Error estimates for SUPG-stabilised Dynamical Low Rank Approximations

Fabio Nobile¹ and Thomas Trigo Trindade¹

¹CSQI, École Polytechnique Fédérale de Lausanne, Switzerland

February 7, 2024

Abstract

We perform an error analysis of a fully discretised Streamline Upwind Petrov Galerkin Dynamical Low Rank (SUPG-DLR) method for random time-dependent advection-dominated problems. The time integration scheme has a splitting-like nature, allowing for potentially efficient computations of the factors characterising the discretised random field. The method allows to efficiently compute a low-rank approximation of the true solution, while naturally "inbuilding" the SUPG stabilisation. Standard error rates in the $\|\cdot\|_{L^2}$ and $\|\cdot\|_{\text{SUPG}}$ -norms are recovered. Numerical experiments validate the predicted rates.

1 Introduction

The simulation of random time-dependent advection-dominated problems

$$\partial_t u - \varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f, \quad \text{in } D \subset \mathbb{R}^d,$$
 (1)

with coefficients ε , **b**, c and data f depending on some random parameter $\omega \in \Omega$, with probability measure μ on Ω , remains a challenge for multiple reasons. These processes often have poorly decaying Kolmogorov n-widths in the time-space domain, even if at each point in time the solution profile is well-approximated by a small subspace. Furthermore, it is well-known that applying the standard Finite Element Method to such problems causes the numerical solution to display unphysical spurious oscillations, in particular when the solution has sharp gradients and/or boundary layers. For practical purposes, it becomes necessary to remove or alleviate these oscillations by using some stabilisation strategy.

The purpose of [11] was to introduce the generalised Petrov-Galerkin Dynamical Low Rank (PG-DLR) framework and its particularisation to the Streamline Upwind/Petrov-Galerkin (SUPG-DLR), which allows to simultaneously tackle both issues. The Dynamical Low Rank (DLR) [8] framework, in this work written in its Dynamically Orthogonal (DO) [13] formalism, consists in seeking an approximation of the form $u_{\text{DLR}} = \sum_{i=1}^R U_i(t,x)Y_i(t,\omega)$ of the solution $u_{\text{true}}(t,x,\omega)$ of (1). The peculiar feature of this framework is that the physical $\{U_i(t,x)\}_{i=1}^R$ and the stochastic modes $\{Y_i(t,\omega)\}_{i=1}^R$ evolve in time to follow a (quasi-)optimal low-rank approximation of u_{true} , making it suited for the type of transport-dominated problems described above. As an extension of that framework, the PG-DLR framework allows to seamlessly import many stabilisation techniques that can be framed as generalised Petrov-Galerkin problems.

The focus of this paper is an error analysis of the SUPG-DLR framework. This work inscribes itself within a growing body of literature addressing the stabilisation of Reduced Order Models, including e.g. [15, 3] for SUPG-stabilised POD methods for advection-dominated problems. An error analysis for the SUPG-stabilised POD method was carried out in [4] for time-dependent advection-diffusion-reaction problems. In the DO setting, a noteworthy alternative to our method is the stabilisation based on Shapiro filters in [2], applied after each time step to smooth out the oscillations.

2 Problem setting & SUPG-DLR approximations

Solutions to random PDEs are function-valued random variables. In this work, we consider the advection-diffusion-reaction problem 1 with homogeneous Dirichlet boundary conditions u=0 on ∂D and initial condition $u_{|t=0}=u_0\in L^2_{\hat{\mu}}(H^0_0(D))$. The coefficients verify the following the Coefficient Assumptions (CoeffA): $\varepsilon>0$, $c\in L^\infty_{\hat{\mu}}(L^\infty(D))$ and $c(x,\omega)\geq c_0>0$ for a.e. $x\in D, \forall \omega\in \hat{\Omega},\ f\in L^2_{\hat{\mu}}(L^2(D)),\ \mathbf{b}\in (L^\infty(D))^d,\ \mathrm{div}\ \mathbf{b}(x)=0$. Therefore the solution $u_{\mathrm{true}}(t,\cdot,\omega)$ belongs to $H^1_0(D)$ for (almost) every t>0 and $\omega\in\Omega$. The probability space is discretised via a collocation method (e.g., the Monte Carlo method), yielding the collocation points $\hat{\Omega}:=\{\omega_i\}_{i=1}^{N_C}\subset\Omega$ and a discrete measure $\hat{\mu}.\ L^2_{\hat{\mu}}(\hat{\Omega})$ denotes the space of random variables, with scalar product $\mathbb{E}_{\hat{\mu}}[YZ]=\sum_{i=1}^{N_C}m_iY_iZ_i$, where $\{m_i\}_{i=1}^{N_C}$ are positive weights summing up to 1, and $Y_i=Y(\omega_i),\ Z_i=Z(\omega_i)$. The random solution $u_{\mathrm{true}}(t,\cdot,\cdot)$ satisfies for almost every $t,\ u\in L^2_{\hat{\mu}}(\hat{\Omega},X):=L^2_{\hat{\mu}}(X)$, where $X=H^1_0(D)$ (with standard H^1_0 -scalar product) or $L^2(D)$. These Bochner spaces admit the scalar product $(u,v)_{L^2_{\hat{\mu}}(X)}=\sum_{i=1}^{N_C}m_i\langle u(\omega_i),v(\omega_i)\rangle_X$. Hereafter, we use the shorthand notation (\cdot,\cdot) and $\|\cdot\|$ to denote the $L^2_{\hat{\mu}}(L^2(D))$ inner product and norm.

