On some properties of suboptimal colorings of graphs

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Abstract

Starting from the trivial observation that in any optimal coloring of a graph, there always exists a node v such that its neighborhood $N(v)$ contains all colors, we examine related properties in suboptimal colorings (i.e., those using more than $\chi(G)$ colors, where $\chi(G)$ is the chromatic number). In particular we show that in any $(\chi(G)+p)$-coloring of G there is a node v such that its generalized neighborhood $N_q(v)$ with $q = \max\{2p-1, 2\}$ contains $\chi(G)$ colors for $p \geq 1$. Additional properties of $(\chi(G)+p)$-colorings are also given.

Key words: suboptimal graph colorings; generalized neighborhood

1 Introduction

We intend to explore some chromatic properties of node colorings in arbitrary graphs. For a graph $G = (V, E)$ a k-coloring (of its nodes) is a partition $(S_1, \ldots, S_k)$ of the node set V into k stable sets S_i (sets of mutually nonadjacent nodes). The smallest k for which G has a k-coloring is the chromatic number $\chi(G)$.

One may wonder why the study of suboptimal colorings may have some interest. Clearly from a purely theoretical point of view this raises new questions which have apparently not been examined earlier; some conjectures are stated in the next sections.

In addition to this, one may also recognize that in many circumstances constructing an optimal coloring is difficult; hence the heuristic procedures used in such cases provide us with suboptimal colorings, i.e., colorings using $\chi(G)+p$ colors (instead of $\chi(G)$ colors). Since some procedures like genetic procedures or tabu search procedures generate sequences of suboptimal colorings (by repeatedly applying exchange routines), it may be of interest to gather some knowledge about properties of suboptimal

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colorings. This paper is a first attempt to characterize such colorings; a few basic results are derived here but there are undoubtedly many more properties of such colorings to be discovered.

As an additional benefit, the knowledge of structural properties of suboptimal colorings may also suggest new recoloring procedures which could be used to eliminate a color in a stepwise procedure.

Let us also mention that in some applications, it may even be required to use more than $\chi(G)$ colors for coloring a graph G; for instance in the case of examination scheduling in a university, one has to schedule all exams within a time interval of $\chi(G)+p$ days (although $\chi(G)$ days would suffice) just to avoid having too many consecutive days with exams for a student. This fact may also be a good motivation for studying suboptimal colorings.

The generalized neighborhood $N_q(v)$ of a node v consists of all nodes in V which can be reached from v by following chains of at most q edges. Hence $N_0(v) = \{v\}$ and $N_1(v)$ consists of v and its neighbors. We also say that $N_q(v)$ is the set of nodes which are at distance at most q from v. Optimal colorings of graphs G (i.e., $\chi(G)$-colorings) have been studied rather extensively. They have many remarkable properties, and let us mention the following: [4] in any $\chi(G)$-coloring of a graph G and for any permutation $a_1, a_2, \ldots, a_k$ of the $k = \chi(G)$ colors, there is an elementary chain of k nodes in G which meets colors $a_1, a_2, \ldots, a_k$ in that order. Another observation should be recalled: in any $\chi(G)$-coloring of a graph G, there is a node v such that $N_1(v)$ contains all $\chi(G)$ colors. What happens if we consider $(\chi(G) + p)$-colorings? It turns out that this last observation can still be made for any $(\chi(G)+p)$-coloring when $\chi(G) \leq 3$. But for $\chi(G) \geq 4$ there are graphs for which $(\chi(G) + p)$-colorings exist where no $N_1(v)$ contains $\chi(G)$ colors.

2 Colorings and neighborhoods

As a motivation for the following study we shall establish here some properties of colorings of the graph pentaK$_{33}$.

The graph G (pentaK$_{33}$) with $\chi(G) = 4$ was constructed by the first author. It was obtained by taking a set $\mathcal{C} = \{1, \ldots, 5\}$ of 5 colors. For every color c and every subset $\{i, j\}$ of size two of the set $\mathcal{C} \setminus \{c\}$ a node $n_{c,i,j}$ is constructed. This produces 6 nodes per color and thus 30 nodes for the whole graph. Note that we consider $n_{c,i,j}$ and $n_{c,j,i}$ to be the same node.

