Capacities, Systoles and Jacobians of Riemann Surfaces

PAR

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To

Yanfei

and

Julian

Happy is the mathematician's lot.
The world forgetting, by the world forgot.
Each wish granted, each claim satisfied.
Eternal sunshine of the spotless mind.

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Abstract

To any compact Riemann surface of genus g one may assign a *principally polarized abelian variety* (PPAV) of dimension g, the *Jacobian* of the Riemann surface. The Jacobian is a complex torus and we call a Gram matrix of the lattice of a Jacobian a *period Gram matrix*. The aim of this thesis is to contribute to the Schottky problem, which is to discern the Jacobians among the PPAVs.

Buser and Sarnak approached this problem by means of a geometric invariant, the first successive minimum. They showed that the square of the first successive minimum, the squared norm of the shortest non-zero vector, in the lattice of a Jacobian of a Riemann surface of genus g is bounded from above by $\log(4g)$, whereas it can be of order g for the lattice of a PPAV of dimension g.

The main goal of this work was to improve this result and to get insight into the connection between the geometry of a compact Riemann surface that is given in hyperbolic geometric terms, and the geometry of its Jacobian. We show the following general findings :

1. For a hyperelliptic surface the first successive minimum is bounded from above by a universal constant.

2. The square of the second successive minimum of the Jacobian of a Riemann surface of genus g is equally of order $\log(g)$.

3. We provide refined upper bounds on the consecutive successive minima if the surface contains several disjoint small simple closed geodesics and a lower bound for the norm of certain lattice vectors of the Jacobian, if the surface contains small non-separating simple closed geodesics.

If the concrete geometry of the Riemann surface is known, more precise statements can be made. In this case we obtain theoretical and practical estimates on all entries of the period Gram matrix. Here we establish upper and lower bounds based on the geometry of the cut locus of simple closed geodesics and also on the geometry of Q-pieces.

In addition the following two results have been obtained :
First, an improved lower bound for the maximum value of the norm of the shortest non-zero lattice vector among all PPAVs in even dimensions. This follows from an averaging method from the geometry of numbers applied to a family of symmetric PPAVs.

Second, a new proof for a lower bound on the number of homotopically distinct geodesic loops, whose length is smaller than a fixed constant. This lower bound applies not only to geodesic loops on Riemann surfaces, but on arbitrary manifolds of non-positive curvature.

Keywords : Riemann surfaces, Jacobians, Schottky problem, principally polarized abelian varieties, harmonic functions, hyperbolic geometry, simple closed geodesics, collars.

Zusammenfassung

Jeder kompakten Riemannschen Fläche vom Geschlecht g kann eine *prinzipal polarisierte abelsche Varietät* (PPAV) der Dimension g zugeordnet werden. Diese Varietät ist ein komplexer Torus und wird *Jacobi-Varietät* der Riemannschen Fläche genannt. Wir nennen eine Gram-Matrix des Gitters einer Jacobi-Varietät *Perioden-Gram-Matrix*. Das Problem, die Jacobi-Varietäten von den übrigen PPAVs zu unterscheiden, wird Schottky Problem genannt.

Buser und Sarnak gelang eine solche Unterscheidung anhand einer geometrischen Invarianten, dem ersten sukzessiven Minimum. Sie zeigten, dass das Quadrat des ersten sukzessiven Minimums, das Quadrat der Länge des kürzesten Gittervektors im Gitter der Jacobi-Varietät einer Riemannschen Fläche vom Geschlecht g nach oben durch $\log(4g)$ beschränkt ist. Hingegen kann diese Invariante in der Menge der PPAVs der Dimension g von der Ordnung g sein.

Das Hauptziel dieser Arbeit war es, dieses Resultat zu verbessern und den Zusammenhang zwischen der Geometrie einer kompakten Riemannschen Fläche, gegeben als hyperbolische Mannigfaltigkeit, und der Geometrie seiner Jacobi-Varietät besser zu verstehen. Hierbei konnten wir folgende generelle Aussagen beweisen :

1. Das erste sukzessive Minimum einer hypereliptischen Fläche ist immer kleiner als eine feste Konstante.

2. Das Quadrat des zweiten sukzessiven Minimums der Jacobi-Varietät einer Riemannschen Fläche vom Geschlecht g ist ebenfalls von der Ordnung $\log(g)$.

3. Wir erhalten weitere obere Schranken für eine gewisse Anzahl sukzessiver Minima, falls die Fläche mehrere disjunkte kurze einfach geschlossene Geodäten enthält. Wir erhalten untere Schranken für die Länge bestimmter Gittervektoren, falls die Fläche mehrere kurze nicht-trennende einfach geschlossene Geodäten enthält.

Ist die konkrete Geometrie der Riemannschen Fläche bekannt, so können genauere Aussagen getroffen werden. In diesem Fall erhalten wir theoretische und praktische Abschätzungen für alle Einträge der Perioden-Gram-Matrix. Diese Schranken können mit Hilfe der Geometrie des Schnittorts einfach geschlossener Geodäten oder der Geometrie von Q-Stücken erhalten werden.

Wir zeigen weiterhin die zwei folgenden Resultate :
Wir erhalten erstens eine verbesserte untere Schranke für das Maximum der Länge des kürzesten Gittervektors über der Menge der PPAVs. Dieses Ergebnis folgt aus einer Mittelwertmethode aus der Geometrie der Zahlen angewandt auf eine Familie symmetrischer PPAVs.

Zweitens geben wir einen neuen Beweis für eine untere Schranke für die Anzahl von geodätischen Schleifen in unterschiedlichen Homotopieklassen, deren Länge kleiner als eine feste Konstante ist. Diese untere Schranke kann nicht nur auf Riemannsche Flächen sondern auch generell auf Mannigfaltigkeiten nicht-positiver Krümmung angewandt werden.

Schlüsselbegriffe : Riemannsche Flächen, Jacobi-Varietäten, Schottky Problem, prinzipal polarisierte abelsche Varietäten, harmonische Funktionen, hyperbolische Geometrie, einfach geschlossene Geodäten, Kragen.

Contents

Notation

General notation

$\bar{A}$	closure of A
∂A	boundary of A
$\mathbb{N}$	natural numbers (without 0)
$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	integers, real and complex numbers
C	continuous functions
C^k	k-times continuously differentiable functions
Lip	Lipschitz functions
$M_n(R)$	$n \times n$ matrices with coefficients in R
$O_n(\mathbb{R})$	$n \times n$ orthogonal matrices

Forms and functions

$\mathcal{E}$	infinitely real differentiable functions, 11
$\mathcal{E}^k$	vector space of k-forms, 11, 12
d	differentiation operator, 12,13
$\wedge$	exterior multiplication of forms, 12
$\bar{}$	complex conjugation operator, 13
$\mathrm{Re}(), \mathrm{Im}()$	real and imaginary part, 13
$*$	star operator, 14
Δ	Laplace operator, 14
Harm	harmonic functions, 14
Harm^1	harmonic 1-forms, 14
$\mathcal{O}^1$	holomorphic 1-forms, 14
$\langle , \rangle$	scalar product, 14
$E_U()$	energy over U, 15
cap	capacity, 25
$\mathrm{cap}_{\partial_2}$	capacity in direction ∂_2, 98

Chains, cycles and homology

Div	group of divisors, 15
deg	degree, 15
C_1, Z_1, H_1	group of 1-chains, cycles and homology group, 16
π_1	fundamental group, 16

Jacobians, PPAVs and lattices

Per	period lattice, 19
$J()$	Jacobian variety, 19, 22
$\mathcal{A}_g$	moduli space of PPAVs, 21
$\mathcal{H}_g$	Siegel upper half space, 21
$Sp_{2n}(R)$	symplectic group of degree $2n$ over R, 21
m_k	k-th successive minimum, 22

Introduction

The field of research of this thesis lies within differential and conformal geometry. The main goal is to contribute to the Schottky problem which is to characterize those principally polarized abelian varieties (PPAVs) that arise as Jacobians of compact Riemann surfaces (R.S.). Here we assume that all Riemann surfaces considered are endowed with the hyperbolic metric of constant curvature -1 and all compact surfaces are of genus $g \geq 2$.

The problem goes back to Schottky, who gave such a characterization in 1888 for the case of a compact Riemann surface of genus 4. Many authors have worked on this problem (see [De] for an overview). A full solution, however, has not been found until today. In 1994, Buser and Sarnak approached the Schottky problem by means of inequalities. In [BS] they describe a large region in the moduli or parameter space of all principally polarized abelian varieties of complex dimension g, of which it is shown that none occurs as a Jacobian.

The aim of this thesis is to improve this result and to get insight into the connection between the geometry of a compact Riemann surface that is given in hyperbolic geometric terms and the geometry of its Jacobian. To this end techniques from hyperbolic geometry, systolic geometry and harmonic functions are being used. The principal idea of this thesis is to find lower or upper bounds for the length of vectors of the Jacobian of a Riemann surface based on the geometry of the surface.

This dissertation is structured as follows :
In **Chapter 1** we will introduce the basic definitions related to the geometry of Riemann surfaces, especially its decomposition into pants, as described in [B]. Then we will present the basic facts about differential forms on Riemann surfaces. In this context we will introduce the homology basis and its relationship with harmonic differential forms. These relationships will enable us to define the Jacobian of a Riemann surface. Then we will introduce the principally polarized abelian varieties and their geometric invariants and show that the Jacobian of a Riemann surface is a PPAV.
Finally we will discuss the energy minimizing properties of harmonic functions and forms and in this context present some results from the calculus of variations (see [Ge]). These will later serve as tools to estimate the energy of certain harmonic forms.

In **Chapter 2** we will present the main results concerning the general relationship of the coarse geometry of a Riemann surface and the geometry of its Jacobian. First we will proof the following result about the first and second successive minimum of the Jacobian $J(S)$ of a R.S. S, $m_1(J(S))$ and $m_2(J(S))$, respectively :

Theorem 1 *Let S be a compact R.S. of genus g and let $J(S)$ be its Jacobian, then*

$$m_1(J(S))^2 \leq \log(4g-2) \quad and \quad m_2(J(S))^2 \leq 3.1\log(8g-7).$$

We show that in the case of a hyperelliptic surface we have :

Theorem 2 *If S is a hyperelliptic R.S. of genus g and $J(S)$ its Jacobian, then*

$$m_1(J(S))^2 \leq \frac{3\log(3+2\sqrt{3}+2\sqrt{5+3\sqrt{3}})}{\pi} = 2.4382... \quad .$$

If a Riemann surface contains small disjoint simple closed geodesics, we obtain :

Theorem 3 *Let S be a compact R.S. of genus g that contains n disjoint simple closed geodesics $(\eta_j)_{j=1,..,n}$ of length smaller than t. If we cut open S along these geodesics, then the decomposition contains m R.S. S_i of signature (g_i, n_i), with $g_i > 0$. There exist m linear independent vectors $(v_i)_{i=1,..,m}$ in the lattice of the Jacobian $J(S)$, such that*

$$\|v_i\|^2 \leq \frac{(n_i+1)\max\{4\log(4g_i+2n_i-3), t\}}{\pi - 2\arcsin(M)} \quad for \quad i \in \{1, ..., m\},$$

where $M = \min\{\frac{\sinh(\frac{t}{2})}{\sqrt{\sinh(\frac{t}{2})^2+1}}, \frac{1}{2}\}$.

The following related theorem will be shown in chapter 3. It provides estimates for the norm of two linear independent, primitive vectors v_1 and v_2 in the lattice of the Jacobian of a Riemann surface S, if S contains a small non-separating simple closed geodesic (scg) :

Theorem 4 *Let S be a R.S. of genus g that contains a small non-separating scg α_2 of length $l(\alpha_2)$ that has a collar $C(\alpha_2)$ of width w_2. Then there exists another scg α_1 that intersects α_2 in exactly one point and a canonical homology basis A with $\alpha_1, \alpha_2 \in A$. Let $(\sigma_i)_{i=1,..,2g}$ be the dual basis of harmonic forms for A. Then for the energy $E(\sigma_1)$ and $E(\sigma_2)$ we have :*

$$\|v_2\|^2 = E(\sigma_2) \geq \frac{\pi - 2\arcsin(\frac{1}{\cosh(w_2)})}{l(\alpha_2)} \quad and \quad \|v_1\|^2 = E(\sigma_1) \leq \frac{l(\alpha_2)}{\pi - 2\arcsin(\frac{1}{\cosh(w_2)})}.$$

Here the upper bound on $\|v_1\|^2$ was shown in [BS]. The last two theorems are related to a result of Fay. In [Fa], chap. III two sequence of Riemann surfaces S_t are constructed, where t denotes the length of a simple closed geodesic η.

In the first case η is a separating geodesic. It divides S_t into two surfaces S_i of signature $(g_i, 1), i \in \{1, 2\}$. If $t \to 0$ then the period Gram matrix for a suitable canonical homology basis converges to a block matrix, where each block is in $M_{2g_i}(\mathbb{R})$. **Theorem 3** shows that indeed the first two successive minima of the Jacobian of the surfaces S_i can only be of order $\log(g_i)$ and gives explicit bounds depending on the length of t.

In the second case a sequence of surfaces is constructed, where η is a non-separating simple close geodesic. If $t \to 0$, the length of one vector in the lattice of the Jacobian converges to zero. **Theorem 4** gives an explicit upper bound for the length of this lattice vector depending on t.

It shows furthermore that the length of a second vector in the lattice of the Jacobian goes to infinity and gives an explicit lower bound depending on t.

In **Chapter 3** we will give estimates for all entries of the period Gram matrix. Here we first give a theoretical upper bound of the entries based on test functions, which can be defined on the whole area of the surface via the cut locus of simple closed geodesics. A theoretical lower bound can be obtained by estimating the projection of the derivative of certain harmonic functions.
Using the same approach we will then establish lower and upper bounds for the entries of the period Gram matrix based on the geometry of Q-pieces.
Finally we will apply the methods developed in this chapter to surfaces that contain small non-separating geodesics and show **Theorem 4**.

While the previous chapters dealt with geometric invariants of the Jacobian of Riemann surfaces, **Chapter 4** deals with Hermite's invariant for PPAVs. We denote by $\mathcal{A}_g$ the moduli or parameter space of PPAVs of dimension g. Then Hermite's invariant for PPAVs, δ_{2g} is the maximal value of the shortest non-zero lattice vector or systole among the PPAVs in $\mathcal{A}_g$.
Inspired by the symmetry of known maximal PPAVs, the question that arose here was, whether symmetric PPAVs tend to have a larger systole in general than the average PPAV and if this can be used to improve the lower bound of δ_{2g}. This idea was pursued to improve the lower bound for δ_{2g} for PPAVs in even dimensions by applying a mean-value argument from the geometry of numbers to a subset of highly symmetric PPAVs. Following this proof we will present newly discovered families of highly symmetric PPAVs.

The result of **Chapter 5** arose from the search for a certain number of short simple closed geodesics on a Riemann surface. Such upper bounds were needed for the proofs in the second chapter. This part was inspired by a theorem of Blichfeldt [Bli] about lattices. We generalize this theorem in the following way :

Theorem 5 *Let M be a manifold of non-positive curvature with* $\mathrm{vol}(M) < \infty$. *Let K be its universal covering and $p : K \to M$ a covering map. If $C \subset K$ is a convex set, such that for a $m \in \mathbb{N}$*

$$\mathrm{vol}(C) > m \, \mathrm{vol}(M),$$

then there exist pairwise distinct points $x_1, ..., x_m \in C$ that map to the same point y in M. The geodesic arcs $(\gamma_{x_i, x_1}) \subset C$ map to $m-1$ homotopically distinct loops $p(\gamma_{x_i, x_1})$ with base point y.

Setting $C = B_r(x)$, where $B_r(x)$ is a geodesic ball of radius r and center x in K, we obtain a lower bound on the number of homotopically distinct geodesic loops, whose length is smaller than a fixed constant.

The results of chapter 2 have been submitted in [Mu].

Chapter 1

Preliminaries

In this chapter we will first introduce the basic definitions related to the geometry of Riemann surfaces. Then we will gather the basic facts about differential forms on Riemann surfaces. In this context we will introduce the homology basis and its relationship with harmonic differential forms. These relationships will enable us to define the Jacobian of a Riemann surface. Afterwards we will introduce the principally polarized abelian varieties (PPAVs) and their main geometric invariants and prove that the Jacobian of a Riemann surface is a PPAV.

Finally, we will discuss some energy-minimizing properties of harmonic functions and forms and present some results from the calculus of variations. These will later serve as tools to estimate the energy of harmonic forms dual to a homology basis.

We will assume in the following chapters that a *surface* is always a connected two-dimensional differentiable manifold.

1.1 Hyperbolic geometry and Riemann surfaces

In this section we first collect the basic facts about the hyperbolic plane and hyperbolic geometry and introduce the Fermi coordinates. Then we give a suitable definition of Riemann surfaces and introduce the Fenchel Nielsen coordinates. A basic reference for this section is [B].

Hyperbolic plane

The Poincare model of the hyperbolic plane is the following subset of the complex plane $\mathbb{C}$:

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

with the hyperbolic metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

The group

$$PSL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}; ad - bc = 1 \right\} \mod \{\pm 1\}$$

acts biholomorphically on $\mathbb{H}$ via the mappings

$$z \mapsto \frac{az + b}{cz + d}.$$

It is the whole group of orientation preserving isometries of $\mathbb{H}$, $\mathrm{Isom}^+(\mathbb{H}) \simeq PSL(2,\mathbb{R})$.

The set of geodesics, $\mathrm{Geod}(\mathbb{H})$, in $\mathbb{H}$ are straight lines parallel to the y-axis and half-circles with center on the x-axis.

The distance between two points $p = p_1 + ip_2$ and $q = q_1 + iq_2$ in this model is given by the following formula :

$$\cosh(\mathrm{dist}(p,q)) = 1 + \frac{\|p - q\|^2}{2p_2 q_2}.$$

Fermi coordinates and cylinders

Fermi coordinates ψ_γ with base line γ and base point p are defined in the following way. The Fermi coordinates are a bijective parametrization of $\mathbb{H}$

$$\psi_\gamma : \mathbb{R}^2 \to \mathbb{H}, \psi_\gamma : (t,s) \mapsto \psi_\gamma(t,s),$$

where $\psi_\gamma(0,0) = p$. Each point $q = \psi_\gamma(t,s) \in \mathbb{H}$ can be reached in the following way. Starting from the base point p we first move along γ the directed distance t to $\psi_\gamma(t,0)$. There is a unique geodesic, ν, intersecting γ perpendicularly in $\psi_\gamma(t,0)$. From $\psi_\gamma(t,0)$ we now move along ν the directed distance s to $\psi_\gamma(t,s)$.

The parametrization $\psi : \mathbb{R}^2 \to \mathbb{H}$

$$\psi(t,s) := \frac{\exp(t)}{\cosh(s)}(\sinh(s) + i)$$

are the Fermi coordinates with base line $\{iy \mid y \in \mathbb{R}_+\}$ and base point i. Fermi coordinates for any geodesic in $\mathbb{H}$ can be obtained by conjugation of $\psi(t,s)$ with the suitable isometries.

A *hyperbolic cylinder* or shortly *cylinder* is a set that is isometric to the set

$$\{\psi(t,s) \mid (t,s) \in [0,a] \times [b_1, b_2]\} \mod \{\psi(0,s) = \psi(a,s) \mid s \in [b_1, b_2]\},$$

with the induced metric from $\mathbb{H}$.

Horocyclic coordinates and cusps

Consider the parametrization $\eta : \mathbb{R}^2 \to \mathbb{H}$

$$\eta(t,s) := \exp(-t)(-\mathrm{sign}(s)\sqrt{2(\cosh(s) - 1)} + i),$$

where $\eta(0,0) = i$. Each point $q = \eta(t,s) \in \mathbb{H}$ can be reached in the following way. Starting from the base point i we first we move along the geodesic $\gamma = \{iy \mid y \in \mathbb{R}_+\}$ the directed distance t to $\eta(t,0)$. There is a unique Euclidean straight line, ν, intersecting γ perpendicularly in $\eta(t,0)$. From $\eta(t,0)$, we now move along ν to the point $q = \eta(t,s)$, situated at the directed distance s from $\eta(t,0)$. These are the *horocyclic coordinates with base line $\{iy \mid y \in \mathbb{R}_+\}$, starting point i and "base point ∞i"*. The horocyclic coordinates for any base line γ in $\mathbb{H}$ can be obtained by conjugation of $\eta(\cdot)$ with the suitable isometries.

A *cusp* is a set that is isometric to the set

$$\{x + iy \in \mathbb{H} \mid (x,y) \in [-a,a] \times [b,\infty[\} \mod \{-a + it = a + it \mid t \in [b,\infty[\},$$

with the induced metric from $\mathbb{H}$. Though it is never reached, we call the limit ∞i or its equivalent a *cusp point* or also *cusp*.

Hyperbolic trigonometry

Hyperbolic trigonometry is a basic tool in almost all chapters of this thesis. We will give here the formulas for the geometry of the most important polygons used in the following :
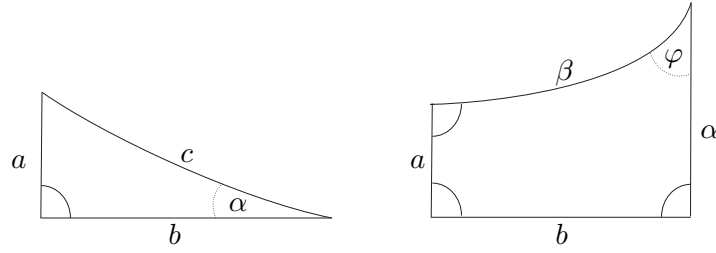
Figure 1.1: Right-angled triangle and trirectangle

1.) Right-angled triangles

$$\cosh(c) = \cosh(a)\cosh(b) \quad and \quad \sinh(a) = \sin(\alpha)\sinh(c)$$

2.) Trirectangles

$$\cosh(a) = \tanh(\beta)\coth(b) \quad and \quad \cosh(a) = \cosh(\alpha)\sin(\varphi)$$
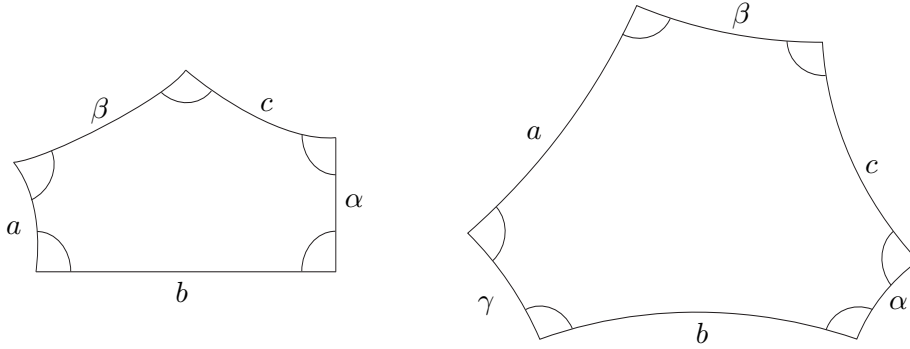$$\sinh(\alpha) = \sinh(a)\cosh(\beta) \quad and \quad \sinh(a) = \coth(b)\cot(\varphi)$$

Figure 1.2: Right-angled pentagon and hexagon

3.) Right-angled pentagons

$$\cosh(c) = \sinh(a)\sinh(b) \quad and \quad \cosh(c) = \coth(\alpha)\coth(\beta)$$

4.) Right-angled hexagons

$$\cosh(c) = \sinh(a)\sinh(b)\cosh(\gamma) - \cosh(a)\cosh(b)$$
$$\coth(\alpha)\sinh(\gamma) = \cosh(\gamma)\cosh(b) - \coth(a)\sinh(b)$$

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We will also allow the limit case of a *degenerated right-angled hexagon*, where one boundary arc has length zero and the limit case, where two boundary arcs, which are connected by exactly one other arc have length zero.

Riemann surfaces

Depending on the approach towards Riemann surfaces, there are many different possibilities for a definition. We will choose a definition that allows us to treat all surfaces, which will occur in the following chapters. We will use the definition of a Riemann surface as hyperbolic surface. We will also allow local charts that are isometric to a cusp. This subsection is based on the definitions in [B], but also introduces some additional notation.

A hyperbolic surface with a cusp can be seen as a limit case of a hyperbolic surface with boundary. Here the length of one boundary goes to zero in the limit case. Therefore we will also call a cusp point of a surface a boundary of the surface, though we are aware that this is incorrect. We define :

Definition 1.1.1 *A Riemann surface S is a complete orientable hyperbolic surface of finite volume. If it has a boundary, then each boundary component is either a smooth simple closed geodesic or a cusp point.*

It follows from the topological classification of compact oriented surfaces that a Riemann surface without boundary is topologically a Pretzel surface with g holes, where g denotes the genus. We further define the signature of a surface in the following way :

Definition 1.1.2 *A Riemann surface S has signature (g, n), if its genus is g and the boundary consists of n disjoint boundary geodesics. We say S has signature (g, n, m), if its genus is g and the boundary consists of n disjoint boundary geodesics and m cusps. If a boundary is a cusp, we say that the boundary is degenerated.*

Y-pieces and Fenchel-Nielsen coordinates

An important type of Riemann surfaces are Y-pieces. Any Riemann surface can be decomposed into or built from these basic building blocks.

Definition 1.1.3 *A R.S. Y of signature $(0, 3)$, $(0, 1, 2)$ or $(0, 2, 1)$ is called a Y-piece. If Y is of signature $(0, 1, 2)$ or $(0, 2, 1)$ then we say that the Y-piece is degenerated.*

Every Y-piece can be obtained by pasting two isometric hexagons. If we paste together two degenerated hexagons, we obtain a Y-piece with one or two cusps.
If Y is a Y-piece with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$, then we can introduce a marking on Y. The marking consists in labeling the boundary components to obtain the *marked Y-piece* $Y[\gamma_1, \gamma_2, \gamma_3]$. For a marked Y-piece $Y[\gamma_1, \gamma_2, \gamma_3]$ we introduce a standard parametrization of the boundaries in the following way.
Let c_{ij} be the geodesic arc going from γ_i to γ_j that meets these boundaries perpendicularly. We set $\mathbb{S}^1 = \mathbb{R} \mod (t \mapsto t + 1)$ and parametrize all boundary geodesics

$$\gamma_i : \mathbb{S}^1 \to \mathbb{H}, \gamma_i : t \mapsto \gamma_i(t),$$

such that each geodesic is traversed once and with the same orientation. We furthermore parametrize the geodesics, such that $\gamma_1(0)$ is the endpoint of c_{31}, $\gamma_2(0)$ is the endpoint of c_{12} and $\gamma_3(0)$ is the endpoint of c_{23} (see *Fig. 1.3*).
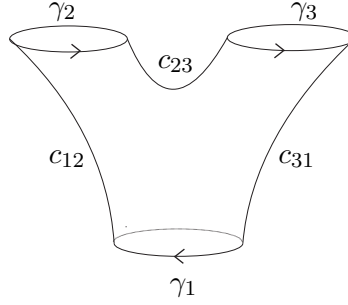

Figure 1.3: A marked Y-piece $Y[\gamma_1, \gamma_2, \gamma_3]$

Two marked Y-pieces Y and Y' that have a boundary geodesic of the same length can be pasted together. If $\gamma_1 \subset Y$ and $\gamma_1' \subset Y'$ are the geodesics of equal length, then we can glue Y and Y' using the identification

$$\gamma_1(t) = \gamma_1'(-t + \tau), t \in \mathbb{S}^1,$$

where $\tau \in \mathbb{R}$ is an additional constant, called the *twist parameter*. We obtain the surface

$$Y + Y' \mod (\gamma_1(t) = \gamma_1'(-t + \tau), t \in \mathbb{S}^1).$$

Here the gluing is restricted to boundaries of non-zero length.

It can be shown that any Riemann surface S can be obtained by gluing together marked Y-pieces. The pasting scheme can be encoded in a graph in the following way. We assign to each Y-piece a vertex and to each of its boundary geodesics a half-edge emanating from this vertex. Pasting along a boundary geodesic corresponds to connecting two half-edges of two different vertices or the same vertex. The pasting scheme is thus encoded in the combinatorial skeleton of a graph $G(S)$. This construction is shown in Figure 1.4.

Every Riemann surface S of signature (g, n, m) can be built from $2g - 2 + n + m$ Y-pieces. The pasting scheme is provided by a graph $G(S)$. The triple

$$((s_i)_{i=1,..,2g-2+n+m}, (b_j)_{j=1,..,3g-3+n+m}, (b_j')_{j=1,..,n+m})$$

lists the $2g - 2 + n + m$ vertices, $3g - 3 + n + m$ edges and $n + m$ half-edges, where the half-edges $(b_j')_{j=n+1,..,n+m}$ correspond to boundaries that are cusp points. Let

$$L(S) = (l_1, ..., l_{3g-3+n+m}) \times (l_1', ..., l_n') \in \mathbb{R}_+^{3g-3+n+m} \times \mathbb{R}_+^n$$

such that each l_i is the length of a geodesic corresponding to the edge b_i and each l_i' the length of a boundary geodesic corresponding to a half-edge b_i' of non-zero length. Let

$$A(S) = (a_1, ..., a_{3g-3+n+m}) \in \mathbb{R}^{3g-3+n+m}$$

be such that a_i is twist parameter from the gluing of the geodesics corresponding to an edge b_i. Then any Riemann surface S can be constructed from the information privided in the triplet $(G(S), L(S), A(S))$.
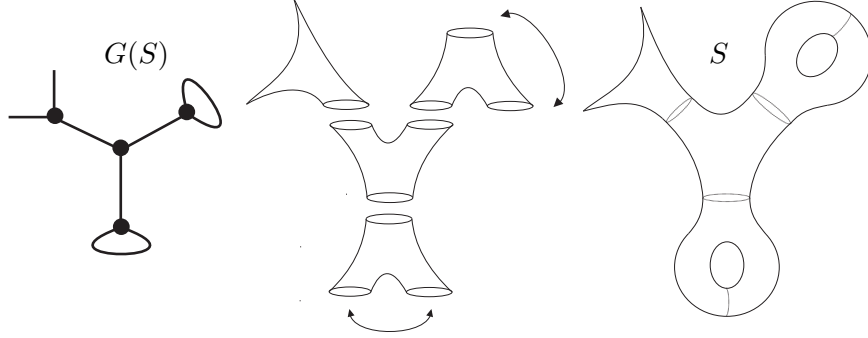
Figure 1.4: Construction of a R.S. S of signature $(2, 0, 2)$ from a graph $G(S)$ and marked Y-pieces

Definition 1.1.4 *(L(S),A(S)) is the sequence of Fenchel-Nielsen coordinates of the Riemann surface S.*

1.2 Differential forms on Riemann surfaces

Differential forms allow us to define integration, such that the result is independent of the choice of the coordinates. We use 1-forms to integrate over curves and 2-forms to integrate over domains.

A Riemann surface in the classical sense is a complex one-dimensional manifold. We will introduce differential forms based on this definition. The connection with hyperbolic surfaces as defined in the previous section is the following.

A classical Riemann surface is a surface with a complex structure, such that the coordinate changes are biholomorphic. As the coordinate changes are holomorphic functions, the surface is orientable. It follows from the general classification of orientable surfaces that any compact Riemann surface is a genus g surface.

It follows furthermore from the Klein-Koebe-Poincare uniformization theorem (see [Po]) that for every compact Riemann surface S of genus $g \geq 2$ we have :

$$S \simeq \mathbb{H} \mod \Gamma, \quad \text{where} \quad \Gamma \subset PSL(2, \mathbb{R})$$

is a fixed point free, discrete subgroup that acts properly and discontinuously on $\mathbb{H}$. As $PSL(2, \mathbb{R})$ is also the group of orientation preserving isometries of $\mathbb{H}$, $\text{Isom}^+(\mathbb{H})$, we can endow S with the hyperbolic metric in a natural way to obtain a hyperbolic surface. As the metric is conformal, the definition of harmonic and holomorphic functions in the hyperbolic plane $\mathbb{H}$ is the usual one.

Finally, as the coordinate changes of a hyperbolic surface of genus g are in $PSL(2, \mathbb{R})$, which is a subgroup of the holomorphic functions on $\mathbb{H}$, any compact orientable hyperbolic surface is a Riemann surface in the classical sense and our definitions coincide in the case of compact Riemann surfaces of genus g.

As we only use differential forms in this setup, we will use the classical definition of a Riemann

surface in the following two sections. Basic references for these sections are [FK] and [Fo].

1-forms

If $U \subset \mathbb{C}$ is open, we can identify $\mathbb{C} \simeq \mathbb{R}^2$ and consider

$$\mathcal{E}(U) = \{f : U \to \mathbb{C} \mid f \text{ infinitely real differentiable}\}.$$

For any $z \in U$, such that $z = x + iy \simeq (x,y)$ and a $f \in \mathcal{E}(U)$, such that $f(z) = u(z) + iv(z) \simeq u(x,y) + iv(x,y)$, set

$$\frac{\partial f}{\partial x} = \frac{\partial u(x,y)}{\partial x} + i \frac{\partial v(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u(x,y)}{\partial y} + i \frac{\partial v(x,y)}{\partial y}.$$

Then $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are differential operators on $\mathcal{E}(U)$. We further consider the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It follows from the Cauchy-Riemann differential equations for all $f \in \mathcal{E}(U)$ that $f \in \mathcal{O}(U) \Leftrightarrow f \in \mathrm{Ker}\left(\frac{\partial f}{\partial \bar{z}}\right)$.
If S is a Riemann surface, then

$$\mathcal{E}(S) = \{f : S \to \mathbb{C} \mid f \circ \varphi^{-1} \in \mathcal{E}(\varphi(U)) \text{ for all local charts } (U, \varphi)\}.$$

We define 1-forms in the following way.

Definition 1.2.1 *Let S be a Riemann surface. A 1-form ω on S is an ordered assignment of two functions $f_1, f_2 \in \mathcal{E}(U)$, to each local chart (U, φ), $\varphi = u_1 + iu_2$, such that*

$$\omega = f_1 du_1 + f_2 du_2$$

is invariant under coordinate changes. We denote by $\mathcal{E}^1(S)$ the vector space of 1-forms.

That means that if (U, φ) and (V, ψ) with $\psi = v_1 + iv_2$ are local charts that intersect, such that on the intersection $U \cap V$

$$\omega = f_1 du_1 + f_2 du_2 = h_1 dv_1 + h_2 dv_2,$$

then we have

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial u_1 \circ \psi^{-1}}{\partial x}\right|_{\psi(\cdot)} & \left.\frac{\partial u_1 \circ \psi^{-1}}{\partial y}\right|_{\psi(\cdot)} \\ \left.\frac{\partial u_2 \circ \psi^{-1}}{\partial x}\right|_{\psi(\cdot)} & \left.\frac{\partial u_2 \circ \psi^{-1}}{\partial y}\right|_{\psi(\cdot)} \end{pmatrix}^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left(\left.D(\varphi \circ \psi^{-1})\right|_{\psi(\cdot)}\right)^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $D(\varphi \circ \psi^{-1})$ is the real Jacobian matrix of the function $\varphi \circ \psi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ (with $\mathbb{R}^2 \simeq \mathbb{C}$). Using complex notation for forms on a chart (U, φ), with $\varphi = u_1 + iu_2$, we can write

$$d\varphi = du_1 + idu_2 \quad \text{and} \quad d\bar{\varphi} = du_1 - idu_2.$$

We may also differentiate functions in $\mathcal{E}(S)$ to obtain forms. To this end we define the operator d locally on a chart (U, φ) :

$$d : \mathcal{E}(U) \to \mathcal{E}^1(U), d : f \mapsto df, \quad \text{such that}$$

$$df := \left. \frac{\partial f \circ \varphi^{-1}}{\partial x} \right|_{\varphi(\cdot)} du_1 + \left. \frac{\partial f \circ \varphi^{-1}}{\partial y} \right|_{\varphi(\cdot)} du_2 = \frac{df}{dx} du_1 + \frac{df}{dy} du_2.$$

Here we use the symbols $\frac{d}{dx}$ and $\frac{d}{dx}$ to distinguish these expressions from the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x}$. The operator d can be extended naturally to an operator $d : \mathcal{E}(S) \to \mathcal{E}^1(S)$.

A 1-form can be integrated over paths. If $c : [a, b] \to S$ is a path and $\omega \in \mathcal{E}^1(S)$, then we can define integration locally on a chart $(U, \varphi = u_1 + iu_2)$, with $\omega = f_1 du_1 + f_2 du_2$ and $c|_U : [a_1, b_1] \to U, c : t \mapsto c(t)$:

$$\int\limits_{c|_U} \omega = \int\limits_{a_1}^{b_1} \left(f_1(c(t)) \frac{\partial u_1 \circ c(t)}{\partial t} + f_2(c(t)) \frac{\partial u_2 \circ c(t)}{\partial t} \right) dt.$$

As there always exists a partition of the interval $[a, b]$, such that each subinterval is contained in a single chart, the integration can be easily extended to S.

2-forms

A 2-form can be defined in the following way.

Definition 1.2.2 *Let S be a Riemann surface. A 2-form ω on S is an assignment of a function $f \in \mathcal{E}(U)$, to each local chart (U, φ), $\varphi = u_1 + iu_2$, such that*

$$\omega = f du_1 \wedge du_2$$

is invariant under coordinate changes. We denote by $\mathcal{E}^2(S)$ the vector space of 2-forms.

That means that if (U, φ) and (V, ψ) with $\psi = v_1 + iv_2$ are local charts that intersect, such that on the intersection $U \cap V$

$$\omega = f \, du_1 \wedge du_2 = h \, du_1 \wedge du_2,$$

then we have

$$h = f \cdot \det \left(D(\varphi \circ \psi^{-1})\big|_{\psi(\cdot)} \right) = f \cdot \left\| (\varphi \circ \psi^{-1})'\big|_{\psi(\cdot)} \right\|^2,$$

where $'$ denotes the complex differentiation.
The *exterior multiplication of forms* $\wedge : \mathcal{E}^1(U) \times \mathcal{E}^1(U) \to \mathcal{E}^2(S)$ satisfies the following rules. For $\omega_1, \omega_2, \omega_3 \in \mathcal{E}^1(U)$ and $\lambda \in \mathbb{C}$, we have

$$
\begin{aligned}
(\omega_1 + \omega_2) \wedge \omega_3 &= \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \\
(\lambda \omega_1) \wedge \omega_2 &= \lambda(\omega_1 \wedge \omega_2) \\
\omega_1 \wedge \omega_2 &= -\omega_2 \wedge \omega_1.
\end{aligned}
$$

Hence, using complex notation for forms on a chart (U, φ), with $\varphi = u_1 + iu_2$, we can write

$$d\varphi \wedge d\bar{\varphi} = -2i \, du_1 \wedge i du_2.$$

We can derive 1-forms in $\mathcal{E}^1(S)$ to obtain 2-forms. Therefore we define the operator d locally on a chart (U, φ), $\varphi = u_1 + iu_2$:

$$d : \mathcal{E}^1(U) \to \mathcal{E}^2(U), d : \omega \mapsto d\omega, \quad \text{such that}$$
$$d\omega = d(f_1 du_1 + f_2 du_2) := df_1 \wedge du_1 + df_2 \wedge du_2.$$

It follows from the rules for the exterior multiplication that

$$d\omega = \left(\frac{df_2}{dx} - \frac{df_1}{dy} \right) du_1 \wedge du_2.$$

As in the case of differentiation of functions, the operator d then can be extended naturally to an operator $d : \mathcal{E}^1(S) \to \mathcal{E}^2(S)$.