We use the Finite Elements Method on a quasi-uniform mesh \mathcal{T}_h with characteristic mesh size h, and consider the space of continuous piece-wise polynomials of degree k, $V_h := \mathbb{P}_k^C(\mathcal{T}_h) \subset H_0^1(D)$ where k denotes the polynomial degree and $N_h := |V_h|$. In this work, we will consider the advection-dominated regime with the condition $\|\mathbf{b}\|_{L^{\infty}} h > 2\varepsilon$ assumed true hereafter.

The numerical approximation $\tilde{u}_{h,\hat{\mu}}$ is sought in $V_h \otimes L^2_{\hat{\mu}}$. The inverse inequality from standard Finite Element theory can be extended to elements in $V_h \otimes L^2_{\hat{\mu}}$, yielding $\|\nabla \tilde{u}_{h,\hat{\mu}}\| \leq C_I h^{-1} \|\tilde{u}_{h,\hat{\mu}}\|$ for some $C_I > 0$ and every $\tilde{u}_{h,\hat{\mu}} \in V_h \otimes L^2_{\hat{\mu}}$, as the inequality holds pointwise in ω . For the same reasons, the standard Poincaré inequality can be extended to $V_h \otimes L^2_{\hat{\mu}}$, yielding $\|\tilde{u}_{h,\hat{\mu}}\| \leq C_P \|\nabla \tilde{u}_{h,\hat{\mu}}\|$, where C_P is the Poincaré constant. Hereafter, to lighten the notation, $\tilde{u} \equiv \tilde{u}_{h,\hat{\mu}} \in V_h \otimes L^2_{\hat{\mu}}$.

The DLR approximation belongs to the differential manifold of R-rank functions, defined as

$$\mathcal{M}_{R} = \{ \tilde{u} \in V_{h} \otimes L_{\hat{\mu}}^{2}(\hat{\Omega}) : \tilde{u} = \sum_{i=1}^{R} U_{i} Y_{i}, \text{ s.t. } \mathbb{E}_{\hat{\mu}}[Y_{i} Y_{j}] = \delta_{ij},$$

$$\{U_{i}\}_{i=1}^{R} \text{ lin. ind. and } \{U_{i}\}_{i=1}^{R} \in V_{h}, \{Y_{i}\}_{i=1}^{R} \in L_{\hat{\mu}}^{2}(\hat{\Omega}) \}.$$
 (2)

Each point $u \in \mathcal{M}_R$ can be equipped with a tangent space, spanned by tangent vectors $\delta u = \sum_{i=1}^R \delta u_i Y_i + U_i \delta y_i$, uniquely identified by imposing the *Dual Dynamically Orthogonal* (Dual DO) condition [10], $\mathbb{E}[Y_i \delta y_j] = 0$ for i, j = 1, ..., R. This leads to the following characterisation

$$\mathcal{T}_{u}\mathcal{M}_{R} = \{\delta u = \sum_{i=1}^{R} \delta u_{i} Y_{i} + U_{i} \delta y_{i}, \text{ such that } \{\delta u_{i}\}_{i=1}^{R} \in V_{h},$$
$$\{\delta y_{i}\}_{i=1}^{R} \in L_{\hat{\mu}}^{2}(\hat{\Omega}), \mathbb{E}_{\hat{\mu}}[\delta y_{i} Y_{j}] = 0, \forall 1 \leq i, j \leq R\}. \quad (3)$$

Given $U = (U_1, \ldots, U_R)$ and $Y = (Y_1, \ldots, Y_R)$ s.t. $u = UY^{\top}$, the tangent space at u is denoted by $\mathcal{T}_{UY^{\top}}\mathcal{M}_R$. Furthermore, for an $L^2_{\hat{\mu}}$ -orthonormal set Y, let $\mathcal{Y} \coloneqq \operatorname{span}(Y_1, \ldots, Y_R)$, and $\mathcal{P}_{\mathcal{Y}}[v] = \sum_{i=1}^R \mathbb{E}[vY_i]Y_i$ and $\mathcal{P}_{\mathcal{Y}}^{\perp}[v] = v - \mathcal{P}_{\mathcal{Y}}[v]$.

