

$$\begin{aligned} A_s(\lambda) &= \begin{bmatrix} A_g(\lambda) - B_g D_k C_g(\lambda) & B_g C_k & 0 \\ -B_k C_g(\lambda) & A_k & 0 \\ -B_w Cg(\lambda) & 0 & A_w \end{bmatrix} \\ B_s &= \begin{bmatrix} B_g D_k \\ B_k \\ B_w \end{bmatrix} \quad C_s = [0 \quad 0 \quad C_w] \quad D_s = 0 \end{aligned} \tag{22}$$

The proposed approach is based on the relation between the infinity norm and quadratic stability according to the following lemma from Chilali et al. (1999):

Lemma 2. The following statements are equivalent.

(1) $\|WS\|_\infty < \gamma$.
(2) $A_s - \gamma^{-1} B_s \Delta C_s$ is quadratically stable for all $\|\Delta\|_\infty \leq 1$.

The approach that we propose is to find a sufficient condition to satisfy Statement (2).

For SISO systems Δ will be a scalar, therefore, if $A_n = A_s - \gamma^{-1} B_s C_s$ and $A_p = A_s + \gamma^{-1} B_s C_s$ are quadratically stable, then Statement (2) and consequently Statement (1) are satisfied. On the other hand, if A_n and A_p are SPR-pair matrices, they will be stable with a common Lyapunov matrix and so Statement (1) is satisfied. However, applying the KYP lemma for these SPR-pair matrices leads to a BMI. Now, suppose that there exists a central stable matrix M_n which is an SPR-pair with A_n and another central matrix M_p which is an SPR-pair with A_p, then M_n will be quadratically stable with A_n and M_p with A_p. Moreover, if we use the same Lyapunov matrix in the KYP lemma associated to each SPR-pair matrices then the quadratic stability of A_n and A_p is ensured. The results can be summarized in the following theorem:

Theorem 2. The fixed-order controller defined in (3) stabilizes the polytopic system in (1) and (2) and guarantees an infinity norm less than γ for the weighted sensitivity function defined in (22) if :

$$\begin{bmatrix} M_n^T P_i + P_i M_n & P_i - M_n^T + (A_n^i)^T \\ P_i - M_n + A_n^i & -2I \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} M_p^T P_i + P_i M_p & P_i - M_p^T + (A_p^i)^T \\ P_i - M_p + A_p^i & -2I \end{bmatrix} < 0 \tag{24}$$

for $i = 1, \ldots, q$, where

$$A_n^i = A_s^i - \gamma^{-1} B_s C_s \tag{25}$$

$$A_p^i = A_s^i + \gamma^{-1} B_s C_s \tag{26}$$

with

$$A_s^i = \begin{bmatrix} A_i - B_g D_k C_i & B_g C_k & 0 \\ -B_k C_i & A_k & 0 \\ -B_w C_i & 0 & A_w \end{bmatrix} \tag{27}$$

and M_n and M_p are stable matrices.

Proof: If the inequalities in (23) and (24) are satisfied, we conclude that A_n^i and A_p^i are stable with P_i as Lyapunov matrix. Therefore, A_n^i and A_p^i are quadratically stable and according to Lemma 2, $\|WS_i\|_\infty < \gamma$, where S_i is the sensitivity function of the i-th vertex. By convex combination of (23) and (24) for all vertices of the closed-loop polytope, we obtain:

$$\begin{bmatrix} M_n^T P + P M_n & P - M_n^T + A_n^T \\ P - M_n + A_n & -2I \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} M_p^T P + P M_p & P - M_p^T + A_p^T \\ P - M_p + A_p & -2I \end{bmatrix} < 0 \tag{29}$$

Then, using the equivalent matrices in (10) and (11) we got:

$$\begin{bmatrix} A_n^T P + P A_n & P - A_n^T + M_n^T \\ P - A_n + M_n & -2I \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} A_p^T P + P A_p & P - A_p^T + M_p^T \\ P - A_p + M_p & -2I \end{bmatrix} < 0 \tag{31}$$

that shows A_n and A_p are quadratically stable with the linear parameter dependent matrix $P = \sum_i^q \lambda_i P_i$ which guarantees $\|WS\|_\infty < \gamma$ for all members of the closed-loop polytopic system. □