A 2-form can be integrated over domains $D \subset S$. If $\omega \in \mathcal{E}^2(S)$, then we can define integration locally over a chart $(U, \varphi = u_1 + iu_2)$, $\omega = f du_1 \wedge du_2$:

$$\int_U \omega = \int_{\varphi(U)} f \circ \varphi^{-1}(x, y)\, dx\, dy,$$

where $(x, y) \in \mathbb{R}^2 \simeq x + iy \in \mathbb{C}$. The integration can be extended over any domain D, using partitions of unity. The most important theorem concerning integration and forms is the theorem of Stokes (see [Fo], p. 78), which states :

Theorem 1.2.3 *(Stokes) If S is a Riemann surface and $D \subset S$ is a domain with piecewise differentiable boundary ∂D, then for every 1-form $\omega \in \mathcal{E}^1(S)$, we have*

$$\int_{\partial D} \omega = \int_D d\omega.$$

We mention also the version for functions :
If $c : [a, b] \to U \subset S$ is a curve and $\omega = df$, with $f \in \mathcal{E}(U)$, then

$$\int_c \omega = \int_c df = f(c(b)) - f(c(a)).$$

Operators on forms

The *complex conjugation operator* $^- : \mathcal{E}^1(S) \to \mathcal{E}^1(S)$ is defined locally in the following way. On a chart (U, φ), $\varphi = u_1 + iu_2$ and $\omega \in \mathcal{E}^1(S)$, with $\omega = f_1 du_1 + f_2 du_2$ we define the operator $^-$ by

$$\bar{\omega} := \bar{f}_1 du_1 + \bar{f}_2 du_2.$$

The *real* and *imaginary part* of ω, $\text{Re}(\omega)$ and $\text{Im}(\omega)$ are

$$\text{Re}(\omega) = \frac{1}{2}(\omega + \bar{\omega}) \quad \text{and} \quad \text{Im}(\omega) = \frac{1}{2i}(\omega - \bar{\omega}).$$

A 1-form is called *real*, if $\omega = \text{Re}(\omega) \Leftrightarrow \omega = \bar{\omega}$.

We introduce furthermore the *star operator* $^* : \mathcal{E}^1(S) \to \mathcal{E}^1(S)$ for 1-forms as follows. We define the operator * locally on a chart (U, φ), $\varphi = u_1 + iu_2$ and $\omega \in \mathcal{E}^1(S)$, with $\omega = f_1 du_1 + f_2 du_2$ by

$$^*\omega := -f_2 du_1 + f_1 du_2.$$

Both operators can be extended naturally to operators on $\mathcal{E}^1(S)$. We will use a definition of harmonic functions based on the star operator.

Definition 1.2.4 *Let S be a Riemann surface and $f \in \mathcal{E}(S)$, then f is called harmonic, if*

$$d^*df = 0.$$

The operator $d^*d = \Delta : \mathcal{E}(S) \to \mathcal{E}^2(S)$ is called the Laplace-operator .
A 1-form $\sigma \in \mathcal{E}^1(S)$ is called harmonic, if locally

$$\sigma = df, \quad \text{such that } f \text{ is a harmonic function.}$$

A 1-form $\omega \in \mathcal{E}^1(S)$ is called holomorphic, if

$$\omega = \sigma + i^*\sigma, \quad \text{such that } \sigma \text{ is a harmonic 1-form.}$$

We denote by $\mathrm{Harm}(S)$ the *vector space of harmonic functions* and by $\mathrm{Harm}^1(S)$ the *vector space of harmonic 1-forms* and by $\mathcal{O}^1(S)$ the *space of holomorphic 1-forms*.
$\mathcal{E}^1(S)$ can be decomposed into different parts, using the following definitions :

Definition 1.2.5 *Let S be a Riemann surface and $\omega \in \mathcal{E}^1(S)$, then ω is called exact, if*

$$\omega = df \quad \text{for some} \quad f \in \mathcal{E}(S).$$

It is called co-exact, if $^*\omega$ is exact.
ω is called closed, if $d\omega = 0$. It is called co-closed, if $^*\omega$ is closed.

If a domain D is simply connected, we have the following useful theorem :

Theorem 1.2.6 *Let S be a Riemann surface and $D \subset S$ be a simply connected domain and $\omega \in \mathcal{E}^1(D)$, such that $d\omega = 0$ then there exists a function $f \in \mathcal{E}(D)$, such that*

$$df = \omega.$$

Furthermore the function is unique up to a constant c.

For a proof see [Fo], p. 72. From **Theorem 1.2.6** and the definitions it is easy to deduce that a 1-form is harmonic if and only if it is closed and co-closed.

With the help of these two operators, we can define a scalar product

$$\langle \, , \rangle : \mathcal{E}^1(S) \times \mathcal{E}^1(S) \to \mathbb{C}, \langle \, , \rangle : (\omega_1, \omega_2) \mapsto \langle \omega_1, \omega_2 \rangle, \quad \text{such that}$$
$$\langle \omega_1, \omega_2 \rangle := \int_S \omega_1 \wedge {}^* \bar{\omega}_2.$$

$(\mathcal{E}^1(S), \langle\,,\rangle)$ is a unitary vector space, but not a Hilbert space. For a given 1-form $\omega \in \mathcal{E}^1(U)$, we denote by

$$E_U(\omega) = \int\limits_U \omega \wedge^* \bar{\omega}$$

the *energy of ω over U*. If $U = S$, then we call $E_S(\omega) = E(\omega)$ simply the *energy of ω*. If $f : S \to \mathbb{C}$ is a differentiable function, we denote by $E_U(f) = E_U(df)$ the *energy of f over U*. Later we will show that harmonic functions and differentials have certain energy-minimizing properties.

We can define orthogonality with the help of the scalar product. As shown in [Fo], p. 157, we have :

Theorem 1.2.7 *On any Riemann surface S, there is an orthogonal decomposition, such that*

$$\mathcal{E}^1(S) = d\mathcal{E}(S) \oplus {}^*d\mathcal{E}(S) \oplus \mathrm{Harm}^1(S)$$

Furthermore, if $d : \mathcal{E}^1(S) \to \mathcal{E}^2(S)$ denotes the differential operator on 1-forms, then

$$\mathrm{Ker}(d) = d\mathcal{E}(S) \oplus \mathrm{Harm}^1(S).$$

1.3   Homology and harmonic 1-forms

In this section, we will first introduce the first homology group $H_1(S)$. There are different, equivalent ways to define $H_1(S)$. For our purposes it is useful to define this group via integration over closed 1-forms. Another approach, using triangulations of surfaces, is given in [FK]. Then we will present the relationship between harmonic 1-forms and the homology group. These relationships will then enable us to define the Jacobian of a Riemann surface.

Chains, cycles and homology

Let S be a Riemann surface. A *divisor* on S is a mapping

$$D : S \to \mathbb{Z},$$

such that for each compact subset $K \subset S$, there are only finitely many points $x \in K$, such that $D(x) \neq 0$. The *group of divisors on S* is denoted by $\mathrm{Div}(S)$.

If S is a compact Riemann surface, then we can define a map, called *degree*, by

$$\deg : \mathrm{Div}(S) \to \mathbb{Z}, \deg : D \mapsto \sum_{\{x \in S | D(x) \neq 0\}} D(x).$$

We denote by $\mathrm{Div}_0(S) = \mathrm{Ker}(\deg)$ the kernel of the mapping.

A *1-chain c* on a compact Riemann surface S is a formal finite linear combination of curves $(c_j)_{j=1,\dots k}$, where $c_j : [0, 1] \to S$:

$$c = \sum_{j=1}^{k} n_j c_j \text{ ,where } n_j \in \mathbb{Z}.$$

We denote by $C_1(S)$ the *set of all 1-chains on S*. It is easy to see that $C_1(S)$ is an abelian group. If $\omega \in \mathcal{E}^1(S)$ is a closed 1-form ($d\omega = 0$), then we define the integration over c by

$$\int_c \omega := \sum_{j=1}^k n_j \int_{c_j} \omega.$$

The *boundary operator* $\partial : C_1(S) \to \mathrm{Div}(S)$ is defined in the following way. If $c_j : [0,1] \to S$, then set

$$\partial c_j = 0 \quad \text{if} \quad c_j(0) = c_j(1) \quad \text{and} \quad \partial c_j(x) = \left\{ \begin{array}{ccc} 1 & & x = c_j(1) \\ -1 & \text{if} & x = c_j(0) \\ 0 & & x \in S \backslash \{c_j(0), c_j(1)\} \end{array} \right. .$$

For a 1-chain $c = \sum_{j=1}^k n_j c_j$, set $\partial c := \sum_{j=1}^k n_j \partial c_j$.

We have $\deg(\partial c) = 0$ for all $c \in C_1(S)$, hence $\partial(C_1(S)) \subset \mathrm{Div}_0(S)$. Conversely, if $D \in \mathrm{Div}_0(S)$, then we can find a finite set of curves $(c_j)_{j=1,\ldots k}$ with start and endpoints in the set $\{x \in S \mid D(x) \neq 0\}$, such that $\partial \left(\sum_{j=1}^k c_j \right) = D$. Hence the mapping is surjective. It follows from the first isomorphism theorem for groups that

$$\mathrm{Div}_0(S) \simeq C_1(S) \mod \mathrm{Ker}(\partial)$$

Denote by $Z_1(S)$ the kernel of the mapping ∂ :

$$Z_1(S) := \mathrm{Ker}(\partial).$$

$Z_1(S)$ is called the *group of 1-cycles* or *group of cycles of S*. An element of $Z_1(S)$ is called a *cycle*. In particular every closed curve is a cycle.

We define an equivalence relation $\overset{\mathrm{hom}}{\sim}$ on two cycles $c, c' \in Z_1(S)$ in the following way :

$$c \overset{\mathrm{hom}}{\sim} c' \Leftrightarrow \int_c \omega = \int_{c'} \omega \text{ for all } \omega, \text{ such that } d\omega = 0$$

If $c \overset{\mathrm{hom}}{\sim} c'$, then we say that c and c' are *homologous*. The set of all homology classes of cycles is the additive group $H_1(S)$, the *(first) homology group of S*.

$$H_1(S) := Z_1(S) \mod \overset{\mathrm{hom}}{\sim}.$$

The integration along elements of $H_1(S)$ over closed forms is well-defined. It is easy to see that there is a surjective group homomorphism F from the fundamental group of S, $\pi_1(S)$ into the first homology group of S :

$$F : \pi_1(S) \to H_1(S).$$

In fact $H_1(S)$ is the abelianization of $\pi_1(S)$, which is in general not abelian. Due to this relationship, we will assume in the following that an element in the homology group is given as a closed curve. If S is a Riemann surface of genus g, then

$$H_1(S) \simeq \mathbb{Z}^{2g}.$$

Homology and harmonic 1-forms

The connection between harmonic forms and homology classes is given in the following theorem, which is proven on p. 163 in [Fo].

Theorem 1.3.1 *Let S be a Riemann surface and $a \in H_1(S)$ be a cycle, then there exists a unique harmonic form $\sigma_a \in \mathrm{Harm}^1(S)$, such that for any closed 1-form $\omega \in \mathrm{Ker}(d)$*

$$\int_a \omega = \int_S \sigma_a \wedge \omega$$

To establish the duality between the homology group and (real) harmonic forms, we will use canonical homology bases.

Definition 1.3.2 *Let S be a Riemann surface of genus g. $(\alpha_i)_{i=1\ldots 2g} \subset H_1(S)$ is a canonical homology basis (chb), if each α_i is a simple closed curve and the curves are paired, such that for each α_i there exists exactly one $\alpha_{\tau(i)} \in (\alpha_j)_{j=1\ldots 2g}$ that intersects α_i in exactly one point and there are no other intersection points.*

We denote by the *standard form of a canonical homology basis* $(\alpha_i)_{i=1\ldots 2g} \subset H_1(S)$ the ordering $(\alpha_1, \ldots, \alpha_g, \alpha_{\tau(1)}, \ldots, \alpha_{\tau(g)})$, such that the matrix of oriented intersections or *intersection matrix* has the form

$$\begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix} := I. \tag{1.1}$$
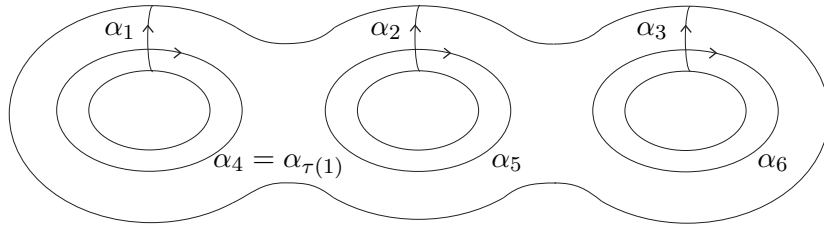
Figure 1.5: A canonical homology basis in standard form for a Riemann surface S of genus 3

An example of a chb in standard form is shown in Figure 1.5. We may assume in the following that a canonical homology basis is always in standard form. We have :

Theorem 1.3.3 *Let S be a Riemann surface of genus g and $(a_i)_{i=1\ldots 2g} \subset H_1(S)$ be a (canonical) homology basis. There exists an unique dual basis of harmonic forms for $\mathrm{Harm}^1(S)$, $(\sigma_k)_{k=1\ldots 2g}$, such that*

$$\int_{a_i} \sigma_k = \delta_{ik}.$$

Furthermore these harmonic forms are real.

proof This is usually stated and shown for a canonical homology basis (see for example [FK], p. 58). However that the basis is canonical is not essential. We will show that the result is true for any homology basis.

Let $(\alpha_i)_{i=1\ldots2g} \subset H_1(S)$ be a canonical homology basis and $(a_i)_{i=1\ldots2g} \subset H_1(S)$ be any homology basis. As both are bases of the homology, there exists a unique

$$M \in M_{2g}(\mathbb{Z}), \det(M) \in \{\pm1\}, \quad \text{such that } (a_1, ..., a_{2g})^T = M(\alpha_1, ..., \alpha_{2g})^T.$$

Let $(u_k)_{k=1\ldots2g}$ be the dual basis for $(\alpha_i)_{i=1\ldots2g} \subset H_1(S)$ and set

$$(s_1, ..., s_{2g}) = (\sigma_1, ..., \sigma_{2g})M^{-1}.$$

Then it follows, from the linearity of the integral that for

$$P = (p_{i,k})_{i,k=1,\ldots2g} \in M_{2g}(\mathbb{C}), \quad \text{such that } p_{i,k} = \int_{a_i} s_k$$

we have $P = M \cdot M^{-1} = Id_{2g}$. Hence the $(s_k)_{k=1\ldots2g}$ fulfill the conditions of the theorem. The basis is unique, due to the uniqueness of the $(\sigma_k)_{k=1\ldots2g}$. $\square$

It follows from this theorem that if S is a Riemann surface of genus g, the vector space of real harmonic 1-forms, $\text{Harm}^1_{\mathbb{R}}(S)$ has real dimension $2g$.

$$\text{Harm}^1_{\mathbb{R}}(S) \simeq \mathbb{R}^{2g}.$$

Jacobian varieties

Given a homology basis of a compact Riemann surface, we define :

Definition 1.3.4 *Let S be a R.S. of genus g and $(a_i)_{i=1\ldots2g} \subset H_1(S)$ a homology basis. Let $(\sigma_k)_{i=1\ldots2g} \subset \text{Harm}^1(S)$ be the dual basis of harmonic 1-forms. A period Gram matrix Q_S of S is the real Gram matrix*

$$Q_S = (\langle \sigma_i, \sigma_j \rangle)_{i,j=1\ldots2g} = \left(\int_S \sigma_i \wedge {}^*\sigma_j \right)_{i,j=1\ldots2g}.$$

We can easily deduce from the definition of the star operator that Q_S is a real symmetric matrix. Changing the homology basis changes the period Gram matrix in the following way.

Lemma 1.3.5 *Let $\left(a_i^1\right)_{i=1\ldots2g} \subset H_1(S)$ and $\left(a_i^2\right)_{i=1\ldots2g} \subset H_1(S)$ be two homology bases of S and Q_S^1 and Q_S^1 be the respective period Gram matrices. There exists a $M \in M_{2g}(\mathbb{Z})$, with $\det(M) \in \{\pm1\}$ such that $(a_1^1, ..., a_{2g}^1)^T = M^{-1}(a_1^2, ..., a_{2g}^2)^T$. Then*

$$Q_S^1 = M^T Q_S^2 M.$$

proof It is shown in the proof of **Theorem 1.3.3** how to change the basis for the dual harmonic 1-forms for the two different homology bases. The result follows from this observation and from the fact that the * operator is $\mathbb{R}$ linear. $\square$

If the canonical homology basis $(\alpha_i)_{i=1...2g}$ is given in standard form, then for

$$Q_S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{with} \quad A, B, C, D \in M_g(\mathbb{R})$$

we have, due to the fact that Q_S is real and symmetric :

$$A = A^T, D = D^T \quad \text{and} \quad B = C^T$$

It is shown in [FK], p. 61 that additionally, A and D are positive definite matrices and

$$B^2 - AD = -Id_g, \quad BA = AB^T \quad \text{and} \quad DB = B^T D \tag{1.2}$$

Furthermore we have

Lemma 1.3.6 *Let S be a Riemann surface of genus g and $(\alpha_j)_{j=1...2g} \subset H_1(S)$ be a chb in standard form and $(\sigma_j)_{j=1...2g}$ a dual basis of real harmonic forms, then for*

$$\omega_j = \sigma_j + i^*\sigma_j \text{ for } j = 1, ..., g$$

the $(\omega_j)_{j=1...g}$ and the $(i\omega_j)_{j=1...g}$ form a basis of the vector space of holomorphic 1-forms $\mathcal{O}^1(S)$.

For a proof, see [FK] p. 62.
With a basis of holomorphic 1-forms we can final define the period lattice and the Jacobian variety of S.

Definition 1.3.7 *S be a Riemann surface of genus g and $(a_j)_{j=1...2g} \subset H_1(S)$ be a homology basis and $(w_j)_{j=1...g} \subset \mathcal{O}^1(S)$ be a basis of holomorphic 1-forms, then*

$$Per \cdot \mathbb{Z}^{2g}, \quad such \ that \quad Per_{j,i} = \int_{a_i} w_j \text{ for } i \in \{1, ..., 2g\}, j \in \{1, ..., g\}$$

is the period lattice $Per(w_1, ..., w_g)$ with basis $(w_j)_{j=1...g}$ and the torus $\mathbb{C}^g \mod Per(w_1, ..., w_g)$ is the Jacobian torus or Jacobian variety of S.

The Jacobian variety is a complex torus or abelian variety. A complex torus is a polarized abelian variety, if it has a polarization, where the polarization is a certain positive definite hermitian form. A Jacobian variety carries a polarization in a natural way. To see this, we introduce a standard basis that provides the hermitian form.
With the conditions as in **Lemma 1.3.6**, we have

Theorem 1.3.8 *$(i\omega_j)_{j=1...g}$ form a basis of the vector space of holomorphic 1-forms $\mathcal{O}(S)$. Let $\Xi = \xi_1, ...\xi_g$ be the basis of $\mathcal{O}(S)$ defined by $(\xi_1, ...\xi_g)^T = D^{-1}(i\omega_1, ..., i\omega_g)^T$, then the period lattice is given by $Per(\Xi)$ is of the form*

$$Per(\Xi) = (Id_g, Z)\mathbb{Z}^{2g}, \quad such \ that \ Z = -D^{-1}B^T + iD^{-1},$$
$$with \ Z = Z^T \ and \ \mathrm{Im}(Z) > 0.$$

We call this form of the period lattice the standard form.

This follows from the definitions and equation (1.2). For a detailed proof see [FK], p. 63. The standard form of the period lattice provides the hermitian form H of the Jacobian variety. It is $H = \text{Im}(Z)^{-1} = D$, the inverse matrix of the imaginary part of Z. We will clarify the relationship with the polarized abelian varieties in the following section.

1.4   Principally polarized abelian varieties

A polarized abelian variety (PAV) of dimension g may be defined as a pair (A, H), where A is a complex torus of dimension g and H, the polarization, is a certain positive definite hermitian form. PAVs play an important role in many areas of mathematics. They occur as a class of manifolds in complex analysis, as a class of varieties in algebraic geometry and play a significant role in the theory of fields in number theory. We will give a full definition in the following. Then we will show that the Jacobian of a Riemann surface of genus g is a PAV. Finally, we will present some important geometric invariants of PAVs, the successive minima. A basic reference for this section is [BL].

Polarized abelian varieties

A *complex lattice L* in $\mathbb{C}^g$ is a subgroup of the form

$$L = \oplus_{i=1}^{2g} \mathbb{Z} \cdot u_i$$

such that the $u_1, u_2, ..., u_{2g} \in \mathbb{C}^g$ are linearly independent over $\mathbb{R}$.
If the $(u_i)_{i=1,...,2g} \in \mathbb{C}^g$ are given with respect to the standard basis of $\mathbb{C}^g$ then the matrix $(u_1, u_2, ..., u_{2g}) = \Pi \in M_{g \times 2g}(\mathbb{C})$ is called the *period matrix of the lattice L. A complex torus*

$$A = \mathbb{C}^g \mod L$$

is the quotient of $\mathbb{C}^g$ with a complex lattice L. A *real lattice L* can be defined analogously. We have :

Definition 1.4.1 *Let $A = \mathbb{C}^g \mod L$ be a complex torus. Then a positive definite hermitian form*

$$H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}, H : (\eta, \nu) \mapsto H(\eta, \nu)$$

is called a polarization, if the imaginary part $\text{Im}(H)$ of H is integral on $L \times L$. The torus A is an abelian variety, if and only if a polarization exists for A. In this case the pair (A, H) is a polarized abelian variety.

If $(A = \mathbb{C}^g \mod L, H)$ is a PAV, then $\text{Im}(H)$ is an alternating bilinear form. By a theorem of Frobenius, there exists a so called *symplectic basis* for the lattice L, such that $\text{Im}(H) = E$ has the following form with respect to this basis :

$$
\begin{aligned}
E \;&=\; \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \quad \text{with } D = \text{diag}(d_1, d_2, ..., d_g) \quad \text{such that} \\
d_i \;&\in\; \mathbb{N}, \quad \text{for all } i \in \{1, ..., g\} \text{ and } d_1 \mid d_2 \mid ... \mid d_g.
\end{aligned}
$$

The numbers $d_1, d_2, ..., d_g$ are chosen to be minimal and are uniquely defined by E. These are the *elementary divisors* of E. A PAV is *principal*, if and only if $D = Id_g$. In this case, it is a *principally polarized abelian variety* (PPAV).

A *moduli* or *parameter space* for PPAVs that contains representatives of all isomorphy classes of PPAVs is $\mathcal{A}_g$. $\mathcal{A}_g$ is defined as $\mathcal{H}_g$ modulo the group $Sp_{2g}(\mathbb{Z})$. Hereby a model of $\mathcal{H}_g$ is the *Siegel upper half space*

$$\mathcal{H}_g = \{Z \in M_g(\mathbb{C}), Z = Z^T \text{ and } \operatorname{Im}(Z) > 0\}$$

and $Sp_{2g}(\mathbb{Z})$ is *Siegel's modular group* or the *symplectic group of degree $2g$ over $\mathbb{Z}$*

$$Sp_{2g}(\mathbb{Z}) = \{R \in M_{2g}(\mathbb{Z}) \mid R^T \begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix} R = \begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix}\}.$$

It acts on $\mathcal{H}_g$ by the *g-dimensional Moebius transformation*

$$Z \mapsto R \cdot Z = (aZ + b)(cZ + d)^{-1}$$

$$\text{for} \quad R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2g}(\mathbb{Z}).$$

Let $Z \in \mathcal{H}_g$ be such that $Z = X + iY$ with real-valued symmetric matrices X and Y such that Y is positive definite. By [BL], chapter 8, Z corresponds to the PPAV (A_Z, H_Z), where

$$A_Z = \mathbb{C}^g \mod (Z, Id_g)\mathbb{Z}^{2g},$$

$$H_Z = Y^{-1}.$$

The *period length* of a lattice vector $x = (Z, Id_g)y$, such that $y \in \mathbb{Z}^{2g}$ is defined as

$$\operatorname{Re}(x^T H_Z \bar{x}) = y^T \operatorname{Re}(Q_Z + iI)y = y^T Q_Z y. \quad (\text{ see } (1.1))$$

Hereby Q_Z is the positive definite real symmetric matrix

$$Q_Z = \begin{pmatrix} XY^{-1}X + Y & XY^{-1} \\ Y^{-1}X & Y^{-1} \end{pmatrix} = \begin{pmatrix} Id_g & X \\ 0 & Id_g \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \quad (1.3)$$

It follows from this equation that $\det(Q_Z) = 1$.

Furthermore $Sp_{2g}(\mathbb{Z})$ acts on the matrices $\{Q_Z \in M_{2g}(\mathbb{R}) \mid Z \in \mathcal{H}_g\}$ in the following way :

$$Q_Z = Q'_Z \mod Sp_{2g}(\mathbb{Z}) \Leftrightarrow Q'_Z = RQ_Z R^T \quad \text{with} \quad R \in Sp_{2g}(\mathbb{Z})$$

It can be easily shown that the association $Z \leftrightarrow Q_Z$ is one-to-one and

$$Z' = Z \mod Sp_{2g}(\mathbb{Z}) \Leftrightarrow Q_Z = Q_{Z'} \mod Sp_{2g}(\mathbb{Z}).$$

This relation follows from the equations in [BL], p. 212.

Q_Z can be decomposed into the matrices $P_Z^T P_Z$, such that

$$P_Z = P_{X,Y} = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \in M_{2g}(\mathbb{R}). \quad (1.4)$$

Hence Q_Z is the Gram-matrix of the real $2g$ dimensional lattice $P_Z\mathbb{Z}^{2g}$.

Jacobian varieties as PPAVs

If S is a compact Riemann surface of genus g, let $J(S)$ be its Jacobian torus in standard form $\mathbb{C}^g$ mod $(Id_g, Z)\mathbb{Z}^{2g}$, as given in **Theorem 1.3.8**. To conform the definition with the definiton of a PPAV, we may assume that the lattice of the Jacobian is given in the form $(Z, Id_g)\mathbb{Z}^{2g}$. It defines the principally polarized abelian variety $(\mathbb{C}^g \mod (Z, Id_g)\mathbb{Z}^{2g}, Y^{-1})$, with $Z = X + iY \in \mathcal{H}_g$. We obtain by equation (1.2), with

$$X = D^{-1}B^T \quad \text{and} \quad Y = D^{-1},$$

that for a lattice vector $x = (Z, Id_g)y$ (with $y \in \mathbb{Z}^{2g}$)

$$x^T H_Z \bar{x} = y^T (Q_Z + iI)y = y^T \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} y^T + iy^T \begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix} y.$$

Hence

$$Q_S = \begin{pmatrix} -Id_g & 0 \\ 0 & Id_g \end{pmatrix} Q_Z \begin{pmatrix} -Id_g & 0 \\ 0 & Id_g \end{pmatrix}$$

Hence Q_S and Q'_Z are equivalent Gram matrices, whose real lattices $P_S\mathbb{Z}^{2g}$ and $P_Z\mathbb{Z}^{2g}$ are equal.

Hermite's constant and Minkowski's first and second theorem

A geometric invariant of the lattice of a PPAVs is the first successive minimum or shortest non-zero lattice vector, whose square is the *minimal period length* of the PPAV. Here the *k-th successive minimum of a complex lattice L* of dimension g is defined by

$$m_k(L) = \min \left\{ r \in \mathbb{R}^+ \,|\, \exists \{l_1, ..., l_k\} \subset L, \text{ lin. independent over } \mathbb{R}, \|l_i\| \leqslant r \forall i \right\}$$

The *k-th successive minimum of a PPAV* $(A = \mathbb{C}^g/L, H)$, $m_k(A, H)$, is defined as the k-th successive minimum of its lattice L. Here the norm $\|\cdot\| = \|\cdot\|_H$ is the norm induced by the hermitian form H. If $(l_i)_{i \in \{1..2g\}}$ is a lattice basis of L, then the corresponding Gram matrix is

$$Q_H = \left(\langle l_i, l_j \rangle_H \right)_{i,j=1...2g}$$

Hence if (A, H) is a PPAV of dimension g,

$$m_k(A, H) = \min \left\{ r \in \mathbb{R}^+ \,|\, \exists \{x_1, ..., x_k\} \subset \mathbb{Z}^{2g}, \text{ lin. independent}, x_i^T Q_H x_i \leqslant r \, \forall i \right\} \qquad (1.5)$$

For every PPAV (A, H) there exists a $Z \in \mathcal{H}_g$, such that $(A, H) \simeq Z$. In this standard form, we have $Q_H = Q_Z$ and Q_Z is the Gram-matrix of the associated real lattice $P_Z\mathbb{Z}^{2g}$. If S is a Riemann surface of genus g and $J(S)$ its Jacobian, then we can replace Q_H by the period Gram matrix Q_S in equation (1.5).

This follows from the previous paragraph and **Lemma 1.3.5**. Q_H has determinant 1, due to the fact that the PAV is principal (see (1.3)). As the determinant is fixed, we can apply the general upper bounds on the successive minima from Minkowski's theorems (see [GL]) to the case

of a PPAV (A, H) of dimension g, whereas the lower bound for Hermite's constant over the PPAVs

$$\delta_{2g} = \max_{(A,H)\in\mathcal{A}_g} m_1(A,H)^2$$

was proven in [BS]. We have :

$$\frac{g}{\pi e} \approx \frac{1}{\pi} \sqrt[g]{2g!} \ \leqslant \delta_{2g} \leqslant \ \frac{4}{\pi} \sqrt[g]{g!} \approx \frac{4g}{\pi e}.$$

The approximation applies to large g. Hence the maximum of the square of the first successive minimum of PPAV of dimension g is of order g. By Minkowski's second theorem we have

$$\prod_{k\,=\,1}^{2g} m_k(A,H)^2 \ \leqslant \ \left(\frac{4}{\pi}\right)^g g!^2.$$

1.5   Energy, capacity and harmonic forms

In this section we will discuss the energy-minimizing properties of harmonic functions and 1-forms. We will first recall that the elements of a dual harmonic basis minimize the energy in certain classes of 1-forms. To obtain an approximation of the energy of certain functions and forms in chapter 3, we will use some results from the calculus of variations. These results will be presented here. Finally, we will use these methods to calculate explicitly the capacities of certain annular regions. These examples will also occur in the following chapters.

Energy-minimizing property of a dual harmonic basis

In the vector space of closed differential 1-forms $\text{Ker}(d)$ on a compact Riemann surface S, we have :

Theorem 1.5.1 *Let S be a Riemann surface of genus g and $(a_i)_{i=1,...,2g} \subset H_1(S)$, be a basis of the homology, represented as closed curves. Let $(\sigma_k)_{k=1,...,2g} \subset \text{Harm}^1(S)$ be a dual basis of harmonic forms, then for all $k \in \{1,...2g\}$:*

$$E(\sigma_k) \leq E(\omega) \quad \text{for all} \quad \omega \in \mathcal{E}^1(S) \ \text{ closed, with} \quad \int_{a_i} \omega = \delta_{ik}.$$

proof Fix $k \in \{1,...2g\}$ and let $\omega \in \text{Ker}(d)$ be a 1-form that satisfies the conditions of the theorem. By **Theorem 1.2.7**, we have that

$$\omega = df + \sigma, \quad \text{with} \quad df \in d\mathcal{E}(S) \ \text{ and } \ \sigma \in \text{Harm}^1(S).$$

By Stokes theorem (**Theorem 1.2.3**) we have for any $a_i : [0,1] \to S$:

$$\int_{a_i} df = f(1) - f(0), \text{ as } a_i(0) = a_i(1) \Rightarrow \int_{a_i} df + \sigma = \int_{a_i} \sigma$$

By **Theorem 1.3.3** $\sigma = \sigma_k$, as the harmonic form satisfying $\int_{a_i} \sigma_k = \delta_{ik}$ is unique. Due to the orthogonality of the decomposition of ω, we have furthermore :

$$E(\omega) = E(df) + E(\sigma_k) \geq E(\sigma_k).$$

which concludes the proof. $\square$

Euler-Lagrange equations

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then the derivative of f in an extremum is equal to zero. This idea is further pursued in the calculus of variations that deals with functionals instead of functions. Hereby a functional is usually a mapping from a set of functions to the real numbers. We will consider functionals involving one and two-dimensional integration.

In the one-dimensional case we consider the functional O, defined in the following way. Let $[a, b] \in \mathbb{R}$, $f \in C^2([a, b])$ and $L : \mathbb{R}^3 \to \mathbb{R}, L : (x_1, x_2, x_3) \mapsto L(x_1, x_2, x_3)$, such that $L \in C^1(\mathbb{R}^3)$. We have :

Theorem 1.5.2 *If O is the functional defined by*

$$O : C^2([a, b]) \to \mathbb{R}, O : f \mapsto O(f) := \int_a^b L(t, f(t), f'(t)) \, dt$$

then an extremum f of O, satisfies the Euler-Lagrange equation

$$\frac{\partial L(x_1, x_2, x_3)}{\partial x_2}\Big|_{(t,f(t),f'(t))} - \frac{\partial}{\partial t}\left(\frac{\partial L(x_1, x_2, x_3)}{\partial x_3}\Big|_{(t,f(t),f'(t))}\right) = 0$$

For a proof see [Ge], p. 14-16.

In the two-dimensional case, we consider the following situation Let $D \in \mathbb{R}^2$ be a domain, $f \in C^2(D)$ and $L : \mathbb{R}^5 \to \mathbb{R}, L : (x_1, ..., x_5) \mapsto L(x_1, ..., x_5)$, such that $L \in C^1(\mathbb{R}^5)$. We have :

Theorem 1.5.3 *If O is the functional defined by*

$$O : C^2(D) \to \mathbb{R}, O : f \mapsto O(f) := \iint_D L(t, s, f(t, s), \frac{\partial f(t, s)}{\partial t}, \frac{\partial f(t, s)}{\partial s}) \, dt \, ds$$

then an extremum f of O, satisfies the Euler-Lagrange equation

$$\frac{\partial L(x_1, ..., x_5)}{\partial x_3}\big|_p - \frac{\partial}{\partial t}\left(\frac{\partial L(x_1, ..., x_5)}{\partial x_4}\big|_p\right) - \frac{\partial}{\partial s}\left(\frac{\partial L(x_1, ..., x_5)}{\partial x_5}\big|_p\right) = 0,$$

for $p = (t, s, f(t, s), \frac{\partial f(t,s)}{\partial t}, \frac{\partial f(t,s)}{\partial s})$.

For a proof see [Ge]. p. 152-154.

If the primitive of a harmonic 1-form ω exists on a certain domain A, then it is a harmonic function. In chapter 3 we will estimate the energy of harmonic functions on annuli. To this end, we will construct test functions, whose energy provides an upper bound for the energy of harmonic forms. We will now gather the necessary facts about harmonic functions on annular regions. As our test functions are not differentiable everywhere, we will also introduce a suitable space of functions that contains these test functions.
Let

$$A_{r_1}^{r_2} = \{x \in \mathbb{R}^2 \mid r_1 < \|x\|_2 < r_2\}, \quad \text{for} \quad r_1 < r_2$$

be the open annulus around $(0,0)$ with inner radius r_1 and outer radius r_2.
We define an *annulus A* as a set in $\mathbb{R}^2 \simeq \mathbb{C}$, such that $A_{r_1}^{r_2}$ can be mapped biholomorphically by a function $\tilde{f}$ onto A. We denote by $\partial_1 A$ and $\partial_2 A$ the two disjoint connected boundary components that constitute ∂A.

Let B be a set in a metric space (X, dist). We denote by a *Lipschitz function* $f : B \to \mathbb{R}$ a function that satisfies

$$|f(x) - f(y)| \leq M \,\text{dist}(x, y) \quad \text{for all} \quad x, y \in B,$$

where $M \in \mathbb{R}^+$ is a constant. We denote by $\text{Lip}(B)$ the *set of Lipschitz functions on B*. We have :

Theorem 1.5.4 *Let A be an annulus and $f_i : \partial_i A \to \mathbb{R}, i \in \{1, 2\}$ be two continuous functions defined on the boundary of A. There exists exactly one harmonic function in $C(\bar{A})$, such that*

$$\Delta f = 0 \quad on \quad A \quad and \quad f|_{\partial_i A} = f_i \quad for \quad i \in \{1, 2\} \tag{1.6}$$

Furthermore the function f is minimizing the energy $E_A(f)$ among all functions in $\text{Lip}(\bar{A})$ (or in $C^2(A) \cap C(\bar{A})$) that satisfy these boundary conditions.

proof We prove the theorem only for the functions in $C^2(A) \cap C(\bar{A})$. The uniqueness follows by contradiction. If f and h are two functions that satisfy equation (1.6), then

$$\Delta(f - h) = \Delta f - \Delta h = 0 \quad \text{and} \quad f - h|_{\partial_i A} = 0 \quad \text{for} \quad i \in \{1, 2\}$$

but the only possible solution for this equation is the function that vanishes on all A, hence $f - h = 0 \Leftrightarrow f = h$. That f is minimizing the energy can be deduced from the Euler-Lagrange equations. If we apply the Euler-Lagrange equations to the energy integral, we obtain that these imply $\Delta f = 0$. The most difficult part is the existence. A proof for the existence of the function follows from [Fo], p. 184. $\square$

Remark : It would be more natural to work in a Sobolev space or Royden algebra rather than in the set of Lipschitz functions. However these functions are dense in these spaces and the energy of the minimizing function is the same.

Definition 1.5.5 *Let A be an annulus. Then the capacity of A, $\text{cap}(A)$ is defined as the energy $E_A(f)$ of the function $f \in C^2(A) \cap C(\bar{A})$ that satisfies*

$$\Delta f = 0 \quad on \quad A \quad and \quad f|_{\partial_1 A} = 0 \quad an \quad f|_{\partial_2 A} = 1.$$

We remark that this definition is independent of the choice of $\partial_1 A$ and $\partial_2 A$. If f satisfies the above definition, then $1 - f$ is the unique harmonic function that satisfies the inverse boundary conditions. Clearly $E_A(1 - f) = E_A(f)$.