Two nodes $n_{c,i,j}$ and $n_{c',i',j'}$ will be joined by an edge iff $c \in \{i', j'\}$ and $c' \in \{i, j\}$. Therefore a node $n_{c,i,j}$ will be adjacent only to nodes associated with colors i and j.

The resulting graph is shown in Fig. 1. From this construction, we can state:

Property 1 *pentaK$_{33}$ has a 5-coloring such that $N_1(v)$ contains 3 colors for each node v.*

Property 2 $\chi(\mathit{pentaK}_{33}) = 4$

Proof: In the following proof G will denote pentaK$_{33}$. A 4-coloring of G can easily be constructed (see Fig. 1). We only have to show that $\chi(G) > 3$. It is not difficult to see that the nodes of G can be covered by 10 disjoint triangles. The triangles are formed by three nodes of the following type: $n_{c,i,j}, n_{i,c,j}, n_{j,i,c}$. Therefore the largest size $\alpha(G)$ of a stable set is at most 10.

Let us show that one cannot find three disjoint stable sets S_1, S_2, S_3 with $|S_i| = 10$ for $i = 1, 2, 3$. We use a different description of G. Start with a clique K_5 on 5 nodes $a_1, \ldots, a_5$ whose edges are numbered $1, \ldots, 10$ as shown in Figure 2. Now each node a_i is replaced by a set A_i of 6 nonadjacent nodes $a_i^1, \ldots, a_i^6$ for $i = 1, \ldots, 5$. Each node a_i of K_5 is incident with 4 edges; let $B(a_i) = \{e, f, g, h\}$ be the set of edges incident with node a_i; to each pair j, k $(j < k)$ of edges in $B(a_i)$ we associate one node a_i^r. This is possible since $\binom{4}{2} = 6$. Let $P(a_i^r)$ be the two indices of the edges of K_5 associated with a_i^r. Two nodes a_i^r and a_j^s are linked if $i \neq j$ and $P(a_i^r) \cap P(a_j^s) \neq \emptyset$.

Let us examine the possible stable sets S with $|S| = 10$. We shall assume that $|S \cap A_1| \geq |S \cap A_2| \geq \ldots \geq |S \cap A_5|$ w.l.o.g.

Case 1 $|S \cap A_1| \leq 3$

$|S \cap A_3| \geq 2$ and $|S \cap A_4| \geq 1$ follow from $|S| = 10$.

Each pair of nodes a_i^r, a_i^s in $S \cap A_i$ is such that $|P(a_i^r) \cup P(a_i^s)| \geq 3$. Furthermore $a_i^r, a_j^t \in S$ with $i \neq j$ implies $P(a_i^r) \cap P(a_j^t) = \emptyset$. Recall that $P(a_i^r) = 2$ for all i, r.

Hence we cannot have a stable set S with $|S| = 10$, since it would mean that $|\bigcup (P(a_i^r) \,|\, a_i^r \in S)\,| \geq 3 + 3 + 3 + 2 = 11 > 10$ which is the number of indices of edges in K_5.

Case 2 $4 \leq |S \cap A_1| \leq 5$

We have $|\bigcup (P(a_1^s) \,|\, a_1^s \in S \cap A_1)\,| = 4$. This implies $|S \cap A_i| \leq 3$ for each $i > 1$.

If we have $|S \cap A_2| = 2$ then we must have $|S \cap A_4| \geq 1$. This is not possible (same as Case 1).

If we have $|S \cap A_2| = 3$ then we must have $S \cap A_4 = \emptyset$. Otherwise we can reapply the argument from Case 1. But if $4 \leq |S \cap A_1| \leq 5$ and $|S \cap A_2| = 3$ then $|S \cap A_3| = 1$ since there is only one node in A_3 that has no edge of K_5 in common with a_1 or a_2. But this would mean that $|S| \leq 9$ which is a contradiction.

Case 3 $|S \cap A_1| = 6$

$|S \cap A_1| = 6$ implies $|\bigcup_{r=1,\ldots,6} P(a_1^r)| = 4$.