To recover dynamic equations for the physical and stochastic modes, the idea is to project Equation (1) onto the tangent space $\mathcal{T}_{UY^{\top}}\mathcal{M}_R$ at each time instant. The SUPG-DLR framework proposes to solve the problem

$$(\dot{u}_{\text{DLR}}, \tilde{v} + \delta \mathbf{b} \cdot \nabla \tilde{v}) + a_{\text{SUPG}}(u_{\text{DLR}}, \tilde{v}) = (f, \tilde{v} + \delta \mathbf{b} \cdot \nabla \tilde{v}).$$

$$\forall \tilde{v} \in \mathcal{T}_{u_{\text{DLR}}} \mathcal{M}_{R}, \text{a.e. } t \in (0, T], \quad (4)$$

with

$$\begin{split} a_{\mathrm{SUPG}}(\tilde{u},\tilde{v}) &= (\varepsilon \nabla \tilde{u}, \nabla \tilde{v}) + (\mathbf{b} \cdot \nabla \tilde{u}, \tilde{v}) + (c\tilde{u}, \tilde{v}) \\ &+ \sum_{K \in \mathcal{T}_{\mathrm{h}}} \delta_{K} (-\varepsilon \Delta \tilde{u} + \mathbf{b} \cdot \nabla \tilde{u} + c\tilde{u}, \mathbf{b} \cdot \nabla \tilde{v})_{K, L_{\tilde{\mu}}^{2}}, \end{split}$$

where $(\cdot,\cdot)_{K,L^2_{\hat{\mu}}} := (\cdot,\cdot)_{L^2_{\hat{\mu}}(L^2(K))}$. Hereafter, we use a uniform stabilisation parameter $\delta \equiv \delta_K$ for each $K \in \mathcal{T}_h$.

Particularising the conditions in [11] to our setting, if (COEFA) and

$$\delta \le \min_{K \in \mathcal{T}_h} \left\{ \frac{1}{2\|c\|_{L^{\infty}_{\hat{a}}(L^{\infty})}}, \frac{h_K^2}{2\varepsilon C_I^2}, \frac{h_K}{\|\mathbf{b}\|_{L^{\infty}} C_I} \right\}$$
 (5)

hold true, then

$$a_{\text{SUPG}}(\tilde{u}, \tilde{u}) \ge \frac{1}{2} \|\tilde{u}\|_{\text{SUPG}}^2,$$
 (6)

where $\|\tilde{u}\|_{\mathrm{SUPG}}^2 = \varepsilon \|\nabla \tilde{u}\|^2 + \delta \sum_{K \in \mathcal{T}_h} \|\mathbf{b} \cdot \nabla \tilde{u}\|_{K, L_{\tilde{\mu}}^2}^2 + \|c^{1/2}\tilde{u}\|^2$. This norm is suitable for advection-dominated problems, as it offers a better control of the stream-line diffusion. As an immediate consequence of (5), $\|\tilde{v} + \delta \mathbf{b} \cdot \nabla \tilde{v}\| \leq 2\|\tilde{v}\|$. Two additional properties of the SUPG setting are summarised below:

Lemma 2.1. Assuming (COEFA), it holds

$$a_{\text{SUPG}}(\tilde{u}, \tilde{v}) \le C_1 \|\nabla \tilde{u}\| \|\tilde{v}\|, \qquad \|\tilde{u}\| \le c_0^{-1} \|\tilde{u}\|_{\text{SUPG}},$$
 (7)

where $C_1 = (C_I + 2) \|\mathbf{b}\|_{L^{\infty}} + 2C_P \|c\|_{L^{\infty}_{\hat{u}}(L^{\infty})}.$

Proof. We detail the proof for some terms, the bounds for the others being direct. Firstly, $\varepsilon|(\nabla \tilde{u}, \nabla \tilde{v})| \leq \|\nabla \tilde{u}\| \|\varepsilon \nabla \tilde{v}\| \leq \frac{C_I \|\mathbf{b}\|_{L^\infty}}{2} \|\nabla \tilde{u}\| \|\tilde{v}\|$, having used $\varepsilon < \frac{1}{2} \|\mathbf{b}\|_{L^\infty} h$ and the inverse inequality. Additionally, letting $C_2 = \frac{C_I}{2} \|\mathbf{b}\|_{L^\infty}$,

$$|\delta \sum_{K \in \mathcal{T}_h} (\varepsilon \Delta \tilde{u}, \mathbf{b} \cdot \nabla \tilde{v})_{K, L_{\hat{\mu}}^2}| \leq C_2 \sum_{K \in \mathcal{T}_h} \|\nabla \tilde{u}\|_{K, L_{\hat{\mu}}^2} \|\tilde{v}\|_{K, L_{\hat{\mu}}^2} \leq C_2 \|\nabla \tilde{u}\| \|\tilde{v}\|.$$

In [11], we use Algorithm 1 reproduced below to sequentially update the physical and stochastic modes in a (potentially) cheap fashion, resulting in a non-linear update on \mathcal{M}_R . The algorithm was originally proposed and analysed in [6] for random uniform coercive problems, and is very similar to the Projector-Splitting algorithm [9]. In this work, we focus on the implicit version of the scheme; however, semi-implicit and fully explicit versions are also possible.