Example 1.5.6 Consider the hyperbolic cylinder C_1 (see chapter 1.1), such that

$$C_1 = \{\psi(t, s) \mid (t, s) \in [0, a] \times [b_1, b_2]\} \mod \{\psi(0, s) = \psi(a, s) \mid s \in [b_1, b_2]\} \quad \text{with}$$
$$\partial_1 C_1 = \{\psi(t, b_1) \mid t \in [0, a]\} \quad \text{and} \quad \partial_2 C_1 = \{\psi(t, b_2) \mid t \in [0, a]\}.$$

To determine the capacity $\mathrm{cap}(C_1)$, we are looking for the harmonic function h that satisfies the boundary conditions of the capacity problem. We then have to evaluate the hyperbolic or Euclidean energy $E_{C_1}(h)$. We will use the Fermi coordinates and **Theorem 1.5.3**. Using the two dimensional chain rule we have

$$D\tilde{h} = D(h \circ \psi) = Dh|_\psi \cdot D\psi \Leftrightarrow D\tilde{h} \cdot D\psi^{-1} = Dh|_\psi.$$

Applying the two-dimensional integration by substitution and the equation above we obtain

$$E_{C_1}(h) = \iint\limits_{C_1} \|Dh\|_2^2 = \int\limits_0^a \int\limits_{b_1}^{b_2} \|Dh|_\psi\|_2^2 \cdot |\det(D\psi)| =$$
$$\int\limits_{t=0}^a \int\limits_{s=b_1}^{b_2} \frac{1}{\cosh(s)} \frac{\partial \tilde{h}(t,s)}{\partial t}^2 + \cosh(s) \frac{\partial \tilde{h}(t,s)}{\partial s}^2 \, dt \, ds. \tag{1.7}$$

We now apply **Theorem 1.5.3** to minimize the value of the integral in (1.7). If $\tilde{h}$ is the minimizing function, we have :

$$\frac{1}{\cosh(s)} \frac{\partial^2 \tilde{h}(t,s)}{\partial t^2} + \sinh(s) \frac{\partial \tilde{h}(t,s)}{\partial s} + \cosh(s) \frac{\partial^2 \tilde{h}(t,s)}{\partial s^2} = 0. \tag{1.8}$$

For $H(s) := 2 \arctan(\exp(s))$ the function

$$\tilde{h}(t, s) := c_1 H(s) + c_2, \quad \text{such that}$$
$$c_1 = \frac{1}{H(b_2) - H(b_1)} \quad \text{and} \quad c_2 = \frac{-H(b_1)}{H(b_2) - H(b_1)}$$

satisfies equation (1.8). Hence $h = \tilde{h} \circ \psi^{-1}$ is a harmonic function that satisfies the boundary conditions of the capacity problem. Evaluating $E_{C_1}(h)$ with the help of equation (1.7), we obtain :

$$\mathrm{cap}(C_1) = E_{C_1}(h) = \frac{a}{2(\arctan(\exp(b_2)) - \arctan(\exp(b_1)))}. \tag{1.9}$$

Example 1.5.7 Let $f : A_{r_1}^{r_2} = A \to \mathbb{R}$ be the function defined by

$$f(x, y) := c_1 \log(x^2 + y^2) + c_2, \quad \text{such that}$$
$$c_1 = \frac{1}{\log(r_2{}^2) - \log(r_1{}^2)} \quad \text{and} \quad c_2 = \frac{-\log(r_1{}^2)}{\log(r_2{}^2) - \log(r_1{}^2)}.$$

Then we have for all $(x, y) \in A$

$$Df(x, y) = 2c_1(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}) \quad \text{and} \quad \Delta f(x, y) = 0.$$

It satisfies the boundary conditions $f|_{\partial_1 A} = 0$ and $f|_{\partial_2 A} = 1$ and hence $E_A(f) = \text{cap}(A)$. Using polar coordinates, we obtain :

$$\text{cap}(A_{r_1}^{r_2}) = \iint\limits_A \|Df(x, y)\|_2^2 \, dx \, dy = 4c_1^2 \iint\limits_A \frac{1}{x^2 + y^2} \, dx \, dy = \frac{2\pi}{\log(r_2) - \log(r_1)}.$$

Lemma 1.5.8 *Consider $\mathbb{C} \simeq \mathbb{R}^2$ and let $A_1, A_2 \subset \mathbb{C}$ be two annuli and $H : A_2 \rightarrow A_1$ be a biholomorphic function, such that $H(A_2) = A_1$, then*

$$\text{cap}(A_1) = \text{cap}(A_2).$$

proof Let f be the function such that $E_{A_1}(f) = \text{cap}(A_1)$. Set $h = f \circ H$. Using the Cauchy-Riemann (C-R) differential equations for H, and the fact that $\text{Re}(H), \text{Im}(H)$ and f are harmonic functions a straightforward calculation shows that h is harmonic and satisfies the boundary conditions of the capacity problem. Hence

$$E_{A_2}(h) = \text{cap}(A_2).$$

Using the chain rule of differentiation and the C-R differential equations we have that

$$\|Dh\|_2^2 = \|Df|_H \cdot DH\|_2^2 = \|Df|_H\|_2^2 \cdot |\det(DH)|.$$

Using integration by substitution we obtain :

$$\text{cap}(A_2) = \iint\limits_{A_2} \|Dh\|_2^2 = \iint\limits_{H^{-1}(A_1)} \|Df|_H\|_2^2 \cdot |\det(DH)| = E_{A_1}(f) = \text{cap}(A_1). \quad \square$$

Example 1.5.9 Consider the flat cylinder C_2, such that

$$C_2 = [0, a] \times [0, b] \mod \{(0, y) = (a, y) \mid y \in [0, b]\}, \quad \text{with}$$
$$\partial_1 C_2 = [0, a] \times \{0\} \quad \text{and} \quad \partial_2 C_2 = [0, a] \times \{b\}.$$

We identify $\mathbb{R}^2 \simeq \mathbb{C}$ and consider the holomorphic map $H : C_2 \mapsto \mathbb{C}, H : z \mapsto \exp(-\frac{2\pi i}{a} z)$. We have that

$$H(C_1) = A_{r_1}^{r_2}, \quad \text{such that} \quad r_1 = 1 \quad \text{and} \quad r_2 = \exp(2\pi\frac{b}{a}).$$

As H satisfies the conditions from **Lemma 1.5.8**, we have by **Example 1.5.7** that

$$\text{cap}(C_2) = \frac{a}{b}.$$

Chapter 2

Global geometry of a Riemann surface and its Jacobian

The main goal of this thesis is to get insight into the relationship between the geometry of a hyperbolic compact Riemann surface and the geometry of its Jacobian. This goal will be pursued in the following two chapters. The principal idea is to find lower and upper bounds for the length of vectors of the Jacobian of a Riemann surface based on the geometry of the surface. Here the upper bounds described in this chapter are more general, whereas the estimates in the following chapter rely on the concrete geometry of the surface.

In this chapter, we will first describe the connection with the Schottky problem. Then we will show, how to relate the length of lattice vectors of the Jacobian to geometric data of the surface. Afterwards we will gather some information about short simple closed geodesics on Riemann surfaces and their collars, topological tubes around these geodesics. This information will be used to prove the following results.

Expanding the result in [BS], we will show that the square of the first two successive minima of a Jacobian of a Riemann surface of genus g is maximally of order $\log(g)$, whereas it can be of order g for the lattice of a PPAV of dimension g. We obtain improved bounds for the k-th successive minima of the Jacobian, if the surface contains small simple closed geodesics. Finally we will show that the square of the first successive minimum of the Jacobian of a hyperelliptic surface is bounded from above by a constant, independent of the genus.

For simplification the following expressions will be abbreviated. A simple closed geodesic will be denoted *scg* and a non-separating simple closed geodesic *nsscg* and a separating simple closed geodesic *sscg*. By abuse of notation we will denote the length of a geodesic arc by the same letter as the arc itself, if it is clear from the context.

2.1   The Schottky problem

The Schottky problem is to characterize those principally polarized abelian varieties that arise as Jacobians of compact Riemann surfaces. The problem goes back to Schottky, who gave such a characterization in 1888 for the case of a compact Riemann surface of genus 4. Many authors have worked on this problem. A full solution, however, has not been found up today. An overview of the problem can be found in [De].

As we have seen in chapter 1.4, the Jacobian of a Riemann surface of genus g is a PPAV of dimension g. The moduli or parameter space $\mathcal{A}_g$ for PPAVs of dimension g has (complex) dimension $\frac{1}{2}g(g+1)$, whereas the moduli space of compact Riemann surfaces of genus g, $\mathcal{M}_g$, has dimension $3g-3$. The assignment of the Jacobian $J(S)$ to the R.S. S provides a mapping

$$t : \mathcal{M}_g \to \mathcal{A}_g.$$

By Torelli's theorem, this mapping is injective. In general, it is not known, if a given PPAV is the image of a Jacobian under t. The Schottky problem is to describe the sublocus $t(\mathcal{M}_g)$ in $\mathcal{A}_g$. The closure of the sublocus of Jacobian varieties $\overline{t(\mathcal{M}_g)}$ in the parameter space $\mathcal{A}_g$ is only equal to $\mathcal{A}_g$ for $g = 2$ and 3. For $g \geq 4$ it is a proper closed subset. Several analytic approaches have been used to further characterize $t(\mathcal{M}_g)$. Notably van Gemen proved in [vG] that $t(\mathcal{M}_g)$ is an irreducible component of the locus $\mathcal{S}_g$ defined by the Schottky-Jung polynomials. However, an exact description of the locus $t(\mathcal{M}_g)$, given in terms of polynomials of theta constants that vanish on $t(\mathcal{M}_g)$ but not on $\mathcal{A}_g$, is only known for $g = 4$ ([Sc]). Shiota [Sh] showed that an indecomposable PPAV is the Jacobian of a Riemann surface, if the corresponding theta function fulfills the K-P differential equation. However, a solution to this equation can not as yet be determined explicitly.

In [BS], it was shown that the Jacobians can be characterized among the PPAVs with the help of a geometric invariant of the lattice of the PPAVs, the first successive minimum or shortest non-zero lattice vector, whose square is also called the *minimal period length* of the PPAV. Buser and Sarnak showed in [BS] that the shortest non-zero lattice vector of the Jacobian of a compact Riemann surface of genus g can be maximally of order $\log(g)$:

Theorem 2.1.1 *If* $\eta_{2g} = \max\limits_{(A,H) \in t(\mathcal{M}_g)} m_1(A,H)^2$ *, then*

$$c \log g \leq \eta_{2g} \leq \frac{3}{\pi} \log(4g-2),$$

where c is a constant.

Among the PPAVs (A, H) of dimension g, however, the maximum of $m_1(A, H)^2$ is of order g (see chapter 1.4.). Hence this inequality describes a large region in the moduli space of all principally polarized abelian varieties of complex dimension g, of which it is shown that none occurs as a Jacobian. We will extend this theorem in the following sections.

2.2 Relating geometric data of the surface to the Jacobian

In [BS] an upper bound for the norm of a certain lattice vector of a Jacobian of a surface S is obtained by linking the norm of the vector to the length of a non-separating simple closed geodesic on S and the width of its collar, a topological tube around this geodesic. This approach can be further expanded.

Let S be a compact R.S. and $(\alpha_i)_{i=1\dots 2g}$ a canonical homology basis on S. The collar of a scg γ, $C(\gamma)$, is defined by

$$C(\gamma) = \{x \in S \mid \operatorname{dist}(x, \gamma) < w\} \,.$$

Here w is the supremum of all ω, such that the geodesic arcs of length ω emanating perpendicularly from γ are pairwise disjoint. For a given α_i, let $\alpha_{\tau(i)}$ be the unique scg in the canonical homology basis that intersects α_i. The closure of the collar $\overline{C(\gamma)}$ is a hyperbolic cylinder centered around γ of width ω on both sides. We define test forms σ_i' on the collar of an α_i that satisfy

$$\int_{\alpha_j} \sigma_i' = \left\{ \begin{array}{ll} 1 & \text{if } j = \tau(i) \\ 0 & \text{if } j \neq \tau(i). \end{array} \right. \tag{2.1}$$

in the following way. Let F_i be the harmonic function that solves the capacity problem for a hyperbolic cylinder $C \subset C(\alpha_i)$ (see **Example 1.5.6**). F_i satisfies the boundary conditions

$$F_i|_{\partial_1 C} = 0 \quad \text{and} \quad F_i|_{\partial_2 C} = 1$$

Consider dF_i. By Stokes theorem, **Theorem 1.2.3**, we have that dF_i satisfies equation (2.1). We can smoothen dF_i in an ϵ-environment $U \subset C$ of the boundary of C. This way we can construct a closed 1-form $\sigma_i' \in \mathcal{E}^1(S)$, such that σ_i' satisfies equally equation (2.1). U can be chosen arbitrarily small, such that WLOG $E(\sigma_i') = E_C(F_i) = \text{cap}(C)$. By **Example 1.5.6**, we have that

$$\frac{l(\alpha_i)}{\pi - 2\arcsin\left(\frac{1}{\cosh(w_i)}\right)} = \text{cap}(C(\alpha_i)) \leq \text{cap}(C), \tag{2.2}$$

, as $2(\arctan(e^{w_i}) - \arctan(e^{-w_i})) = \pi - 2\arcsin\left(\frac{1}{\cosh(w_i)}\right)$. Here $l(\alpha_i)$ denotes the length of α_i and w_i the width of the collar.

By **Theorem 1.5.1**, we have that among all closed differential forms on S that satisfy equation (2.1), the corresponding harmonic form $\sigma_{\tau(i)}$ in the dual basis for the homology basis $(\alpha_i)_{i=1\dots2g}$ has minimal energy $E(\cdot)$. For $C = C(\alpha_i)$, we have

$$\int_S \sigma_{\tau(i)} \wedge {}^*\sigma_{\tau(i)} = E(\sigma_{\tau(i)}) < E(\sigma_i') = \text{cap}(C(\alpha_i)) \quad \text{for all} \quad i \in \{1\dots2g\}. \tag{2.3}$$

The same approach works, if we define a suitable test function on a topological tube A in S, which is a continuous deformation of $C(\alpha_i)$. More precisely A must satisfy the condition that there exists an isotopy

$$H : C \times [0,1] \leftarrow C, \quad \text{such that} \quad H(\cdot, 0) = id \quad \text{and} \quad H(C, 1) = A.$$

This approach will be pursued in chapter 3.

If (A, H) is a Jacobian, where $A = \mathbb{C}^g \mod L$ is the Jacobian torus $J(S)$ of the surface S, then with the appropriate base change (see chapter 1.5.), we have for a basis $(l_i)_{i=1\dots2g}$:

$$\left(\langle l_i, l_j \rangle_H\right)_{i,j=1\dots2g} = Q_H = Q_S = \left(\int_S \sigma_i \wedge {}^*\sigma_j \right)_{i,j=1\dots2g}$$

Therefore we have for all i

$$\langle l_i, l_i \rangle_H = \int_S \sigma_i \wedge {}^*\sigma_i = E(\sigma_i).$$

The $(l_i)_{i=1...2g}$ are linear independent vectors of the lattice L and if we can obtain an upper bound for the test forms $(\sigma'_i)_{i=1...k}$, we obtain an upper bound on $m_k(A, H)^2$ by equation (2.3),

$$m_k(A, H)^2 \leq \max_{i \in \{1,...,k\}} E(\sigma'_i) = \max_{i \in \{1,...,k\}} \text{cap}(C(\alpha_i)) \qquad (2.4)$$

$\text{cap}(C(\alpha_i))$ is a strictly increasing function with respect to w_i. The following values W and W' for the width of a collar occur frequently in our proof :

$$W = \text{arccosh}(2) = 1.3169... \quad \text{and} \quad W' = \text{arctanh}(2/3) = 0.8047...$$

If $w_i = W$, we have that

$$\text{cap}(C(\alpha_i)) = \frac{3l(\alpha_i)}{2\pi} \leq \frac{l(\alpha_i)}{2}.$$

If $w_i = W'$, we obtain that

$$\text{cap}(C(\alpha_i)) = \frac{l(\alpha_i)}{\pi - 2\arcsin(\frac{\sqrt{5}}{3})} \leq \frac{7l(\alpha_i)}{10}.$$

The upper bound in **Theorem 2.1.1** follows from the fact that a canonical homology basis $(\alpha_i)_{i=1...2g}$ can always be constructed, such that α_1 is the shortest nsscg on a Riemann surface S. It was shown in [BS] that the length of the shortest nsscg α_1 is smaller than $2\log(4g-2)$ for any R.S. of genus g and that its collar width w_1 is bounded from below by W'. It follows from the above equations that $m_1(J(S))^2$ is bounded from above. A more refined analysis shows that $m_1(J(S))^2 \leq \frac{3}{\pi}\log(4g-2)$.
We will extend this result to $m_2(J(S))$ by showing that there exist two short nsscg, α_1 and α_2, whose collar widths are bounded from below and which can be incorporated together into a canonical homology basis.
In principle this approach would provide further bounds for the consecutive $m_k(J(S))$, but finding bounds for both collar width and length of the nsscgs has already proven to be very technical for $k = 2$.
From the fact that the non-separating systole of a hyperelliptic surface is bounded from above by a constant, independent of the genus, we will obtain a refined result for the minimal period length of hyperelliptic surfaces.

2.3 Upper bounds for the length of scgs on a Riemann surface

To prove the main theorems we will have to consider on many occasions the configuration in which the closure of the collar of a scg self-intersects. The closure of the collar of a scg γ, $\overline{C(\gamma)}$ self-intersects in a finite number of points. Let p be such an intersection point. There exist two geodesic arcs of length w emanating from γ and perpendicular to γ having the endpoint p in common. These two arcs, δ' and δ'', form a smooth geodesic arc δ. Two cases are possible - either δ arrives at γ on opposite sites of γ or it arrives on the same side (*see Fig. 2.1.*).

Definition 2.3.1 *The closure of the collar of a scg γ, $\overline{C(\gamma)}$ self-intersects in a point p. We say that $C(\gamma)$ **is in configuration 1** if the shortest geodesic arcs δ' and δ'' emanating from the intersection point p and meeting γ perpendicularly arrive at γ on opposite sides. We say that $C(\gamma)$ **is in configuration 2**, if they arrive on the same side of γ.*
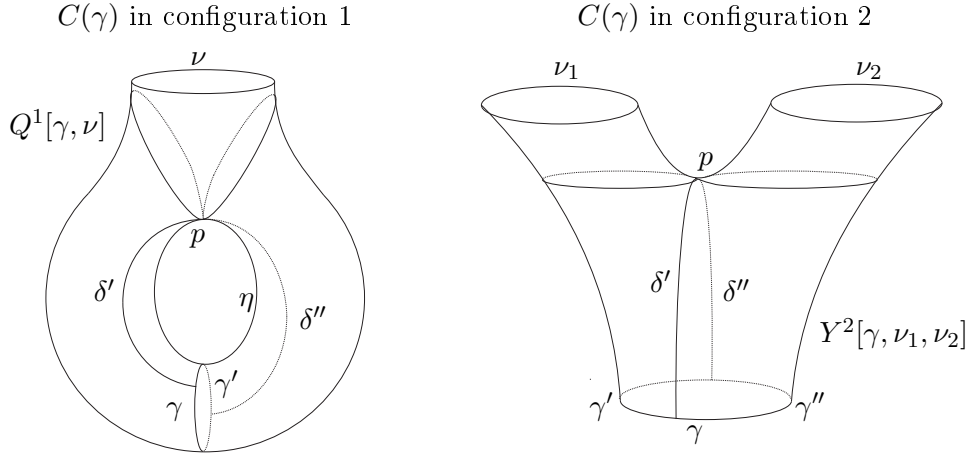
Figure 2.1: $C(\gamma)$ in configuration 1 and 2

For both configurations we have a corresponding Y-piece, a topological three-holed sphere, whose interior is isometrically embedded in S. If $C(\gamma)$ is in configuration 1, we cut open S along γ. We call S' the surface obtained in this way from S. Let γ^1 and γ^2 the two scg in S' corresponding to γ in S. Let ν be the shortest scg in the free homotopy class of $\gamma^1\delta\gamma^2\delta^{-1}$. Then γ^1, γ^2 and ν bound a three-holed sphere Y^1, whose interior lies in S'. As this decomposition occurs frequently, we will refer to it as $Y^1[\gamma, \nu]$, *the Y-piece for γ from configuration 1*. If we close $Y^1[\gamma, \nu]$ again at γ, we obtain a R.S. of signature $(1,1)$, $Q^1[\gamma, \nu] \subset S$ (*see Fig. 2.1.*). Note that in this case we obtain

$$\nu < 2\gamma + 2\delta = 2\gamma + 4w, \tag{2.5}$$

as ν is in the free homotopy class of $\gamma^1\delta\gamma^2\delta^{-1}$. However, we can also calculate the exact value of ν by decomposing $Y^1[\gamma, \nu]$ into two isometric hexagons, H_1 and H_2. Here we cut open $Y^1[\gamma, \nu]$ along the shortest geodesic arcs connecting the boundary curves. In H_1 δ is the shortest geodesic arc connecting $\frac{\gamma^1}{2}$ and $\frac{\gamma^2}{2}$ and $\frac{\nu}{2}$ is the side opposite of δ. From the geometry of right-angled hexagons (see [B], p. 454) we obtain

$$\cosh(\frac{\nu}{2}) = \sinh(\frac{\gamma}{2})^2 \cosh(\delta) - \cosh(\frac{\gamma}{2})^2$$

As $\cosh(x)^2 = \sinh(x)^2 + 1$ this is equal to

$$\nu = 2\operatorname{arccosh}(\sinh(\frac{\gamma}{2})^2(\cosh(2w) - 1) - 1). \tag{2.6}$$

We note furthermore that there exists a geodesic arc γ' in γ connecting the two endpoints of δ on γ, whose length is restricted by $\gamma' \leq \frac{\gamma}{2}$. The shortest scg η in the free homotopy class of $\gamma'\delta$ has length smaller than

$$\eta < \frac{\gamma}{2} + \delta = \frac{\gamma}{2} + 2w. \tag{2.7}$$

If $C(\gamma)$ is in configuration 2, δ divides γ in two parts, γ' and γ''. Let ν_1 and ν_2 be the scg in the free homotopy class of $\gamma'\delta$ and $\gamma''\delta$. The three scg γ, ν_1 and ν_2 then bound a Y-piece, we will refer to it as $Y^2[\gamma, \nu_1, \nu_2]$, *the Y-piece for γ from configuration 2* (*see Fig. 2.1.*). Note that

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$\nu_1 < \gamma' + \delta$ and $\nu_2 < \gamma'' + \delta$. Let WLOG $\gamma' \le \gamma''$. As $\gamma = \gamma' \cup \gamma''$, we have

$$\nu_1 < \gamma' + \delta \le \frac{\gamma}{2} + \delta = \frac{\gamma}{2} + 2w \quad \text{and} \quad \nu_2 < \gamma'' + \delta < \gamma + 2w. \tag{2.8}$$

For small values of γ, we obtain a better bound for ν by decomposing $Y^2[\gamma, \nu_1, \nu_2]$ into two isometric hexagons, H_1 and H_2, by cutting it open along the shortest geodesic arcs connecting the boundary curves. Here $\frac{\delta}{2}$ is the unique geodesic arc in H_1 perpendicular to $\frac{\gamma}{2}$ and the geodesic arc between $\frac{\nu_1}{2}$ and $\frac{\nu_2}{2}$ and with endpoints on both arcs. $\frac{\delta}{2}$ divides H_1 into two pentagons, P_1 and P_2. Let P_1 be the pentagon that contains $\frac{\gamma'}{2}$ as a boundary arc. From the geometry of right-angled pentagons (see [B], p. 454), we get

$$\cosh(\frac{\nu_1}{2}) = \sinh(\frac{\gamma'}{2})\sinh(\frac{\delta}{2}) \quad \text{and} \quad \nu_1 \le 2\operatorname{arccosh}(\sinh(\frac{\gamma}{4})\sinh(w)), \tag{2.9}$$

as sinh and arccosh are strictly increasing functions on $\mathbb{R}^+$.
With the help of this decomposition, the following lemma is proven in [BS], p. 40-42 :

Lemma 2.3.2 *Let S be a compact R.S. of genus g and γ a scg in S. Let $C(\gamma)$ be the collar of γ of width w. If $C(\gamma)$ is in configuration 1, let δ be the geodesic arc emanating from both sides of γ and perpendicular to γ. δ divides γ into two arcs. Let γ' be the shorter of the two. Let furthermore η be the scg in the free homotopy class of $\gamma'\delta$. If $\eta \ge \gamma$, then*

$$w \ge \max\left\{ \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\gamma}{2})}\right), \operatorname{arccosh}\left(\frac{\cosh(\frac{\gamma}{2})}{\cosh(\frac{\gamma}{4})}\right) \right\} \ge W'$$

If $C(\gamma)$ is in configuration 2, let $Y^2[\gamma, \nu_1, \nu_2]$ be the Y-piece for γ from configuration 2. If either ν_1 or ν_2 is bigger than γ, then

$$w \ge \operatorname{arccosh}(2) = W$$

The lower bound for the width of the geodesic γ depends on the constant K, where

$$K = 3.326.$$

As a consequence of this lemma we have that if $C(\gamma)$ is in configuration 1 and $\gamma < 3.326 = K$ then its width w is bigger than $\arctanh(2/3) = W'$. If $C(\gamma)$ is in configuration 1 and $\gamma > K$ than its width w is bigger than $\operatorname{arccosh}(2) = W$. If $C(\gamma)$ is in configuration 2 its width w is always bigger than W.

For the proof of **Theorem 2.4.1** we also need the following lemma from [BS], p. 38 :

Lemma 2.3.3 *Let F be a compact Riemann surface of signature $(h, 1)$, such that $1 \le h$ and assume that the boundary η of F has length $\eta < 2\log(8h - 2)$ Then F contains a nsscg α of length smaller than $2\log(8h - 2)$ in its interior.*

A consequence of this lemma is that every compact R.S. of genus g contains a nsscg of length smaller than $2\log(4g - 2)$ in its interior (see [BS], p. 38). With the help of this lemma, we prove the following :

Lemma 2.3.4 *Let F be a compact Riemann surface of signature $(h, 1)$ and assume that the boundary η of F has length η. Then F contains a nsscg α of length smaller than*
$L = \max \left\{ \frac{\eta}{2} + \log(8h - 2), 2 \log(8h - 2) \right\}$ *in its interior.*

proof of Lemma 2.3.4 The collar of η, $C(\eta)$ of width w in F is in configuration 2. Let $Y^2[\eta, \nu_1, \nu_2]$ be the Y-piece for η from configuration 2 and $\nu_1 \leq \nu_2$. We have by (2.8)

$$\nu_1 < \frac{\eta}{2} + 2w$$

We now show that either F contains a nsscg α of length $2 \log(8h - 2)$ in its interior or that $2w < \log(8h - 2)$, from which follows that $\nu_1 < L$. If ν_1 is non-separating, then we are done. If not, we cut open F along ν_1 into two parts. The part F^1 that does not contain η has signature $(h^1, 1)$, where $h^1 \leq h - 1$ and its boundary is $\nu_1 \leq L$. In this case we argue as before and divide F^1 again into two parts. As long as the shorter scg in the Y-piece for the boundary geodesic from configuration 2 is separating, we can successively cut off pieces F^k from F. Let $(h^k, 1)$ be the signature of F^k, where $h^k \leq h - k$. Repeating the argument for ν_1 we obtain that the boundary geodesic of a F^k has length smaller than L. This procedure ends at least, when F^k is a Q-piece, a Riemann surface of signature (1,1). Then the decomposition of F^k yields a nsscg α of length smaller than L in the interior of $F^k \subset F$.

To conclude the proof, we have to show that $2w < \log(8h-2)$. Consider the surface $F' = F + F/\eta$, which is obtained by attaching the mirror image of F along the boundary η. It has genus $2h$. As a consequence of **Lemma 2.3.3** there exists a nsscg α of length smaller than $2 \log(8h - 2)$ in the interior of F'. Note that $\alpha \neq \eta$, as η is separating in F'. If $\alpha \cap \eta = \emptyset$, then α is contained in F and we are done. If $\alpha \cap \eta \neq \emptyset$, then it has to traverse the collar of η, $C(\eta)$ in F' at least twice and therefore $2 \log(8h - 2) > \alpha > 4w$, from which follows that $2w < \log(8h - 2)$. $\square$

With the help of the previous lemmata we establish an upper bound for the second shortest scg on a compact R.S. in the following lemma. Other methods were applied in [B], p. 123 to obtain such an upper bound, however the one obtained here is lower.

Lemma 2.3.5 *Let S be a compact Riemann surface of genus $g \geq 2$ and let γ_1 be the systole of S and γ_2 be the second shortest scg on S. Then $\gamma_1 \leq 2 \log(4g - 2)$ and $\gamma_2 \leq 3 \log(8g - 7)$.*

proof of Lemma 2.3.5 By an area argument (see [B], p. 124) the length of the shortest scg, γ_1 of a compact Riemann surface S of genus g is bounded from above by $2 \log(4g - 2)$. If γ_1 is separating, we cut open S along γ_1 into two parts S^1 and S^2. Let WLOG S^1 be the part, such that S^1 is of signature $(h, 1)$, such that $h \leq \frac{g}{2}$. By **Lemma 2.3.4** there exists a nsscg α of length smaller or equal to $2 \log(4g - 2)$ in the interior of S^1. In this case we have $\gamma_2 \leq 2 \log(4g - 2)$. If γ_1 is non-separating, we have to take another approach. The collar of γ_1, $C(\gamma_1)$ intersects in the point p_1 and has width w_1. We furthermore know that the interior of $C(\gamma_1)$ is isometrically embedded into S and therefore its area can not exceed the area of S. Therefore

$$2\gamma_1 \cdot \sinh(w_1) = \operatorname{area} C(\gamma_1) < \operatorname{area} S = 4\pi(g - 1)$$

Hence

$$w_1 \leq \operatorname{arcsinh}\left(\frac{2\pi(g-1)}{\gamma_1}\right)$$

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If $\frac{\pi}{2} \leq \gamma_1 \leq 2\log(4g-2)$, we obtain an upper bound for w_1, using that $\operatorname{arcsinh}(x) \leq \log(2x+1)$.

$$w_1 \leq \log(8(g-1)+1) < \log(8g-7)$$

In this case, we can conclude that there is a scg $\gamma_2 \neq \gamma_1$ in S of length smaller than

$$\gamma_2 < \frac{\gamma_1}{2} + 2w_1 < 3\log(8g-7).$$

To see this, we apply either equation (2.7) or equation (2.8), depending on, whether the collar of γ_1 is in configuration 1 or 2, respectively.

If $\gamma_1 < \frac{\pi}{2}$, we have to consider again both possible configurations. If γ_1 is in configuration 1, we obtain by equation (2.6), using the decomposition of the Y-piece from configuration 1, $Y^1[\gamma_1, \nu]$ into hexagons and as $w_1 \leq \operatorname{arcsinh}(\frac{2\pi(g-1)}{\gamma_1})$ that

$$\nu \leq 2\operatorname{arccosh}((\sinh(\frac{\gamma_1}{2})^2(\cosh(2\operatorname{arcsinh}(\frac{2\pi(g-1)}{\gamma_1}))-1))-1) \leq 4\log(8g-7). \qquad (2.10)$$

Here the upper bound of $4\log(8g-7)$ was determined using MAPLE (see appendix A). If we cut open S along ν, we obtain two pieces, one corresponding to $Y^1[\gamma_1, \nu]$ and a second piece S' of signature $(g-1, 1)$. Applying **Lemma 2.3.4** to S', we conclude that there exists a scg γ_2 in $S' \subset S$, whose length is bounded from above by $\frac{\nu}{2} + \log(8(g-1)-2) \leq 3\log(8g-7)$.

If γ_1 is in configuration 2, we obtain from the decomposition of the Y-piece from configuration 2, $Y^2[\gamma_1, \nu_1, \nu_2]$ into pentagons (equation (2.9)) and as $w_1 \leq \operatorname{arcsinh}(\frac{2\pi(g-1)}{\gamma_1})$ that

$$\nu_1 \leq 2\operatorname{arccosh}\left(\sinh(\frac{\gamma}{4})\frac{2\pi(g-1)}{\gamma_1}\right) \leq 3\log(8g-7). \qquad (2.11)$$

Again the upper bound of $3\log(8g-7)$ was determined using MAPLE (see appendix A). $\square$

A useful result for Riemann surfaces with boundary was obtained in [Gen] :

Lemma 2.3.6 *Let S be a Riemann surface of signature (g,n). Let γ_1 be the systole of S and $l(\partial S)$ be the length of the boundary of S. Then $\gamma_1 \leq 4\log(4g+2n+3) + l(\partial S)$.*

For a Q-piece, a R.S. of signature $(1,1)$ we have the following inequalities for a short canonical homology basis (α_1, α_2) by [Pa1], p. 59-62 :

Lemma 2.3.7 *Let Q be a Riemann surface of signature $(1,1)$ and γ be the boundary geodesic of Q. There exists a canonical homology basis (α_1, α_2) , $\alpha_1 \leq \alpha_2$ of Q satisfying the following inequalities :*

$$\cosh(\frac{\alpha_1}{2}) \leq \cosh(\frac{\gamma}{6}) + \frac{1}{2} \quad and$$

$$\cosh(\frac{\alpha_2}{2}) \leq \sqrt{\frac{\cosh^2(\frac{\gamma}{4}) + \cosh^2(\frac{\alpha_1}{2}) - 1}{2(\cosh(\frac{\alpha_1}{2}) - 1)}}.$$

The result is stated differently in [Pa1]. In [Pa1] α_1 is the shortest scg in the interior of Q and α_2 the shortest scg in Q that intersects α_1. But due to this construction, both α_1 and α_2 are non-separating. Furthermore α_2 intersects α_1 only once due to its minimality. Hence α_1 and α_2 have the required properties for a canonical homology basis.

Another lemma needed for the proof of the main theorem concerns comparison surfaces and can be found in [Pa2], p. 234 :

Lemma 2.3.8 *Let S be a Riemann surface of signature (g, n) with $n > 0$. Let $\beta_1, ..., \beta_n$ be the boundary geodesics of S. For $(\epsilon_1, ..., \epsilon_n) \in (\mathbb{R}^+)^n$ with at least one $\epsilon_i > 0$, there exists a comparison surface S_c with boundary geodesics of length $\beta_1 + \epsilon_1, ..., \beta_n + \epsilon_n$ such that for each simple closed geodesic γ_c in the interior of the comparison surface S_c, there exists a geodesic γ in the interior of S, such that $\gamma < \gamma_c$.*

We finally state a consequence of the collar lemma stated in [BS], p. 106 :

Lemma 2.3.9 *Let S be a Riemann surface of genus g with $g \geq 2$. Let γ be a simple closed geodesic in S. If η is another scg that does not intersect γ, then*

$$\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\gamma}{2})}\right) < \operatorname{dist}(\eta, \gamma)$$

and if w is the width of the collar of γ, $C(\gamma)$, then $w > \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\gamma}{2})}\right)$.

2.4 The second successive minimum of the Jacobian

To extend **Theorem 2.1.1**, we are going to prove the following :

Theorem 2.4.1 *Let S be a compact R.S. of genus g and let $J(S)$ be its Jacobian. Then*

$$m_1(J(S))^2 \leq \log(4g - 2) \quad and \quad m_2(J(S))^2 \leq 3.1 \log(8g - 7)$$

For the second successive minimum of a PPAV (A, H) of dimension g we obtain by Minkowski's second theorem :
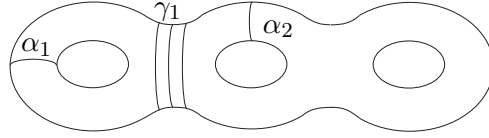
$$m_2(A, H)^2 \leqslant \frac{1}{\sqrt[g]{m_1(A, H)}} \left(\frac{4\sqrt[g]{g!}}{\pi}\right)^{2g/(2g-1)} \approx \frac{4g}{\sqrt[g]{m_1(A, H)}\pi e},$$

where the approximation applies for large g. Furthermore there exist examples of PPAVs where $m_2(A, H)^2$ is of order g. This follows from the fact that PPAVs, whose shortest lattice vector is maximal, have a basis of minimal non-zero vectors (see [Ber]). In this case all $m_k(A, H)^2$ are of order g. In contrast, we have for the Jacobian of a Riemann surface, $J(S)$ that $m_1(J(S))^2$ and $m_2(J(S))^2$ are both of order $\log(g)$, independent of the length of the shortest non-zero lattice vector.

proof of Theorem 2.4.1 It is well known that two nsscg α_1 and α_2 can be incorporated together into a canonical homology basis, if $\alpha_1 \cup \alpha_2$ does not separate S into two parts and if α_1
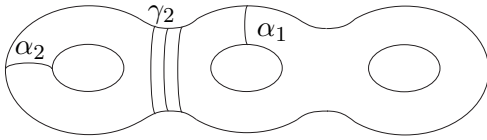
and α_2 have either exactly one or no intersection point. To prove **Theorem 2.4.1** we have to show that there exist two short nsscg, α_1 and α_2, with these properties and whose collar width is bounded from below. Then we can obtain **Theorem 2.4.1** from equation (2.4). The proof of **Theorem 2.4.1** depends on whether the shortest scg γ_1 in S, the systole, is separating or non-separating. We will distinguish several cases. These cases are depicted in Fig. 2.2.

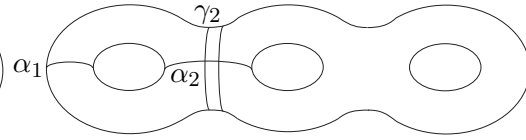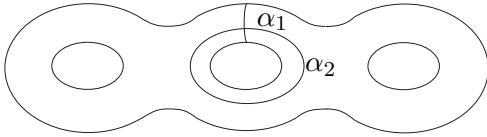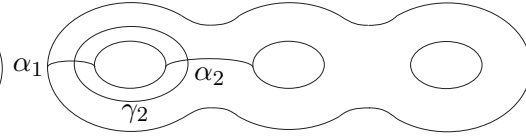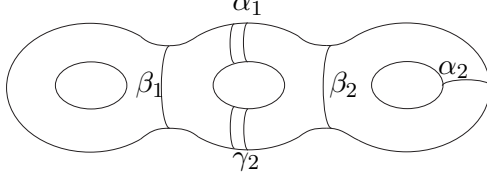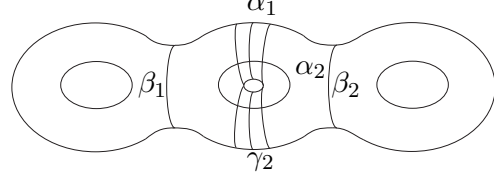Figure 2.2: Relative positions of α_1 and α_2 in the different cases of the proof of Theorem 2.4.1

Case 1. : *The systole γ_1 of S is a separating scg*

By **Lemma 2.3.5**, γ_1 has length smaller than $2\log(4g-2)$. We cut open S along γ_1, which yields two R.S., S_1 and S_2 of signature $(h_1, 1)$ and $(h_2, 1)$, respectively, such that $h_1 \leq h_2$. In both surfaces the half-collar of γ_1 is in configuration 2. By **Lemma 2.3.2** the width of a half-collar of γ_1 is bigger than W. Let α_1 and α_2 be the shortest nsscg in S_1 and S_2, respectively. We will show that both have a collar in S, whose width is at least W'. As $\alpha_1 \cup \alpha_2$ cannot divide S into two parts and as α_1 and α_2 do not intersect, they can be both together incorporated into a canonical homology basis of S. By **Lemma 2.3.3** and **Lemma 2.3.4**, we have that

$$\alpha_1 < 2\log(4g-2) \quad \text{and} \quad \alpha_2 < 2\log(8(g-1)-2) = 2\log(8g-10).$$

We now show that each $\alpha_i, i \in \{1, 2\}$ has a collar, whose width w_i is bounded from below. Namely, if $\alpha_i < K$ then $w_i > W'$ and if $\alpha_i > K$, then $w_i > W$.

Figure 2.3: Lift of $C(\gamma_1)$ in the universal covering

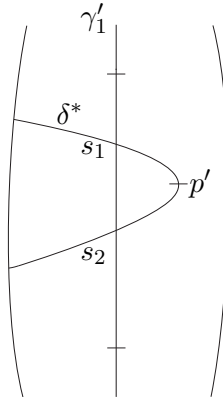Consider WLOG the collar of α_1 in S_1. Its closure self-intersects in a point $p \in S_1$ or a geodesic arc δ of length smaller than w_1, emanating perpendicularly from α_1 meets the boundary of S_1 first.

In the first case, we apply **Lemma 2.3.2**. We obtain that if $\alpha_1 < K$ then $w_1 > W'$ and if $\alpha_1 > K$, then $w_2 > W$. In the second case, we show that $\overline{C(\alpha_1)}$ can not self-intersect in $C(\gamma_1) \cap S_2$. Therefore it self-intersects a point $p \in S_2 \backslash C(\gamma_1)$. In this case every geodesic arc δ' emanating perpendicularly from α_1 with endpoint p has to traverse $C(\gamma_1) \cap S_2$ and hence has length bigger than W in S.

To prove that $\overline{C(\alpha_1)} \subset S$ can not self-intersect in $S_1 \cup C(\gamma_1)$, we lift $C(\gamma_1)$ into the universal covering. Here γ_1 lifts to γ_1' and $\delta \cap C(\gamma_1)$ to δ^* (*see Fig. 2.3.*). The lift δ^* is a geodesic. Let s_1 and s_2 be the first intersection points of δ^* and γ_1', the lift of γ_1 on opposite sides of p', the lift of p. There exists an unique geodesic arc connecting s_1 and s_2, which is an arc in γ_1', as γ_1' is a geodesic. But s_1 and s_2 also lie on δ^*, which implies that δ^* is contained in γ_1', a contradiction.

Summary of Case 1: If the systole γ_1 of S is a separating scg, then we can always find two short nsscg $\alpha_1 < 2\log(4g - 2)$ and $\alpha_2 < 2\log(8g - 10)$ for a homology basis of S. Let w_1 and w_2 be the collar width of α_1 and α_2, respectively. If $\alpha_i < K$ then $w_i > W'$ and if $\alpha_i > K$, then $w_i > W$, for $i \in \{1, 2\}$. It follows from equation (2.2) and (2.4) and the subsequent remark that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from **Theorem 2.4.1**.

Case 2. : *The systole γ_1 of S is a non-separating scg*

In this case we can find a homology basis of S, such that $\gamma_1 = \alpha_1$. As α_1 is the shortest nsscg, it follows from equation (2.2) and (2.4) that $m_1(J(S))^2$ satisfies the inequalities from **Theorem 2.4.1**.

To find a second short scg that does not separate S together with α_1, we have to consider several subcases. Let γ_2 be the second shortest scg on S. By **Lemma 2.3.5** its length smaller than $3\log(8g - 7)$. We will have to examine different cases, depending on whether γ_2 is separating, non-separating and non-separating with α_1 or non-separating but separating together with α_1.