If $|S \cap A_2| = 1$ then $|S \cap A_i| = 1$ for $i = 2, \ldots, 5$ and therefore $|\bigcup (P(a_i^r) \,|\, a_i^r \in S)\,| \geq 4 + 2 + 2 + 2 + 2 = 12 > 10$.

If $|S \cap A_2| = 2$ then $|\bigcup_{j=r,s,t} P(a_2^j)| = 3$ and there is only one possible node in A_3 which has no index of an edge of K_5 in common with the nodes in $S \cap A_1$ or in $S \cap A_2$. (The node in A_3 associated with edges 3 and 7 is not linked to any node in A_1 or A_2.) Therefore $|S \cap A_3| = |S \cap A_4| = 1$

and $|\bigcup(P(a_i^r)\,|\,a_i^r \in S)\,| \geq 4+3+2+2 = 11 > 10$.

For the case $|S\cap A_2| = 3$ we can find 3 nodes a_2^r, a_2^s, a_2^t in A_2 such that $\big(P(a_2^r) \cup P(a_2^s) \cup P(a_2^t)\big) \cap (\bigcup_{r=1,\ldots,6} P(a_1^r)) = \emptyset$; We have $|\bigcup_{j=r,s,t} P(a_2^j)| = 3$. Hence in A_3 we can find a node a_3^u which has no index of an edge of K_5 in common with the nodes in $S \cap A_1$ or in $S \cap A_2$. (The node in A_3 associated with edges 3 and 7 is not linked to any node in A_1 or A_2.) Hence one can find a stable set S with $|S \cap A_1| = 6$, $|S \cap A_2| = 3$, $|S \cap A_3| = 1$ and this is the only way to form a stable set of cardinality 10.

Notice that one cannot cover the nodes of G with three stable sets of that form.

So pentaK$_{33}$ cannot be 3-colored and hence $\chi(G) > 3$. It follows that $\chi(G) = 4$.

□

In the next section we will derive a property of $(\chi(G)+p)$-colorings; this suggests the study of suboptimal colorings in general. While optimal colorings (i.e., $\chi(G)$-colorings) have some structure, it turns out that for suboptimal colorings, the search for some structures is less immediate.

3 Properties of $(\chi(G)+p)$-colorings

We shall examine here the following question: how big must be the integer q, in order to be sure that in any $(\chi(G)+p)$-coloring of a graph G there exists a node v such that $N_q(v)$ contains at least $\chi(G)$ different colors? If $p = 0$ then $q = 1$ is sufficient according to the observation given in Section 1. Furthermore it also holds for $q \geq \lfloor \frac{\chi(G)}{2} \rfloor$. This can be seen easily, if we notice that in any coloring there exists a chain of $\chi(G)$ nodes where no two have the same color [4]. Taking the central node of such a chain for v gives the result.

Theorem 1

Let $c: V \to \{1,\ldots,\chi(G)+p\}$ be a $(\chi(G)+p)$-coloring of a graph G. Then there exists a node v such that $N_q(v)$ with $q = \max\{2p-1, 2\}$ contains at least $\chi(G)$ colors.

Before proving the theorem, we introduce some notation and definitions that will be needed.

$$\mathcal{K}_q(v) := \{c(w) | w \in N_q(v)\}$$

will be the set of colors which occur on nodes at distance at most q from v. $\bar{\mathcal{K}}_q(v)$ will denote the set of colors in $\{1,\ldots,\chi(G)+p\}$ which do not occur in $\mathcal{K}_q(v)$.

$$\mathcal{A}(v) := \{w \in N_1(v)\,|\,c(w) < c(v)\}$$

is the set of neighbors of v which have colors smaller than the color of v.

The set $\bar{E}(v)$ is defined as follows:

$$\bar{E}(v) := \begin{cases} \bar{\mathcal{K}}_q(v) & \text{if } \mathcal{A}(v) = \emptyset \\ \\ \displaystyle\bigcup_{w \in \mathcal{A}(v)} \bar{E}(w) & \text{otherwise} \end{cases}$$

This means that if v has no neighbors with a smaller color, then $\bar{E}(v)$ contains all colors which do not occur (on nodes located) at distance at most q from v. Let $E(v)$ denote the complement of $\bar{E}(v)$ with respect to the ground set $\{1, \ldots, \chi(G) + p\}$.