The matrices M_n and M_p can be computed based on a set of initial controllers computed for each vertex from the following LMIs:

$$\begin{bmatrix} (\bar{A}_n^i)^T P_i + P_i \bar{A}_n^i & P_i - (\bar{A}_n^i)^T + M_n^T \\ P_i - \bar{A}_n^i + M_n & -2I \end{bmatrix} < 0 \quad (32)$$

$$\begin{bmatrix} (\bar{A}_p^i)^T P_i + P_i \bar{A}_p^i & P_i - (\bar{A}_p^i)^T + M_p^T \\ P_i - \bar{A}_p^i + M_p & -2I \end{bmatrix} < 0 \quad (33)$$

for $i = 1, \ldots, q$, where $\bar{A}_n^i = \bar{A}_s^i - \gamma^{-1} B_s C_s$ and $\bar{A}_p^i = \bar{A}_s^i + \gamma^{-1} B_s C_s$ and $\bar{A}_s^i$, B_s and C_s are computed from (22) by replacing the initial controllers for A_k, B_k, C_k and D_k.

3.1 Simulation results

In this part, the control objective is to design a second-order controller such that $\|WS\|_\infty < \gamma$ for the mechanical polytopic system of Subsection 2.4. The low-pass weighting filter W is as follows:

$$W(s) = \frac{1.2}{s + 0.04} \quad (34)$$

and $k = 0.2$ is chosen. In the first step, some initial controllers are required for computing the central matrices M_n and M_p. For this purpose, $K_0(s)$ in (19) is used. This controller ensures the following weighted infinity norm of the sensitivity functions for four vertices of the polytopic system: 0.6200, 1.7356, 4.4813, 2.4660. The central matrices are computed by minimizing γ in the LMIs in (32) and (33) using the bisection algorithm. In the next step, an H_∞ controller is computed using the LMIs in (23) and (24). Note that the H_∞ performance of this controller will be never worse than the initial one, because the initial controller is in the feasible set of these LMIs. Therefore, using an iterative algorithm the infinity norm of WS_i will monotonically converge to a local minimum that depends on the initial controller. Fig. 1 shows that the value of γ converges to $\gamma = 0.8743$ after 13 iterations. The final controller is :

$$K_2(s) = \frac{-13743(s + 0.4598)(s + 16.71)}{(s + 0.04193)(s + 879)} \quad (35)$$

which results in the following infinity norms at the vertices: 0.6559, 0.7149, 0.8145, 0.7374. The Bode magnitude diagrams of WS_i are shown only for the four vertices in Fig.2, however, the H_∞ constraint is satisfied for the whole polytope according to Theorem 2.

The results can only be compared with fixed-order H_∞ controller designed for simultaneous stabilization of multi-model systems, since, to the best of the authors knowledge, there is no fixed-order H_∞ controller design method for polytopic systems.

The frequency-domain robust control toolbox (FDRC) [Karimi (2012)] can be used to compute a second order controller that minimizes the infinity norm of the weighted sensitivity functions of all vertices. This method is based on the loop shaping in the Nyquist diagram with constraints on the infinity norm of the sensitivity functions and uses Laguerre basis functions to obtain linearly parameterized controllers (the denominator of the controller is fixed) [Karimi and Galdos (2010)]. The resulting controller is given by:

$$K_{\text{fdrc}}(s) = \frac{-454.7689(s^2 + 1.195s + 3.967)}{(s + 1)^2} \quad (36)$$