Case 2.a) γ_2 *is separating*

Note that γ_1 and γ_2 can not intersect. It is easy to see that otherwise we could find a scg in S that is smaller than γ_2. We separate S into two parts, S_1 and S_2 along γ_2. Let S_1 be the part, which contains α_1 and S_2 be the remaining part of signature $(h_2, 1)$, such that $h_2 \leq g - 1$. In this case γ_2 is smaller than $2 \log(8g - 10)$, due to the minimality of γ_2. Otherwise we would arrive again at a contradiction, if we apply **Lemma 2.3.4** to S_2. The collar of γ_2 is in configuration 2. Let $Y^2[\gamma_2, \nu_1, \nu_2]$ be the Y-piece for γ_2 from configuration 2. We have to distinguish two cases for the choice of α_2, where the choice depends on $Y^2[\gamma_2, \nu_1, \nu_2]$.

Case 2.a.1.) $Y^2[\gamma_2, \nu_1, \nu_2] \neq Y^2[\gamma_2, \alpha_1, \alpha_1]$

In this case let α_2 be the shortest nsscg in S_2. As $\gamma_2 < 2 \log(8g - 10)$ it follows from **Lemma 2.3.4** that $\alpha_2 < 2 \log(8g - 10)$. α_1 and α_2 can be incorporated together into a canonical homology basis. As α_1 does not occur twice in the boundary curves of $Y^2[\gamma_2, \nu_1, \nu_2]$ and as γ_2 is the second shortest scg in S, we conclude by **Lemma 2.3.2** that the collar of γ_2 has width $w' \geq W$. We now determine a lower bound for the width of $C(\alpha_2)$, w_2. α_2 is the shortest nsscg in the interior of S_2. Hence we can argue as in the case of the collar of α_2 in **Case 1** to obtain a lower bound for the width of $C(\alpha_2)$, w_2. If $\alpha_2 < K$ then $w_2 > W'$ and if $\alpha_2 > K$, then $w_2 > W$.

Case 2.a.2.) $Y^2[\gamma_2, \nu_1, \nu_2] = Y^2[\gamma_2, \alpha_1, \alpha_1]$

If $\nu_1 = \nu_2 = \alpha_1$, then the interior of $Y^2[\gamma_2, \nu_1, \nu_2]$ is embedded in the Q-piece $Q_1 = S_1$, a R.S. of signature $(1, 1)$. This case can not occur, if

$$2.1 \leq \alpha_1 = \gamma_1 \leq \gamma_2 \quad \text{(see appendix A)} \tag{2.12}$$

, because otherwise there would exist a scg $\alpha_2' \neq \alpha_1$ in Q_1 that is smaller than γ_2 by **Lemma 2.3.7**. In this case let β be the shortest nsscg in S_2. As $\gamma_2 < 2 \log(8g - 10)$ it follows from **Lemma 2.3.4** that $\beta < 2 \log(8g - 10)$. Let α_2 be the shortest nsscg in S that does not intersect α_1. We have $\alpha_2 \leq \beta < 2 \log(8g - 10)$.

α_2 has a collar $C(\alpha_2)$, whose width w_2 is bounded from below. To see this, we cut open S along α_1 to obtain S'. Consider the collar of α_2 in S'. Its closure self-intersects in a point $p \in S'$ or a geodesic arc δ of length smaller than w_2, emanating perpendicularly from α_2 meets the boundary of S' first.

By **Lemma 2.3.9** $\mathrm{dist}(\alpha_1, \alpha_2) > \mathrm{arcsinh}(\frac{1}{\sinh(\frac{\alpha_1}{2})})$, as $\alpha_1 \leq 2.1$. It follows from the same arguments as in **Case 1.** that w_2 has the lower bound

$$w_2 > \min \left\{ \mathrm{arcsinh}(\frac{1}{\sinh(\frac{2.1}{2})}), W' \right\} > 0.73.$$

Summary of Case 2.a.) : We can always find two short nsscg α_1 and α_2 for a homology basis of S, whose lengths satisfy the same upper bounds as in as in **Case 1.** and whose collar width is bounded from below, such that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from **Theorem 2.4.1**.

Case 2.b) γ_2 *is non-separating and non-separating with* $\gamma_1 = \alpha_1$

In this case we have to distinguish two cases, $\alpha_2 = \gamma_2$ and $\alpha_2 \neq \gamma_2$.

Case 2.b.1.) $\alpha_2 = \gamma_2$

We have $\alpha_2 = \gamma_2 < 3\log(8g-7)$. Note that α_2 can not intersect α_1 more than once, as otherwise there would exist a scg, which is shorter than α_2. We now determine a lower bound for the width of the collar of α_2, $C(\alpha_2)$.
If $C(\alpha_2)$ is in configuration 2, let $Y^2[\alpha_2, \nu_1, \nu_2]$ be the Y-piece for α_2 from configuration 2. If ν_1 and ν_2 are both smaller than α_2, then both must be α_1. If $Y^2[\alpha_2, \nu_1, \nu_2] = Y^2[\alpha_2, \alpha_1, \alpha_1]$, then $Y^2[\alpha_2, \alpha_1, \alpha_1]$ is embedded in S as a Q-piece with boundary α_2 and α_2 would be separating, a contradiction. Hence we conclude by **Lemma 2.3.2** the width of $C(\alpha_2)$ is bigger than W.

If $C(\alpha_2)$ is in configuration 1, $\overline{C(\alpha_2)}$ self-intersects in a point p. There exist two geodesic arcs of length w_2 emanating from α_2 and perpendicular to α_2 having the endpoint p in common. These two arcs form a smooth geodesic arc δ_2. We lift α_2 and δ_2 in the universal covering. Here α_2 lifts to α_2' and α_2^* and δ_2 to δ_2'. In the covering there exist two points, $s' \in \alpha_2'$ and $s^* \in \alpha_2^*$, on opposite sites of δ_2' and at the same distance $r_2 \leq \frac{\alpha_2}{4}$ from δ_2', such that s' and s^* are mapped to the same point $s \in \alpha_2$ by the covering map. By drawing the geodesic λ' from s' to s^*, we obtain two isometric right-angled geodesic triangles. (*see Fig. 2.4.*)
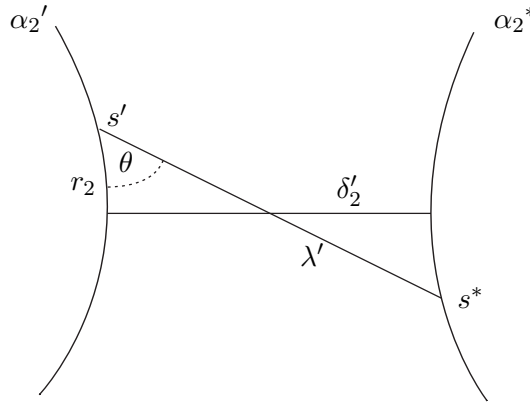
Figure 2.4: Two lifts of α_2 in the universal covering

We have to consider two subcases, $\lambda' \neq \alpha_1$ and $(\lambda' = \alpha_1) \wedge (\alpha_1 > 1.1)$. In the case $\lambda' = \alpha_1 \leq 1.1$, we will switch to **Case 2.b.2.)**

$\lambda' \neq \alpha_1$

If $\lambda' \neq \alpha_1$, then we can again argue as in **Case 1**. We obtain that if $\alpha_2 < K$ then $w_2 > W'$ and if $\alpha_2 > K$, then $w_2 > W$.

$\lambda' = \alpha_1$ *and* $\alpha_1 > 1.1$

To intersect α_1, α_2 has to traverse the collar of α_1. We can use this fact to derive a lower bound for the width of the collar of α_2, $C(\alpha_2)$.

Figure 2.5: Lift of $C(\alpha_1)$ in the universal covering

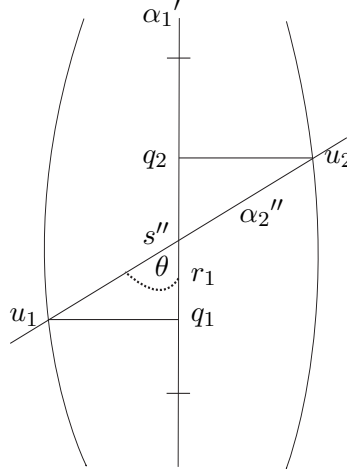Lift $C(\alpha_1)$ in the universal covering and let $C'(\alpha_1)$ be the lift of $C(\alpha_1)$ (*see Fig. 2.5*). α_2 traverses the collar $C(\alpha_1)$ of width w_1. It lifts to α_2'' and in the lift it enters $C'(\alpha_1)$ at a point u_1 and leaves at a point u_2. Consider the geodesic arcs emanating from u_1 and u_2 respectively and meeting the lift of α_1, α_1' perpendicularly. Their length is w_1. Let q_1 and q_2 be the endpoints of these geodesic arcs on α_1'. α_2'' intersects α_1' in the midpoint s'' of the geodesic arc between q_1 and q_2 under angle θ. Here s'' is a lift of $s \in Q_1$. Set $r_1 = \text{dist}(q_1, s'') = \text{dist}(q_2, s'')$. Then r_1 is smaller or equal to $\frac{\alpha_1}{4}$, as α_2 is the third shortest scg in Q_1 and otherwise there exists another point u_2', such that u_2 and u_2' map to the same point on Q_1 under the universal covering map, such that $\text{dist}(u_1, u_2') < \text{dist}(u_1, u_2)$, a contradiction to the fact that α_2 is minimal. Consider the right-angled triangle with vertices u_1, q_1 and s''. From the geometry of hyperbolic triangles (see [B], p. 454) we have for θ :

$$\sin(\theta) = \frac{\sinh(w_1)}{\sinh(\text{dist}(u_1, s''))} \quad \text{and} \quad \cosh(\text{dist}(u_1, s'')) = \cosh(w_1) \cdot \cosh(r_1)$$

from which follows, as $\sinh^2(x) = \cosh^2(x) - 1$ that

$$\sin(\theta) = \frac{\sinh(w_1)}{\sqrt{\cosh^2(r_1) \cdot \cosh^2(w_1) - 1}}$$

The point s'' corresponds to s' in the other lift of α_2 (*see Fig. 2.4.*) and the angle θ to the interior angle of the right-angled geodesic triangle at the vertex s'. From the geometry of this triangle we get :

$$\sin(\theta) = \frac{\sinh(w_2)}{\sinh(\frac{\alpha_1}{2})}$$

and therefore, as $\sinh(w_2)$ is decreasing with increasing $r_1 \leq \frac{\alpha_1}{4}$

$$\sinh(w_2) \geq \frac{\sinh(w_1) \cdot \sinh(\frac{\alpha_1}{2})}{\sqrt{\cosh^2(\frac{\alpha_1}{4}) \cdot \cosh^2(w_1) - 1}} \tag{2.13}$$

Note that the left hand side in (2.13) is increasing with increasing w_1 and increasing α_1. As the width of $C(\alpha_1)$, w_1 is bigger than W', we get a lower bound for w_2, if we set $w_1 = W'$. In this case we obtain from equation (2.13)

$$w_2 \geq w_2{}^Q = \operatorname{arcsinh}\left(\frac{\frac{2\sqrt{5}}{5} \cdot \sinh(\frac{\alpha_1}{2})}{\sqrt{\frac{9}{5}\cosh^2(\frac{\alpha_1}{4}) - 1}}\right) \tag{2.14}$$

In this case we obtain, with $\alpha_1 > 1.1$:

$$m_2(J(S))^2 < \frac{3\log(8g-7)}{\pi - 2 \cdot \arcsin(\frac{1}{\cosh(w_2{}^Q)})} \leq 3.1\log(8g-7) \quad \text{(see appendix A).} \tag{2.15}$$

Summary of Case 2.b.1.) : We can always find two short nsscg $\alpha_1 = \gamma_1 < 2\log(4g-2)$ and $\alpha_2 = \gamma_2 < 3\log(8g-7)$ for a homology basis of S. Their collar width is bounded from below, such that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from **Theorem 2.4.1**.

Case 2.b.2.) $\alpha_2 \neq \gamma_2$

This case treats the remaining case for $C(\gamma_2)$ in configuration 1, and the geodesic λ' (see Fig. 2.4.) is α_1, with $\alpha_1 \leq 1.1$.
In this case we cut open S along α_1 to obtain the surface S' of signature $(g-1, 2)$. Let α_1' and α_1'' be the boundary. In this case we let α_2 be the shortest nsscg in S' that does not intersect α_1. We first show the following claim.

Claim 2.4.2 *The shortest nsscg $\alpha_2 \subset S'$ has length smaller than $2\log(24g-23) + 2.2$.*

proof of Claim 2.4.2 We first show that there exists a scg of length smaller than $2\log(24g-23) + 2.2$ in the interior of S'. It is sufficient to proof this statement for the case $\alpha_1 = 1.1$. It follows from **Lemma 2.3.8** that this is also true for $\alpha_1 < 1.1$. If there exists a scg of length smaller than $2\log(24g-23) + 2.2$ in S' and this geodesic is non-separating, we are done. If it is separating, we apply **Lemma 2.3.4** and conclude that there exists a nsscg in S' that is smaller than $\log(24g-23) + 1.1 + \log(8g-10) < 2\log(24g-23) + 2.2$, which proves the claim.
Let $\alpha_1' = 1.1$. The closure of the half-collar $\overline{C(\alpha_1')} \subset S'$ self-intersects in a point in S' or a geodesic arc emanating perpendicularly from α_1' meets α_1'' perpendicularly in a point p_1 before $\overline{C(\alpha_1')}$ self-intersects in S'. We have to examine these two cases.

i) The closure of the half-collar of α_1' intersects α_1'' in p_1 before self-intersecting in S'

A geodesic arc $\sigma \subset S'$ meets α_1' and α_1'' perpendicularly on both endpoints where p_1 is the endpoint on α_1''. We now define for a scg γ in S and an $r > 0$, the distance set of distance r of γ, $Z_r(\gamma)$ by

$$Z_r(\gamma) = \{x \in S \mid \operatorname{dist}(x, \gamma) < r\}.$$

As long as r is small enough, such that $Z_r(\gamma) \subset C(\gamma)$, we have from hyperbolic geometry that $\operatorname{area}(Z_r(\gamma)) = 2\gamma \cdot \sinh(r)$. Consider $Z_\sigma(\alpha_1') \cap S'$. It is embedded into S' and therefore its area can not exceed the area of $S' = S$, which is smaller than $4\pi(g-1)$. Therefore

$$\alpha_1' \cdot \sinh(\sigma) = \operatorname{area}(Z_\sigma(\alpha_1') \cap S') < \operatorname{area}(S) = 4\pi(g-1)$$

As $\alpha_1{}' = 1.1$ and as $\operatorname{arcsinh}(x) \le \log(2x + 1)$, we obtain an upper bound for σ.

$$\sinh(\sigma) \le \frac{4\pi(g - 1)}{1.1} \Rightarrow \sigma \le \log(24g - 23)$$

Hence we conclude that the shortest scg $\beta_1 \subset S'$ in the free homotopy class of $\alpha_1{}' \sigma \alpha_1{}'' \sigma^{-1}$ has length smaller than $2\log(24g - 23) + 2.2$.

ii) The closure of the collar of $\alpha_1{}'$ self-intersects in $p_1 \in S'$

A geodesic arc σ passes through p_1 and meets $\alpha_1{}'$ perpendicularly on both endpoints. Let $Y^2[\alpha_1{}', \nu_1, \nu_2] \subset S'$ be the Y-piece for $\alpha_1{}'$ from configuration 2.
As $\alpha_1{}' = 1.1$, we conclude by the same area argument as in case *i)* that $\sigma < 2\log(24g - 23)$. From equation (2.8) it follows that both ν_1 and ν_2 are smaller than

$$\alpha_1{}' + \sigma \le 2\log(24g - 23) + 1.1.$$

At least one of these geodesics is not $\alpha_1{}''$. Hence there exists a scg of length smaller than $2\log(24g - 23) + 2.2$ in S'. Hence we have proven the claim. $\square$

Let w_2 be the width of the collar of α_2. In this case we conclude from **Lemma 2.3.9** that $\operatorname{dist}(\alpha_1, \alpha_2) > \operatorname{arcsinh}(\frac{1}{\sinh(\frac{\alpha_1}{2})}) > \operatorname{arcsinh}(\frac{1}{\sinh(\frac{1.1}{2})})$, as $\alpha_1 \le 1.1$. It follows from the same arguments as in **Case 2.a.2.)** that w_2 has the lower bound

$$w_2 > \min\left\{ \operatorname{arcsinh}(\frac{1}{\sinh(\frac{1.1}{2})}), W' \right\} > W'.$$

Summary of Case 2.b.2.) : We have that $\alpha_2 < 2\log(24g - 23) + 2.2$ and $w_2 > W'$. We obtain that

$$m_2(J(S))^2 < \frac{2\log(24g - 23) + 2.2}{\pi - 2\arcsin(\frac{1}{\cosh(W')})} \le 3.1\log(8g - 7)$$

Case 2.c) *γ_2 is non-separating, but separating with $\gamma_1 = \alpha_1$*

By **Lemma 2.3.5** we know that the length of γ_2 is bounded by $3\log(8g - 7)$. It is easy to see that γ_2 can not intersect α_1. It can not intersect α_1 more than once, due to the minimality of the two geodesics and it can not intersect α_1 once due to the fact that it is separating with α_1. As γ_2 is separating with α_1, we conclude by **Lemma 2.3.2** that its collar width is bounded from below. If $\gamma_2 < K$ then the width of its collar is bigger than W' and if $\gamma_2 > K$, then the width of its collar is bigger than W.
We cut open S along γ_2 and α_1. The two geodesics divide S into S_1 and S_2. We first show, the following claim :

Claim 2.4.3 *The shortest nsscg $\alpha_2{}^i \subset S_i$ has length smaller than $4.5\log(8g - 7)$ for $i \in \{1, 2\}$.*

The proof is similar to the proof of **Claim 2.4.2**.
proof of Claim 2.4.3 Consider WLOG S_1. We proof the claim for the cases $\alpha_1 < \pi$ and $\alpha_1 \ge \pi$:

a) $\alpha_1 \geq \pi$

$\overline{C(\alpha_1)}$ self-intersects in a point in S_1 or a geodesic arc emanating perpendicularly from α_1, of length smaller than w_1 meets γ_2 perpendicularly in a point p_1. We examine two cases, which depend on how $\overline{C(\alpha_1)}$ intersects itself.

i) The closure of the collar of α_1 intersects γ_2 in p_1 before self-intersecting in S_1

A geodesic arc $\sigma \subset S_1$ meets α_1 and γ_2 perpendicularly on both endpoints where p_1 is the endpoint on γ_2. Consider $Z_\sigma(\alpha_1) \cap S_1$. It is embedded into S_1 and therefore its area can not exceed the area of S_1, which is smaller than $4\pi((g-2)-1)$. Therefore

$$\alpha_1 \cdot \sinh(\sigma) = \text{area}\, Z_\sigma(\alpha_1) \cap S_1 < \text{area}\, S_1 = 4\pi(g-3).$$

As $\pi \leq \alpha_1$ and as $\text{arcsinh}(x) \leq \log(2x+1)$, we obtain an upper bound for σ.

$$\sinh(\sigma) \leq \frac{4\pi(g-3)}{\pi} \Rightarrow \sigma \leq \log(8(g-3)+1) < \log(8g-7)$$

Hence we conclude that the shortest scg β_1 in the free homotopy class of $\alpha_1\,\sigma\gamma_2\sigma^{-1}$ has length smaller than $7\log(8g-7)$. It is a separating scg. Applying **Lemma 2.3.4** we conclude that there exists a nsscg of length smaller than $4.5\log(8g-7)$ in S_1. Note that, using the hexagon decomposition (see [BS], p. 454) of the Y-piece Y' with boundary geodesics β_1, α_1 and γ_2, we can obtain the exact value of the length of β_1, which will be useful later for small values of α_1. It is

$$\cosh(\frac{\beta_1}{2}) = \sinh(\frac{\alpha_1}{2})\sinh(\frac{\gamma_2}{2})\cosh(\sigma) - \cosh(\frac{\alpha_1}{2})\cosh(\frac{\gamma_2}{2}). \tag{2.16}$$

ii) The closure of the collar of α_1 self-intersects in $p_1 \in S_1$

A geodesic arc σ passes through p_1 and meets α_1 perpendicularly on both endpoints. Let $Y^2[\alpha_1, \nu_1, \nu_2]$ be the Y-piece for α_1 from configuration 2.
if $\alpha_1 \geq \pi$, we conclude by the same area argument as in case *i)* that $\sigma < \log(8g-7)$. From equation (2.8) it follows that both ν_1 and ν_2 are smaller than

$$\alpha_1 + \sigma \leq 3\log(8g-7).$$

At least one of them is not γ_2. Therefore, if this scg is non-separating, we are done. If it is separating, we cut off the part of S_1 that contains α_1 and conclude by **Lemma 2.3.4** that this part contains a nsscg of length smaller than $3\log(8g-7) < 4.5\log(8g-7)$. This settles the claim in the case $\alpha_1 \geq \pi$.

b) $\alpha_1 < \pi$

If $\alpha_1 < \pi$, we use the fact that there exists a comparison surface S_1^c for S_1, as described in **Lemma 2.3.8**, such that one boundary geodesic has length π and the other has length γ_2 and conclude that it contains a scg of length smaller than $4.5\log(8g-7)$ in its interior. Therefore there exists a scg of length smaller than $4.5\log(8g-7)$ in S_1, by **Lemma 2.3.8**. If this geodesic

is separating, we apply again **Lemma 2.3.4** and conclude that there exists a nsscg in S_1 that is smaller than $4.5 \log(8g - 7)$. $\square$

In total, we obtain that the shortest nsscg $\alpha_2{}^1$ in S_1 and the shortest nsscg $\alpha_2{}^2$ in S_2 are both smaller than $4.5 \log(8g - 7)$. Both can be incorporated with α_1 into a canonical homology basis. Consider the sets $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$, with $W' = \operatorname{arctanh}(2/3)$. We now choose a nsscg α_2 that is non-separating with α_1. The choice depends on how $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect. We distinguish two cases.

Case 2.c.1.) $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_1 = \emptyset$ *or* $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_2 = \emptyset$

If $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_1 = \emptyset$, then we choose $\alpha_2 = \alpha_2{}^2 \subset S_2$ and if $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_2 = \emptyset$ we choose $\alpha_2 = \alpha_2{}^1 \subset S_1$. Consider WLOG the first case. We show that the collar of $\alpha_2{}^2$ has width bigger than W'. If the closure of the collar of $\alpha_2{}^2$ self-intersects in S_1, it has to traverse either $S_1 \cap Z_{W'}(\alpha_1)$ or $S_1 \cap Z_{W'}(\gamma_2)$ and hence its width is bigger than W'. This follows from the same arguments as in **Case 1**. If $\overline{C(\alpha_2{}^2)}$ self-intersects in S_2, we conclude by **Lemma 2.3.2** that $\alpha_2{}^2$ has a collar of with bigger than W'.

Summary of Case 2.c.1.) : α_2 is the shortest nsscg in either S_1 or S_2, its length is restricted by $\alpha_2 < 4.5 \log(8g - 7)$, the width of its collar is bigger than W'. It follows from equation (2.2) and (2.4) that
$$m_2(J(S))^2 < 3.1 \log(8g - 7).$$

Case 2.c.2.) $Z_{W'}(\alpha_1)$ *and* $Z_{W'}(\gamma_2)$ *intersect both in* S_1 *and* S_2

If $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect both in S_1 and S_2 we have to argue in a different way. We choose another small nsscg to be α_2. We may assume that

$$\alpha_1 \geq 1.5 \quad \text{and} \quad \gamma_2 \geq 2.1 \quad \text{(see appendix A).} \tag{2.17}$$

Otherwise we arrive at a contradiction to the fact that $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ do not intersect. This contradiction follows from equation (2.16) , as $\sigma < 2W'$.
We now choose α_2. Let δ' and δ'' be the shortest geodesic arcs in S_1 and S_2, respectively, connecting α_1 and γ_2. Their length is bounded from above by $2W'$ as $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect. The endpoints of δ' and δ'' divide each α_1 and γ_2 into two geodesic arcs. Let $\alpha_1{}^*$ and γ_2^* be the shorter of these arcs. We define α_2 to be the shortest scg in the free homotopy class of $\delta' \alpha_1{}^* \delta'' \gamma_2^*$. It intersects α_1 only once. The length of α_2 is restricted by

$$\alpha_2 < \frac{\alpha_1}{2} + \frac{\gamma_2}{2} + 4W' < 2.5 \log(8g - 7) + 4W'.$$

However, if the length of γ_2 is small, the upper bound for α_2 given above is not sufficient to establish an appropriate lower bound for the collar of α_2. Therefore we will establish a better upper bound for the length of α_2.
Let $Y' \subset S_1$ be the Y-piece with boundaries α_1, γ_2 and the shortest scg β_1 in the free homotopy class of $\alpha_1 \delta' \gamma_2 \delta'^{-1}$. Let $Y'' \subset S_2$ be the Y-piece constructed in the same way in S_2, having as third boundary β_2. The union $Y' \cup Y''$ in S is embedded as a Riemann surface F of signature (1,2) *(see Fig. 2.6.)*.
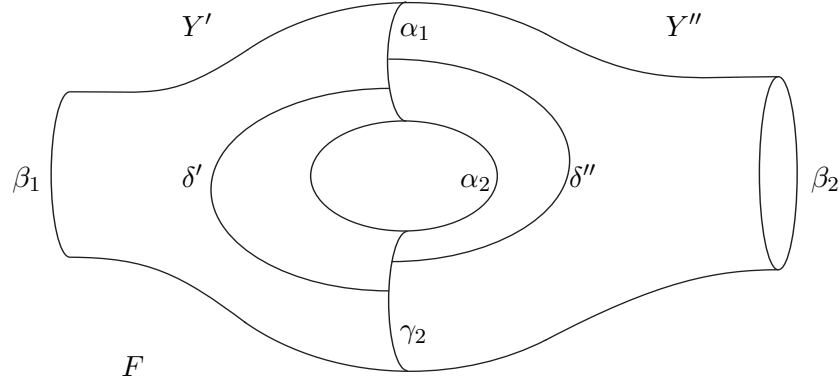
Figure 2.6: The Riemann surface F of signature (1,2)

Lift α_1 to α_1' and equally δ' and δ'' into the universal covering *(see Fig. 2.7.)*. By abuse of notation we will denote the lift of these two arcs by the same letter. To δ' and δ'' attach the adjacent lifts of γ_2, γ_2' and γ_2'' on opposite sides of α_1'. Let q' be the endpoint of δ' on α_1' and q'' be the endpoint of δ'' on α_1', such that $\mathrm{dist}(q', q'') \leq \frac{\alpha_1}{2}$. Let furthermore be s' be the endpoint of δ' on γ_2' and s'' be the endpoint of δ'' on γ_2''. Let s^* be the point on γ_2'' that maps to the same point on γ_2 under the covering map as s', such that $\mathrm{dist}(s'', s^*) \leq \frac{\gamma_2}{2}$. Let equally be s^{**} be the point on γ_2' that maps to the same point on γ_2 under the covering map as s'', such that $\mathrm{dist}(s^{**}, s') = \mathrm{dist}(s'', s^*)$. Let η' be the geodesic arc connecting the midpoint of s' and s^{**} on γ_2' and midpoint of s'' and s^* on γ_2''.
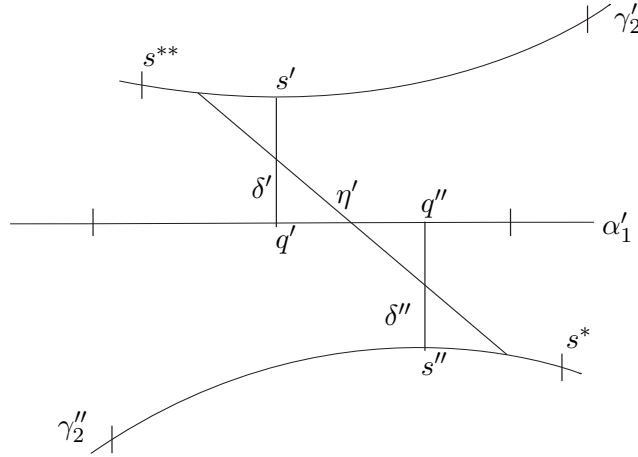
Figure 2.7: Lifts of α_2 and γ_2 in the universal covering

The image of η' under the covering map, η forms a closed geodesic arc on S. As η is in the same free homotopy class as α_2, its length provides an upper bound for the length of α_2. In Fig. 2.7., the points s^{**} and s^* lie on opposite sides of δ' and δ''. We will derive an upper bound for this case. In any other case the length of η' is either shorter or the situation is a mirror image of the depicted one. It is clear that η' is maximal, if $\mathrm{dist}(s^{**}, s') = \mathrm{dist}(s'', s^*) = \frac{\gamma_2}{2}$ and $\mathrm{dist}(q', q'') = \frac{\alpha_1}{2} = \frac{\gamma_2}{2}$. Therefore it is sufficient to discuss this case.

In this case we obtain from the geometry of hyperbolic triangles :

$$\cosh(\frac{\alpha_2}{4}) < \cosh(\frac{\eta'}{4}) = \cosh(\frac{\gamma_2}{4}) \cdot \cosh(\frac{\delta'}{2}) \leq \cosh(\frac{\gamma_2}{4}) \cosh(W') \tag{2.18}$$

as $\delta' \leq 2W'$. We now determine a lower bound for the width of the collar of α_2. We have to distinguish several subcases :

Case 2.c.2.a) *The collar of α_2 is in configuration 1*

$\overline{C(\alpha_2)}$ has width w_2. It self-intersects in a point p. Lift α_2 into the hyperbolic plane as described **Case 2.b.1.)** (*see Fig. 2.4.*) We have to discuss two cases, $\lambda' = \alpha_1$ and $\lambda' \neq \alpha_1$.

$\lambda' = \alpha_1$

This case was discussed in **Case 2.b.1.)**. We may assume that $\alpha_1 \geq 1.5$ from equation (2.16). Hence we can apply equation (2.13) with $\alpha_1 = 1.5$ and $w_1 = W'$ and obtain

$$w_2 > 0.66$$

$\lambda' \neq \alpha_1$

If $\lambda' \geq \alpha_2$ we can apply **Lemma 2.3.2** and conclude that $w_2 \geq W'$. If $\lambda' < \alpha_2$, we conclude from equation(2.18) and as γ_2 is the second shortest scg in S that

$$\gamma_2 \leq \lambda' < \alpha_2 < 4\operatorname{arccosh}(\cosh(\frac{\gamma_2}{4})\cosh(W')). \tag{2.19}$$

From the geometry of right-angled hyperbolic triangles (see [B], p. 454), we obtain from Fig. 2.4. that

$$\cosh(\frac{\alpha_2}{4})\cosh(w_2) \geq \cosh(r_2)\cosh(w_2) = \cosh(\frac{\lambda'}{2}).$$

Using the upper bound for α_2 and the lower bound for λ' from equation (2.19) in this inequality we obtain :

$$\cosh(w_2) \geq \frac{\cosh(\frac{\gamma_2}{2})}{\cosh(\frac{\gamma_2}{4})\cosh(W')}. \tag{2.20}$$

Case 2.c.2.b) *The collar of α_2 is in configuration 2*

Let $Y^2[\alpha_2, \nu_1, \nu_2]$ be the Y-piece for α_2 in configuration 2. $\overline{C(\alpha_2)}$ self-intersects in the point p_2, such that $\operatorname{dist}(p_2, \alpha_2) = w_2 = \frac{\delta_2}{2}$. The geodesic arc δ_2 emanating perpendicularly from α_2 passes through this point and its endpoints divide α_2 into two parts, α_2' and α_2''. The common perpendiculars of the boundary geodesics of $Y^2[\alpha_2, \nu_1, \nu_2]$ separate the Y-piece into two isometric hexagons and δ_2 decomposes these hexagons into pentagons. By the pentagon formula (see [B], p. 454) we have

$$\sinh(\frac{\delta_2}{2})\sinh(\frac{\alpha_2'}{2}) = \cosh(\frac{\nu_1}{2}) \quad \text{and} \quad \sinh(\frac{\delta_2}{2})\sinh(\frac{\alpha_2''}{2}) = \cosh(\frac{\nu_2}{2})$$

None of these boundary geodesics can be α_1, as α_1 intersects α_2. If either ν_1 or ν_2 is bigger, than α_2, we obtain from **Lemma 2.3.2** that $w_2 > W$. If not, then both must be bigger than γ_2.

Additionally $\frac{\alpha_2'}{2} + \frac{\alpha_2''}{2} = \frac{\alpha_2}{2}$. Therefore either $\frac{\alpha_2'}{2}$ or $\frac{\alpha_2''}{2}$ is bigger than $\frac{\alpha_2'}{4}$. Let WLOG α_2' be the bigger one. We obtain from equation (2.18) :

$$\sinh(w_2)\sinh(\operatorname{arccosh}(\cosh(\frac{\gamma_2}{4})\cosh(W'))) \geq \sinh(\frac{\delta_2}{2})\sinh(\frac{\alpha_2'}{2}) = \cosh(\frac{\nu_1}{2}) \geq \cosh(\frac{\gamma_2}{2}).$$

or equally , as $\sinh(x) = \sqrt{\cosh^2(x) - 1}$ for $x \geq 0$:

$$\sinh(w_2) \geq \frac{\cosh(\frac{\gamma_2}{2})}{\sqrt{\cosh^2(\frac{\gamma_2}{4})\cosh^2(W') - 1}}.$$

It follows from this equation that in **Case 2.c.2.b)**

$$w_2 > 0.96.$$

Summary of Case 2.c.2.) : α_2 to be the shortest scg in the free homotopy class of $\delta' \alpha_1{}^* \delta'' \gamma_2^*$ *(see Fig.2.6)*. Its length is restricted by

$$\alpha_2 < 4\operatorname{arccosh}(\cosh(\frac{\gamma_2}{4})\cosh(W')).$$

From the discussion of the subcases **Case 2.c.2.a)** we conclude that the width of the collar w_2 is bounded from below by

$$w_2 \geq \min\left\{0.66, \operatorname{arccosh}\left(\frac{\cosh(\frac{\gamma_2}{2})}{\cosh(\frac{\gamma_2}{4})\cosh(W')}\right)\right\}$$

As a consequence of equation (2.16),we have that $2.1 \leq \gamma_2$. With the help of this lower bound it follows from the above equation that w_2 is bounded from below. As α_2 is bounded from above, it follows from equation (2.4) and (2.2) that $m_2(J(S))^2$ is bounded from above. A refined analysis shows that

$$m_2(J(S))^2 < 3.1\log(8g - 7) \quad \text{see appendix A.} \tag{2.21}$$

This proves that **Theorem 2.4.1** is valid. $\square$

2.5   The Jacobian of a surface with small simple closed geodesics

If a R.S. contains a certain number of mutually disjoint small simple closed geodesics, we obtain the following corollary of **Theorem 2.4.1** :

Corollary 2.5.1 *Let S be a compact R.S. of genus g that contains n disjoint simple closed geodesics $(\eta_j)_{j=1,..,n}$ of length smaller than t. If we cut open S along these geodesics, then the decomposition contains m R.S. S_i of signature (g_i, n_i), with $g_i > 0$. There exist m linear independent vectors $(l_i)_{i=1,..,m}$ in the lattice of the Jacobian $J(S)$, such that*

$$\|l_i\|_H{}^2 \leq \frac{(n_i + 1)\max\{4\log(4g_i + 2n_i - 3), t\}}{\pi - 2\arcsin(M)} \quad \text{for} \quad i \in \{1, ..., m\},$$

where $M = \min\{\frac{\sinh(\frac{t}{2})}{\sqrt{\sinh(\frac{t}{2})^2 + 1}}, \frac{1}{2}\}$

As the vectors l_i are linearly independent, the corollary implies improved bounds for a certain number of $m_k(J(S))$. This corollary is related to a Theorem of Fay. In [Fa], chap. III a sequence of Riemann surfaces S_t is constructed, where t denotes the length of a separating simple closed geodesic η. η divides S_t into two surfaces S_i of signature $(g_i, 1), i \in \{1, 2\}$. If $t \to 0$ then the period Gram matrix for a suitable canonical homology basis converges to a block matrix, where each block is in $M_{2g_i}(\mathbb{R})$.

If η is any separating geodesic that separates a R.S. S into two surfaces S_i of signature $(g_i, 1), i \in \{1, 2\}$.is small enough. Applying **Lemma 2.3.4**, we obtain a slightly better bound than in the corollary :

$$m_i(J(S))^2 \le \log(8g_i - 2) \quad \text{for} \quad i \in \{1, 2\}.$$

The corollary shows that indeed the first two successive minima of the Jacobian of the surfaces S_i can only be of order $log(g_i)$ and gives explicit bounds depending on the length of t.

It shows that $m_1(J(S))^2$ and $m_2(J(S))^2$ of a R.S. with a sufficiently small separating scg, is at most of the order of the first successive minimum of a R.S. of genus g_1 and g_2, respectively.

proof of Corollary 2.5.1 The proof is very similar to the discussion of **Case 1.** of **Theorem 2.4.1**.

Let $\eta_i \le t$ be one of the simple closed geodesics that divide S. By **Lemma 2.3.9** the width of a half-collar of η_i is bigger than $\operatorname{arcsinh}(\frac{1}{\sinh(\frac{t}{2})})$ on both sides of η_i. It follows also from the collar theorem that any other scg in S has a distance greater than $\operatorname{arcsinh}(\frac{1}{\sinh(\frac{t}{2})})$ from η_i.

Let S_i be a surface of genus (g_i, n_i), $g_i > 0$ from the decomposition of S. Let WLOG $\eta_1, ... \eta_{n_i}$ be its boundary geodesics. We first prove that the shortest nsscg, α_i in S_i is smaller than $(n_i + 1) \max\{4 \log(4g_i + 2n_i + 3), t\}$. Then we show that it has a collar in S whose width is bounded from below. By **Lemma 2.3.6**, there exists a scg γ_i^1 in S_i of length $\gamma_i^1 \le 4 \log(4g_i + 2n_i + 3) + n_i t$. We have that

$$4 \log(4g_i + 2n_i + 3) + n_i t \le (n_i + 1) \max\{4 \log(4g_i + 2n_i + 3), t\}.$$

Hence, if γ_i^1 is non separating, we are done. If γ_i^1 is separating, we cut open S_i along γ_i^1. S_i decomposes into two surfaces, such that one of these two, S_i^2 has signature (g_i', n_i'), with $g_i' > 0$ and $n_i' \le n_i - 1$. The length of its boundary is smaller than $4 \log(4g_i + 2n_i + 3) + (n_i - 1)t$. We can again apply **Lemma 2.3.6** to this surface to obtain an upper bound for the length of a scg in S_i. Repeating this process iteratively we obtain that there exists a nsscg in S_i, whose length is smaller than $(n_i + 1) \max\{4 \log(4g_i + 2n_i + 3), t\}$.

Each $\alpha_i, i \in \{1, ..., m\}$ has a collar, whose width w_i is bounded from below. Namely, if $\alpha_i < K$ then $w_i > \min\left\{\operatorname{arcsinh}(\frac{1}{\sinh(\frac{t}{2})}), W'\right\}$ and if $\alpha_i > K$, then $w_i > \min\left\{\operatorname{arcsinh}(\frac{1}{\sinh(\frac{t}{2})}), W\right\}$. This follows from the same line of argumentation as in **Case 1.** of **Theorem 2.4.1**. The $(\alpha_i)_{i=1,...,m}$ can be together incorporated into a canonical homology basis of S. From the bounds on the length and the width of the collars of the geodesics follows the bound on the norm of the lattice vectors of the Jacobian of S. In total we obtain **Corollary 2.5.1**. $\square$

2.6 The Jacobian of hyperelliptic surfaces

Using the same methods as in [BS], we will show that

Theorem 2.6.1 *If S is a hyperelliptic R.S. of genus g and $J(S)$ its Jacobian, then*

$$m_1(J(S))^2 \le \frac{3\log(3 + 2\sqrt{3} + 2\sqrt{5 + 3\sqrt{3}})}{\pi} = 2.4382...$$

It is worth mentioning that this result follows from a simple refinement of the proof that the systole of hyperelliptic surfaces is bounded from above by a constant, which was shown in [Ba1] and [Je].

proof of Theorem 2.6.1 For the proof of **Theorem 2.6.1** we first give a suitable definition of a hyperelliptic surface.