In addition, let $\alpha(v)$ be the length (number of edges) of a longest chain meeting nodes with strictly increasing colors and ending in v; that is

$$\alpha(v) := \begin{cases} 0 & \text{if } \mathcal{A}(v) = \emptyset \\ \\ 1 + \displaystyle\max_{w \in \mathcal{A}(v)} \alpha(w) & \text{otherwise} \end{cases}$$

Finally let $\mathcal{W}(v)$ be the set of all origins of maximal chains which meet strictly increasing colors and which end in v. We can write $\bar{E}(v) = \displaystyle\bigcup_{w \in \mathcal{W}(v)} \bar{\mathcal{K}}_q(w)$.

Define $\min_n X = x_n$ if $X = \{x_1, x_2, \ldots, x_{|X|}\}$ is ordered smallest first.

Proof of Theorem 1

Case 1 Let us start with the case $p \geq 2$ and therefore $q = 2p - 1$.

We shall assume there is a graph $G = (V, E)$ and a $(\chi(G)+p)$-coloring c such that $\forall v \in V: \ |\mathcal{K}_q(v)| < \chi(G)$. We shall show that G can be recolored with $\chi(G) - 1$ colors. This will give the required contradiction.

Claim 1

For each node $v \in V$ such that $c(v) \leq p + 1$ we have

$$\bar{E}(v) \ \subset \ \bar{\mathcal{K}}_1(v)$$

Proof: Every $w \in \mathcal{W}(v)$ is linked to v by a chain of length $< c(v) \leq p + 1 \leq q$. So we have $\mathcal{K}_1(v) \subset \mathcal{K}_q(w)$. Hence we have $\bar{\mathcal{K}}_q(w) \subset \bar{\mathcal{K}}_1(v)$ for all $w \in \mathcal{W}$ which proves the Claim.

□

Claim 2

For each node $v \in V$ such that $c(v) \leq p + 1$ and for each $w \in N_1(v)$ with $c(w) < c(v)$ (which is equivalent to $w \in \mathcal{A}(v)$) we have

$$\alpha(w) < \alpha(v)$$

and

$$\bar{E}(w) \subseteq \bar{E}(v)$$

Proof: Since $w \in \mathcal{A}(v)$ these properties follow directly from the definitions.

□

Claim 3

Given two sets X, Y of integers such that $X \subseteq Y$ and two integers n, m such that $n > m$, $n \leq |X|$ and $m \leq |Y|$ then

$$\min{}_n X > \min{}_m Y$$

Proof: We may assume that the elements of $X = \{x_1, \ldots, x_{|X|}\}$ and those of $Y = \{y_1, \ldots, y_{|Y|}\}$ are ordered in increasing order. Then $x_n = y_i$ for some $i \geq n$ since Y contains X. So $y_m < y_i = x_n$ because $i \geq n > m$.

$\square$

Claim 4

$\forall v \in V$ such that $c(v) \leq p+1$ we have

$$|E(v) \cap \{1, \ldots, c(v)\}| > \alpha(v)$$

Proof: Every node $w \in \mathcal{W}(v)$ is linked to v by a chain (with increasing colors) containing no more than $\alpha(v)$ edges. From the definition of $\alpha(v)$ there exists a chain C from v which meets $\alpha(v) + 1$ strictly decreasing colors. We have to consider two cases:

Case 1 $\alpha(v) < p$
Then $2\alpha(v) < q = 2p - 1$. Any node $u \in C$ is linked to any other $w' \in \mathcal{W}(v)$ by a chain no longer than $2\alpha(v)$. Therefore for node $u \in C$ and any $w' \in \mathcal{W}(v)$, we have $u \in N_q(w')$. It follows that $c(u) \in \mathcal{K}_q(w')$, i.e., $c(u) \notin \bar{\mathcal{K}}_q(w')$. So the colors on chain C are not contained in $\bar{E}(v)$, and hence they are in $E(v)$. Since C meets $\alpha(v)+1$ different colors which are at most $c(v)$ the result holds in this case.