Algorithm 1. Given the solution $u_{h,\hat{\mu}}^n = \sum_{i=1}^R U_i^n Y_i^n$:

1. Find \tilde{U}_j^{n+1} , $j = 1, \ldots, R$, such that

$$\Delta t^{-1} (\tilde{U}_j^{n+1} - U_j^n, v_h + \delta \mathbf{b} \cdot \nabla v_h)_{L^2(D)} + a_{\text{SUPG}} (u_{h,\hat{\mu}}^{n+1}, v_h Y_j^n)$$

$$= (f^{n+1}, v_h Y_j^n + \mathbf{b} \cdot \nabla v_h Y_j^n), \quad \forall v_h \in V_h. \quad (8)$$

2. Find \tilde{Y}_{j}^{n+1} , $j=1,\ldots,R$ such that $(\tilde{Y}_{j}^{n+1}-Y_{j}^{n})\in\mathcal{Y}^{\perp}=\mathcal{P}_{\mathcal{Y}}^{\perp}(L_{\hat{\mu}}^{2})$ and

$$\Delta t^{-1} \sum_{i=1}^{R} \mathbb{E}[(\tilde{Y}_{i}^{n+1} - Y_{i}^{n})z] \tilde{W}_{ij}^{n+1} + a_{\text{SUPG}}(u_{h,\hat{\mu}}^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z)$$

$$= (f^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z + \delta \mathbf{b} \nabla \tilde{U}_{j}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z), \quad \forall z \in L_{\hat{\mu}}^{2}. \quad (9)$$

where $\tilde{W}_{ij}^{n+1} = (\tilde{U}_i^{n+1}, \tilde{U}_j^{n+1} + \delta \mathbf{b} \nabla \tilde{U}_j^{n+1})_{L^2(D)}$.

- 3. Reorthonormalise \tilde{Y}^{n+1} such that $\mathbb{E}[Y_i^{n+1}Y_j^{n+1}] = \delta_{ij}$ and modify $\{\tilde{U}_i^{n+1}\}_{i=1}^R$ such that $\sum_{i=1}^R \tilde{U}_i^{n+1} \tilde{Y}_i^{n+1} = \sum_{i=1}^R U_i^{n+1} Y_i^{n+1}$.
- 4. The new solution is given by $u_{h,\hat{\mu}}^{n+1} = \sum_{i=1}^R U_i^{n+1} Y_i^{n+1}$.

When applying Algorithm 1, the update verifies a variational formulation (Proposition 2.1) which allows to analyse the scheme using variational methods and, among others, prove norm-stability of the scheme (Proposition 2.2).

Proposition 2.1. (from [11]) The numerical solution by Algorithm 1 satisfies

$$\frac{1}{\Delta t}(u_{h,\hat{\mu}}^{n+1} - u_{h,\hat{\mu}}^{n}, v_{h,\hat{\mu}} + \delta \mathbf{b} \cdot \nabla v_{h,\hat{\mu}}) + a_{\text{SUPG}}(u_{h,\hat{\mu}}^{n+1}, v_{h,\hat{\mu}}) = (f^{n+1}, v_{h,\hat{\mu}} + \delta \mathbf{b} \cdot \nabla v_{h,\hat{\mu}}),$$

$$\forall v_{h,\hat{\mu}} \in \mathcal{T}_{\tilde{U}^{n+1}(Y^n)^{\top}} \mathcal{M}_R. \quad (10)$$

Proposition 2.2. (from [11]) Assuming δ verifies (5) and $\delta \leq \triangle^t/4$, then it holds for the numerical solution computed by Algorithm 1

$$\|u_{h,\hat{\mu}}^N\|^2 + \sum_{n=1}^N \Delta t \|u_{h,\hat{\mu}}^n\|_{\text{SUPG}}^2 \le \|u_{h,\hat{\mu}}^0\|^2 + \Delta t \left(\frac{4}{c_0} + 4\delta\right) \sum_{j=1}^N \|f^j\|^2.$$
(11)

3 Error estimate

The idea of the SUPG method is to skew the test space by $\mathcal{H} = (I + \delta \mathbf{b} \cdot \nabla)$. Its ajoint is given by $\mathcal{H}^* = I - \delta \mathbf{b} \cdot \nabla$ thanks to the zero-divergence of **b**. Denote $\mathcal{P}_{\mathcal{H}^*} : V_h \otimes L_{\hat{\mu}}^2 \to \mathcal{T}_u \mathcal{M}_R$ the oblique projection on the tangent space:

$$(\mathcal{P}_{\mathcal{H}^*}\tilde{u}, \mathcal{H}^*w) = (\tilde{u}, \mathcal{H}^*w) \quad \forall w \in \mathcal{T}_u \mathcal{M}_R.$$
 (12)