Definition 2.6.2 *Let S be a compact Riemann surface of genus g. An involution is an isometry $\phi : S \to S, \phi \ne id$, such that $\phi^2 = id$. The surface S is hyperelliptic, if it has an involution that has exactly $2g + 2$ fixed points. These fixed points are called the Weierstrass points (WPs).*

It is well known that the above definition is equivalent to the usual one. Let S be a hyperelliptic surface of genus g with involution ϕ. We will show that the shortest nsscg α_1 of S is bounded by a constant, independent of the genus. It was shown in [BS] that the width of the collar of the shortest nsscg on a R.S. S is bounded from below. It then follows from equation (2.4) and (2.2) that **Theorem 2.6.1** holds.

Consider the quotient surface $S\backslash\phi$. This surface is a topological sphere with $2g+2$ cones of angle π, whose vertices $\{p_i\}_{i=1..2g+2}$ are the images of the WPs $\{p_i^*\}_{i=1..2g+2}$ under the projection *(see Fig. 2.8.)*.
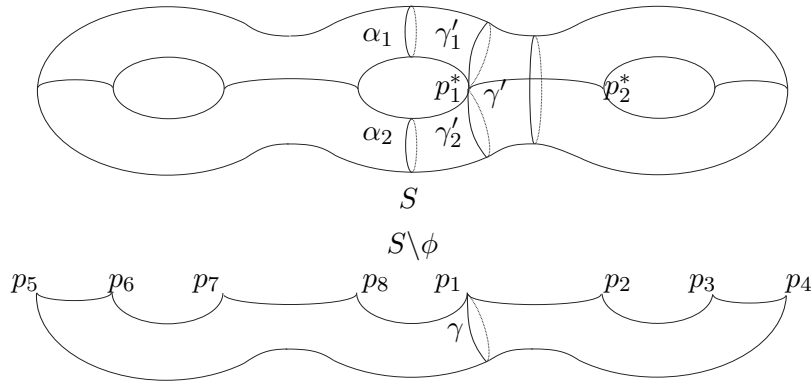

Figure 2.8: A hyperelliptic surface S and the quotient surface $S\backslash\phi$

Let $B_r(p_i)$ be a disk of radius r around a vertex of a cone. As long as these disks are small enough, they are embedded in $S\backslash\phi$. In this case the area of a disk of radius r around a vertex of a cone p_i, $B_r(p_i)$ is half the area of a disk of radius r in the hyperbolic plane, area$(B_r(p_i)) = \pi(\cosh(r)-1)$. Now expand all disks around the cone points until either a disk self-intersects or two different disks intersect for the first time at radius R. In this limit case we still obtain :

$$(2g + 2)\pi(\cosh(R) - 1) = \text{area}\left(\bigcup_{i=1}^{2g+2} B_R(p_i)\right) < \text{area}(S\backslash\phi) = 2\pi(g - 1).$$

As $\frac{g-1}{g+1} < 1$ we conclude that $R < \operatorname{arccosh}(2)$.

When the radii of the disks reach R and two different disks intersect the geodesic arc that forms lifts to a simple closed geodesic in S. When a disk self-intersects at radius R, the geodesic arc that forms lifts to a figure 8 geodesic in S. This figure 8 geodesic consists of two loops. The scg in the free homotopy class of such a loop is smaller than the loop itself. Hence there exists a scg of length smaller than $4R$ in S. It follows, for the systole γ_1 in S that $\gamma_1 < 4\operatorname{arccosh}(2) = 5.2678....$ By a refinement of this area estimate Bavard obtains a better upper bound in [Ba1], which is

$$\gamma_1 < 4\operatorname{arccosh}\left(\left(2\sin\left(\tfrac{\pi(g+1)}{12g}\right)\right)^{-1}\right) < 2\log(3 + 2\sqrt{3} + 2\sqrt{5 + 3\sqrt{3}}) = 5.1067... \qquad (2.22)$$

We now show that this upper bound is equally valid for the shortest non-separating scg in S. Consider the case, where two different disks intersect at radius R. In this case a geodesic arc of length smaller than $2R$ connects WLOG p_1 and p_2. It is easy to see that it lifts to a scg α_1 of length $4R$ in the double covering S *(see Fig. 2.8.)*. α_1 is non-separating, which we will prove by contradiction. Assume that α_1 is separating. There exists a third WP p_3^*. If it would lie on α_1, it would follow that $\phi(p_1^*) \neq p_1^*$, as ϕ acts by a rotation of angle π around p_3^* and thus $\phi(p_1^*)$ must lie on α_1 and must have distance $\frac{\alpha_1}{2}$ from p_3^*. But this is already the only possible location for p_2^* and we arrive at a contradiction. We know furthermore that α_1 separates S into two parts, S_1 and S_2. If p_3^* does not lie on α_1, then it lies WLOG on S_1. But then, as ϕ acts by a rotation of angle π around p_1^*, $\phi(p_3^*)$ lies in S_2, which is impossible.

Consider now the case, where WLOG $B_R(p_1)$ self-intersects. The geodesic arc γ that passes p_1 and the intersection point, lifts to a figure 8 geodesic γ' in S *(see Fig. 2.8.)*. Let γ_1' and γ_2' be the two different lifts of γ in S with intersection point p_1^*. Both lifts are not null-homotopic. Let α_1 and α_2 be the scg in the free homotopy class of γ_1' and γ_2', respectively. The length of both is bounded from above by $2R$. We will show that either α_1 or α_2 are non-separating. We prove this by contradiction. Suppose both are separating and consider the curve that we obtain by splitting γ' at p_1^* in a way that we obtain a non-self intersecting curve. Let β be the scg in the free homotopy class of this curve. It follows from surfaces topology that β, α_1 and α_2 are the boundary curves of a Y-piece Y', with γ' in its interior and whose interior is embedded in S. As ϕ acts by a half-turn around p_1^*, it follows from the geometry of hyperbolic Y-pieces that $\phi(Y') = Y'$ and $\phi(\alpha_1) = \alpha_2$.

If $\alpha_1 = \alpha_2$ in S, then the geodesic can not be separating, hence we may assume that $\alpha_1 \neq \alpha_2$. As both are separating and S is compact, α_1 and α_2 separate the surfaces S_1 and S_2 of signature $(k_1, 1)$ and $(k_2, 1)$, $k_1, k_2 \geq 1$ respectively on the side opposite of γ'. Every geodesic arc ν with starting point on in S_1 and endpoint p_1^* passes α_1. Consider $\phi(\nu)$. As the hyperbolic involution acts by a half-turn around p_1^*, $\phi(\nu)$ passes α_2 and extends into S_2. As $\phi^2 = id$, we must have $\phi(S_1) = S_2$. Now if S_1 contains another WP p_2^*, then $\phi(p_2^*) \neq p_2^*$, a contradiction. If S_1 contains no WP, then $S \backslash \phi$ would be at least of genus $k_1 \geq 1$, a contradiction to the fact that it is a topological sphere. Therefore one of the two geodesics α_1, α_2 must be non-separating. Let WLOG α_1 be non-separating. As α_2 is the image of α_1 under ϕ, we conclude that in fact both are non-separating.

In any case there exists a nsscg α_1 in S, whose length is smaller than the constant from equation (2.22). In any case there exists a nsscg α_1 in S, whose length is smaller than the constant from equation (2.22). Hence we obtain the upper bound for $m_1(J(S))^2$ in **Theorem 2.6.1** :

$$m_1(J(S))^2 < \frac{3\log(3 + 2\sqrt{3} + 2\sqrt{5 + 3\sqrt{3}})}{\pi} \leq 2.4382... \quad \square$$

Chapter 3

Estimates for the period Gram matrix based on geometric data

In the previous chapter, we obtained upper bounds on a certain number of lattice vectors of the Jacobian. The upper bounds on the first and second shortest lattice vector apply universally to all Riemann surfaces of genus g. We obtain improved bounds, if there is supplementary information available about the surface, i.e. the existence of a hyperelliptic involution or of short separating geodesics.

In this chapter we will first present a theoretical approach how to estimate all entries of the period Gram matrix of a Riemann surface, if the concrete geometry of the surface is known. Then we will give more practical estimates based on the geometry of the Q-pieces that contain a canonical homology basis.

Finally we will apply these methods to surfaces that contain small non-separating geodesics.

In this chapter we will make use of inequalities for the capacity of annuli on hyperbolic cylinders. This topic will be treated in more depth in appendix B.

3.1 Theoretical estimates for the period Gram matrix

Let S be a Riemann surface of genus g and A a canonical homology basis, which we assume to be given in the form

$$\text{A} = (\alpha_1, \alpha_{\tau(1)}, ..., \alpha_g, \alpha_{\tau(g)}) \quad (\text{see } \textbf{Def. 1.3.2}) \ .$$

If $(\sigma_i)_{i=1,..,2g}$ is a dual basis of real harmonic forms for A, then the period Gram matrix Q_S is the matrix

$$Q_S = (q_{ij})_{i,j=1...2g} = \left(\int_S \sigma_i \wedge {}^*\sigma_j \right)_{i,j=1...2g} \ .$$

We have seen in chapter 2.2 how to obtain an upper bound on the diagonal entries of Q_S by evaluating the capacity of a cylinder around the elements of the canonical homology basis. This approach can be expanded to obtain better estimates of these entries and finally of all entries of Q_S. The approach relies on the premise that the cut locus of a given simple closed geodesic on a Riemann surface can be (partially) calculated.

Estimates for the diagonal entries of Q_S

We first show how to obtain upper and lower bounds on the diagonal entries of Q_S. Consider WLOG $E(\sigma_1) = q_{11}$. It is clear that the capacity of a topological tube in S, which is obtained by a continuous deformation of the collar $C(\alpha_2)$ in S gives an upper bound for the energy of σ_1, $E(\sigma_1)$ – see chapter 2.2 for details. We obtain such a tube by cutting open S along the cut locus $CL(\alpha_2)$ of α_2. The *cut locus* of a subset $X \subset S$, $CL(X)$ is defined in the following way, where $\gamma_{a,b}$ denotes a geodesic arc connecting the points a and b :

$$CL(X) := \{y \in S \mid \exists \gamma_{x,y}, \gamma_{x',y}, \gamma_{x,y} \neq \gamma_{x',y}, \text{ with } x, x' \in X \text{ and } \operatorname{dist}(x,y) = l(\gamma_{x,y}) = l(\gamma_{x',y})\}.$$

For more information about the cut locus, see [Ba2]. We denote by S_X the surface, which we obtain by cutting open S along $CL(X)$. For a set $X \subset S$, set

$$Z_r(X) = \{x \in S \mid \operatorname{dist}(x,X) \leq r\}.$$

Let U be a set of disjoint simple closed geodesics $(\gamma_i)_{i=1,\ldots,n}$ then for sufficiently small r, $Z_r(U)$ consists of disjoint cylinders around these geodesics. We obtain $CL(U)$ by letting r grow continuously until $Z_r(U)$ self-intersects. We stop the expansion in the points of intersection, but continue expanding the rest of the set, until the process halts. The points of intersection then form $CL(U)$. It follows from this process that the surface S_U, which we obtain by cutting open S along $CL(U)$ can be retracted onto the union of small cylinders around the $(\gamma_i)_{i=1,\ldots,n}$. If $U = \gamma$, then S_U can be embedded into an sufficiently large cylinder C around γ.

Consider an embedding of $S_{\alpha_2} = S_2$ in a cylinder C, which, by abuse of notation, we also call S_2. This is shown in Figure 3.1. We have that the boundary ∂S of $S_2 \subset C$ consists of the two connected components $\partial_1 S$ and $\partial_2 S$. Fixing a base point $x \in S_2 \subset C$, we can construct a
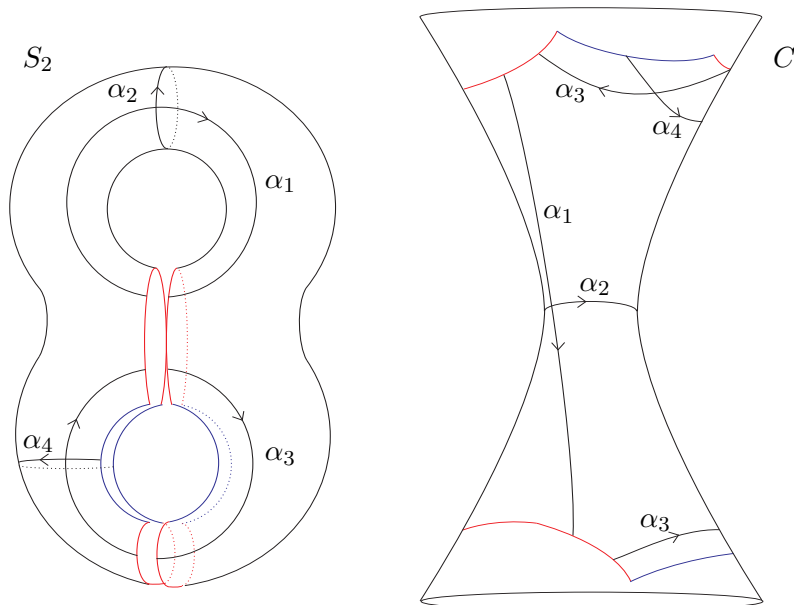

Figure 3.1: Embedding of $S_2 = S_{\alpha_2}$ in a cylinder around α_2 .

primitive F_1 of σ_1 by integrating σ_1 along paths starting from the base point x. As $\int\limits_{\alpha_2} \sigma_1 = 0$, the value of the integral is independent of the chosen path in S_2. Therefore there exists a primitive F_1 of σ_1 on $S_2 \subset C$. Furthermore F_1 is a real harmonic function, as σ_1 is a real harmonic 1-form. We remind here that the value of the integral of a σ_1 over a closed curve depends only on the homology class of the curve. Especially the value of the integral is the same for two curves in the same free homotopy class.

The conditions on the homology basis A imply the following boundary conditions for F_1. For each point p_1 on the boundary $\partial S_2 \subset C$, there exists a point p_2 that maps to the same point p on S as p_1, such that

$$F_1(p_2) - F_1(p_1) = 0 \quad \text{or} \quad F_1(p_2) - F_1(p_1) = 1.$$

We color p_1 and p_2 blue in the first case and red in the second case and call such a decomposition a *red-blue decomposition*. Such a decomposition is shown in Figure 3.1.

For the red-blue decomposition that is obtained via the cut locus $CL(\alpha_2)$, the following holds. If p_1 and p_2 are blue, then p_1 and p_2 lie on the same side of the boundary ∂S_2. If p_1 and p_2 are red then they lie on different sides of ∂S_2. There are points on ∂S_2 that are both red and blue. However the number of those points is finite.

Upper bound

The capacity of $S_2 \subset C$, $\text{cap}(S_2)$ provides an upper bound on the energy of σ_1. We obtain an upper bound on $\text{cap}(S_2)$ by evaluating the energy of any test function F_{1t}, which is a Lipschitz function on S_2 and satisfies the boundary conditions of the capacity problem (see chapter 1.5 or [Go]) :

$$E(F_{1t}) \geq \text{cap}(S_2) \geq E(\sigma_1) = q_{11}.$$

In **Theorem B.4.1** we obtain general upper and lower bounds on the capacity of annuli A on a cylinder of constant curvature. Here an annulus is a region that is homeomorphic to the cylinder itself. The idea for the upper bound is to construct a test function in the following way. We adapt the harmonic function that solves the capacity problem for cylinders of constant width (see **Example 1.5.6**) to the boundary using the parametrization of a cylinder in Fermi coordinates. If the annulus $A \subset C$ is given in Fermi coordinates by

$$A = \psi\{(t,s) \mid s \in [a_1(t), a_2(t)], t \in [0, l(\alpha_2)]\},$$

where $a_1(\cdot)$ and $a_2(\cdot)$ are piecewise differentiable functions with respect to t, then it follows :

Theorem 3.1.1 *There exists a Lipschitz function $\tilde{F} \in \text{Lip}(A)$, such that for $H(s) = 2\arctan(\exp(s))$ and $q_i(t) = \frac{\partial H(s_0)}{\partial s}|_{s_0=a_i(t)} \cdot a_i'(t)$ for $i \in \{1, 2\}$, we have :*

$$\int\limits_0^{l(\alpha_2)} \frac{1 + \frac{q_1(t)^2 + q_1(t)q_2(t) + q_2(t)^2}{3}}{H(a_2(t)) - H(a_1(t))} \, dt = E(\tilde{F}) \geq \text{cap}(A) \geq \int\limits_0^{l(\alpha_2)} \frac{1}{H(a_2(t)) - H(a_1(t))} \, dt$$

As $S_2 \subset C$ is an annulus that satisfies the conditions of **Theorem 3.1.1**, we can apply this theorem to obtain an estimate of the capacity $\text{cap}(S_2)$.

Lower bound

We obtain a lower bound on $q_{11} = E(F_1)$ in the following way. Connect the endpoints of two corresponding opposite red boundary segments in the red-blue composition of $S_2 \subset C$ with differentiable curves, such that these curves do not mutually intersect. Then the curves, together with the boundary segments of S_2, enclose a subset of S_2. We denote the union of all enclosed areas that can be obtained in this way by S_2^{red}. We have :

$$E(\sigma_1) = E(F_1) \geq \int\limits_{S_2^{red}} \|DF_1\|_2^2.$$

Let I be a disjoint union of intervals in $\mathbb{R}$ and

$$\varphi : I \times [a_1, a_2] \to S_2^{red}, \varphi : (t, s) \mapsto \varphi(t, s)$$

a bijective function that parametrizes S_2^{red} in the following way :

$$\varphi(I \times \{a_1\}) = S_2^{red} \cap \partial_1 S_2 \quad \text{and} \quad \varphi(I \times \{a_2\}) = S_2^{red} \cap \partial_2 S_2$$

and for a fixed $c \in I$, $\varphi(\{c\} \times [a_1, a_2])$ is a differentiable curve in S_2^{red}, such that

$$F_1(\varphi(c, a_2)) - F_1(\varphi(c, a_1)) = 1.$$

For a $x = \varphi(c, a) \in S_2^{red}$ denote by $p_\varphi : T_x(S_2^{red}) \to \{\lambda \cdot \frac{\partial \varphi(c,a)}{\partial s} \mid \lambda \in \mathbb{R}\}$ the orthogonal projection of a tangent vector in x onto the subspace spanned by $\frac{\partial \varphi(c,a)}{\partial s}$. Then we have :

$$E(F_1) \geq \int\limits_{S_2^{red}} \|DF_1\|_2^2 \geq \int\limits_{S_2^{red}} \|p_\varphi(DF_1)\|_2^2.$$

Denote by $\mathcal{F}_1$ the set of functions :

$$\mathcal{F}_1 = \{f : S_2^{red} \to \mathbb{R} \mid f \in \text{Lip}(S_2^{red}) \ \text{and} \ f(\varphi(c, a_2)) - f(\varphi(c, a_1)) = 1 \ \forall \ c \in I\}.$$

We obtain a lower bound on $q_{11} = E(\sigma_1) = E(F_1)$ if we can find a function f_1, such that

$$\int\limits_{S_2^{red}} \|p_\varphi(Df_1)\|_2^2 = \inf_{f \in \mathcal{F}_1} \int\limits_{S_2^{red}} \|p_\varphi(Df)\|_2^2. \tag{3.1}$$

Then $q_{11} = E(F_1) \geq \int\limits_{S_2^{red}} \|p_\varphi(Df_1)\|_2^2.$

Note that the problem of finding the function f_1 is in general easier than finding the function F_1. We will apply these ideas in the following section using Fermi coordinates and results from the calculus of variations.

Estimates for the non-diagonal entries of Q_S

We now show, how we can estimate the remaining entries of the period Gram matrix Q_S. For $i \neq j$, we have, as $\int\limits_S \cdot \wedge {}^*\cdot$ is a scalar product :

$$
q_{ij} \quad = \quad \int\limits_S \sigma_i \wedge {}^*\sigma_j = \frac{1}{2} \left(E(\sigma_i + \sigma_j) - E(\sigma_i) - E(\sigma_j) \right) \quad \text{and} \tag{3.2}
$$

$$
q_{ij} \quad = \quad \frac{1}{2} \left(E(\sigma_i) + E(\sigma_j) - E(\sigma_i - \sigma_j) \right). \tag{3.3}
$$

We have shown how to find upper and lower bounds on $E(\sigma_i)$ and $E(\sigma_j)$. Hence we obtain an estimate for q_{ij}, if we can find upper and lower bounds on either $E(\sigma_i + \sigma_j)$ or $E(\sigma_i - \sigma_j)$.
Consider the harmonic 1-forms ω_1 and ω_2 that satisfy the following equations on the cycles :

$$
\int\limits_{\alpha_k} \omega_1 \quad = \quad \delta_{ik} + \delta_{jk} \quad \text{for all} \quad k \in \{1, .., 2g\}. \tag{3.4}
$$

$$
\int\limits_{\alpha_k} \omega_2 \quad = \quad \delta_{ik} - \delta_{jk} \quad \text{for all} \quad k \in \{1, .., 2g\}. \tag{3.5}
$$

It follows easily from **Theorem 1.5.1** that $\sigma_i + \sigma_j$ is the unique 1-form that minimizes the energy among all closed 1-forms that satisfy equation (3.4) and that $\sigma_i - \sigma_j$ is the unique 1-form that minimizes the energy among all closed 1-forms that satisfy equation (3.5).
In the homotopy class of either $\alpha_{\tau(i)} + \alpha_{\tau(j)}$ or $\alpha_{\tau(i)} - \alpha_{\tau(j)}$ there is a simple closed curve α. There is a primitive of either $\sigma_i + \sigma_j$ or $\sigma_i - \sigma_j$ on S_α. Therefore we can proceed as in the previous subsection to obtain estimates on $\sigma_i + \sigma_j$ or $\sigma_i - \sigma_j$ and hence for q_{ij}.
We will present this approach in the case, where $\alpha_j = \alpha_{\tau(i)}$. We present these estimates in *Case 1*. If $\alpha_j \neq \alpha_{\tau(i)}$, we will present an alternative approach in *Case 2*. We will make use of these two methods in section 4.2.

Case 1 : Estimates for $q_{i\tau(i)}$

Consider WLOG q_{12}. Consider the simple closed geodesic α_{12} in the free homotopy class of $\alpha_1 \alpha_2^{-1}$. Due to the relationships in equation (3.4), $\sigma_1 + \sigma_2$ has a primitive on $S_{\alpha_{12}} = S_{12}$. We embed S_{12} into a cylinder C. We also denote the embedded surface by S_{12}.
As in the previous subsection, the capacity $\text{cap}(S_{12})$ provides an upper bound for $E(\sigma_1 + \sigma_2)$:

$$
\text{cap}(S_{12}) \geq E(\sigma_1 + \sigma_2).
$$

We obtain a lower bound for $E(\sigma_1 + \sigma_2)$ by applying the same methods used for a lower bound on $E(\sigma_1)$ presented in the previous subsection.
With the estimates for $E(\sigma_1 + \sigma_2)$, $E(\sigma_1)$ and $E(\sigma_2)$ applied to equation (3.2), we obtain an upper and lower bound on q_{12}.

Case 2 : Estimates for $q_{ij}, j \neq \tau(i)$

In this case α_i and α_j do not intersect. Consider WLOG q_{13}. For $\alpha_{\tau(1)} = \alpha_2$ and $\alpha_{\tau(3)} = \alpha_4$

consider $S_{\alpha_2 \cup \alpha_4} = S_{24}$. S_{24} consists out of two connected parts. Let S_{24}^1 be the part that contains α_2 and let S_{24}^3 be the part that contains α_4. We embed S_{24}^1 into a cylinder C_1 around α_2 and S_{24}^3 into a cylinder C_3 around α_4 and denote the embedded surfaces by the same name. Due to the relationships in equation (3.4), $\sigma_1 + \sigma_3$ has a primitive on both $S_{24}^1 \subset C_1$ and $S_{24}^3 \subset C_3$. Such a decomposition is shown in Figure 3.2.
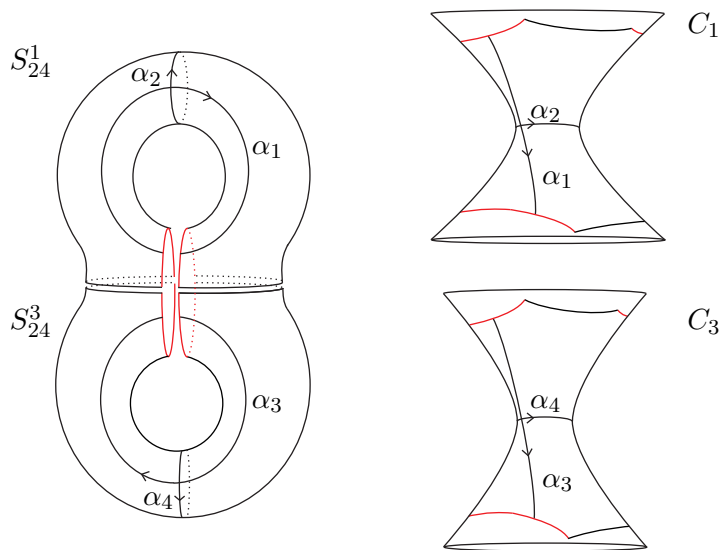
Figure 3.2: Embedding of S_{24}^i in a cylinder C_i around $\alpha_{\tau(i)}$ for $i \in \{1, 3\}$.

For $i \in \{1, 3\}$, let F_i on S_{24}^i be a function that satisfies boundary conditions for the capacity problem on S_{24}^i. Then these functions together define naturally a function $\tilde{F}_{13}$ on S_{24}. By smoothing $\tilde{F}_{13}$ in an inner environment of the boundary of S_{24}, we obtain a function F_{13} on S, whose derivative dF_{13} is a closed differential form that satisfies the same integral conditions on the cycles as $\sigma_1 + \sigma_3$. Due to the energy-minimizing property of $\sigma_1 + \sigma_3$ we have that $E(F_{13}) \geq E(\sigma_1 + \sigma_3)$. Hence the sum of the capacities of S_{24}^1 and S_{24}^3 provides an upper bound for $E(\sigma_1 + \sigma_3)$:

$$\text{cap}(S_{24}^1) + \text{cap}(S_{24}^3) \geq E(\sigma_1 + \sigma_3).$$

We obtain a lower bound for $E(\sigma_1 + \sigma_3)$ by applying the methods used to obtain a lower bound on $E(\sigma_1)$ presented in the previous subsection. We can apply these methods to S_{24}^1 in C_1 and S_{24}^3 in C_3. The only difference here is that we have some segments of the boundary, where the red-blue decomposition does not apply. Here we disregard these pieces in the construction of S_{24}^{1red} and S_{24}^{1red}.

With the estimates for $E(\sigma_1 + \sigma_3)$, $E(\sigma_1)$ and $E(\sigma_3)$ applied to equation (3.2), we obtain an upper and lower bound on q_{13}.

In the following section, we will apply these methods to a decomposition of the Riemann surface, where the elements of the canonical homology basis are contained in Q-pieces.

3.2 Estimates for the period Gram matrix based on Q-pieces

Let S is a Riemann surface of genus $g > 2$ and A a canonical homology basis. Then the pairs $((\alpha_i, \alpha_{\tau(i)}))_{i=1,3,..2g-1}$ are contained in disjoint surfaces of signature $(1,1)$ or Q-pieces. If $g = 2$, then the Q-pieces intersect at the boundary, but this will not concern us here. Let $(Q_i)_{i=1,3,..,2g-1}$ be the set of Q-pieces, such that

$$(\alpha_i, \alpha_{\tau(i)}) \subset Q_i \quad \text{for} \quad i \in \{1, 3, .., 2g - 1\}$$

We will present estimates on the entries of the period Gram matrix Q_S using a partition of S that contains these pieces. To establish these estimates, the following information about the geometry of the Q-pieces $(Q_i)_{i=1,3,..,2g-1}$ must be known.

Let β_i be the boundary geodesic of Q_i and let $\alpha_{i\tau(i)} \subset Q_i$ be the simple closed geodesic in the free homotopy class of $\alpha_i(\alpha_{\tau(i)})^{-1}$. If we cut open Q_i along one of these three geodesics, Q_i decomposes into a Y-piece. Let t_j be the twist parameter for the decomposition with α_j, where $j \in \{i, \tau(i), i\tau(i)\}$. To obtain practical estimates on all entries of the matrix Q_S, we assume that the following parameters are known for each Q_i:

- the length of β_i
- the length of $\alpha_i, \alpha_{\tau(i)}$ and $\alpha_{i,\tau(i)}$
- the twist parameters t_j for $j \in \{i, \tau(i), i\tau(i)\}$

For practical reasons, we will first give a parametrization of trirectangles. The formulas developed here, will be used in the following subsection.

Upper bounds based on trirectangles

Let C be a hyperbolic cylinder with base line γ. Let $T \subset C$ be a trirectangle, which has one side d on $\gamma \subset C$ and such that the two adjacent sides meet d perpendicularly. We assume that T is given in Fermi coordinates by

$$T = \psi(\{(t, s) \mid s \in [0, a_2(t)], t \in [0, d]\}),$$

where, by abuse of notation, d denotes the length of d. We assume furthermore that the shortest side of T, which is perpendicular to d has length w and intersects d in $\psi(0, 0)$. For $t \in [0, d]$, we have

$$a_2(t) = \operatorname{arctanh}\left(\frac{\cosh(t)}{\coth(w)}\right), \quad \text{where} \quad d < \operatorname{arcsinh}\left(\frac{1}{\sinh(w)}\right). \tag{3.6}$$

This follows from the geometry of trirectangles (see chapter 1.1.). We remind the notation of **Theorem 3.1.1**, where $H(s) = 2\arctan(\exp(s))$ and $q_i(t) = \frac{\partial H(s_0)}{\partial s}\big|_{s_0=a_i(t)} \cdot a_i'(t)$. For the boundary line of T, we obtain with equation (3.6), as $\frac{1}{\cosh(\operatorname{arctanh}(x))} = \sqrt{1 - x^2}$ and as $\exp(\operatorname{arctanh}(x)) = \sqrt{\frac{1+x}{1-x}}$:

$$H(a_2(t)) = 2\arctan\left(\sqrt{\frac{\coth(w) + \cosh(t)}{\coth(w) - \cosh(t)}}\right) \quad \text{and} \quad q_2(t)^2 = \frac{\sinh(t)^2}{\coth(w)^2 - \cosh(t)^2}. \tag{3.7}$$

Let D be a topological tube around γ, whose boundary segments consist of either boundary arcs of trirectangles or curves contained in a distance set $\partial Z_r(\gamma)$ for some r. Then we obtain an upper bound for the capacity $\mathrm{cap}(D)$ of such a tube D using **Theorem 3.1.1** and equation (3.7). Finally, if $\gamma = \alpha_{\tau(i)}$ for some $\alpha_{\tau(i)} \in \mathrm{A}$ and D is contained in $S_{\alpha_{\tau(i)}} \subset C$, then the capacity of D provides an upper bound for the entry q_{ii} of the period Gram matrix Q_S (see chapter 2.2).

$$\mathrm{cap}(D) > q_{ii}.$$

Upper bounds for the entries of Q_S based on the geometry of Q-pieces

We will establish estimates for all entries of the period Gram matrix based on the geometry of the Q-pieces $(Q_i)_{i=1,3,..,2g-1}$. Following the approach given in section 3.1, it is sufficient to construct suitable functions on

$$S_\gamma \cap Q_i, \quad \text{where} \quad \gamma \in \{\alpha_i, \alpha_{\tau(i)}, \alpha_{i\tau(i)}\}, \quad \text{for} \quad i \in \{1, 3, .., 2g - 1\}.$$

We will exemplify this for the entry q_{11}. Here Q_1 is the Q-piece that contains (α_1, α_2). Let $\beta = \beta_1$ be the boundary geodesic of Q_1. We embed $S_{\alpha_2} \cap Q_1$ into a hyperbolic cylinder C and denote this embedding by the same name. To obtain an estimate on q_{11} we will give a parametrization of

$$S_{\alpha_2} \cap Q_1 \subset C$$

based on a decomposition into trirectangles. We will give a description of $S_{\alpha_2} \cap Q_1$ that depends only on the length of α_2 and β and the twist parameter t_2.

To obtain this parametrization, we first cut open Q_1 along α_2 to obtain the Y-piece Y_1 with boundary geodesics β, α_2' and α_2''. α_2' and α_2'' have length α_2 (see *Fig. 3.3*). Denote by b the shortest geodesic arc connecting α_2' and α_2''. Now we cut open Y_1 along the shortest geodesic arcs connecting β and the other two boundary geodesics. We call Y_1' the surface, which we obtain by cutting open Y_1 along these lines. By abuse of notation, we denote the geodesic arcs in Y_1' by the same letter as in Y_1. The geodesic arc b divides Y_1' into two isometric hexagons H_1 and H_2. This decomposition is also shown in Figure 3.3.
In H_1 b is the boundary geodesic connecting $\frac{\alpha_2'}{2}$ and $\frac{\alpha_2''}{2}$. Denote by δ' the shortest geodesic arc in H_1 connecting b and the side opposite of b of length $\frac{\beta}{2}$. By abuse of notation we denote this side by $\frac{\beta}{2}$. We denote by δ'' the arc in H_2 corresponding to δ' in H_1. Let δ be the geodesic arc in Y_1', such that $\delta = \delta' \cup \delta''$. It is easy to see that the geodesic arc in Q_1 that maps to δ in Y_1' constitutes the intersection of the cut locus of α_2 with Q_1. We denote this geodesic arc in Q_1 equally by δ :
$$\delta = CL(\alpha_2) \cap Q_1.$$

We denote furthermore by a' the geodesic arcs connecting $\frac{\alpha_2'}{2}$ and $\frac{\beta}{2}$ in H_1 and by a'' the corresponding arc in H_2. Let a be the length of such an arc.
δ' divides H_1 into two isometric right-angled pentagons P_1 and P_2. Let P_1 be the pentagon that has $\frac{\alpha_2'}{2}$ as a boundary. To establish the parametrization for $S_{\alpha_2} \cap Q_1$, we divide P_1 into two trirectangles.
Let c be the geodesic arc in P_1 that emanates from the vertex, where $\frac{\beta}{2}$ and δ intersect and that meets $\frac{\alpha_2'}{2}$ perpendicularly. It divides $\frac{\alpha_2'}{2}$ into two parts, α' and α'' (see *Fig. 3.3*). c divides P_1 into two trirectangles, T_1, which has the boundary α' and T_2, which has the boundary α''.
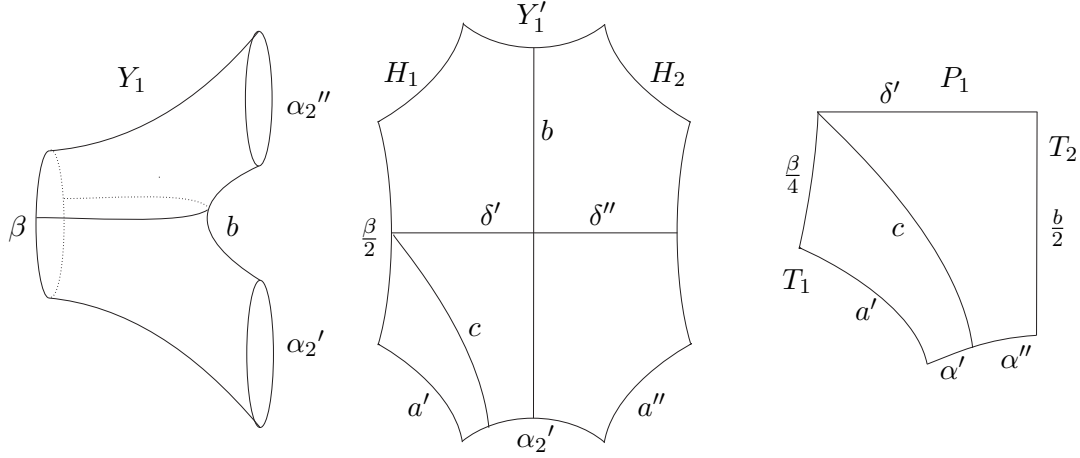
Figure 3.3: Decomposition of Y_1 into isometric hexagons H_1 and H_2.

To obtain an upper bound for q_{11}, we have to know the geometry of T_1 and T_2. To apply the results from the previous subsection, we have to know the lengths a and α' and $\frac{b}{2}$ and α''.
As we obtain this result along the way, we will express the lengths of all boundary geodesics of T_1 and T_2 in terms of α_2 and β. To obtain this description, we will use the geometry of H_1, P_1, T_1 and T_2. The following formulas can be found in chapter 1.1.

From the geometry of the hyperbolic hexagon H_1 and as $\sinh(b) = 2\sinh(\frac{b}{2})\cosh(\frac{b}{2})$ and $\cosh(b) = 2\cosh(\frac{b}{2})^2 - 1$, we obtain :

$$\coth(a) = \frac{\sinh(b)\sinh(\frac{\alpha_2}{2})}{1 + \cosh(b)} = \tanh(\frac{b}{2})\cosh(\frac{\alpha_2}{2}). \tag{3.8}$$

From the geometry of the hyperbolic pentagon P_1 we have :

$$\cosh(\frac{\beta}{4}) \quad = \quad \sinh(\frac{\alpha_2}{2})\sinh(\frac{b}{2}). \tag{3.9}$$

$$\cosh(\delta') \quad = \quad \sinh(\frac{\alpha_2}{2})\sinh(a). \tag{3.10}$$

From equation (3.10) and (3.8) and as $\sinh(\text{arccoth}(x)) = \frac{1}{\sqrt{x^2-1}}$ we obtain :

$$\cosh(\delta') = \frac{\sinh(\frac{\alpha_2}{2})}{\sqrt{\tanh(\frac{b}{2})^2\cosh(\frac{\alpha_2}{2})^2 - 1}}. \tag{3.11}$$

Finally from the geometry of hyperbolic trirectangles T_1 and T_2 we obtain :

$$\coth(\alpha') \quad = \quad \frac{\cosh(a)}{\tanh(\frac{\beta}{4})} \quad \text{and} \quad \coth(\alpha'') = \frac{\cosh(\frac{b}{2})}{\tanh(\delta')}. \tag{3.12}$$

$$\sinh(c) \quad = \quad \sinh(\delta')\cosh(\frac{b}{2}). \tag{3.13}$$

Then we obtain for c from equation (3.13) and (3.11) and as $\sinh(x)^2 = \cosh(x)^2 - 1$:

$$\sinh(c) = \cosh(\frac{b}{2})\sqrt{\frac{\sinh(\frac{\alpha_2}{2})^2}{\tanh(\frac{b}{2})^2\cosh(\frac{\alpha_2}{2})^2 - 1} - 1}. \tag{3.14}$$

We now express also α'' and α' in terms of α_2 and b. From equation (3.12) and (3.11) we obtain with $\frac{1}{\tanh(\mathrm{arccosh}(x))} = \frac{x}{\sqrt{x^2-1}}$ that

$$\coth(\alpha'') = \frac{\cosh(\frac{b}{2})^2 \sinh(\frac{\alpha_2}{2})}{\sqrt{\cosh(\frac{b}{2})^2 \sinh(\frac{\alpha_2}{2})^2 - (\sinh(\frac{b}{2})^2\cosh(\frac{\alpha_2}{2})^2 - \cosh(\frac{b}{2})^2)}}.$$

As $\sinh(x)^2 = \cosh(x)^2 - 1$, this simplifies to

$$\coth(\alpha'') = \cosh(\frac{b}{2})^2 \tanh(\frac{\alpha_2}{2}). \tag{3.15}$$

For α' we obtain from equation (3.12) and (3.8) we obtain with $\frac{1}{\tanh(\mathrm{arccosh}(x))} = \frac{x}{\sqrt{x^2-1}}$ that with $\cosh(\mathrm{arccoth}(x)) = \frac{1}{\tanh(\mathrm{arccosh}(x))} = \frac{x}{\sqrt{x^2-1}}$:

$$\coth(\alpha') = \frac{\sinh(\frac{b}{2})^2 \sinh(\frac{\alpha_2}{2})\cosh(\frac{\alpha_2}{2})}{\sqrt{(\sinh(\frac{b}{2})^2\cosh(\frac{\alpha_2}{2})^2 - \cosh(\frac{b}{2})^2) \cdot (\sinh(\frac{\alpha_2}{2})^2\sinh(\frac{b}{2})^2 - 1)}}.$$

As $\sinh(x)^2 = \cosh(x)^2 - 1$, this simplifies to

$$\coth(\alpha') = \frac{\sinh(\frac{b}{2})^2 \sinh(\frac{\alpha_2}{2})\cosh(\frac{\alpha_2}{2})}{\sinh(\frac{\alpha_2}{2})^2\sinh(\frac{b}{2})^2 - 1}. \tag{3.16}$$

$$\alpha' = \frac{\alpha_2}{2} - \alpha''. \tag{3.17}$$

Finally it follows for $\frac{b}{2}$ from equation (3.9) that

$$\sinh(\frac{b}{2}) = \frac{\cosh(\frac{\beta}{4})}{\sinh(\frac{\alpha_2}{2})}. \tag{3.18}$$

Using equation (3.18) we can express the length of all boundary geodesics in T_1 and T_2 in terms of α_2 and β. For our purposes it is sufficient to know the length of $a, \alpha', \frac{b}{2}$ and α'' - see equation (3.8),(3.17),(3.18) and (3.15), respectively.