Case 2 $\alpha(v) \geq p$
Since $\alpha(v) < c(v) \leq p+1$, we must have $\alpha(v) = p$ and $c(v) = p+1$. So every $w' \in \mathcal{W}(v)$ is either at distance p from v (in this case every color in $\{1, \ldots, p+1\}$ is not contained in $\bar{\mathcal{K}}_q(w')$ since w' is linked to v by a chain meeting all colors from 1 to $p+1$) or at distance $d(v, w') \leq p-1$. But all nodes $u \in C$ are at distance at most $2p-1$ from such a w' (at most $p-1$ edges from w' to v and at most p to go from v to u). So $\{1, \ldots, p+1\} \subseteq \mathcal{K}_q(w')$ for every $w' \in \mathcal{W}(v)$. Therefore $\bar{E}(v)$ does not contain any color in $\{1, \ldots, p+1\}$, so the Claim holds.

$\square$

Claim 5

If $c(v) \leq p+1$ then

$$|\bar{E}(v) \cap \{1, \ldots, p+1\}| < p+1-\alpha(v)$$

Proof: Claim 4 gives

$$|E(v) \cap \{1, \ldots, p+1\}| \;\geq\; |E(v) \cap \{1, \ldots, c(v)\}| \;>\; \alpha(v)$$

Taking the complement gives the Claim.

$\square$

Claim 6

$$|\bar{E}(v)| \geq p+1 \qquad \text{for any node } v \in V \text{ with } c(v) \leq p+1$$

Proof: Notice that $\bar{E}(v)$ contains $\bar{\mathcal{K}}_q(w)$ as a subset for some $w \in \mathcal{W}(v)$. By the assumption $|\mathcal{K}_q(w)| < \chi(G)$ one has $|\bar{\mathcal{K}}_q(w)| \geq p+1$.

□

Recoloring

For all $v \in V$ such that $c(v) \leq p+1$ define a new coloring by

$$c'(v) = \begin{cases} \min_{p+1-\alpha(v)} \bar{E}(v) & \text{if } c(v) \leq p+1 \\ c(v) & \text{otherwise} \end{cases}$$

Observe that $\min_{p+1-\alpha(v)} \bar{E}(v)$ is well defined since, by Claim 6, $\bar{E}(v)$ contains at least $p+1$ colors and $\alpha(v) < p+1$.

Claim 7

$$c'(v) > p+1$$

Proof: This is clearly true for nodes v with $c(v) > p+1$. Consider then a node $v \in V$ with $c(v) \leq p+1$. According to Claim 5 the $(p+1-\alpha(v))^{\text{th}}$ smallest color in $\bar{E}(v)$ has to be strictly bigger than $p+1$.

□

Claim 8

For every edge $(a,b) \in E$, $c'(a) \neq c'(b)$.

Proof: Let D be the set of nodes v with $c'(v) \neq c(v)$. Clearly the nodes in $\bar{D}$ are not recolored, so for an edge (a,b) with $a, b \in \bar{D}$, we will still have $c'(a) \neq c'(b)$.

According to Claim 1, nodes $v \in D$ are recolored with some color in $\bar{E}(v) \subset \bar{\mathcal{K}}_1(v)$, so for every edge (a,b) with $a \in D$ and $b \notin D$, we will again have $c'(a) \neq c'(b)$.

It now remains to show that for every edge (a,b) with $a, b \in D$ we have $c'(a) \neq c'(b)$. Assume w.l.o.g. that $c(b) < c(a)$. By Claim 2, we have $\bar{E}(b) \subseteq \bar{E}(a)$. Let $X = \bar{E}(b)$ and $Y = \bar{E}(a)$ and $n = p+1-\alpha(b)$ and $m = p+1-\alpha(a)$. Also by Claim 2 we have $\alpha(b) < \alpha(a)$ and therefore $n > m$. Then from Claim 3, we have $\min_n X = c'(b) > c'(a) = \min_m Y$.

□

So we have shown that G can be recolored with $\chi(G)-1$ colors, which is the desired contradiction. Hence there must be a node v such that $N_q(v)$ contains at least $\chi(G)$ colors. This ends the proof of the case $p \geq 2$.