Its well-posedness is ensured by the coercivity of $(u, \mathcal{H}^*u) = ||u||^2$ on $V_h \otimes L^2_{\hat{\mu}}$. Hereafter, we use the shorthand notation $\tilde{v}^{\perp} := \mathcal{P}_{\mathcal{H}^*}^{\perp} \tilde{v} = \tilde{v} - \mathcal{P}_{\mathcal{H}^*} \tilde{v}$ for any $\tilde{v} \in V_h \otimes L^2_{\hat{\mu}}(\hat{\Omega})$. By definition of the projection,

$$(\mathcal{P}_{\mathcal{H}^*}^{\perp}\tilde{v},\mathcal{H}^*w) = 0 \quad \forall w \in \mathcal{T}_u \mathcal{M}_R. \tag{13}$$

A useful property of the oblique projection is the following:

Lemma 3.1.

$$||I - \mathcal{P}_{\mathcal{H}^*}|| = ||\mathcal{P}_{\mathcal{H}^*}|| \le 3.$$
 (14)

Proof. The first equality is a standard result of projectors [14]. Consider the orthogonal projector $\Pi: V_h \otimes L^2_{\tilde{\mu}} \to \mathcal{T}_u \mathcal{M}_R$ verifying $(\Pi \tilde{u}, w) = (\tilde{u}, w)$ for $w \in \mathcal{T}_u \mathcal{M}_R$, we have

$$\|(\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u}\|^2 = ((\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u}, (I - \delta \mathbf{b}\nabla)(\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u})$$
$$= (\tilde{u} - \Pi\tilde{u}, (I - \delta \mathbf{b}\nabla)(\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u}) \le 2\|\Pi^{\perp}\tilde{u}\|\|(\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u}\| \quad (15)$$

From there, we conclude $\|\mathcal{P}_{\mathcal{H}^*}\tilde{u}\| \leq \|(\mathcal{P}_{\mathcal{H}^*} - \Pi)\tilde{u}\| + \|\Pi\tilde{u}\| \leq 3\|\tilde{u}\|$.

We will make use of the following assumptions to analyse the convergence of the SUPG-DLR method. The first is the standard Model Error Assumption, particularised to the SUPGcontext. It asks that the dynamics neglected by the DLR approximation is negligible. This is a standard assumption made to analyse the convergence of DLR approximations [1, 7, 8].

Assumption 3.1. (Model Error Assumption) For n = 0, ..., N-1, let $\hat{u}^n = \tilde{U}^{n+1}Y^n$ be the "intermediate" point obtained by Algorithm 1. For $\nu \ll 1$, it holds

$$|a_{\text{SUPG}}(\hat{u}^n, v_{h,\hat{u}}^{\perp}) - (f, \mathcal{H}v_{h,\hat{u}}^{\perp})| \le \nu \|\tilde{v}\|, \quad \forall \tilde{v} \in V_h \otimes L_{\hat{u}}^2, \quad \text{for } \nu \ll 1.$$
 (16)

The second is an assumption on the H^1 -stability of the physical basis.

Assumption 3.2. (Local basis inverse inequality) Given the DLR iterates $\{u_{h,\hat{\mu}}^n = \tilde{U}^n \tilde{Y}^n\}_{n=1}^N$ obtained via Algorithm 1, and denoting $(\mathbb{S}_n)_{ij} = (\nabla \tilde{U}_i^{n+1}, \nabla \tilde{U}_j^{n+1})_{L^2(D)}$ and $(\mathbb{M}_n)_{ij} = (\tilde{U}_i^{n+1}, \tilde{U}_j^{n+1})_{L^2(D)}$ the stiffness and mass matrices associated to the physical basis $\{\tilde{U}_i^n\}_{i=1}^R$, there exists a constant $C_{\text{lbi}} < \infty$ such that

$$\max_{n=0,\dots,N} \left(\sup_{x \in \mathbb{R}^R} \frac{x^{\top} \mathbb{S}_n x}{x^{\top} \mathbb{M}_n x} \right) \le C_{\text{lbi}}. \tag{17}$$

The functions (U_1^n, \ldots, U_R^n) are typically globally supported and display regularity, justifying a moderate value for C_{lbi} . Assumption 3.2 implies that, for any $n \geq 0$,

$$\|\nabla \tilde{U}^n Z^\top\| \le C_{\text{lbi}} \|\tilde{U}^n Z^\top\| \quad \text{for } Z \in [L^2_{\hat{\mu}}]^R.$$
(18)

The elliptic projection operator $\pi: L^2_{\hat{\mu}}(\hat{\Omega}, H^1_0(D)) \to V_h \otimes L^2_{\hat{\mu}}$ is defined by

$$(\nabla(u - \pi u), \nabla v_{h,\hat{\mu}}) = 0, \quad \forall v_{h,\mu} \in V_h \otimes L^2_{\hat{\mu}}. \tag{19}$$