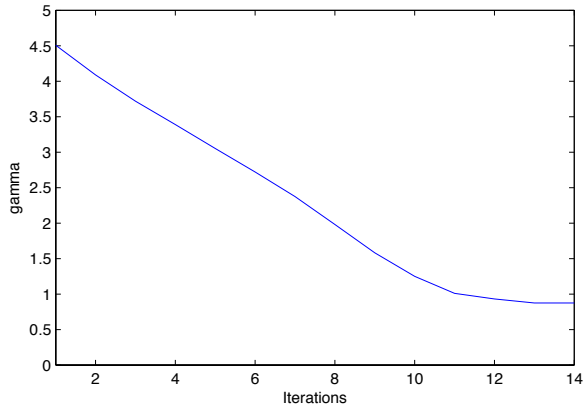

Fig. 1. Evolution of the infinity norm versus the iteration number

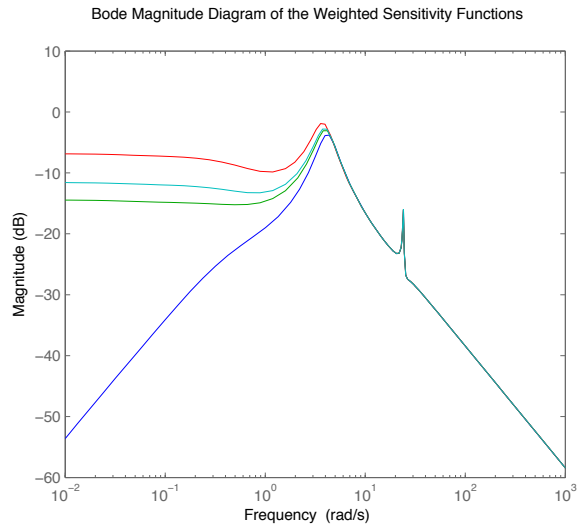

Fig. 2. Bode magnitude diagrams of WS_i

which leads to the following infinity norms: 0.6604, 0.6868, 0.7286, 0.6956.

If the above controller is used as an initial controller for the method proposed in this paper, the algorithm converges to $\gamma = 0.7353$ after four iterations and leads to the following controller:

$$K_3(s) = \frac{-609.9401(s^2 + 3.1s + 3.804)}{(s + 5.726)(s + 0.1234)} \quad (37)$$

with the following infinity norms: 0.4636, 0.4813, 0.5078, 0.4872. It can be observed that the results can be significantly improved because of more degree of freedom in the parameters of the controller's denominator.

The results are also compared with HIFOO [Burke et al. (2006)] which is a public-domain MATLAB package for fixed-order H_∞ controller design of multi-model systems using non-smooth non-convex optimization algorithms [Gumussoy and Overton (2008)]. Since HIFOO uses randomly generated starting points, three sequences of optimized H_∞ norm with 10 iterations are generated. To improve the results with HIFOO, the designed controller in previous iteration is used as an initial guess. HIFOO converges into the following controller after 10 iterations

in each run:

$$K_{\text{hifoo}}(s) = \frac{-1506(s+2363)(s+13.01)}{(s+1865)(s+14.88)} \tag{38}$$

This controller can minimize $\|WS_i\|_\infty$ for all vertices of the polytope (not for the whole polytope). The norms $\|WS_i\|_\infty$ achieved at the four vertices are as follows: 0.7795, 0.7837, 0.7895, and 0.7895, respectively.

The results of HIFOO controller are very close to those of $K_2(s)$ designed based on Theorem 2 and initialized with a stabilizing controller with the difference that $K_2(s)$ guarantees the performance for the whole polytope. On the other hand, it can be seen that the final results depends on the quality of the initial controller. In this example, an initialization using the FDRC toolbox seems to give the best results.

4. CONCLUSIONS

In this paper, the design of fixed-order stabilizing controllers for multivariable systems with polytopic parameter uncertainty in their state space representation is investigated. An inner convex approximation of fixed-order stabilizing controllers as a set of LMIs is given. The approach is based on the new definition of SPR-pair matrices that can help to decouple the Lyapunov matrix variables from the controller variables. It is shown that this concept can be applied to compute fixed-order H_∞ controllers for SISO polytopic systems. The convex approximation is based on the choice of a central state matrix. A method based on a set of initial stabilizing controllers to compute the central matrix is proposed. The simulation results have demonstrated the effectiveness of the proposed method. The extension of the proposed idea to design of fixed-order MIMO H_∞ and H_2 controllers is under investigation.

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