Case 2 We now examine the case $p = 1$ and therefore $q = 2$. We will use the same framework as for the proof in Case 1:

Claim 1 still holds since it only requires $p < q$. Claim 2 does not depend on q and therefore it holds. Claim 3 involves neither p nor q and so it holds. Claim 4 requires a new proof under these new conditions.

Claim 4' ($p = 1$, $q = 2$)

$\forall v \in V$ such that $c(v) \leq 2$ we have

$$|E(v) \cap \{1, \ldots, c(v)\}| > \alpha(v)$$

Proof: Since $\alpha(v) < c(v)$ there are two possible values for $\alpha(v)$:

Case 1 $\alpha(v) = 0$
That means $\bar{E}(v) = \bar{\mathcal{K}}_2(v)$ or simply $E(v) = \mathcal{K}_2(v)$ and of course $c(v) \in \mathcal{K}_2(v)$ and $|E(v) \cap \{1, \ldots, c(v)\}| \geq 1 > 0 = \alpha(v)$.

Case 2 $\alpha(v) = 1$
That means that $c(v) = 2$ and for each $w \in \mathcal{W}(v)$ we have $c(w) = 1$ and $(w, v) \in E$. Therefore for each $w \in \mathcal{W}(v)$ we have $\{1, 2\} \subseteq \mathcal{K}_2(w)$. Consequently $\{1, 2\} \subseteq E(v) = \bigcap_{w \in \mathcal{W}(v)} \mathcal{K}_2(v)$ which proves the Claim.

□

Claim 5 follows from Claim 4. Claim 6 does not depend on q and therefore holds. The recoloring is not changed. Claim 7 is a consequence of Claim 5 and of the definition of c'. Claim 8 follows from Claims 1 to 3 and the definition of c'. This completes the proof of Case 2.

□

Remark

We started from the well known observation that for any $\chi(G)$-coloring of G there is a node v such that $N_1(v)$ contains $\chi(G)$ colors. For $p = 1$, we have proved that in any $(\chi(G) + 1)$-coloring there is a node v such that $N_2(v)$ contains at least $\chi(G)$ colors. We know that the bound for the size of the neighborhood is tight since for pentaK_{33} $N_1(v)$ meets fewer than $\chi(G)$ colors. For $p = 2$, Theorem 1 implies that for any $(\chi(G)+2)$-coloring there is a node v for which $N_3(v)$ contains at least $\chi(G)$ colors.

We are thus led to conjecture that one could replace q by $p+1$ (instead of $2p - 1$) and the result could be true for all $p \geq 0$:

Conjecture 1

For a given graph G and a $(\chi(G) + p)$-coloring there exists a node v such that $N_{p+1}(v)$ contains at least $\chi(G)$ colors.

The conjecture is true for $p = 0, 1, 2$. Notice that Theorem 1 and its corollary only imply that in any $(\chi(G)+p)$-coloring there is a node v such that $N_{\max(p+1, 2p-1)}(v)$ contains at least $\chi(G)$ colors.

4 Other properties of colorings

We have examined until now some properties of colorings related to the occurrence of $\chi(G)$ colors in the generalized neighborhood of a node in an arbitrary graph G. We next raise a few additional questions related to the number of colors appearing in some neighborhood of a node.

For any $(\chi(G)+p)$-coloring $c(p)$ of a graph $G=(V,E)$ let $k(c(p),v,q)$ denote the number of colors occurring in the neighborhood $N_q(v)$ of v. We are interested in $\max_{v\in V(G)} k(c(p),v,q)$ (i.e., the largest number of colors which occur at distance at most q from a node).

Finally if $\mathcal{C}(p)$ is the family of all $(\chi(G)+p)$-colorings of G, we want to estimate

$$A_G(p,q) = \min_{c(p)\in\mathcal{C}(p)} \left(\max_{v\in V(G)} k\left(c(p),v,q\right) \right)$$

We know at this stage $A_G(0,1)=\chi(G)$ (i.e., there exists in any optimal coloring of G a node v such that $N_1(v)$ contains all $\chi(G)$ colors). Besides this $A_G(p,2p-1)\geq \chi(G)$ from Section 2 (for $p\geq 2$). For $p=1$ we can write $A_G(1,2)\geq\chi(G)$. Trivially we have $A_G(p,1)\leq \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of G.