For brevity, denote $\pi^n u = \pi u(t_n)$. We split $u_{h,\hat{\mu}}^n - u(t_n) = (u_{h,\hat{\mu}}^n - \pi^n u) + (\pi^n u - u(t_n)) = \tilde{e}^n + \eta^n$. The interpolation error η^n is bounded using standard estimates which, assuming $u(t_n) \in L^2_{\hat{\mu}}(H^{k+1})$ for any n, yields (see e.g. [12])

$$\mathcal{E}^{N}(\eta) := \|\eta^{N}\|^{2} + \frac{\Delta t}{4} \sum_{j=1}^{N} \|\eta^{j}\|_{\text{SUPG}}^{2} \lesssim h^{2k+1}.$$
 (20)

For the other error term, Proposition 2.1 allows to derive

$$\Delta t^{-1} \left(\tilde{e}^{n+1} - \tilde{e}^{n}, \tilde{v} \right) + a_{\text{SUPG}} \left(\tilde{e}^{n+1}, \tilde{v} \right) = a_{\text{SUPG}} \left(u(t_{n+1}) - \pi^{n+1} u, \tilde{v} \right)$$

$$+ \left(\dot{u}(t_{n+1}) - \Delta t^{-1} (\pi^{n+1} u - \pi^{n} u), \mathcal{H} \tilde{v} \right) - \delta \Delta t^{-1} \left(\tilde{e}^{n+1} - \tilde{e}^{n}, \mathbf{b} \cdot \nabla \tilde{v} \right) + a_{\text{SUPG}} \left(\hat{u}^{n}, \tilde{v}^{\perp} \right)$$

$$- \left(f^{n+1}, \mathcal{H} \tilde{v}^{\perp} \right) - a_{\text{SUPG}} \left(\hat{u}^{n} - u_{h,\hat{\mu}}^{n+1}, \tilde{v}^{\perp} \right) + \Delta t^{-1} \left(u_{h,\hat{\mu}}^{n+1} - u_{h,\hat{\mu}}^{n}, \mathcal{H} \tilde{v}^{\perp} \right), \quad \forall \tilde{v} \in V_{h} \otimes L_{\hat{\mu}}^{2}. \quad (21)$$

Note that the last term in (21) vanishes by (13). One last technical lemma is needed before presenting the main result:

Lemma 3.2. Let $\tilde{\delta}Y^n := \tilde{Y}^{n+1} - Y^n$. It holds

$$\Delta t^{-1} \|\tilde{U}^{n+1} \tilde{\delta} Y^n\|^2 = a_{\text{SUPG}}(u_{h,\hat{u}}^{n+1}, \tilde{U}^{n+1} \tilde{\delta} Y^n) + (f^{n+1}, \mathcal{H} \tilde{U}^{n+1} \tilde{\delta} Y^n).$$

Proof. Start from (9). Using the definition of \tilde{W}_{ij}^{n+1} , we rewrite it as

$$\Delta t^{-1}(\sum_{i=1}^{R} \tilde{U}_{i}^{n+1} \tilde{\delta} Y_{j}^{n}, \tilde{U}_{j}^{n+1} z_{j} + \delta \mathbf{b} \cdot \nabla \tilde{U}_{j}^{n+1} z_{j}) = a_{\text{SUPG}}(u_{h,\hat{\mu}}^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z_{j})$$

$$+ (f^{n+1}, \tilde{U}_{i}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z_{j} + \delta \mathbf{b} \cdot \nabla \tilde{U}_{i}^{n+1} \mathcal{P}_{\mathcal{Y}}^{\perp} z_{j}) \quad \text{for } j \in 1, \dots, R, \forall z_{j} \in L_{\hat{\mu}}^{2}.$$

Set $z_j = \tilde{Y}_j^{n+1} - Y_j^n$, the result is obtained by summing over j since $\tilde{Y}^{n+1} - Y^n \in (\mathcal{Y}^n)^{\perp}$ (the l.h.s. becomes $\|\tilde{U}^{n+1}\tilde{\delta}Y^n\|^2$ by zero-divergence of **b**).