With these formulas we are now able to obtain a description of the boundary of $S_{\alpha_2} \cap Q_1 \subset C$. Consider now $\delta \subset Y_1'$. δ divides Y_1' into two isometric hexagons. Let H_1' be the hexagon that contains α_2' as a boundary geodesic and H_2' be the hexagon that contains α_2'' as a boundary geodesic.
δ forms $CL(\alpha_2) \cap Q_1$ in Q_1. Denote by Q_1' the surface that we obtain if we cut open Q_1 along δ. Q_1' is a topological cylinder around α_2. A lift of Q_1' in the universal covering is depicted in Fig. 3.4.
By abuse of notation we denote by H_1' and H_2' two hexagons in this lift, which are isometric to the hexagons with the same name in Y_1' and that are adjacent along the lift of α_2. We keep the
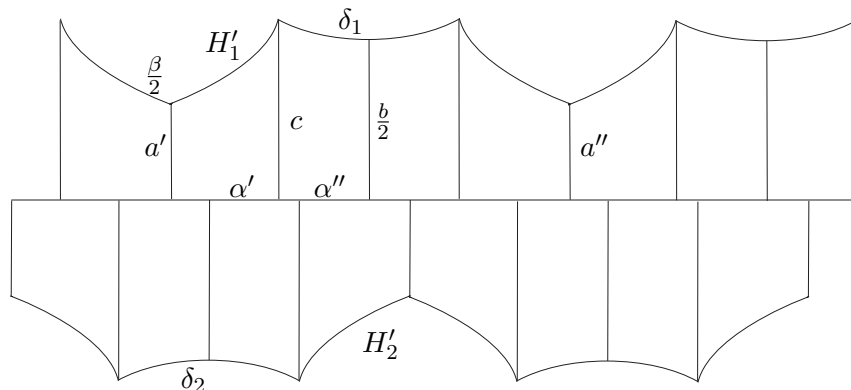
Figure 3.4: Lift of Q_1' into the universal covering.

notation from Y_1', but denote by $\delta_i \subset H_i'$, for $i \in \{1, 2\}$, the two sides corresponding to δ in Y_1'. In the lift of Q_1' the two hexagons H_1' and H_2' are shifted against each other according to the twist parameter t_2 at α_2. It is easy to see from Fig. 3.4, how to parametrize $S_{\alpha_2} \cap Q_1$ in a cylinder C around α_2. Here all boundaries are boundaries of trirectangles, which are isometric to either T_1 or T_2. Using the above formulas, we can express all parameters that occur in terms of t_2, α_2 and $\beta = \beta_1$. With equation (3.6), (3.7) and **Theorem 3.1.1**, we obtain :

$$\text{cap}(S_{\alpha_2} \cap Q_1) \geq q_{11}.$$

Remark : From numerical simulations we get the following impression. If the length of α' or α'' is near the maximal possible value $\text{arcsinh}(\frac{1}{\sinh(a)})$ or $\text{arcsinh}(\frac{1}{\sinh(\frac{b}{2})})$, respectively, then our test function is not well suited for the capacity problem and overestimates the capacity of $S_{\alpha_2} \cap Q_1$ and therefore also $E(\sigma_1)$.

We also obtain an upper bound for $E(\sigma_1)$ in chapter 2.2. In chapter 2.2 we get an upper bound from the capacity of a collar around α_2, $\text{cap}(C(\alpha_2))$. In this chapter we get an upper bound from the capacity $\text{cap}(S_{\alpha_2} \cap Q_1)$ of $S_{\alpha_2} \cap Q_1$. This upper bound seems to be strongly improving the estimate from chapter 2.2 if $\min\{a, \frac{b}{2}\}$ is small. For $\min\{a, \frac{b}{2}\} > 4$ hardly any improvement can be obtained.

Lower bounds on the entries of Q_S based on Q-pieces

Consider the primitive F_1 of σ_1 in $Q_1' = S_{\alpha_2} \cap Q_1 \subset C$. The two geodesic arcs corresponding to $\delta \subset Q_1$ constitute $S_2^{red} \cap \partial Q_1'$.

We now lift Q_1' into the universal covering as in the previous subsection (see *Fig. 3.4*). We use the same notation for the geodesic arcs that occur. Let $\tilde{\alpha}_2$ be the lift of α_2 in the lift of Q_1'. This is depicted in Fig. 3.5.

Let λ be the geodesic arc connecting the midpoints of δ_1 and δ_2. The midpoint m of λ and the endpoints of $\frac{b}{2}$ are the vertices of a right-angled triangle D (see also *Fig. 3.6*). It follows from the geometry of right-angled triangles (see chapter 1.1) that

$$\cosh(\frac{\lambda}{2}) = \cosh(\frac{b}{2}) \cosh(\frac{\alpha_2 t_2}{2}), \tag{3.19}$$
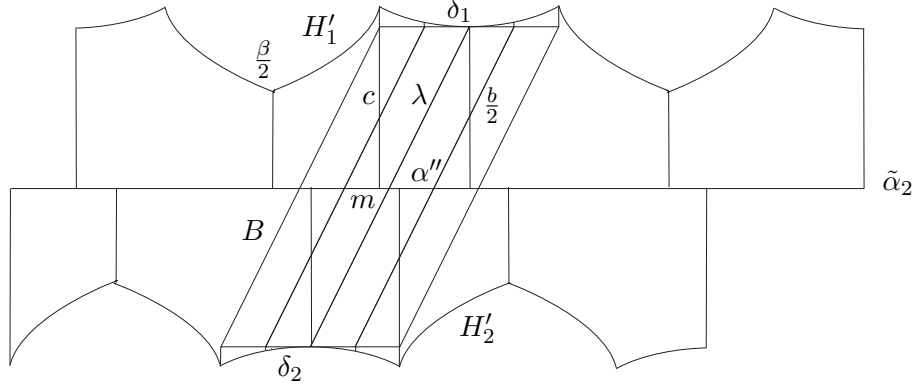
Figure 3.5: A lift of Q_1' in the universal covering.

where we assume WLOG that the twist parameter t_2 is in the interval $[0, \frac{1}{2}]$. Otherwise the situation is symmetric to the depicted one. Let ν be the angle of D at the midpoint m of λ. We have :

$$\sinh(\frac{\lambda}{2}) = \frac{\sinh(\frac{b}{2})}{\sin(\nu)}.$$

The equations for D imply with $\sinh(x)^2 = \cosh(x)^2 - 1$:

$$\sin(\nu) = \frac{\sinh(\frac{b}{2})}{\sqrt{\cosh(\frac{b}{2})^2 \cosh(\alpha_2 \frac{\alpha_2 t_2}{2})^2 - 1}}. \tag{3.20}$$

We will use equation (3.20) to obtain a parametrization of the hatched subset B in the lift of Q_1' in Fig. 3.5. We will describe B in the following.

The boundary of B contains the lines δ_1 and δ_2. For each point $p_1 \in \delta_1$ there exists a point $p_2 \in \delta_2$, such that p_1 and p_2 map to the same point p on $\delta \subset Q_1$. We may assume WLOG that

$$F_1(p_2) - F_1(p_1) = 1 \;\; \text{for all} \;\; p_1 \in \delta_1$$

We will describe B as a set of lines, where each line l_p connects p_1 and p_2. l_p can be described in the following way. From p_1 we go along the geodesic that meets $\tilde{\alpha}_2$ perpendicularly until we meet $\partial Z_{\frac{b}{2}}(\tilde{\alpha}_2)$. We call this intersection point p_1' and the geodesic arc that forms γ_p^1. Let p_2' be the point on $\partial Z_{\frac{b}{2}}(\tilde{\alpha}_2)$ on the other side of $\tilde{\alpha}_2$ that can be reached analogously, starting from p_2. We now go along the geodesic arc that connects p_1' and p_2'. We call this arc γ_p. Then from p_2', we move along the geodesic arc connecting p_2' and p_2. We call this arc γ_p^2. We define l_p as the line traversed in this way. Let B be the disjoint union of these lines :

$$B = \biguplus_{p \in \delta} \{l_p\}$$

We will use a bijective parametrization φ of B. We want a $\varphi : (t, s) \mapsto \varphi(t, s)$ such that for a fixed $t_0 \in [-\alpha'', \alpha'']$, $\varphi(t_0, \cdot)$ parametrizes the line l_p that traverses $\tilde{\alpha}_2$ in a point with directed

distance t_0 from m. We parametrize the sets $\bigcup_{p \in \delta} \{\gamma_p^1\}$ and $\bigcup_{p \in \delta} \{\gamma_p^2\}$ in Fermi coordinates with base line $\tilde{\alpha}_2$. The proper parametrization can be deduced from the geometry of the trirectangle T_2.

We will parametrize $Z_{\frac{b}{2}}(\tilde{\alpha}_2) \cap B = \bigcup_{p \in \delta} \{\gamma_p\}$ using *skewed Fermi coordinates* ψ^ν with angle ν and base line $\tilde{\alpha}_2$. These are defined in the same way as the usual Fermi coordinates ψ, but instead of moving along geodesics emanating perpendicularly from the base line, we move along geodesics that meet the base line under the angle ν. We will not give these coordinates explicitly, but will derive the essential information from the usual Fermi coordinates ψ (see chapter 1.1). Consider the following geodesic arcs in $Z_{\frac{b}{2}}(\tilde{\alpha}_2) \cap B$. For a $n \in \mathbb{N}$ let λ' be a geodesic arc of length $\frac{2\alpha''}{n}$ on $\tilde{\alpha}_2$ with midpoint m. λ intersects λ' in m under the angle ν. This is depicted in Fig. 3.6. Let η' be a geodesic intersecting λ perpendicularly in m. Let furthermore μ_1 and μ_2 be two
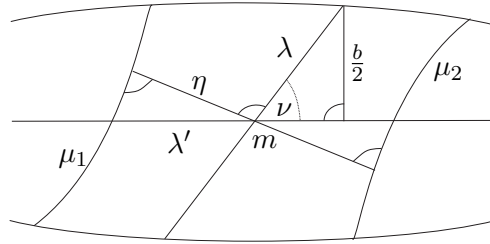

Figure 3.6: Construction of skewed Fermi coordinates ψ^ν.

geodesic arcs with endpoints on $Z_{\frac{b}{2}}(\tilde{\alpha}_2)$ that intersect η' perpendicularly, such that each of the arcs passes through an endpoint of λ' on each side of λ. Let furthermore η be the geodesic arc on η' with endpoints on μ_1 and μ_2. For fixed $n \in \mathbb{N}$, we denote by η_n the length of η and by μ^n the length of μ_1 and μ_2 :

$$\eta_n = l(\eta) \quad \text{and} \quad \mu^n = l(\mu_1) = l(\mu_2).$$

By choosing usual Fermi coordinates with baseline η, we can parametrize the strip, whose boundary lines are μ_1 and μ_2 and two segments of $\partial Z_{\frac{b}{2}}(\tilde{\alpha}_2)$ (see *Fig. 3.6*).
n such strips can be aligned next to each other to obtain a parametrization of $Z_{\frac{b}{2}}(\tilde{\alpha}_2) \cap B$. For $n \to \infty$ we obtain a parametrization ψ^ν of $Z_{\frac{b}{2}}(\tilde{\alpha}_2) \cap B$. We have :

$$\lim_{n \to \infty} n \cdot \eta_n = \sin(\nu) 2\alpha'' \quad \text{and} \quad \lim_{n \to \infty} \mu^n = \lambda.$$

Combining the parametrizations for the several pieces of B, we may assume that we have a parametrization φ that satisfies our conditions. For practical purposes, we extend the parametrization φ to the geodesics meeting $\partial Z_{\frac{b}{2}} \cap B$ perpendicularly in the direction opposite of $\tilde{\alpha}_2$.
Consider a point $p_1 = \varphi(t_0, -x) \in \delta_1$ and $p_2 = \varphi(t_0, x) \in \delta_2$. The function F_1 satisfies the

boundary conditions $F_1(p_2) = 1 + \tilde{c}$ and $F_1(p_2) = \tilde{c}$, where $\tilde{c}$ is a constant. As we will see in the following, the constant $\tilde{c}$ is not important for our estimate and we assume that $\tilde{c} = 0$.
We consider the strip V, where

$$V = \varphi([t_0 - \epsilon, t_0] \times [-x, x]), \quad \text{where} \quad \epsilon > 0 \text{ if } t_0 < 0 \text{ and } \epsilon < 0 \text{ if } t_0 > 0.$$

We will show, how to obtain a lower bound for the energy of $F_1|_V$, $E_V(F_1)$ for sufficiently small ϵ. We can align these strips to obtain a lower bound for $E_B(F_1) \leq E(F_1)$. We derive a lower bound for the energy of $F_1|_V$, assuming that

$$\begin{aligned} F_1|_{\varphi([t_0 - \epsilon, t_0] \times \{-x\})} &= F_1(p_1) = 0 \quad \text{and} \quad F_1|_{\varphi([t_0 - \epsilon, t_0] \times \{x\})} = F_1(p_2) = 1 \quad \text{and} \\ F_1|_{\varphi([t_0 - \epsilon, t_0] \times \{\frac{-\lambda}{2}\})} &= F_1(p_1') = a_1' \quad \text{and} \quad F_1|_{\varphi([t_0 - \epsilon, t_0] \times \{\frac{\lambda}{2}\})} = F_1(p_2') = a_2' \end{aligned}$$

Consider the set $\varphi([t_0 - \epsilon, t_0] \times [\frac{-\lambda}{2}, \frac{\lambda}{2}]) = V' \subset V$. Let $F_{t_0} = f_{t_0} \circ \psi^\nu$ be a function defined on V' that realizes the minimum

$$\min\{\int_{V'} \|p_\varphi(Df)\|_2^2 \mid f \in \text{Lip}(V'), f|_{\varphi([t_0 - \epsilon, t_0] \times \{\frac{-\lambda}{2}\}} = a_1 \text{ and } f|_{\varphi([t_0 - \epsilon, t_0] \times \{\frac{\lambda}{2}\}} = a_2\} \quad \text{(see eq. (3.1))}.$$

It follows from **Theorem 1.5.3** or by considering skewed Fermi coordinates as a limit case that f_{t_0} is given by

$$f_{t_0}(t, s) = \frac{a_2 - a_1}{H(\frac{\lambda}{2}) - H(-\frac{\lambda}{2})} H(s) + \frac{a_1 H(\frac{\lambda}{2}) - a_2 H(-\frac{\lambda}{2})}{H(\frac{\lambda}{2}) - H(-\frac{\lambda}{2})}.$$

The energy $E_{V'}(p_\varphi(DF_{t_0}))$ is

$$E_{V'}(p_\varphi(DF_{t_0})) = \frac{(a_2 - a_1)^2 \sin(\nu)|\epsilon|}{2(\arctan(\exp(\frac{\lambda}{2})) - \arctan(\exp(-\frac{\lambda}{2})))} = k_1(a_2 - a_1)^2|\epsilon|. \quad (3.21)$$

We can extend F_{t_0} to a function on V that satisfies the boundary conditions

$$F_{t_0}|_{\varphi([t_0 - \epsilon, t_0] \times \{\pm x\})} = F_1|_{\varphi([t_0 - \epsilon, t_0] \times \{\pm x\})}.$$

Again we choose F_{t_0}, such that it minimizes $E_{V \setminus V'}(p_\varphi(D(\cdot)))$ with the given boundary conditions. We have with $E_{V \setminus V'}(p_\varphi(DF_{t_0})) = E_{V \setminus V'}(F_{t_0})$:

$$E_{V \setminus V'}(F_{t_0}) = \frac{(a_1^2 + (1 - a_2)^2)|\epsilon|}{2(\arctan(\exp(x)) - \arctan(\exp(\frac{b}{2})))} = k_2(a_1^2 + (1 - a_2)^2)|\epsilon|. \quad (3.22)$$

For $a_1 = a_1'$ and $a_2 = a_2'$, we have by construction $E_V(F_1) \geq E_V(p_\varphi(DF_{t_0}))$.
Though we do not know the values a_1' and a_2', we obtain a lower bound of the energy of F_1 on V, if we determine the values $F_{t_0}(p_1') = c_1 = c_1(t_0)$ and $F_{t_0}(p_2') = c_2 = c_2(t_0)$, respectively, such that these values are minimizing the energy $E_V(p_\varphi(F_{t_0}))$. As the two arcs γ_p^1 and γ_p^2 have the same length, we have to solve the following problem :
Find c_1, c_2, such that $1 - c_2 = c_1 \Leftrightarrow (c_2 - c_1) = 1 - 2c_1$, such that

$$E_V(p_\varphi(DF_{t_0})) = E_{V'}(p_\varphi(DF_{t_0})) + E_{V \setminus V'}(F_{t_0})$$

is minimal. We obtain from equation (3.21) and (3.22) that $c_1 = \frac{k_1}{k_2 + 2k_1}$ and

$$E_V(F_1) \geq E_V(p_\varphi(DF_{t_0})) = \frac{|\epsilon|k_1 k_2}{k_2 + 2k_1}. \tag{3.23}$$

We can cover B with a set of such strips $V_{t_0} = V$, such that these intersect only on the boundary and combine the $F_{t_0}|_{V_{t_0}}$ to a function f_1 on B. We have $E_V(F_1) \geq E_V(p_\varphi(Df_1))$.
As we consider only the energy $E_B(p_\varphi(Df_1))$ of the projection, the approximation is true in the limit case, where $\epsilon \to 0$. Hence

$$q_{11} = E(F_1) \geq E_B(p_\varphi(Df_1)). \tag{3.24}$$

$E_{V \setminus V'}(p_\varphi(DF_{t_0}))$ is monotonously decreasing if the length is $x - \frac{b}{2}$ increasing. Hence we find a simpler approximation for $E(F_1)$, setting $x = c$ in equation (3.22). In this case, we call $k_i = k_i(c)$, for $i \in \{1, 2\}$. We obtain a simplified upper bound, if we define our test function only on $Z_{w'}(\alpha_2)$, where $w' = \min\{a, \frac{b}{2}\}$ We obtain :

$$\frac{2\alpha'' k_1(c)k_2(c)}{k_2(c) + 2k_1(c)} \leq E(F_1) = q_{11} \leq \frac{\alpha_2}{2(\arctan(\exp(w') - \arctan(\exp(-w')))}.$$

The lower bound that depends only on α_2, t_2 and β_1 can be obtained by expressing $\frac{b}{2}$ and λ and ν in terms of these variables (see equation (3.18),(3.19) and (3.20)). Using the parametrization of T_2 in equation (3.22) we can express the lower bound in terms of these variables α_2, t_2 and β_1 using (3.23). This way we obtain explicit values in equation (3.24).

Remark : From numerical simulations we get the impression that the lower bound for $E(F_1)$ is only usable, if the twist parameter t_2 is small. The reason for this seems to be that using the projection strongly underestimates the energy. The result gets worse with increasing t_2 and increasing length of α_2.

Using these methods, we obtain upper and lower bounds on all entries of the period Gram matrix Q_S (see section 3.1).

3.3   Surfaces with small non-separating simple closed geodesics

Consider a surface S containing a small non separating scg α_2. α_2 can be extended to a canonical homology basis A = $(\alpha_1, \alpha_{\tau(1)}, ..., \alpha_g, \alpha_{\tau(g)})$. We have by the collar lemma (**Theorem 2.3.9**) that the width w_2 of a collar $C(\alpha_2)$ around α_2 in S has the lower bound

$$w_2 \geq \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\alpha_2}{2})}\right). \tag{3.25}$$

For $|w| < w_2$, consider the set of points a_w with constant directed distance w from α_2

$$a_w = \{x \in C(\alpha_2) \mid \operatorname{dist}(x, \alpha_2) = w\}.$$

The set a_w forms a simple closed curve freely homotopic to α_2. This implies that

$$\int_{a_w} \sigma_2 = \int_{\alpha_2} \sigma_2 = 1 \tag{3.26}$$

We can find a curve system $A' = (\alpha_1', \alpha_{\tau(1)}', ..., \alpha_g', \alpha_{\tau(g)}')$ that satisfies the following conditions :
$\alpha_2 = \alpha_2'$ and given a parametrization ψ of the closure $\overline{C(\alpha_2)}$ in Fermi coordinates with base line α_2, α_1 contains a segment α' that is given by

$$\alpha' = \psi(\{0\} \times [-w_2, w_2]), \alpha' \subset \alpha_1$$

and $\alpha_1 \setminus \alpha'$ does not intersect $Z_{w_2-\epsilon}(\alpha_2)$ for all $\epsilon > 0$. Furthermore for each $i \in \{3, ..., 2g\}$, α_i' is a curve that represents the same homology class as α_i, but such that α_i' does not intersect $C(\alpha_2)$. We further demand that we obtain a simply connected surface S_o by cutting open S along $A' \setminus \alpha_2$. It is easy to see that this is indeed possible.
On S_o σ_2 has a primitive F_2. For two boundary points p_1 and p_2 on $S_o \cap C(\alpha_2)$ that map to the same point p in S, the integral conditions in equation (3.26) on the a_w imply WLOG

$$F_2(p_2) - F_2(p_1) = 1.$$

We obtain a lower bound of the energy $E(\sigma) = E(F_2)$ in the following way. Let $F_2 = f_2 \circ \psi$ be given in Fermi coordinates of α_2. We have by equation (1.7) :

$$
\begin{aligned}
E(F_2) &\geq E_{C(\alpha_2)}(F_2) = \int\limits_0^{\alpha_2} \int\limits_{-w_2}^{w_2} \frac{\partial f_2(t,s)}{\partial t}^2 \cdot \frac{1}{\cosh(s)} + \frac{\partial f_2(t,s)}{\partial s}^2 \cdot \cosh(s) \, dt \, ds \\
&\geq \int\limits_0^{\alpha_2} \int\limits_{-w_2}^{w_2} \frac{\partial f_2(t,s)}{\partial t}^2 \cdot \frac{1}{\cosh(s)} \, dt \, ds.
\end{aligned}
$$

Let $D = [0, \alpha_2] \times [-w_2, w_2]$. Using **Theorem 1.5.2** or **Theorem 1.5.3** we can find the function $\tilde{f}$ that realizes the minimum

$$\min\{\int\limits_0^{\alpha_2} \int\limits_{-w_2}^{w_2} \frac{\partial f(t,s)}{\partial t}^2 \cdot \frac{1}{\cosh(s)} \, dt \, ds \mid f \in \mathrm{Lip}(D), f(\alpha_2, s) - f(0, s) = 1 \;\; \forall s \in [-w_2, w_2]\}.$$

f is the function given by $f(s,t) = \frac{t}{\alpha_2} + b(s)$.
Set $H(s) = 2 \arctan(\exp(s))$. As $H(s)' = \frac{1}{\cosh(s)}$ and $H(s) - H(-s) = \pi - 2 \arcsin(\frac{1}{\cosh(s)})$, we have :

$$E(\sigma_2) = E(F_2) \geq \frac{\pi - 2 \arcsin(\frac{1}{\cosh(w_2)})}{l(\alpha_2)}.$$

We have proven the following theorem, where the upper bound on $E(\sigma_1)$ was shown in [BS].

Theorem 3.3.1 *Let S be a Riemann surface that contains a small non-separating scg α_2 of length $l(\alpha_2)$ that has a collar $C(\alpha_2)$ of width w_2. Then there exists a canonical homology basis A, such that $\alpha_1 = \alpha_{\tau(2)}$ and a dual basis of real harmonic forms $(\sigma_i)_{i=1,..,2g}$, such that :*

$$E(\sigma_2) \geq \frac{\pi - 2 \arcsin(\frac{1}{\cosh(w_2)})}{l(\alpha_2)} \quad and \quad E(\sigma_1) \leq \frac{l(\alpha_2)}{\pi - 2 \arcsin(\frac{1}{\cosh(w_2)})}.$$

For small α_2, we can also apply equation (3.25), to obtain inequalities that depend only on the length of α_2.

Remark : That the upper bound is the inverse of the lower bound can be explained in the following way. The area $\psi([0, \alpha_2] \times [-w, w])$ can be mapped biholomorphically onto a real flat cylinder. It is easy to see that, translating the boundary conditions to the flat cylinder, we obtain this relationship (see **Example 1.5.9**).

For $i \in \{1, 2\}$, $E(\sigma_i)$ is the squared norm of a vector v_i in the lattice of the Jacobian $J(S)$. Therefore **Theorem 3.3.1** provides estimates for the squared norm of the two primitive, linear independent vectors v_1 and v_2.

This way the theorem is related to a result of Fay. In [Fa], chap. III a sequence of Riemann surfaces S_t are constructed, where t denotes the length of a simple closed geodesic η. Here η is a non-separating simple close geodesic. If $t \to 0$, the length of one vector in the lattice of the Jacobian converges to zero. **Theorem 3.3.1** gives an explicit upper bound for the length of this lattice vector depending on t. It shows furthermore that the length of a second vector in the lattice of the Jacobian goes to infinity and gives an explicit lower bound depending on t.

Chapter 4

A lower bound for Hermite's constant among PPAVs

It was shown in [BS], p. 53-54 that the corresponding $2g$ dimensional real lattices of PPAVs of dimension g (see chapter 1.4) are exactly the symplectic lattices. Based on this finding, a number of symplectic lattices are presented, whose shortest non-zero lattice vector - or systole - is large. These lattices all share in common that they exhibit a high degree of symmetry, i.e. have a large automorphism group. This was the incentive to investigate, whether symmetric PPAVs tend to have a larger systole in general than the average PPAV, and if this can be used to improve the lower bound of Hermite's constant δ_{2g}.

This idea will be pursued in the first section of this chapter. Here we could improve the lower bound for δ_{2g} in even dimensions $(g = 2n)$ by applying a mean-value argument from the geometry of numbers to a subset of symmetric PPAVs. Here we obtain only a slight improvement. However, we believe that the method applied has further potential.
In the course of research for a proof, we also discovered families of highly symmetric PPAVs in dimensions of power of two. Here a family in dimension 2^n is constructed with the help of a multiplicative matrix group isomorphic to $(\mathbb{Z}_2{}^n, +)$. These families will be presented in the second section of this chapter. We conjecture that we can use these or similar families to improve the lower bound for δ_{2g} further.

4.1 An improved lower bound for Hermite's constant among the PPAVs

We have seen in chapter 1.4 that any PPAV of dimension g can be represented as (A_Z, H_Z), such that $Z = X + iY \in \mathcal{H}_g$. The corresponding real Gram matrix with respect to the polarization is

$$Q_Z = \begin{pmatrix} XY^{-1}X + Y & XY^{-1} \\ Y^{-1}X & Y^{-1} \end{pmatrix} = \begin{pmatrix} Id_g & X \\ 0 & Id_g \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \qquad (4.1)$$

which is the Gram matrix of the real $2g$ dimensional symplectic lattice $P_Z \cdot \mathbb{Z}^{2g}$, such that

$$P_Z = P_{X,Y} = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \in M_{2g}(\mathbb{R}). \qquad (4.2)$$

We recall that Hermite's invariant δ_{2g} among PPAVs of dimension g, is given by

$$\delta_{2g} = \max_{(A,H)\in\mathcal{A}_g} m_1(A,H)^2 = \max_{Z\in\mathcal{H}_g} \min_{l\in\mathbb{Z}^{2g}\backslash\{0\}} \|P_Z \cdot l\|_2^2.$$

We call a real g dimensional lattice L *symmetric*, if there is an $O \in O_g(\mathbb{R})\backslash\{\pm Id_g\}$, such that

$$OL = L.$$

We call a PPAV (A_Z, H_Z) *symmetric*, if its corresponding real $2g$ lattice $P_Z \cdot \mathbb{Z}^{2g}$ is symmetric. The motivation to investigate symmetric PPAVs is the following result by Voronoï :
Among all lattices of fixed determinant, a lattice is called *extreme*, if the length of its shortest non-zero lattice vector is a local maximum. Voronoï showed in [Vo] that this definition is equivalent to the lattice being *perfect* and *eutactic*. Especially, for a lattice to be extreme it has to have many systoles of the same length and often these lattices exhibit a high degree of symmetry.
Explicit examples of PPAVs or symplectic lattices, where the lengths of the systoles of the corresponding lattices are a local maxima are given in [Ber] and [BS]. Buser and Sarnak show in [BS] :

Theorem 4.1.1 *Among the PPAVs of dimension g one has for Hermite's invariant δ_{2g} :*

$$\frac{1}{\pi}\sqrt[g]{2g!} \leq \delta_{2g}.$$

We now give a slightly improved bound in the case, where the moduli space $\mathcal{A}_g$ has an even dimension.

Theorem 4.1.2 *If $g = 2n$, where $n \in \mathbb{N}$ one has for Hermite's invariant for PPAVs of dimension g :*

$$\frac{1}{\pi}\sqrt[g]{4g!} \leq \delta_{2g}.$$

The following proof is similar to the proof of **Theorem 4.1.1**, given in [BS].

proof Be $Z = X + iY \in \mathcal{H}_g$. To simplify the following calculations we will work with another version of the lattice $P_Z\mathbb{Z}^{2g}$. It is the lattice, where the basis P_Z' is given in matrix form by

$$P_Z' = P_{X,Y}' = \begin{pmatrix} \sqrt{Y^{-1}} & \sqrt{Y^{-1}}X \\ 0 & \sqrt{Y} \end{pmatrix} = \begin{pmatrix} 0 & Id_g \\ Id_g & 0 \end{pmatrix} P_Z \begin{pmatrix} 0 & Id_g \\ Id_g & 0 \end{pmatrix}. \tag{4.3}$$

For a positive real number r and a PPAV (A,H) we define $n_r(A,H)$ as the number of non-zero lattice vectors of the lattice L whose squared length is smaller than or equal to r^2. One has

$$n_r(A,H) = \#\{l \in L\backslash\{0\} \mid H(l,l) \leq r^2\}. \tag{4.4}$$

For a representant $Z \in \mathcal{H}_g$, representing the isomorphy class of PPAVs (A_Z, H_Z) we obtain

$$n_r(A_Z, H_Z) = \sum_{l\in\mathbb{Z}^{2g}\backslash\{0\}} \chi_{r^2}(l^T Q_Z' l),$$

where $Q'_Z = P'^T_Z P'_Z$ and χ_{r^2} is the characteristic function of the interval $[0, r^2]$. We now consider the following subset of matrices in $\mathcal{H}_g = \mathcal{H}_{2n}, n \in \mathbb{N}$ of the form

$$Z = X + i\frac{1}{y^2} Id_g,$$

such that $y > 0$, $X = X^T$ and $KXK = X$, where K is the orthonormal matrix

$$K = \begin{pmatrix} 0 & ... & 0 & 1 \\ ... & & -1 & 0 \\ 0 & 1 & ... & \\ -1 & 0 & ... & 0 \end{pmatrix} \in M_g(\mathbb{R}).$$

The matrix $KXK = X = (X_{i,j})_{i,j=1,..,g}$ has the following shape:

$$\begin{aligned} X_{i,j} &= X_{j,i} \\ X_{i,j} &= \begin{cases} -X_{g+1-i,g+1-j} & \text{if } i+j \text{ is even} \\ X_{g+1-i,g+1-j} & \text{if } i+j \text{ is odd} \end{cases} \end{aligned}$$

This is equivalent to that the matrix $|X|$ is symmetric with respect to the first and second main diagonal and the diagonals below the second main diagonal of X are alternatively the positive and negative image of the corresponding diagonals above the second main diagonal with respect to the reflection along this diagonal. The matrix X is completely determined by the $\frac{g}{4}(g+2)$ entries $(X_{i,j})_{(i \leq j) \wedge (i \leq g+1-j)}$.
Under these conditions the matrix $P'_Z = P'_{X, \frac{1}{y^2} Id_g}$ has the following property :

$$K' \cdot P'_Z = P'_Z \cdot K', \quad \text{where} \quad K' = \begin{pmatrix} K & 0 \\ 0 & K^T \end{pmatrix}.$$

This means that the corresponding lattice $P'_Z \cdot \mathbb{Z}^{2g}$ is closed under the symmetry induced by the orthonormal matrix K', whose only fixed point is zero. It follows that the lattice vectors $P'_Z \cdot l$ and $P'_Z \cdot K'l$ of the lattice $P'_Z \cdot \mathbb{Z}^{2g}$ have the same length and $K'l = l$ if and only if $l = 0$. Therefore there exist always 4 distinct non-zero lattice vectors in $P'_Z \cdot \mathbb{Z}^{2g}$ of the same length.

We now consider the set V_g for $g = 2n, n \in \mathbb{N}$:

$$\begin{aligned} V_g &= \{X \in M_g(\mathbb{R}) \mid X = X^T, KXK = X, X_{ij} \in [0,1]\} \quad \text{and} \quad (4.5) \\ \dim(V_g) &= v_g = \frac{g(g+2)}{4}. \end{aligned}$$

Hereby $\mathrm{vol}(V_g) = 1$ as we integrate over the interval $[0, 1]$ in each variable. For a fixed y we obtain the mean value $I(y)$ of $n_{r^2}(A_Z, H_Z)$ over the compact set V_g :

$$I(y) = \frac{1}{\mathrm{vol}(V_g)} \int_{V_g} \sum_{l \in \mathbb{Z}^{2g} \setminus \{0\}} f(P'_{X, \frac{1}{y^2} Id_g} \cdot l) \ dX$$

Hereby $f : \mathbb{R}^{2g} \to \mathbb{R}$ is the function $f(x) = \chi_{r^2}(x^T \cdot x)$ and $dX = \prod_{(i \leq j) \wedge (i \leq g+1-j)} dX_{ij}$. With the help of the following **Lemma 4.1.3** and the volume formula for unit balls in dimension $2g$ we obtain

$$\lim_{y \to +\infty} I(y) = \int_{\mathbb{R}^{2g}} f(x)\, dx = \sigma_{2g} \cdot r^{2g} = \frac{\pi^g \cdot r^{2g}}{g!}.$$

For $r^2 < \frac{1}{\pi} \sqrt[g]{4g!} = c$ one has

$$\lim_{y \to +\infty} I(y) < 4.$$

As $I(y)$ is the mean number of non-zero lattice vectors with squared length smaller than or equal to r^2 in V_g and as $I(y)$ is decreasing for increasing values of y, there exists for a $r^2 < c$ a y', such that the mean value $I(y')$ is smaller than 4. As in our case the number of non-zero lattice vectors of fixed length is always a multiple of four, there exists an X' in V_g, such that the PPAV with period matrix $(X', \frac{1}{y'^2} Id_g)$ for $r^2 < c$ has no non-zero lattice vector with squared length smaller than r^2. Therefore we have found a lower bound for δ_{2g} and we obtain :

$$\delta_{2g} \geq \frac{1}{\pi} \sqrt[g]{4g!}. \quad \square$$

Lemma 4.1.3 *Let $g = 2n, n \in \mathbb{N}$ and $f : \mathbb{R}^{2g} \to \mathbb{R}$ an integrable function of compact support. Consider the function $I_f : \mathbb{R}^+ \to \mathbb{R}$ which is defined by*

$$I_f(y) = \int_{V_g} \sum_{l \in \mathbb{Z}^{2g} \setminus \{0\}} f(P'_{X, \frac{1}{y^2} Id_g} \cdot l) \ \ dX,$$

such that P'_Z and V_g as in equation (4.3) and (4.5) respectively and $dX = \prod_{(i \leq j),(i \leq g+1-j)} dX_{ij}$. One has :

$$\lim_{y \to +\infty} I_f(y) = \int_{\mathbb{R}^{2g}} f(x)\, dx.$$

proof Let $l = (m, n)^T \in \mathbb{Z}^g \times \mathbb{Z}^g$. We now evaluate the integral $I_f(y)$:

$$I_f(y) = \int_{V_g} \sum_{\substack{m,n \in \mathbb{Z}^g \\ (m,n) \neq (0,0)}} f\left(\begin{array}{c} y(m + Xn) \\ \frac{1}{y}n \end{array} \right) \ \ dX$$

Hereby the sum and the integral are interchangeable. We split the sum over n into the part where $n = 0$ and the part where $n \neq 0$. This way we obtain for $I_f(y)$:

$$I_f(y) = \int_{V_g} \sum_{n=0} \sum_{m \in \mathbb{Z}^g \setminus \{0\}} f\left(\begin{array}{c} y(m + Xn) \\ \frac{1}{y}n \end{array} \right) \ \ dX + \sum_{n \in \mathbb{Z}^g \setminus \{0\}} c_n(X)$$

For the first summand we obtain:

$$\int_{V_g} \sum_{n=0} \sum_{m \in \mathbb{Z}^g \setminus \{0\}} f\left(\begin{array}{c} y \cdot m \\ 0 \end{array} \right) \ \ dX = \sum_{m \in \mathbb{Z}^g \setminus \{0\}} f\left(\begin{array}{c} y \cdot m \\ 0 \end{array} \right)$$

In the second summand we sum over $n \in \mathbb{Z}^g \backslash \{0\}$. To structure this part of the proof, we first have to fix some notations concerning the matrix X.

Let M be the set of index-pairs

$$M = \{(i', j') \mid (i' \leq j') \wedge (i' \leq g + 1 - j')\}.$$

For $(i, j) \in M$ we have

$$
\begin{aligned}
X_{i,j} = X_{j,i} &= \sigma(i,j) X_{i,g+1-j} = \sigma(i,j) X_{g+1-j,i}, \quad \text{where} \qquad (4.6) \\
\sigma(i,j) &= \begin{cases} -1 & \text{if } i+j \text{ even} \\ 1 & i+j \text{ odd} \end{cases}
\end{aligned}
$$

For the proof we will have to consider pairs of rows and columns of the matrix X of P'_Z simultaneously. We call $X_1, ..., X_g$ the rows of the matrix X and use the same notation for P'_Z. For the proof, we will have to consider the pairs X_i and X_{g+1-i}, respectively. To abbreviate the notations we denote

$$h(i) = g + 1 - i, \quad \text{for all } i \in \{1, ..., g\}.$$

Considering the terms of n_1 and $n_{h(1)} = n_g$ in the sum $\sum_{n \in \mathbb{Z}^g \backslash \{0\}} c_n(X)$ above, one has :

$$
\begin{aligned}
\sum_{n \in \mathbb{Z}^g \backslash \{0\}} c_n(X) &= \sum_{(n_1^\vee, n_2, .., n_g^\vee) \backslash \{0\}} \Big(\sum_{n_1 = 0 \wedge n_g = 0} \int_{V_g} \sum_{m \in \mathbb{Z}^g} f \begin{pmatrix} y(m + Xn) \\ \frac{1}{y} n \end{pmatrix} \ dX \Big) + \\
& \quad \sum_{(n_1^\vee, n_2 ..., n_g^\vee)} \Big(\sum_{n_1 \neq 0 \vee n_g \neq 0} \int_{[0,1]^{v_g - 2}} \sum_{(m_1^\vee, m_2, ..., m_g^\vee)} F_{1,1}(m, n, X) \ dX \backslash (X_{1,1}, X_{1,g}) \Big)
\end{aligned}
$$

Hereby $F_{1,1}(m, n, X)$ is defined as :

$$
F_{1,1}(m, n, X) = \sum_{m_1, m_g} \int_{[0,1]^2} f \begin{pmatrix} y(m + Xn) \\ \frac{1}{y} n \end{pmatrix} \ dX_{1,1} dX_{1,g}
$$

Analogously we define $\{F_{i,j}(m, n, X)\}_{(i,j) \in M}$ by

$$
F_{i,j}(m, n, X) = \sum_{m_i, m_{h(i)}} \int_{[0,1]^2} f \begin{pmatrix} y(m + Xn) \\ \frac{1}{y} n \end{pmatrix} \ dX_{i,j} dX_{i,h(i)}.
$$

We now evaluate $F_{1,1}(m, n, X)$. We consider only the first and g-th entry of $P'_Z \cdot \begin{pmatrix} m \\ n \end{pmatrix}$ and note that $X_{1,1}$ and $X_{1,g}$ only occur in these two entries. $F_{1,1}(m, n, X)$ can now be written as

$$
F_{1,1}(m, n, X) = \sum_{m_1, m_g} \int_{[0,1]^2} f \begin{pmatrix} y \cdot (m_1 + \sum_{j \neq 1, j \neq g} X_{1,j} n_j + X_{1,1} n_1 + X_{1,g} n_g) \\ ... \\ y \cdot (m_g + \sum_{j \neq 1, j \neq g} X_{g,j} n_j - X_{1,1} n_g + X_{1,g} n_1) \end{pmatrix} \ dX_{1,1} dX_{1,g},
$$

due to the relations in (4.6).