Theorem 2

For a graph G with $\chi(G)\geq 2$ and $p\geq 1$

$$A_G(p,1) \geq \left\lceil \frac{\chi(G)}{p+1} \right\rceil + 1$$

Proof: Assume to the contrary that there exists a graph G and a $(\chi(G)+p)$-coloring c such that for every node v we have $|\mathcal{K}_1(v)| \leq \left\lceil \frac{\chi(G)}{p+1} \right\rceil = \left\lfloor \frac{\chi(G)-1}{p+1} \right\rfloor + 1$. For graphs G with $\chi(G)\leq 2(p+1)$ the restriction on $\mathcal{K}_1(v)$ implies that the graph G is bipartite and therefore the proposition is true, because it is either trivial (for $\chi(G)=2$) or we get immediately the desired contradiction (for $\chi(G)\geq 3$).

For graphs G with $\chi(G)>2(p+1)$ let $P_1,\dots,P_{p+1}$ be disjoint subsets of $X=\{p+2,\dots,\chi(G)+p\}$ with $|P_i|\geq \left\lfloor \frac{\chi(G)-1}{p+1} \right\rfloor$ for $i=1,\dots,p+1$ (note that $|X|=\chi(G)-1$). Let V' be the set of nodes v with $c(v)\leq p+1$ such that there is no neighbor w of v with $c(w)\leq p+1$. Then define the new coloring

$$c'(v) = \begin{cases} \max \bar{\mathcal{K}}_1(v) & \text{if } v\in V' \\ \max\left(\bar{\mathcal{K}}_1(v)\cap P_{c(v)}\right) & \text{if } v\notin V' \text{ and } c(v)\leq p+1 \\ c(v) & \text{otherwise (i.e., } c(v)\geq p+2) \end{cases}$$

In the first and third case, we create no conflicts in the recoloring procedure. In the second case ($v\notin V'$ and $c(v)\leq p+1$) we have to check that $\bar{\mathcal{K}}_1(v)\cap P_{c(v)}\neq\emptyset$. This is the case since $|\mathcal{K}_1(v)\cap\{1,\dots,p+1\}|\geq 2$ and therefore $|\mathcal{K}_1(v)\cap X|\leq \left\lfloor \frac{\chi(G)-1}{p+1} \right\rfloor - 1$ which implies that $P_{c(v)}\not\subseteq \mathcal{K}_1(v)$ and therefore $\bar{\mathcal{K}}_1(v)\cap P_{c(v)}\neq\emptyset$.

By definition of $P_{c(v)}$ it is clear that $c'(v)\geq p+1$. It remains to check that c' is a feasible coloring. For nodes $v\in V'$ there are no conflicts since they are not adjacent to nodes that change color and they are recolored with a color in $\bar{\mathcal{K}}_1(v)$. The remaining nodes are colored with colors chosen in $p+1$ disjoint sets of colors and therefore they do not conflict with each other. Hence c' colors G with $\chi(G) - 1$ colors which is the desired contradiction.

□

For $\chi(G) \leq 5$ the bound given in Theorem 2 is tight for $p = 1$. The tightness for $\chi(G) = 5$ is shown by proving that the graph hexa_N_1_4, a generalization of pentaK_{33} (see [2]) has chromatic number 5. For $6 \leq \chi(G) \leq 10$ we have graphs (generalizations of pentaK_{33}) that seem to satisfy Theorem 2 with equality. The problem is that we only have heuristics to determine their chromatic number due to the size of the graphs. This leads us to the following conjecture:

Conjecture 2

For each integer ν there exists a graph G with $\chi(G) = \nu$ for which
$$A_G(1,1) = \left\lceil \frac{\chi(G)}{2} \right\rceil + 1$$

For a general p the bound given in Theorem 2 might be tight as well. Generalizations of pentaK_{33} seem to provide such examples but unfortunately we do not know their exact chromatic number.