Theorem 3.1. Let $\mathbf{b} \in (L^{\infty}(D))^d$ such that $\operatorname{div} \mathbf{b} = 0$, $c \in L^{\infty}_{\hat{\mu}}(L^{\infty}(D))$ and assume the true solution verifies $u, \partial_t u \in L^{\infty}(0, T; L^{\infty}_{\hat{\mu}}(H^{k+1}(D))), \partial_t^2 u \in L^2(0, T; L^{\infty}_{\hat{\mu}}(H^1))$. Under (Coeff A), (16), (5) as well as $\delta \leq \triangle^t/4$, the DLR iterates $\{u^n_{h,\hat{\mu}}\}_{n=0}^N$ of Algorithm 1 satisfy

$$||u(t_N) - u_{h,\hat{\mu}}^N|| + \left(\sum_{i=1}^N \triangle t ||u(t_i) - u_{h,\hat{\mu}}^i||_{\text{SUPG}}^2\right)^{1/2}$$

$$\lesssim h^{k+1} + \triangle t + \delta^{1/2} h^k + \delta^{-1/2} h^{k+1} + ||\pi^0 u - u_{h,\hat{\mu}}^0|| + \nu. \quad (22)$$

Proof. The proof largely follows the structure of the proof in [5, pp. 10-12]. Testing against \tilde{e}^{n+1} , the first two terms in the r.h.s of (21) verify

$$a_{\text{SUPG}}(u(t_{n+1}) - \pi^{n+1}u, \tilde{e}^{n+1}) + (\dot{u}(t_{n+1}) - \triangle t^{-1}(\pi^{n+1}u - \pi^{n}u), \mathcal{H}\tilde{e}^{n+1})$$

$$= \delta \sum_{K \in \mathcal{T}_{\bullet}} (\tilde{T}_{\text{stab},K}^{n+1}, \mathbf{b} \cdot \nabla \tilde{e}^{n+1})_{K, L_{\hat{\mu}}^{2}} + (T_{\text{zero}}^{n+1}, \tilde{e}^{n+1}) + (T_{\text{conv}}^{n+1}, \tilde{e}^{n+1}),$$

where

$$T_{\text{zero}}^{n+1} = (\dot{u}(t_{n+1}) - \pi^{n+1}\dot{u}) + c(u(t_{n+1}) - \pi^{n+1}u) + \left(\pi^{n+1}\dot{u} - \frac{\pi^{n+1}u - \pi^{n}u}{\Delta t}\right),$$

$$T_{\text{conv}}^{n+1} = \mathbf{b} \cdot \nabla(u(t_{n+1}) - \pi^{n+1}u),$$

$$\tilde{T}_{\text{stab},K}^{n+1} = \left(T_{\text{zero}}^{n+1} + T_{\text{conv}}^{n+1} + \varepsilon\Delta(\pi^{n+1}u - u(t_{n+1}))\right)_{|K|}.$$

Counter-integrating $(T_{\text{conv}}^{n+1}, \tilde{e}^{n+1})$ and using the zero-divergence of **b** yields

$$(T_{\text{conv}}^{n+1}, \tilde{e}^{n+1}) = -\delta \sum_{K \in \mathcal{T}_h} (\delta^{-1}(\pi^{n+1}u - u(t_{n+1})), \mathbf{b} \cdot \nabla \tilde{e}^{n+1})_{K, L_{\hat{\mu}}^2},$$

which can then be included in $T_{\text{stab},K}^{n+1}$, defining

$$T_{\text{stab},K}^{n+1} = \tilde{T}_{\text{stab},K}^{n+1} - \delta^{-1}(\pi^{n+1}u - u(t_{n+1})).$$

We then bound the terms via Young's inequality, suitably balancing the coefficients such that the \tilde{e}^{n+1} -quantities on the r.h.s can be absorbed by $\frac{1}{2}||e^{n+1}||^2_{\text{SUPG}}$ on the l.h.s. To this end, let

 $0 < \gamma \le 1/16$. As $(2\triangle t)^{-1} (\|\tilde{e}^{n+1}\|^2 - \|\tilde{e}^n\|^2 + \|\tilde{e}^{n+1} - e^n\|^2) + 1/2 \|\tilde{e}^{n+1}\|_{\text{SUPG}}^2$ lower-bounds the l.h.s of (21), it holds

$$\begin{split} (2\triangle t)^{-1}(\|\tilde{e}^{n+1}\|^2 - \|\tilde{e}^n\|^2 + \|\tilde{e}^{n+1} - \tilde{e}^n\|^2) + 1/2\|\tilde{e}^{n+1}\|_{\mathrm{SUPG}}^2 \\ & \leq \delta \sum_{K \in \mathcal{T}_h} (T_{\mathrm{stab},K}^{n+1} - \triangle t^{-1}(\tilde{e}^{n+1} - \tilde{e}^n), \mathbf{b} \cdot \nabla \tilde{e}^{n+1})_{K,L_{\hat{\mu}}^2} + (T_{\mathrm{zero}}^{n+1}, \tilde{e}^{n+1}) \\ & + a_{\mathrm{SUPG}}(u_{h,\hat{\mu}}^{n+1} - \hat{u}^n, (\tilde{e}^{n+1})^{\perp}) + a_{\mathrm{SUPG}}(\hat{u}^n, (\tilde{e}^{n+1})^{\perp}) - (f^{n+1}, \mathcal{H}(\tilde{e}^{n+1})^{\perp}) \\ & \leq C\delta \sum_{K \in \mathcal{T}_h} \|T_{\mathrm{stab},K}^{n+1}\|_{K,L_{\hat{\mu}}^2}^2 + \delta \gamma \|\mathbf{b} \cdot \nabla \tilde{e}^{n+1}\|_{K,L_{\hat{\mu}}^2}^2 + C\triangle t^{-1} \|\tilde{e}^{n+1} - \tilde{e}^n\|^2 + C\|T_{\mathrm{zero}}^{n+1}\|^2 \\ & + \gamma \|\tilde{e}^{n+1}\|^2 + a_{\mathrm{SUPG}}(u_{h,\hat{\mu}}^{n+1} - \hat{u}^n, (\tilde{e}^{n+1})^{\perp}) + a_{\mathrm{SUPG}}(\hat{u}^n, (\tilde{e}^{n+1})^{\perp}) - (f^{n+1}, \mathcal{H}(\tilde{e}^{n+1})^{\perp}), \end{split}$$