These two rows can be written as

$$y \left(A_{1g} \left(\begin{array}{c} X_{1,1} \\ X_{1,g} \end{array} \right) + \left(\begin{array}{c} m_1 \\ m_g \end{array} \right) + \left(\begin{array}{c} \lambda_{11} \\ \lambda_{1g} \end{array} \right) \right) \qquad (4.7)$$

where A_{1g} is the matrix $\left(\begin{array}{cc} n_1 & n_g \\ -n_g & n_1 \end{array} \right)$. As $n_1 \neq 0$ or $n_g \neq 0$, $\det(A_{1g}) = n_1^2 + n_g^2 \neq 0$ and we apply the two dimensional formula for integration by substitution to these two rows.

Note that $A_{1g} \cdot [0,1]^2$ is a parallelogram. Furthermore the translates of the parallelogram $A_{1g} \cdot [0,1]^2$ by $(m_1, m_g) \in \mathbb{Z}^2$ cover disjointly $\det(A_{1g})$ copies of $\mathbb{R}^2$. By integration by substitution we therefore obtain for $F_{1,1}(m,n,X)$:

$$F_{1,1}(m,n,X) = \frac{\det(A_{1g})}{\det(A_{1g})} \int\limits_{\mathbb{R}^2} f \left(\begin{array}{c} y \cdot X_{1,1} \\ ... \\ y \cdot X_{1,g} \end{array} \right) \, dX_{1,1} dX_{1,g}.$$

We proceed the same way with $X_{1,2} = X_{2,1}, ..., X_{1,\frac{g}{2}} = X_{\frac{g}{2},1}$ by successively integrating over $F_{1i}(m,n,X)$. In each step the variables $X_{1,i}$ and $X_{1,h(i)}$ occur only twice and we use again integration by substitution. Hereby the determinants of the corresponding transformation matrices A_{1i} are strictly positive, due to the shape of the matrix X.

Therefore we obtain by successive integration :

$$\sum_{(n_1^{\vee}, n_2..., n_g^{\vee})} \sum_{n_1 \neq 0 \vee n_g \neq 0} \int\limits_{[0,1]^{v_g-2}} \sum_{(m_1^{\vee}, m_2..., m_g^{\vee})} F_{11}(m,n,X) \, dX \backslash (X_{1,1}, X_{1,g})$$

$$= \sum_{(n_1^{\vee}, n_2..., n_g^{\vee})} (\sum_{n_1 \neq 0 \vee n_g \neq 0} \int\limits_{[0,1]^{v_g-g}} (\frac{1}{y^g} \int\limits_{\mathbb{R}^g} f \left(\begin{array}{c} X_1 \\ \frac{1}{y} n \end{array} \right) \, dX_1) \, dX \backslash X_1)$$

$$= \frac{1}{y^g} \sum_{(n_1^{\vee}, n_2..., n_g^{\vee})} (\sum_{n_1 \neq 0 \vee n_g \neq 0} \int\limits_{\mathbb{R}^g} f \left(\begin{array}{c} X_1 \\ \frac{1}{y} n \end{array} \right) \, dX_1) \qquad (4.8)$$

Hereby $X_1 = (X_{11}, ..., X_{1g})^T$. For the set $\mathbb{Z}^g \backslash \{0\}$ we obtain :

$$\mathbb{Z}^g \backslash \{0\} = \bigcup_{i=1}^{g/2} \{n \in \mathbb{Z}^g \mid (n_i \neq 0) \vee (n_{h(i)} \neq 0) \text{ and } n_j = 0 \; \forall j, \text{ such that } (j < i) \vee (j > h(i))\}$$

We now divide the set $\mathbb{Z}^g \backslash \{0\}$ successively into disjoint unions of these subsets and sum up these subsets after integration over the $\{F_{i,j}(m,n,X)\}_{(i,j) \in M}$. As in equation (4.8) we obtain that

$$\sum_{n \in \mathbb{Z}^g \backslash \{0\}} c_n(X) = \sum_{n \in \mathbb{Z}^g \backslash \{0\}} y^{-g} \int\limits_{\mathbb{R}^g} f \left(\begin{array}{c} t \\ \frac{n}{y} \end{array} \right) \, dt.$$

As f is continuous of compact support we obtain in total :

$$
\begin{aligned}
\lim_{y \to +\infty} I_f(y) &= \lim_{y \to +\infty} \left(\sum_{n \in \mathbb{Z}^g \backslash \{0\}} c_n(X) + \sum_{m \in \mathbb{Z}^g \backslash \{0\}} f \begin{pmatrix} y \cdot m \\ 0 \end{pmatrix} \right) \\
&= \lim_{y \to +\infty} \left(\sum_{n \in \mathbb{Z}^g \backslash \{0\}} y^{-g} \int_{\mathbb{R}^g} f \begin{pmatrix} t \\ \frac{n}{y} \end{pmatrix} dt \right) \\
&\quad + \lim_{y \to +\infty} \sum_{m \in \mathbb{Z}^g \backslash \{0\}} f \begin{pmatrix} y \cdot m \\ 0 \end{pmatrix} \\
&= \int_{\mathbb{R}^{2g}} f(x) \ dx + 0 = \int_{\mathbb{R}^{2g}} f(x) \ dx,
\end{aligned}
$$

as for a continuous function $h : \mathbb{R}^g \to \mathbb{R}$ of compact support one has

$$
\lim_{y \to +\infty} \sum_{n \in \mathbb{Z}^g \backslash \{0\}} y^{-g} \cdot h \left(\frac{n}{y} \right) = \int_{\mathbb{R}^g} h(t) \, dt,
$$

by integration over Riemann sums. $\square$

Remark : The lower bound given for $\delta_{4n}, n \in \mathbb{N}$ could be further improved, if we could find PPAVs, which have additional symmetries. Though this idea was pursued in section 4.2, we could not evaluate the corresponding integral. This problem occurs in the case, where the determinant of the transformation matrices used in (4.7) is zero. If we could find a way to evaluate this integral it might be possible to improve the result (see **Conjecture 4.2.2**).

4.2   Families of highly symmetric PPAVs

In the following section we will construct PPAVs, represented by $Z \in \mathcal{H}_g$ of dimension $g = 2^n, n \in \mathbb{N}$, whose corresponding real lattices $P_Z \cdot \mathbb{Z}^{2g}$ (see equation (4.2)) have at least $2^n - 1$ pairwise distinct symmetries. We will then show that these lie in the stabilizer of a certain subgroup $G'_g \simeq \mathbb{Z}_2{}^n$ of $Sp_{2g}(\mathbb{Z})$.

We remind that we call a real lattice L of dimension g symmetric, if it is closed under a nontrivial element of the orthonormal group. This means that there exists an orthogonal matrix $O \in O_g(\mathbb{R}) \backslash \{\pm Id_g\}$ such that :
$$ OL = L $$

If $L = A \cdot \mathbb{Z}^g$, where A is a matrix, whose columns form a basis of L, then this means that the matrix O induces a change of basis. Therefore the above equality is equal to :

$$ O \cdot A = A \cdot R, \quad \text{such that } R \in M_g(\mathbb{Z}), \det(M) \in \{\pm 1\}. \tag{4.9} $$

In the previous section we have constructed lattices that have always at least four vectors of the same length. In general, we have the following result, if the lattice is symmetric.
Let $l \in L, l = Az, z \in \mathbb{Z}^g$. If the lattice L is symmetric , the vector Ol, has the same length as l

and not only 2 but 4 vectors of the same length exist, except if $Ol = l$. By equation (4.9), this is equal to the fixed point equality :

$$O \cdot l = l \Leftrightarrow O \cdot Az = Az \Leftrightarrow Rz = z \tag{4.10}$$

We will construct our families of symmetric PPAVs with the help of the following matrices. Consider $J_g \in O_g(\mathbb{R})$, such that

$$J_g = \begin{pmatrix} 0 & ... & 0 & 1 \\ ... & & 1 & 0 \\ 0 & 1 & ... & \\ 1 & 0 & ... & 0 \end{pmatrix}, \quad \text{we have } J_g = J_g^T = J_g^{-1}, \tag{4.11}$$

Be $A \in M_g(\mathbb{R})$. We denote by A_T, the matrix that we obtain by reflecting A along its second main diagonal. We have

$$J_g^T A J_g = A \Leftrightarrow A = A^T = A_T.$$

It is easy to see that the set of matrices

$$R_g = \{A \in M_g(\mathbb{R}) \mid J_g^T A J_g = A\}$$

is closed under addition, multiplication and contains the matrix Id_g and therefore forms a ring. If A is invertible, $A^{-1} \in R_g$. We also have that $J_g \in R_g$.
We now construct matrices A of dimension $g = 2^n$, whose corresponding lattices $A \cdot \mathbb{Z}^g$ have at least $2^n - 1$ pairwise distinct symmetries. Let $\mathbb{A}_2$ be the set of matrices.

$$\mathbb{A}_2 = \{A_2 \in M_2(\mathbb{R}) \mid A_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \text{ such that } a, b \in \mathbb{R}\}.$$

We define inductively $\mathbb{A}_{2^n}$ to be the set of matrices

$$\mathbb{A}_{2^n} = \{A_{2^n} \in M_{2^n}(\mathbb{R}) \mid A_{2^n} = \begin{pmatrix} A_{2^{n-1}} & B_{2^{n-1}} \\ B_{2^{n-1}} & A_{2^{n-1}} \end{pmatrix}, \text{ such that } A_{2^{n-1}}, B_{2^{n-1}} \in \mathbb{A}_{2^{n-1}}\}. \tag{4.12}$$

For $n \in \mathbb{N}$, set

$$J_{2^n}^n = J_{2^n}, J_{2^n}^{n-1} = \begin{pmatrix} J_{2^{n-1}} & 0 \\ 0 & J_{2^{n-1}} \end{pmatrix}, ..., J_{2^n}^1 = \begin{pmatrix} J_2 & 0 & ... & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & ... & 0 & J_2 \end{pmatrix}.$$

It is easy to prove by induction that

$$A \in \mathbb{A}_{2^n} \Leftrightarrow J_{2^n}^k A J_{2^n}^k = A \text{ for all } k \in \{1, .., n\}. \tag{4.13}$$

The symmetries among the matrix entries of $\mathbb{A}_k$, for $k = 2, 4$ and 8 are depicted in Figure 4.1. Here each box represents a matrix entry and each diagonal line an axis of symmetry. That the matrices exhibit these symmetries follows from equation (4.12). These symmetries equally define $\mathbb{A}_{2^n}$.
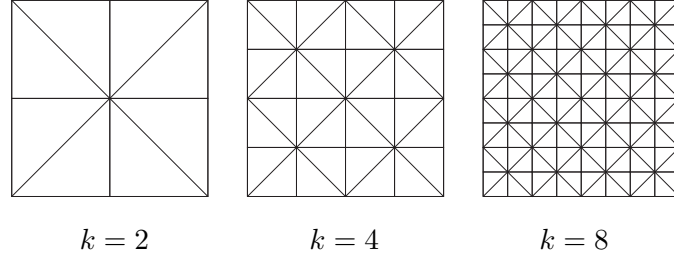
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$$k = 2 \qquad\qquad k = 4 \qquad\qquad k = 8$$

Figure 4.1: Symmetries of the matrices in $\mathbb{A}_k$ for $k = 2, 4$ and 8.

We denote by G_{2^n} the multiplicative subgroup of $O_{2^n}(\mathbb{R})$ generated by the matrices $\left(J_{2^n}^k \right)_{k=1,..n}$:

$$G_{2^n} = \{ B \in O_{2^n}(\mathbb{R}) \mid B \in \left\langle \left(J_{2^n}^k \right)_{k=1,..n} \right\rangle \}.$$

It is a commutative group. This follows from the observation that $\left(J_{2^n}^k \right)_{k=1,..n} \subset \mathbb{A}_{2^n}$. The commutativity follows then from equation (4.13). As every element in G_{2^n} has order 2, it follows from the classification of finite commutative groups that $G_{2^n} \simeq \mathbb{Z}_2{}^n$.

$$B = B^{-1} = B^T = B_T \quad \text{for all} \ \ B \in G_{2^n},$$

as the generators of the group G_{2^n} satisfy these equalities and as the group is commutative. G_{2^n} has exactly 2^n elements. This can be easily seen, as G_{2^n} can be embedded in $G_{2^{n+1}}$ and the multiplication with $J_{2^{n+1}}$ doubles the number of elements. We think that $G_{2^n} \simeq \mathbb{Z}_2{}^n$, but there was no time to verify this claim. Clearly

$$BAB = A \ \text{for all} \ A \in \mathbb{A}_g, B \in G_{2^n}.$$

The matrices in $\mathbb{A}_{2^n}$ have $2^n - 1$ pairwise distinct symmetries, which can be easily seen by setting $O = R^{-1} = B$, for a $B \in G_{2^n}$ in equation (4.9).

We now apply this construction to the matrices in the moduli spaces for PPAVs of dimension 2^n. Remember that for $Z = X + i \cdot Y \in \mathcal{H}_g$ we have

$$P_Z = P_{X,Y} = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \in M_{2g}(\mathbb{R}).$$

We now consider the following set of PPAVs defined by $Z = X + i \cdot Y \in \mathcal{H}_{2^n}$:

$$\mathbb{K}_{2^n} = \{ Z = X + i \cdot Y \in \mathcal{H}_{2^n} \mid \sqrt{Y}, X \in \mathbb{A}_{2^n} \}$$

We define G'_{2^n} to be the group of matrices

$$G'_{2^n} = \begin{pmatrix} G_{2^n} & 0 \\ 0 & G_{2^n} \end{pmatrix}.$$

We have that $G_{2^n} \simeq G'_{2^n}$. As the generators of G'_{2^n} are contained in $Sp_{2^{n+1}}(\mathbb{Z})$ - see chapter 1.4 for a definition of $Sp_{2^{n+1}}(\mathbb{Z})$ - we have that G'_{2^n} is a subgroup of $Sp_{2^{n+1}}(\mathbb{Z})$. For the elements of $\mathbb{K}_{2^n} \subset \mathcal{H}_{2^n}$, we have

Theorem 4.2.1 *For $g = 2^n$, we have :*
1.) $\mathbb{K}_g$ is contained in the stabilizer of the subgroup G'_g of $Sp_{2g}(\mathbb{Z})$.
2.) If $Z \in \mathbb{K}_g$, then the corresponding real lattice $P_Z \cdot \mathbb{Z}^{2g}$ has $g - 1$ symmetries.

proof 1.) We have seen that G'_g is a subgroup of $Sp_{2g}(\mathbb{Z})$. We remind that $Sp_{2g}(\mathbb{Z})$ acts on $\mathcal{H}_g$ by the g-dimensional Moebius transformation (see chapter 1.4). As $\sqrt{Y} \in \mathbb{A}_g$, we have that $\sqrt{Y} = \sqrt{Y}^T$ and

$$BYB = B\sqrt{Y}^T \sqrt{Y} B = B\sqrt{Y}^T BB\sqrt{Y}B = \sqrt{Y}^T \sqrt{Y} = Y \quad \text{for all} \ \ B \in G_g,$$

by the properties of the group G_g. As $X \in \mathbb{A}_g$, we have for all $B = B^{-1} \in G_g$:

$$BZB = Z \Leftrightarrow (BZ + 0) \cdot (0 \cdot Z + B)^{-1}, \quad \text{for all} \ \ Z \in \mathbb{K}_{2^n}.$$

This means that the elements of $\mathbb{K}_g$ are fixed points of the action of the group $G'_g \subset Sp_{2g}(\mathbb{Z})$. In other words, $\mathbb{K}_g$ is contained in the stabilizer of the subgroup G'_g of $Sp_{2g}(\mathbb{Z})$.

2.) To prove that P_Z has $g - 1$ symmetries, we note that for the generators $\left(J_{2^n}^k\right)_{k=1,..n}$ of G_g, we have

$$\begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}, \quad \text{for all } k \in \{1,..,n\}$$

and

$$\begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix} \begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} = \begin{pmatrix} Id_g & 0 \\ X & Id_g \end{pmatrix}, \quad \text{for all } k \in \{1,..,n\}$$

it follows that

$$\begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} \cdot P_Z \cdot \begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} = P_Z, \quad \text{for all } k \in \{1,..,n\}$$

, as

$$\begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} \cdot \begin{pmatrix} J_g^k & 0 \\ 0 & J_g^k \end{pmatrix} = Id_g, \quad \text{for all } k \in \{1,..,n\}.$$

As this is true for the generators of $G_g \simeq G'_g$, these equalities are true for all $B \in G_g$. Therefore P_Z has the demanded symmetries. (Set $P_Z = A, O = R^{-1} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$ in equation (4.9). $\square$

Conjecture 4.2.2 *If g is a power of 2 we obtain for Hermite's invariant for PPAVs of dimension g :*

$$\frac{1}{\pi} \sqrt[g]{2g \cdot g!} \leq \delta_{2g}.$$

The idea is to use a subset of the symmetric lattices represented by $\mathbb{K}_{2^n}$ presented in **Theorem 4.2.1** in **Theorem 4.1.2**. Due to the symmetry of these lattices the number of lattice vectors of a certain length is always a multiple of 2^{n+1} for a lattice of dimension 2^n. The problem is to evaluate the integral in the case, where a linear transformation as in equation (4.7) is not possible.

For real lattices of dimension n, a better lower bound for Hermite's constant γ_n than in **Theorem 4.1.2** was given by Rogers in [Ro]. It might be possible to combine this approach with the results presented in [Ro]. In this case, we might be able to use the set of lattices $\mathbb{A}_{2^n}$.

Chapter 5

A generalization of a theorem of Blichfeldt

The result of this chapter arose from the search for a certain number of short simple closed geodesics on Riemann surfaces. These were needed for the proofs in the second chapter. The following part was inspired by a Theorem of Blichfeldt [Bli] about real tori, which can be easily transferred to Riemann surfaces.

As a consequence of his theorem Blichfeldt gives a lower bound for the number of lattice vectors of length smaller than a fixed constant. This lower bound depends only on the volume of the corresponding torus of the lattice. As the proof relies only on the pigeonhole principle, the theorem can be easily generalized.

In this chapter we first give a further generalization of this theorem to manifolds of non-positive curvature. Then we compare the result with known results about the asymptotic growth rate of closed geodesics. Finally we will apply the theorem to Riemann surface of signature $(g, 0, n)$.

5.1 A theorem of Blichfeldt

In [Bli] Blichfeldt states the following theorem :

Theorem 5.1.1 *Let $L \subset \mathbb{R}^n$ be a n dimensional lattice and $T = \mathbb{R}^n \mod L$ be the corresponding torus. Let $m \in \mathbb{N}$ and $C \subset \mathbb{R}^n$ be a measurable set, such that*

$$\mathrm{vol}(C) > m \, \mathrm{vol}(T)$$

Then there exist $x_1, ..., x_m \subset C$, such that $l_i = x_i - x_1 \in L \backslash \{0\}, i \in \{2, ..., m\}$. Furthermore the $(l_i)_{i \in \{2,...,m\}}$ are pairwise distinct.

The situation is exemplified for a two dimensional lattice in Fig. 5.1. We note that if C is convex, then the straight line between x_1 and x_i of length $\|l_i\|_2$ is always contained in C, which is not true in general.

Setting $C = B_r(0) \subset \mathbb{R}^n$, a ball of radius r around 0 in Theorem 5.1.1, we obtain :

Corollary 5.1.2 *Let $B_r(0) \subset \mathbb{R}^n$ such that for a $m \in \mathbb{N}$, $\mathrm{vol}(B_r(0)) > m \, \mathrm{vol}(T)$ then there exist disjoint $l_2, ..., l_m \in L \backslash \{0\}$, such that*

$$\|l_i\|_2 \leq 2r \quad for \ all \quad i \in \{2, .., m\}$$
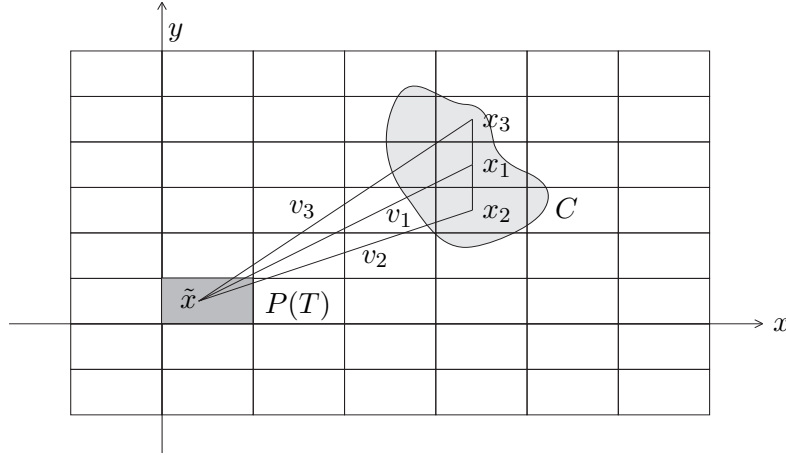
Figure 5.1: A measurable set C in a two dimensional lattice L

proof   Let $T = \mathbb{R}^n \mod L$ be a torus and $B_r(0)$ be the ball of volume bigger than $m\,\mathrm{vol}(T)$. It follows from **Theorem 5.1.1** and the subsequent remark that it contains the points $x_1, ...x_m$, such that $l_i = x_i - x_1 \in L\backslash\{0\}$ and $\|l_i\|_2 \leq 2r$ for all $i \in \{2, .., m\}$. $\square$

We now give a proof of **Theorem 5.1.1**.

proof of Theorem 5.1.1 Lift T into the universal covering $\mathbb{R}^n$ and let $P(T)$ be a fundamental parallelepiped of T. For $l \in L$ define

$$C_l = C \cap (l + P(T)).$$

We have that C is the disjoint union of the $(C_l)_{l \in L}$ and $C_l - l \subset P(T)$.
To prove the theorem, we have to show that there exists a point $\tilde{x} \in P(T)$ and pairwise distinct

$$(v_i)_{i \in \{1, .., m\}} \subset L, \quad \text{such that} \quad x_i = \tilde{x} + v_i \in C_{v_i}.$$

It follows then, as L is a lattice that $l_i = (\tilde{x} + v_i) - (\tilde{x} + v_1) = v_i - v_1 \in L$.

We show that there exists a $\tilde{x} \in P(T)$ that is covered at least m times by disjoint $(C_{v_i} - v_i)_{i \in \{1, .., m\}}$. We prove the result by contradiction. If every point in $P(T)$ was covered only m times by the disjoint $(C_l - l)_{l \in L}$, then we would obtain :

$$\mathrm{vol}(C) = \sum_{l \in L} \mathrm{vol}(C_l) = \sum_{l \in L} \mathrm{vol}(C_l - l) \leq \int_{P(T)} m\,d\mu \leq m\,\mathrm{vol}(T).$$

But $\mathrm{vol}(C) > m\,\mathrm{vol}(T)$, hence this is a contradiction. $\square$

The proof of the theorem depends only on the pigeonhole principle. We will generalize the theorem and the corollary in the following section.

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5.2   Generalization to manifolds of non-positive curvature

We assume in the following two sections that a manifold M is a connected complete Riemannian manifold of non-positive (sectional) curvature. By covering theory, the universal covering space of M, K is a simply connected space of non-positive curvature. There exists a group G of Deck transformations with the following properties :

$$M \simeq K \mod G, \quad G \subset \mathrm{Isom}(K) \text{ and } G \simeq \pi_1(M).$$

A geodesic loop $\delta \subset M$ with base point q defines a homotopy class $[\delta] \in \pi_1(M, q)$. We denote by $l([\delta])$, the *length of* $[\delta]$, which we define as the length of the shortest geodesic loop in the homotopy class of δ :

$$l([\delta]) = \min_{\delta' \in [\delta]} l(\delta').$$

It is a well known fact that - as the curvature of M is non-positive - there exists a unique shortest loop in each homotopy class. We call a non-trivial geodesic loop δ *primitive*, if

$$[\delta] \neq [\alpha]^n, \text{ for } [\alpha] \in \pi_1(M, q), \text{ with } n \in \mathbb{N}\backslash\{1\}.$$

We say that a geodesic loop η is not primitive, if it is a *multiple* of another geodesic loop. We define *primitive* and *multiple* analogously for closed geodesics and their free homotopy classes.

Theorem 5.2.1 *Let M be a manifold of non-positive curvature with $\mathrm{vol}(M) < \infty$. Let K be its universal covering. If $C \subset K$ is a convex set, such that for a $m \in \mathbb{N}$*

$$\mathrm{vol}(C) > m \, \mathrm{vol}(M),$$

then there exist pairwise distinct $x_1, ..., x_m \subset C$, and geodesic arcs $(\gamma_{x_i, x_1}) \subset C$ connecting x_i and x_1. The universal covering map p maps these geodesic arcs to homotopically pairwise distinct geodesic loops $\delta_i = p(\gamma_{x_i, x_1})$ with base point $p(x_1)$.

Let $B_r(x) \subset K$ be a ball of radius r with center x. As K has non-positive curvature, the ball is convex. In this more general setting we obtain the following corollary.

Corollary 5.2.2 *Let $B_r(x)$ be a ball in the universal covering K of M, such that for a $m \in \mathbb{N}$, $\mathrm{vol}(B_r(x)) > m \, \mathrm{vol}(M)$. Then there exist a point $q \in M$ and non-trivial geodesic loops $\delta_2, ..., \delta_m \subset M$, with common base point q such that*

$$l(\delta_i) \leq 2r \quad \text{for all} \quad i \in \{2, .., m\}.$$

Furthermore these geodesic loops are all in different homotopy classes.

In the free homotopy class of each geodesic loop δ_i of the corollary, there is a unique closed geodesic γ_i. We note that we can not conclude from the theorem that the $(\gamma_i)_{i=2,...,m}$ are pairwise distinct. We can also not exclude that such a geodesic γ_i is the multiple of another geodesic.

proof of Theorem 5.2.1 Lift M into the simply connected universal covering K and let

$G \subset \mathrm{Isom}(K)$ be the group of Deck transformations. Let $F(M)$ be a fundamental domain of the group action of G. For $g \in G$ define

$$C_g = C \cap g(F(M)).$$

We have that C is the disjoint union of the $(C_g)_{g \in G}$ and $g^{-1}(C_g) \subset F(M)$.
To prove the theorem, we have to show that there exists a point $x \in F(M)$ and pairwise distinct

$$(g_i)_{i \in \{1,..,m\}} \subset G, \quad \text{such that} \quad x_i = g_i(x) \in C_{g_i}.$$

It follows then from the convexity of C that the geodesic arc γ_{x_i,x_1} connecting x_i and x_1 is in C. As the $(x_i)_{i=2,..m}$ are pairwise distinct, the $(\gamma_{x_i,x_1})_{i=2,..m}$ map to homotopically pairwise distinct geodesic loops $(\delta_i)_{i=2,..m}$ under the covering map p. Their common base point is $p(x_1)$.

We show that there exists a $x \in F(M)$ that is covered at least m times by pairwise distinct $g_i^{-1}(C_{g_i})_{i \in \{1,..,m\}}$. We prove the result by contradiction. If every point in $F(M)$ was covered only m times by the disjoint $(g^{-1}(C_g))_{g \in G}$, then we would obtain :

$$\mathrm{vol}(C) = \sum_{g \in G} \mathrm{vol}(C_g) = \sum_{g \in G} \mathrm{vol}(g^{-1}(C_g)) \leq \int_{F(M)} m \, d\mu \leq m \, \mathrm{vol}(M).$$

But $\mathrm{vol}(C) > m \, \mathrm{vol}(M)$, hence we obtain a contradiction. $\square$

We will compare this result with known results about the length spectrum of closed geodesics and geodesic loops in the following section.

5.3   Asymptotic growth rate of closed geodesics and loops

Let M be a manifold of non-positive curvature and K its universal covering. Let $p : K \to M$ be the universal covering map. For $q \in M$, we chose a $x \in p^{-1}(q) \subset K$. The *volume entropy* of M is defined as

$$h_{vol}(M) = \lim_{t \to \infty} \frac{\log(\mathrm{vol}(B_t(x)))}{t}$$

If M is compact, the limit exists and does not depend on the point $q \in M$ (see [Ma]). If M is a n dimensional manifold, such that the curvature of M is bounded between $-K_1^2$ and $-K_2^2$, where $0 \leq K_1 \leq K_2$, then

$$(n-1) \cdot K_1 \leq h_{vol}(M) \leq (n-1) \cdot K_2$$

We denote by $v(t)$ the number of primitive (closed) geodesics in M of length smaller or equal to t :

$$v(t) = \#\{\gamma \subset M \mid \gamma \text{ primitive geodesic, such that } l(\gamma) \leq t\}.$$

For $q \in M$, we denote furthermore

$$P_q(t) = \#\{\delta \subset M \mid \delta \text{ geodesic loop with base point } q, \text{ such that } l([\delta]) \leq t\}.$$

It follows immediately from **Corollary 5.2.2** :

Corollary 5.3.1 *Let M be a manifold of non-positive curvature, such that M has finite volume* $\mathrm{vol}(M)$. *Then there exists a $x \in K$ and a $q = p(x) \in M$, such that*

$$P_q(t) \geq \frac{\mathrm{vol}(B_{\frac{t}{2}}(x))}{\mathrm{vol}(M)} - 2$$

This corollary is related to the following result stated in [Kn].

Theorem 5.3.2 *Let M be a compact manifold of non-positive curvature, such that M contains a geodesic, which is not the boundary of a flat half-plane. Then we have for all $q \in M$ that*

$$\lim_{t \to \infty} \frac{\log(v(t))}{t} = \lim_{t \to \infty} \frac{\log(P_q(t))}{t} = h_{vol}(M).$$

In comparison, **Corollary 5.3.1** implies the following. If M is a compact manifold then it follows that
$$\lim_{t \to \infty} \frac{\log(P_q(t))}{t} \geq \frac{1}{2} h_{vol}(M).$$
Therefore we underestimate $\lim\limits_{t \to \infty} \frac{\log(P_q(t))}{t}$, if we use the corollary.
Theorem 5.3.2 is additionally stronger in as far, as it makes also a statement about the number of prime geodesics and not only about geodesic loops emanating from a fixed base point. However it treats the asymptotic limit and applies only to compact manifolds, whereas **Corollary 5.2.2** gives a lower bound for the number of homotopically different geodesic loops of length smaller than a certain constant and does not rely on the compactness of M.
Theorem 5.3.2 has been generalized to Hadamard manifolds without the neccessity of the assumption of compactness or even finite volume of the manifold in [Li].

The limit case for geodesic loops on compact manifolds is not stated as a theorem in [Kn], but can be deduced from the proof. As this part is easy to prove, we will give a proof here :

proof of Theorem 5.3.2 $\left(\lim\limits_{t \to \infty} \frac{\log(P_q(t))}{t} = h_{vol}(M)\right)$.
Let $p : K \to M$ be the universal covering map. For a $q \in M$, we choose a $x \in K$, such that $p(x) = q$. We denote by $D = \mathrm{diam}(M)$ the diameter of M. We first show that $\mathrm{vol}(B_{t+D}(x)) \geq P_q(t) \, \mathrm{vol}(M)$ and then that $\mathrm{vol}(B_{t-D}(x)) \leq P_q(t) \, \mathrm{vol}(M)$. Then the proof follows by applying the logarithm.

1.) $\mathrm{vol}(B_{t+D}(x)) \geq P_q(t) \, \mathrm{vol}(M)$

Let K be the universal covering of M and G be the group of Deck transformations. Let $F(M)$ be a fundamental domain of $K \mod G$. Let H be the set

$$H = \{ g \in G \mid \mathrm{dist}(x, g(x)) \leq t \}$$

Consider the geodesic arcs $(\gamma_{x,h(x)})_{h \in H}$. Their images under the universal covering map p are in one to one correspondence with the elements of $P_q(t)$, hence $P_q(t) = \#H$. We will show that

$$\bigcup_{h \in H} h(F(M)) \subset B_{t+D}(x).$$

For any $y \in h(F(M))$, we have by the triangle inequality

$$\mathrm{dist}(y,x) \leq \mathrm{dist}(y,h(x)) + \mathrm{dist}(h(x),x) \leq D + t,$$

hence $h(F(M)) \subset B_{t+D}(x)$. It follows that $\mathrm{vol}(B_{t+D}(x)) \geq P_q(t) \, \mathrm{vol}(M)$.
With $\log((P_q(t) \cdot \mathrm{vol}(M)) = \log((P_q(t)) + \log(\mathrm{vol}(M))$, we obtain

$$\lim_{t \to \infty} \frac{\log(\mathrm{vol}(B_t(x)))}{t} = \lim_{t \to \infty} \frac{\log(\mathrm{vol}(B_{t+D}(x)))}{t} \geq$$
$$\lim_{t \to \infty} \frac{\log(P_q(t) \cdot \mathrm{vol}(M))}{t} = \lim_{t \to \infty} \frac{\log(P_q(t))}{t}.$$

2.) $\mathrm{vol}(B_{t-D}(x)) \leq P_q(t) \, \mathrm{vol}(M)$

We will show that

$$B_{t-D}(x) \subset \bigcup_{h \in H} h(F(M)).$$

For any $y \in B_{t-D}(x)$, we have that $y \in h'(F(M))$, for some $h' \in G$ and $\mathrm{dist}(y, h'(x)) \leq D$. It follows that

$$\mathrm{dist}(h'(x), x) \leq \mathrm{dist}(h'(x), y) + \mathrm{dist}(y, x) \leq D + (t - D) = t.$$

Hence $h' \in H$ and $y \in \bigcup_{h \in H} h(F(M))$ and therefore $B_{t-D}(x) \subset \bigcup_{h \in H} h(F(M))$. It follows that

$$\lim_{t \to \infty} \frac{\log(P_q(t))}{t} \geq \lim_{t \to \infty} \frac{\log(\mathrm{vol}(B_t(x)))}{t},$$

which proves the theorem. $\square$

In the case of Riemann surfaces, we obtain from **Corollary 5.3.1** :

Corollary 5.3.3 *Let S be a Riemann surface of signature $(g, 0, n)$. We have that $\mathrm{vol}(S) = 4\pi(g-1) + 2\pi n$. There exists a point $q \in S$, such that*

$$P_q(t) \geq \frac{\cosh(\frac{t}{2}) - 1}{2(g-1) + n} - 2 \geq \frac{\exp\left(\frac{t}{2}\right) - 1}{4(g-1) + 2n} - 2.$$

If S contains cusps, however, this corollary is trivial. In this case, we can always choose a point $q \in S$ sufficiently close to a cusp point. Then the geodesic loops with base point q, which wind around the cusp satisfy the conditions of the corollary. If S is a compact Riemann surface the prime number theorem for compact Riemann surfaces (see [B], p. 246) asserts that

$$v(t) \sim \frac{\exp(t)}{t} \text{ for } t \to \infty \quad \Leftrightarrow \quad \lim_{t \to \infty} \frac{tv(t)}{\exp(t)} = 1.$$

The prime number theorem implies **Theorem 5.3.2** in the case of Riemann surfaces. The proof however requires more advanced methods from harmonic analysis.

The aim of future research will be to generalize Blichfeldt's theorem to groups acting on measurable spaces and explore its various implications.

Appendix A

Numerical estimates for Chapter 2

Proof of Lemma 2.3.5 : In this proof we first have to verify equation (2.10). We have that

$$\nu \leq 2\operatorname{arccosh}((\sinh(\frac{\gamma_1}{2})^2(\cosh(2\operatorname{arcsinh}(\frac{2\pi(g-1)}{\gamma_1}))-1))-1) = f_\nu^1(\gamma_1, g).$$

We have to show that for $\gamma_1 < \frac{\pi}{2}$, $f_\nu^1(\gamma_1, g) \leq 4\log(8g-7)$.