Another question is how does $A_G(p,1)$ behave as a function of p for a fixed graph G? Intuitively one might expect that its value increases with p, but pentaK_{33} and its generalizations show that this is not always the case. We notice also that $A_G(n - \chi(G), 1) = \Delta(G) + 1$ where n is the number of nodes of G.

Proposition

$$A_G(p,q) \leq 2q \quad \Longrightarrow \quad \chi(G) \leq 2q$$

Proof: Let G be a graph and c be a $(\chi(G) + p)$-coloring of G such that $|\mathcal{K}_q(v)| \leq 2q$ for every node v of G. Orient each edge of G from the node with the smaller color to the node with the larger color. The longest path in that orientation contains strictly fewer than $2q$ edges. Otherwise if there were a path P of length $2q$ then for the middle node m of P we would have $|\mathcal{K}_q(m)| \geq 2q + 1$ since there are $2q + 1$ nodes of increasing (and therefore different) colors in P. Using the Roy-Gallai theorem [3] we conclude that $\chi(G) \leq 2q$.

□

5 Conclusion

The properties studied in Section 3 are not only interesting in themselves, but they also suggested some original proof techniques, such as in our case a direct recoloring technique where explicit coloring formulas can be derived.

The results of this paper exhibit the interest of exploring in depth the properties of suboptimal colorings and in general of suboptimal solutions to combinatorial optimization problems.

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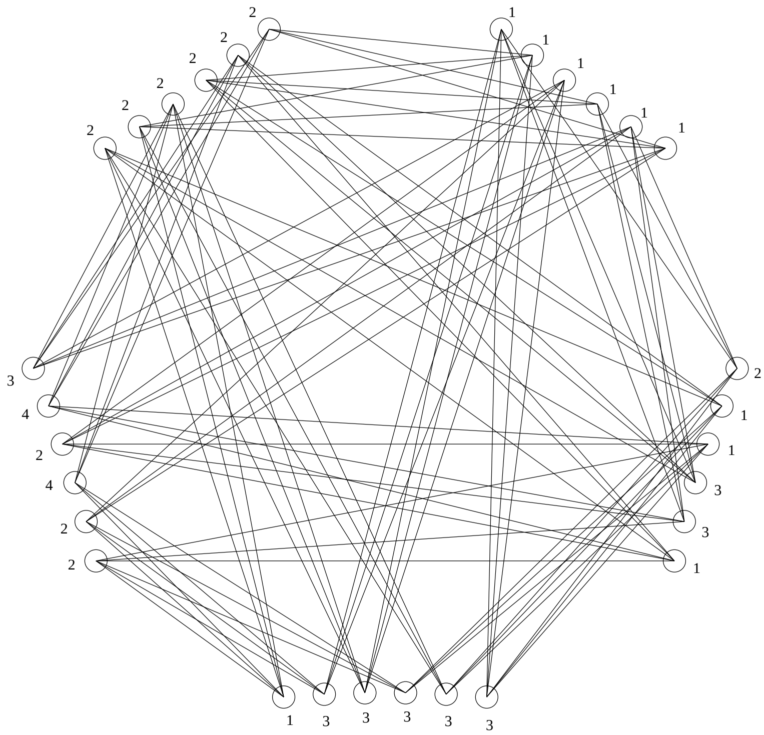

Figure 1: pentaK$_{33}$, can be colored with 5 colors (by assigning to every group of 6 nodes a color) such that for every node v, $N_1(v)$ contains only 3 colors, but $\chi(\text{pentaK}_{33}) = 4$. An example of a 4-coloring is given.

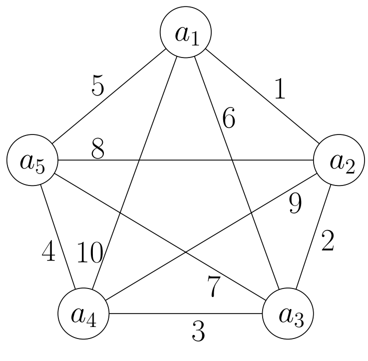

Figure 2: K_5 used to construct pentaK$_{33}$.