Figure A.1 shows a plot of $f_\nu^1(\gamma_1, g)$ and $4\log(8g-7)$ over $D_1 = [0.005, \frac{\pi}{2}] \times [2, 20]$. The figure shows that the inequality is true on D_1.
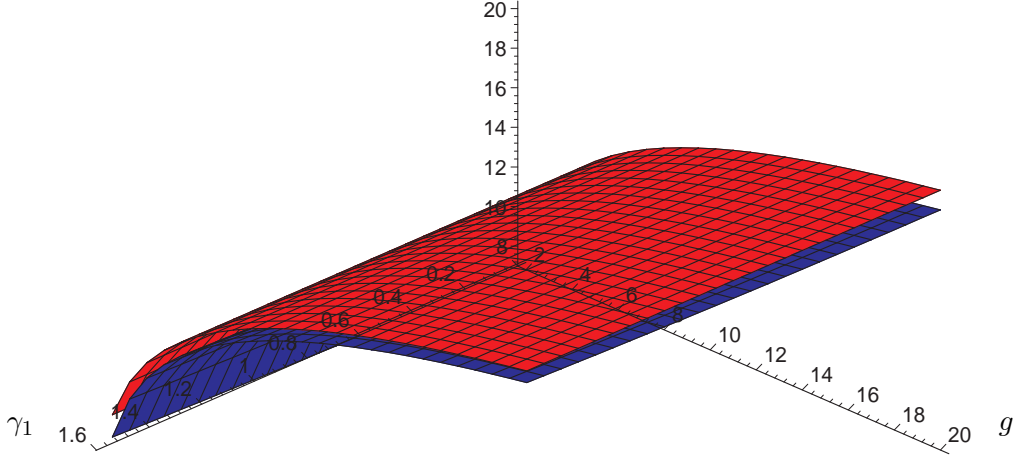
Figure A.1: Plot of $f_\nu^1(\gamma_1, g)$ *(blue)* and $4\log(8g-7)$ *(red)* over D_1

We have also calculated the limit $\lim_{\gamma_1 \to 0} f_\nu^1(\gamma_1, g) = f_\nu^1(0, g)$ using MAPLE. However this function is too complicated to be displayed here. Again a plot of $f_\nu^1(0, g)$ and $4\log(8g-7)$ in the interval $[2, 40]$ shows that the inequality is valid in the limit case (see *Fig. A.1*).
Then we have to show inequality (2.11). We have that

$$\nu \leq 2\operatorname{arccosh}\left(\sinh(\frac{\gamma}{4})\frac{2\pi(g-1)}{\gamma_1}\right) = f_\nu^2(\gamma_1, g).$$

We have to show that for $\gamma_1 < \frac{\pi}{2}$, $f_\nu^2(\gamma_1, g) \leq 3\log(8g-7)$.
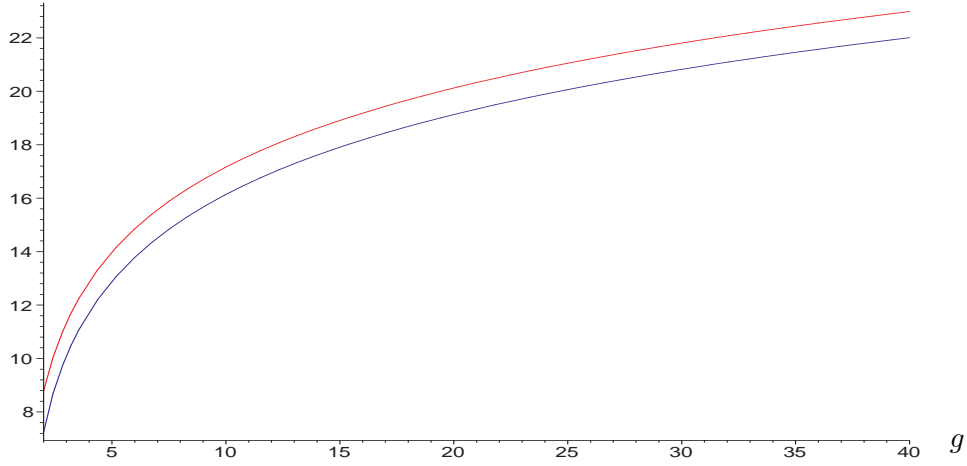
Figure A.2: Plot of $f_\nu^1(0, g)$ *(blue)* and $4\log(8g - 7)$ *(red)* in the interval $[2, 40]$

Figure A.3 shows a plot of $f_\nu^2(\gamma_1, g)$ and $3\log(8g - 7)$ over $D_2 = [0.005, \frac{\pi}{2}] \times [2, 20]$.
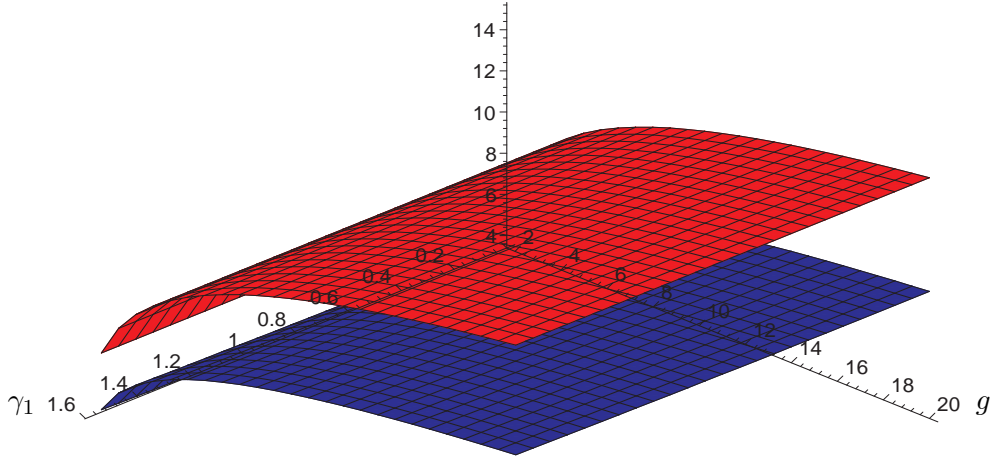

Figure A.3: Plot of $f_\nu^2(\gamma_1, g)$ *(blue)* and $3\log(8g - 7)$ *(red)* over D_2

We have also calculated the limit $\lim_{\gamma_1 \to 0} f_\nu^2(\gamma_2, g) = f_\nu^2(0, g)$ using MAPLE. Here we obtain that

$$f_\nu^2(0, g) = 2\log(\sqrt{(\pi g - \pi)^2 - 1} + \pi g - \pi) < 3\log(8g - 7).$$

Statement in Case 2.a.2.) : We have to verify the statement concerning equation (2.12). By **Lemma 2.3.7**, we have that there is a scg α_2' in the Q-piece Q_1, such that

$$\alpha_2' \leq 2\operatorname{arccosh}\left(\sqrt{\frac{\cosh^2(\frac{\gamma_2}{4}) + \cosh^2(\frac{\alpha_1}{2}) - 1}{2(\cosh(\frac{\alpha_1}{2}) - 1)}}\right) = f(\alpha_1, \gamma_2).$$

Here $\alpha_1 = \gamma_1$ is the shortest and γ_2 the second shortest simple closed geodesic S. We have to show that $\alpha_2{}' < \gamma_2$, if $2.1 < \alpha_1$.

Figure A.4 shows a plot of $f(\alpha_1, \gamma_2)$ and γ_2 in $D_3 = \{(\alpha_1, \gamma_2) \in \mathbb{R}^2 \mid \gamma_2 \in [\alpha_1, 8], \alpha_1 \in [0.05, 4]\}$. It can be seen in the figure that $\alpha_2{}' < \gamma_2$, if $2.1 < \alpha_1$.
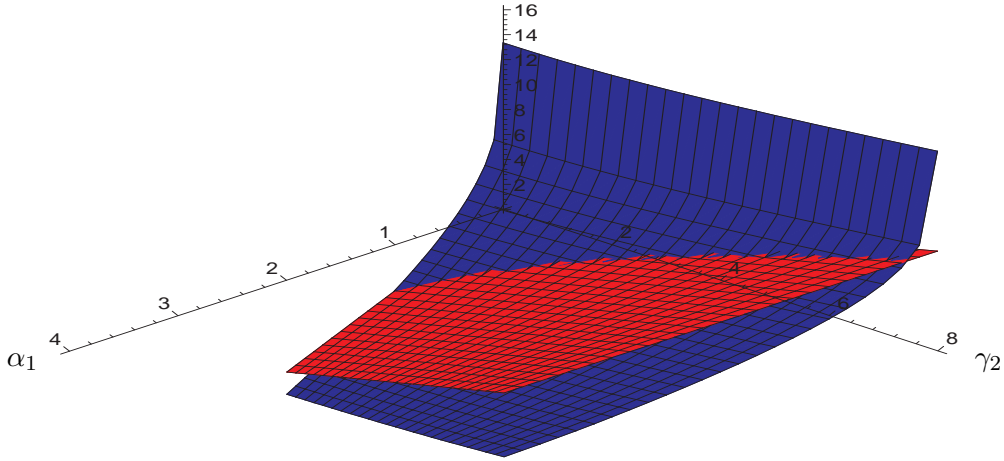

Figure A.4: Plot of $f_(\alpha_1, \gamma_2)$ *(blue)* and γ_2 *(red)* over D_3

Statement in Case 2.b.1.) We have to verify inequality (2.15). For

$$w_2{}^Q = \operatorname{arcsinh}\left(\frac{\frac{2\sqrt{5}}{5} \cdot \sinh(\frac{\alpha_1}{2})}{\sqrt{\frac{9}{5}\cosh^2(\frac{\alpha_1}{4}) - 1}}\right)$$

we have to show that

$$f(\alpha_1, g) = \frac{3\log(8g - 7)}{\pi - 2 \cdot \arcsin(\frac{1}{\cosh(w_2{}^Q)})} \leq 3.1\log(8g - 7),$$

if $1.1 < \alpha_1$.

Figure A.5 shows a plot of $f(\alpha_1, g)$ and $3.1\log(8g - 7)$ over $D_4 = [1.1, 5] \times [2, 40]$. It can be deduced from this figure that the statement is true for all $(\alpha_1, g) \in D_4$.

Statement in Case 2.c.2.) We have to verify inequality (2.17). We have to show that

$$\alpha_1 \geq 1.5 \quad \text{and} \quad \gamma_2 \geq 2.1.$$

This is true, because otherwise we arrive at a contradiction to the fact that $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect. Consider the Y-piece $Y' \subset S_1$ with boundary geodesics β_1, α_1 and γ_2 (see *Fig. 2.6*). By construction σ is the shortest geodesic arc connecting α_1 and γ_2 in S_1. It follows that $\sigma \leq 2W'$. It follows from the collar lemma that α_1 has to be bigger than 0.8. Otherwise we have in S_1 that
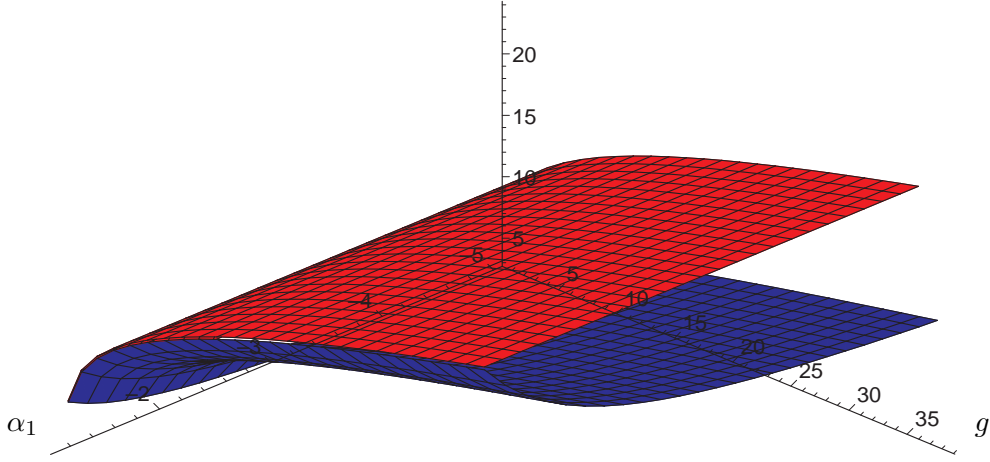
Figure A.5: Plot of $f(\alpha_1, g)$ *(blue)* and $3.1 \log(8g - 7)$ *(red)* over D_4
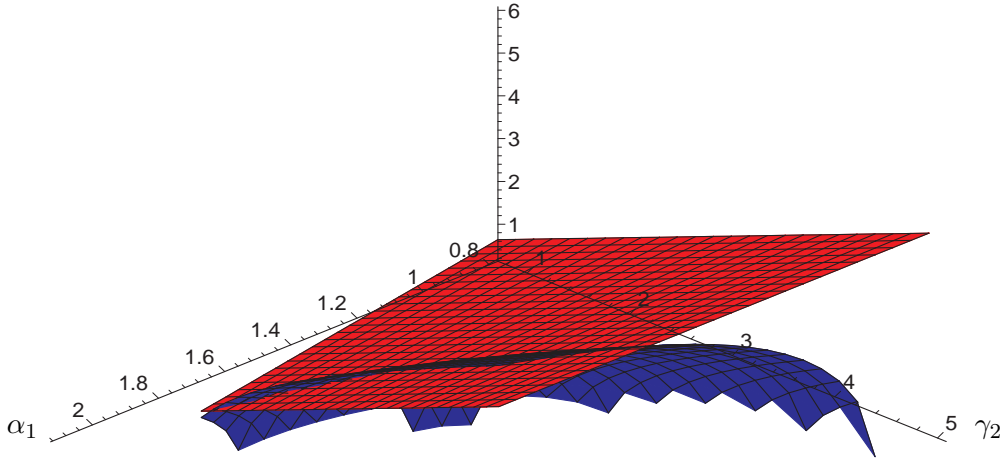
$\text{dist}(\alpha_1, \gamma_2) > 2W'$, a contradiction.

It follows from equation (2.16), as $\sigma < 2W'$:

$$\beta_1 \leq 2 \operatorname{arccosh}\left(\sinh(\frac{\alpha_1}{2})\sinh(\frac{\gamma_2}{2})\cosh(2W') - \cosh(\frac{\alpha_1}{2})\cosh(\frac{\gamma_2}{2})\right) = f^\sigma(\alpha_1, \gamma_2).$$

We have to show that $\beta_1 < \gamma_2$, if $\neg(\alpha_1 \geq 1.5 \; \text{ and } \; \gamma_2 \geq 2.1)$.

Figure A.6 shows a plot of $f^\sigma(\alpha_1, \gamma_2)$ and γ_2 over $D_5 = \{(\alpha_1, \gamma_2) \in \mathbb{R}^2 \mid \gamma_2 \in [\alpha_1, 5], \alpha_1 \in [0.8, 2.1]\}$. It can be deduced from this figure that the statement is true for all $(\alpha_1, \gamma_2) \in D_5$. The plot does not show some values of $f^\sigma(\alpha_1, \gamma_2)$, as for some small values of (α_1, γ_2), no hyper-

Figure A.6: Plot of $f^\sigma(\alpha_1, \gamma_2)$ *(blue)* and γ_2 *(red)* over D_5

bolic hexagon exists.

Discussion of the summary of Case 2.c.2.) We have to show inequality (2.21). We have that

$$m_2(J(S))^2 \leq \frac{\alpha_2}{\pi - 2\arcsin(\frac{1}{\cosh(w_2)})},$$

where $\alpha_2 < 4\operatorname{arccosh}(\cosh(\frac{\gamma_2}{4})\cosh(W')) = \alpha_2(\gamma_2)$ and

$$w_2 \geq w(\gamma_2) = \min\left\{0.66, \operatorname{arccosh}\left(\frac{\cosh(\frac{\gamma_2}{2})}{\cosh(\frac{\gamma_2}{4})\cosh(W')}\right)\right\}.$$

We have to show that for $3\log(8g-7) \geq \gamma_2 > 2.1$, we have that

$$f(\gamma_2) = \frac{\alpha_2(\gamma_2)}{\pi - 2\arcsin(\frac{1}{\cosh(w(\gamma_2))})} \leq 3.1\log(8g-7).$$

Figure A.7 shows a plot of $f(\gamma_2)$ and $3.1\log(8g-7)$ over $D_6 = \{(g, \gamma_2) \in \mathbb{R}^2 \mid \gamma_2 \in [2.1, 3\log(8g-7)], g \in [2, 40]\}$. It can be deduced from this figure that the statement is true for all $(g, \gamma_2) \in D_6$.
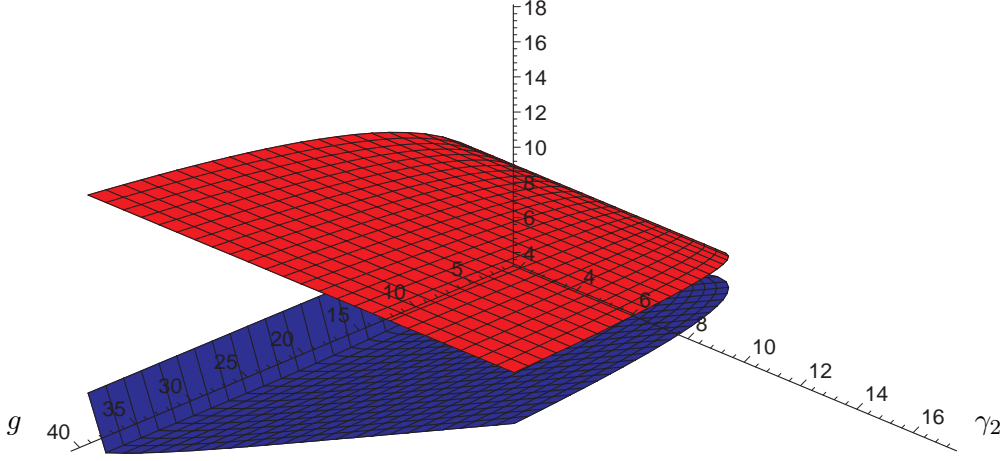
Figure A.7: Plot of $f(\gamma_2)$ *(blue)* and $3.1\log(8g-7)$ *(red)* over D_6

Appendix B

Inequalities for capacities on cylinders of constant curvature

The content of this section is not directly related to Riemann surfaces. However, we have made use of the inequalities presented here in **Theorem 3.1.1** of chapter 3. This part was meant as an independent publication and is written in this style. We hope to extend the presented results to cylinders of variable non-positive curvature.

The initial impulse to study this topic came from a paper of F.W. Gehring :
In [G], Gehring provides elementary estimates for the capacities of rings in $\mathbb{R}^n$, taken with respect to an arbitrary metric. The lower bound, however does not apply to certain annuli on cylinders. Using different methods, we give elementary estimates for the capacity of these annular regions on cylinders of constant curvature and give examples, where these inequalities are sharp.

B.1   Introduction

We will first give the definition of a cylinder of constant curvature.
Let E_K be a two dimensional simply connected complete space of constant (sectional) curvature K. A model of E_K is the hyperbolic plane, if $K < 0$, the Euclidean plane , if $K = 0$ and the sphere, if $K > 0$.
Let $\gamma \subset E_K$ be a geodesic. γ divides E_K into two parts, E_K^+ and E_K^-. Let $\gamma' \subset \gamma$ be a geodesic arc of length $l(\gamma') < \text{diam}(E_K)$ with endpoints p_1 and p_2. There exists a unique isometry $M \in \text{Isom}^+(E_K)$ that preserves γ and such that $M(p_1) = p_2$.
For each point $p \in \gamma'$ there exists an unique geodesic δ_p that is perpendicular to γ and that passes p. The *Fermi coordinates with base point p_1 and base line γ'* in E_K are an injective parametrization

$$\psi : D = [0, l(\gamma')] \times]b, a[\to E_K, \psi : (t, s) \mapsto \psi(t, s)$$

with $b < a$ and $|a| \leq |b| \leq \frac{\text{diam}(E_K)}{2}$, such that the parametrization satisfies the following conditions :
Each point $q = \psi(t, s) \in \psi(D)$ can be reached in the following way. Starting from the base point p_1 we first we move along γ' the directed distance t to $\psi(t, 0) = p$ and then from $\psi(t, 0) = p$, we now move along δ_p the directed distance s to $\psi(t, s) = q$.

We call an image $\psi(D) = S$ a *strip* of constant length. A *cylinder of constant curvature K with baseline γ'* or shortly *cylinder*, C, is

$$C = S \mod M,$$

such that the side of S containing p_1 is identified by M with the side of the strip S containing p_2. An *annulus $A \subset C$ on a cylinder C* is a set that can be obtained by a continuous deformation of C. More precisely there exists an isotopy

$$J : C \times [0,1] \leftarrow C, \quad \text{such that} \quad J(\cdot, 0) = id \quad \text{and} \quad J(C, 1) = A.$$

We denote by $\partial_1 A$ and $\partial_2 A$ its two connected boundary components that constitute the boundary ∂A.
Let

$$\phi : [0, c_1] \times]c_2, c_3[\mod \{(0, t) \sim (c_1, t) \mid t \in]c_2, c_3[\} \to C$$

be a bijective parametrization of C and G the corresponding metric tensor. Let $F \in \mathrm{Lip}(\bar{A})$ a Lipschitz function on the closure of A (see chapter 1.5). Then the *energy of F on A*, $E_A(F)$ is given by

$$E_A(F) = \iint\limits_{\phi^{-1}(A)} \|D(F \circ \phi)\|_{G^{-1}}^2 \sqrt{\det(G)}$$

The *capacity* of an annulus A, $\mathrm{cap}(A)$ is given by

$$\mathrm{cap}(A) = \inf\{E_A(F) \mid \{F \in \mathrm{Lip}(\bar{A}) \mid F|_{\partial_1(A)} = 0, F|_{\partial_2(A)} = 1\}\}.$$

For further information about the definition of the capacity in metric spaces, see [Go]. With the help of an approximation for the energy using Fermi coordinates and a symmetrization process, we obtain the following theorem :

Theorem B.1.1 *Let E_K be a two dimensional space of constant curvature $K = -k^2$ and $C \subset E_K$ a cylinder with base line γ' of length $l(\gamma') = l$. Let $A \subset C$ be an annulus of finite area $\mathrm{area}(A)$. Then for $W = \mathrm{arcsinh}(\frac{k\,\mathrm{area}(A)}{2 \cdot l})$ and $h_1(x) = 2\arctan(\exp(x))$, we have*

$$\mathrm{cap}(A) \geq \frac{k \cdot l}{h_1(W) - h_1(-W)}.$$

In the limit case $k \to 0$, we obtain $\mathrm{cap}(A) \geq \frac{l^2}{A}$ (see **Example 1.5.9**).
The theorem says that among all annuli of fixed area on a cylinder of constant non-positive curvature with base line γ', the annulus with constant width, centered around γ' has minimal capacity. We think that this is also the annulus, whose boundary line has minimal length among all annuli of fixed area. This means that for this annulus the isoperimetric inequality for annuli on cylinders is sharp. On the contrary, there is no such lower bound depending on the area for annular regions contained in $\mathbb{R}^2$.
In the case of constant positive curvature, we obtain

Theorem B.1.2 *Let E_K be a two dimensional space of constant curvature $K = k^2$ and $C \subset E_K$ a cylinder with base line γ' of length $l(\gamma') = l$. Let $A \subset C$ be an annulus of finite area $\mathrm{area}(A)$. Then for $W = \arcsin(\frac{k\,\mathrm{area}(A) - l\sin(kb)}{l})$ and $h_2(x) = \log\left(\frac{1+\sin(x)}{\cos(x)}\right)$, we have*

$$\mathrm{cap}(A) \geq \frac{k \cdot l}{h_2(W) - h_2(b)}.$$

The theorem says that among all annuli of fixed area on a cylinder of constant positive curvature with base line γ', the annulus with constant width, where one boundary is the shorter boundary of the cylinder itself, has minimal capacity.

We will prove these theorems in section 3. Using the same methods, we will derive lower and upper bounds of the capacities of annuli on cylinders of constant curvature in section 4.

B.2 Preliminaries

We will use the following models for the two dimensional spaces of constant negative curvature and positive curvature :

For $K < 0$, set $E_K = E_{-k^2} = \mathbb{H}$, where $\mathbb{H}$ is the Poincare model of the hyperbolic plane. It is the following subset of the complex plane $\mathbb{C}$:

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

with the hyperbolic metric

$$ds^2 = \frac{1}{(ky)^2}(dx^2 + dy^2).$$

As the metric is conformal, the hyperbolic energy of a function F on a set L, $F : L \subset \mathbb{H} \to \mathbb{R}$ is equal to the Euclidean energy of F.

The parametrization $\psi : [0, l] \times \mathbb{R} \mod \{(0, t) \sim (l, t) \mid t \in \mathbb{R}\} \to \mathbb{H}$

$$\psi(t, s) := \frac{\exp(kt)}{\cosh(ks)}(\sinh(ks) + i)$$

parametrizes a cylinder $C \subset \mathbb{H}$ with base line $\gamma' = \{iy \mid y \in [1, \exp(kl)]\}$. Choosing the correct parameters, any cylinder of constant curvature K in $\mathbb{H}$ is isometric to a cylinder C.

For $K > 0$, set $E_K = E_{k^2} = \mathbb{S}^2_k$, where $\mathbb{S}^2_k$ is the sphere $\partial B_{\frac{1}{k}}(0) \subset \mathbb{R}^3$, the boundary of the ball of radius $\frac{1}{k}$. The parametrization $\psi : [0, l] \times [b, a] \mod \{(0, t) \sim (l, t) \mid t \in [b, a]\} \to \mathbb{S}^2_k$

$$\psi(t, s) := \frac{1}{k}(\cos(kt)\cos(ks), \sin(kt)\cos(ks), \sin(ks))$$

parametrizes a cylinder $C \subset \mathbb{S}^2_k$ with base line $\gamma' = \{(\frac{1}{k}\cos(ky), 0, 0) \mid y \in [0, l]\}$. Choosing the correct parameters, any cylinder of constant curvature K in $\mathbb{S}^2_k$ is isometric to a cylinder C. Using these models, we obtain the metric tensor G with respect to the Fermi coordinates :

$$G(t, s) = \begin{pmatrix} h(s)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $h(s) = \cosh(ks)$, if $K = -k^2$ and $h(s) = \cos(ks)$, if $K = k^2$.
Using the formulas for the metric tensor G, the area of an annulus A, area(A) is given by

$$\text{area}(A) = \iint\limits_{\psi^{-1}(A)} \sqrt{\det(G(t,s))} \, ds \, dt = \iint\limits_{\psi^{-1}(A)} h(s) \, ds \, dt. \tag{B.1}$$

Now let $A \subset C$ be an annulus of constant curvature $K \neq 0$ and $F : A \rightarrow \mathbb{R}$ be an Lip$(\bar{A})$ function. For a $x = \psi(t_0, s_0) \in C$ denote by

$$p_\psi : T_x(C) \rightarrow \{\lambda \cdot \frac{\partial \psi(t_0, s_0)}{\partial s} \mid \lambda \in \mathbb{R}\}$$

the orthogonal projection of a tangent vector in x onto the subspace spanned by $\frac{\partial \psi(t_0, s_0)}{\partial s}$. We denote by $E_A(\partial_2 F) = E_A(p_\psi(DF))$ the energy of this orthogonal projection of DF. Let $L \subset C$ be a set. For technical purposes, we also define the *capacity of L in direction ∂_2*, cap$_{\partial_2}(L)$ by

$$\text{cap}_{\partial_2}(L) = \inf\{E_L(\partial_2 F) \mid \{F \in \text{Lip}(\bar{L}) \mid F|_{\partial_1(L)} = 0, F|_{\partial_2(L)} = 1\}.\}$$

Using Fermi coordinates, we obtain for the energy of F on A, $E_A(F)$, with $F \circ \psi = f$:

$$E_A(F) = \iint\limits_{\psi^{-1}(A)} \frac{1}{h(s)} \frac{\partial f(t,s)}{\partial t}^2 + h(s) \frac{\partial f(t,s)}{\partial s}^2 \, ds \, dt \geq \iint\limits_{\psi^{-1}(A)} h(s) \left(\frac{\partial f(t,s)}{\partial s}\right)^2 \, ds \, dt = E_A(\partial_2 F).$$
$$\tag{B.2}$$

Using equation (B.2), we obtain the following lemma. For a definition of h_1 and h_2 see **Theorem B.1.1** and **B.1.2**:

Lemma B.2.1 *For $K \neq 0$, let $S \subset E_K$ be a strip in E_K and $C = S \mod M$ be a cylinder with baseline γ' of length $l(\gamma') = l$. Then we have*

$$\text{cap}(C) = \text{cap}_{\partial_2}(S) = E_S(P) = \frac{l}{H(a) - H(b)}, \quad \text{where} \quad H(s) = \begin{cases} \frac{h_1(ks)}{k} \\ \frac{h_2(ks)}{k} \end{cases} \quad if \quad \begin{matrix} K = -k^2 \\ K = k^2 \end{matrix}.$$

Here $P \in \text{Lip}(\bar{C})$ is the harmonic function that satisfies the boundary conditions for the capacity of the cylinder C and that satisfies $\frac{\partial P \circ \psi}{\partial t} = 0$. Furthermore $H(s) = \int \frac{1}{h(s)}$.

proof of Lemma B.2.1 For any function $F \in \text{Lip}(\bar{C}), F|_{\partial_1 C} = 0, F|_{\partial_2 C} = 1$, with $F \circ \psi = f$, we obtain by inequality (B.2) that

$$E_C(F) \geq \iint\limits_{\psi^{-1}(C)} h(s) \left(\frac{\partial f(t,s)}{\partial s}\right)^2 \, ds \, dt.$$

Using the calculus of variations (see **Theorem 1.5.3**), we can determine the function $P = p \circ \psi^{-1} \in \text{Lip}(\bar{C})$ that satisfies the boundary conditions on C and such that p minimizes second integral in the above inequality (B.2). Hence for all $F \in \text{Lip}(\bar{C}), F|_{\partial_1 C} = 0, F|_{\partial_2 C} = 1$:

$$E_C(F) \geq \iint\limits_{\psi^{-1}(C)} h(s) \left(\frac{\partial f(t,s)}{\partial s}\right)^2 \, ds \, dt \geq \int\limits_{t=0}^{l} \int\limits_{s=b}^{a} h(s) \left(\frac{\partial p(t,s)}{\partial s}\right)^2 \, ds \, dt.$$

Additionally, p satisfies $\frac{\partial p}{\partial t} = 0$, hence $E_S(\partial_2 P) = E_S(P) = \text{cap}(C)$.

We have $E_S(P) = \dfrac{l}{\int\limits_b^a \frac{1}{h(s)}\, ds}$, from which follows the lemma $\square$.

We note that the energy $E_S(P) = \text{cap}_{\partial_2}(S)$ is decreasing, if the length of the strip S is increased.

B.3 Lower bounds on the capacity depending on the area

We will prove **Theorem B.1.1** and **B.1.2** in the case, where an annulus A consists of a set of strips of uniform width ϵ. It will be clear from the proof that the proof applies to any annulus that satisfies the definition. The proof consists in replacing in three steps the set of strips of the initial annulus $A = A_0$, with a set of strips A_i such that

$$\text{cap}_{\partial_2}(A_i) \geq \text{cap}_{\partial_2}(A_{i+1}) \quad \text{and} \quad \text{area}(A_i) = \text{area}(A).$$

In the final step, we obtain an annulus A_3 of constant width that satisfies $\text{cap}_{\partial_2}(A_3) = \text{cap}(A_3)$.

Step 1 : Reduction of the number of strips

We first consider the case, where a section of a A in Fermi coordinates is given by $\psi([0,\epsilon] \times (]a_1, a_2[\cup]a_3, a_4[) = S_1 \cup S_2$, where $a_2 < a_3$. It is sufficient to consider the two configurations of type I and II depicted in Fig. B.1. Here a section of type I or II consists of two disjoint strips. The two types differ by the boundary values for these strips given by the capacity problem. If the section is of type I, then the boundary conditions on the first strip imply that the boundary values on both sides of the strip are equal. The boundary conditions on the second strip imply that the values are different. If the strip is of type II, then the boundary conditions on both strips imply that the boundary values on the two sides are different. If the section is of type I,
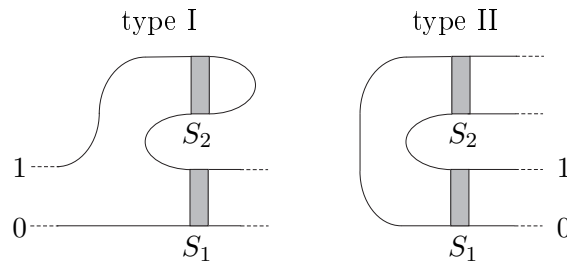

Figure B.1: Configuration of a section of A of type I and type II

we replace $S_1 \cup S_2$ it with a strip S of the same area, which we obtain by elongating S_1 in the direction of S_2. We have $\text{cap}_{\partial_2}(S_2) = 0$. As S is longer than S_1, it follows from **Lemma B.2.1** that $\text{cap}_{\partial_2}(S_1) > \text{cap}_{\partial_2}(S)$.

If the section is of type II, we replace $S_1 \cup S_2$ it with one strip S of the same area. If $K > 0$, we replace it with a strip $S = \psi([0,\epsilon]\times]b, W[)$ of the same area, situated near the shorter boundary of C.

If $K < 0$, we replace it with a strip $S = \psi([0, \epsilon] \times] - W, W[)$ of the same area, centered around γ'. We obtain :

$$\text{cap}_{\partial_2}(S_1) + \text{cap}_{\partial_2}(S_2) \geq \text{cap}_{\partial_2}(S). \tag{B.3}$$

We obtain this inequality in the following way. By equation (B.1), we have to solve the equations

$$\text{area}(S_1 \cup S_2) = \epsilon \cdot \int_{a_1}^{a_2} h(s) \, ds + \epsilon \cdot \int_{a_3}^{a_4} h(s) \, ds = \epsilon \cdot \int_{-W}^{W} h(s) \, ds = \text{area}(S) \quad \text{if} \quad K < 0,$$

$$\text{area}(S_1 \cup S_2) = \epsilon \cdot \int_{a_1}^{a_2} h(s) \, ds + \epsilon \cdot \int_{a_3}^{a_4} h(s) \, ds = \epsilon \cdot \int_{b}^{W} h(s) \, ds = \text{area}(S) \quad \text{if} \quad K > 0 \tag{B.4}$$

with respect to W to obtain the length coordinates of the strip S. Then the inequality (B.3) follows from **Lemma B.2.1**.

If the section consists of more than two strips, we can argue in a similar fashion and reduce the number of strips in a section to one. The resulting set of strips A_1 might not be an annulus, but this is not important for our proof, as $\text{cap}_{\partial_2}(\cdot)$ is well defined on A_1.

Step 2 : Positioning the strips

A section of A_1 in Fermi coordinates is WLOG given by $\psi([j\epsilon, (j+1)\epsilon] \times]a_1, a_2[) = S_1$. We replace S_1 with one strip S of the same area. If $K > 0$, we replace it with a strip $S = \psi([j\epsilon, (j+1)\epsilon] \times]b, W[)$ situated near the shorter boundary of C.

If $K < 0$, we replace it with a strip $S = \psi([j\epsilon, (j+1)\epsilon] \times] - W, W[)$ centered around γ'. We set $\text{area}(S_2) = 0$ and solve the corresponding equation in (B.4), to obtain the length coordinates of the strip S. Then the inequality (B.3) follows from **Lemma B.2.1**. The resulting set of strips A_2 is now an annulus.

Step 3 : Adjusting the length of the strips

A section of A_2 in Fermi coordinates is WLOG given by $\psi([j\epsilon, (j+1)\epsilon] \times]a_1, a_2[)$, where $a_1 = b$, if $K > 0$ and $-a_1 = a_2 = W$, if $K < 0$. Let S_1 and S_2 be two different sections of A_2, which share a common boundary. If these sections have different length, we replace S_1 and S_2 by two strips, S_1' and S_2' of equal length and equal area $\frac{\text{area}(S_1 \cup S_2)}{2}$. We position these strips as in **Step 2**. It follows from **Lemma B.2.1**, with $S = S_1' \cup S_2'$ that $\text{cap}_{\partial_2}(S_1) + \text{cap}_{\partial_2}(S_2) > \text{cap}_{\partial_2}(S)$. If A_2 consists of more than two strips of different length, we can argue in a similar fashion to obtain an annulus A_3 of constant width that satisfies

$$\text{cap}(A) \geq \text{cap}_{\partial_2}(A) \geq \text{cap}_{\partial_2}(A_3) = \text{cap}(A_3).$$

We obtain always the same annulus A_3, independent of the starting annulus A.

If $\epsilon \to 0$ in each of these steps, the area of the comparison strips and $\text{cap}_{\partial_2}(\cdot)$ of the comparison strips depend linearly on the width ϵ. Hence we can apply the comparison arguments to any annulus A in C. This argument completes the proof of **Theorem B.1.1** and **B.1.2** □

B.4 Inequalities for the capacity of an annulus on a cylinder

The upper and lower bound for the capacity of an annulus A of constant curvature can be obtained in the following way. To obtain an upper bound, we can evaluate the energy of any test function $F_T \in \mathrm{Lip}(\bar{A})$ that satisfies the boundary conditions for the capacity problem for A. We can easily construct a test function by adjusting the minimizing function P from **Lemma B.2.1** to the boundary, such that P is the minimizing function for $\mathrm{cap}_{\partial_2}(A)$. We can evaluate our choice by evaluating $\mathrm{cap}_{\partial_2}(A)$, which provides the lower bound for $\mathrm{cap}(A)$. This approach works immediately for an annulus A that can be parametrized in Fermi coordinates in the following way :

$$A = \psi\{(t,s) \mid s \in [a_1(t), a_2(t)], t \in [0,l]\},$$

where $a_1(\cdot)$ and $a_2(\cdot)$ are piecewise derivable functions with respect to t.

In this case, we say that our annulus is of *type A*. As $a_1(\cdot)$ and $a_2(\cdot)$ are functions sections of type I or II (see *Fig. B.1*) can not occur.

We say an annulus is of *type B* if it is not of type A and if its boundary is a piecewise differentiable curve. In this case the approach can be adapted to obtain an upper or lower bound. Here the lower bound can be constructed by the same method, which we present in the following. Though the method can also be adapted to obtain an upper bound for any annulus of type B, we think that this upper bound deviates too much from the real value of the capacity and we will not present this approach here.

It is noteworthy that in [G] different test functions are used to obtain an upper bound for the capacity of annuli. These bounds also apply here. However, the upper bound presented here is more practical and explicit. It is also noteworthy that the lower bound from [G] does not apply in our case.

Annuli of type A

The following theorem can be concluded from the discussion above. For a definition of H see **Lemma B.2.1** :

Theorem B.4.1 *For $K \neq 0$, let $C \subset E_K$ be a cylinder with base line γ' of length $l(\gamma') = l$. Let $A \subset C$ be an annulus of type A and $P \in \mathrm{Lip}(\bar{A})$ be the function that minimizes $\mathrm{cap}_{\partial_2}(A)$, then*

$$E_A(P) \geq \mathrm{cap}(A) \geq \mathrm{cap}_{\partial_2}(A) = E_A(\partial_2 P).$$

For $q_i(t) = \frac{\partial H(s_0)}{\partial s}\big|_{s_0 = a_i(t)} \cdot a_i'(t)$ for $i \in \{1,2\}$, we have :

$$\int_{t=0}^{l} \frac{1 + \frac{q_1(t)^2 + q_1(t)q_2(t) + q_2(t)^2}{3}}{H(a_2(t)) - H(a_1(t))} \, dt \geq \mathrm{cap}(A) \geq \int_{t=0}^{l} \frac{1}{H(a_2(t)) - H(a_1(t))} \, dt.$$

proof of Theorem B.4.1 The first inequality follows from the previous paragraph. It remains to prove the second inequality. The lower bound follows from the representation of the optimal

function P for $\text{cap}_{\partial_2}(A)$ in Fermi coordinates. For the upper bound, we will calculate $p = P \circ \psi^{-1}$ explicitly. For fixed $t \in [0, l]$, the boundary conditions imply

$$p(t,s) = c_1(t)H(s) - c_2(t), \quad \text{where}$$
$$c_1(t) = \frac{1}{H(a_2(t)) - H(a_1(t))} \quad \text{and} \quad c_2(t) = \frac{H(a_1(t))}{H(a_2(t)) - H(a_1(t))}.$$

We now calculate the energy of P on A, $E_A(P)$. It follows with $H(s)' = \frac{1}{h(s)}$:

$$\frac{\partial p(t,s)}{\partial s} = \frac{c_1(t)}{h(s)} \quad \text{and} \quad \frac{\partial p(t,s)}{\partial t} = c_1'(t) \cdot H(s) - c_2'(t).$$

Hence we obtain :

$$E_A(P) = \int\limits_{t=0}^{l} \int\limits_{s=a_1(t)}^{a_2(t)} (c_1'(t) \cdot H(s) - c_2'(t))^2 \cdot \frac{\partial H(s)}{\partial s} + c_1(t)^2 \cdot \frac{\partial H(s)}{\partial s} \, dt \, ds.$$

Evaluating the integral with respect to s, we have

$$E_A(P) = \int\limits_{t=0}^{l} \frac{(c_1'(t)u - c_2'(t))^3}{3c_1'(t)} + c_1(t)^2 u \Bigg|_{u=H(a_1(t))}^{H(a_2(t))} dt.$$

We have for $c_1'(t)$ and $c_2'(t)$, as $q_i(t) = \frac{\partial H(s_0)}{\partial s}|_{s_0=a_i(t)} \cdot a_i'(t)$ for $i \in \{1, 2\}$:

$$c_1'(t) = \frac{q_1(t) - q_2(t)}{(H(a_2(t)) - H(a_1(t)))^2} \quad \text{and} \quad c_2'(t) = \frac{q_1(t)H(a_2(t)) - q_2(t)H(a_1(t))}{(H(a_2(t)) - H(a_1(t)))^2}.$$

With these equations $E_B(p) = E_A(P)$ simplifies to $\displaystyle\int\limits_{t=0}^{l} \frac{1 + \frac{q_1(t)^2 + q_1(t)g_2(t) + q_2(t)^2}{3}}{H(a_2(t)) - H(a_1(t))} dt.$

This is the upper bound in **Theorem B.4.1**. $\square$

It is clear that the upper and lower bound are nearly optimal, if $\displaystyle\int\limits_{t=0}^{l} |a_1'(t)|^2 + |a_2'(t)|^2 \, dt$ is small.

If the integral has a high value, it might be possible to choose an annulus A'' in the interior of A, whose boundary line varies less. Then a test function can be constructed on this annulus as above. We can then evaluate the energy of this test function, to obtain a better upper bound.

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Curriculum Vitae

I was born on August 13th, 1977 in Hammelburg, Germany. I attended primary school in Fuchsstadt and secondary school in Hammelburg and finally obtained the German secondary school degree 'Abitur' in 1996.

In 1997 I undertook to study biology at Marburg university in Germany and one year later I started studying mathematics at the same institution. I obtained a diploma degree in biology with specialization in evolutionary genetics in 2003 and a diploma degree in mathematics with main subject algebraic geometry in 2006.

I was then hired as a teaching and research assistant at the EPFL in Switzerland and I began working on the present thesis in November 2006.