having used $\delta \lesssim \Delta t$ in the last inequality, and where C depends on γ^{-1} . Lemma 3.2 with (18) and (16) respectively yield

$$a_{\text{SUPG}}(\tilde{U}^{n+1}\tilde{\delta}Y, (\tilde{e}^{n+1})^{\perp}) \lesssim \|\tilde{U}^{n+1}\tilde{\delta}Y\| \|\tilde{e}^{n+1}\| \leq C \Delta t^{2} (\|u_{h,\hat{\mu}}^{n+1}\|^{2} + \|f\|^{2}) + \gamma \|\tilde{e}^{n+1}\|^{2},$$

$$a_{\text{SUPG}}(\hat{u}^{n}, (\tilde{e}^{n+1})^{\perp}) - (f^{n+1}, \mathcal{H}(\tilde{e}^{n+1})^{\perp}) \leq \nu \|\tilde{e}^{n+1}\| \leq C\nu^{2} + \gamma \|\tilde{e}^{n+1}\|^{2}.$$

Note that $\sum_{j=0}^{N-1} \triangle t^2(\|u_{h,\hat{\mu}}^j\|^2 + \|f\|^2) \lesssim \triangle t$ by Proposition 2.2. Cancelling, rearranging a few terms and summing over $j=0,\ldots,N-1$, we obtain

$$\|\tilde{e}^{n+1}\|^2 + \sum_{n=1}^N \|\tilde{e}^n\|_{\mathrm{SUPG}}^2 \lesssim \|\tilde{e}^0\|^2 + \Delta t \sum_{n=1}^N \|T_{\mathrm{zero}}^n\|^2 + \Delta t \sum_{n=1}^N \sum_{K \in \mathcal{T}_b} \delta_K \|T_{\mathrm{stab},K}^n\|_K^2 + \nu^2 + \Delta t^2.$$

As in [5], the regularity assumptions on u and its derivatives allow to bound

Denoting $\gamma^n := u_{h,\hat{\mu}}^n - u(t_n)$, the claim follows as $\mathcal{E}^N(\gamma) \lesssim \mathcal{E}^N(\tilde{e}) + \mathcal{E}^N(\eta)$.

4 Numerical experiments

We solve problem (1) on D = [0, 1] with

$$\varepsilon = 10^{-8},$$
 $\mathbf{b} = 1,$ $c(x, \omega) = 1 + \omega,$ $\omega \sim \mathcal{U}[0, 1]$

and choose the right-hand-side such that the true solution is given by

$$u_{\text{true}}(t, x, \omega) = e^{x \sin(2\pi\omega(t+1))} \sin(2\pi x). \tag{23}$$

The stochasticity therefore resides in the initial conditions, reaction term and forcing term. The sample space $\Omega = [0,1]$ is then approximated with the discrete set $\hat{\Omega} = \{i/N_C\}_{i=1}^{N_C}$ with $N_C = 15$ and equal probabilities in all the sample points. The physical space is discretised using a regular mesh with increasingly fine mesh size $h_i \sim 2^{-i}$. For the initial conditions, we compute a (generalised) SVD of (23) at time t=0.

As in [5], the terms $\triangle t$, $\delta^{1/2}h^k$ and $h^{k+1}\delta^{-1/2}$ in the error estimate need to be balanced to yield the best possible decay rate for a fixed h. Since Proposition 2.2 requires $\delta \sim \triangle t$, this imposes the condition $\triangle t \sim \mathcal{O}(h^{\frac{2(k+1)}{3}})$.

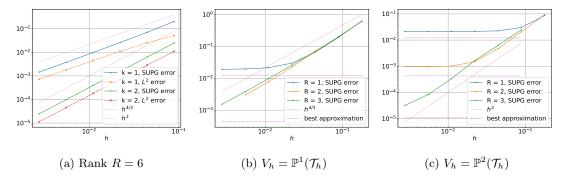


Figure 1: SUPG error for k = 1, 2 and small approximation rank R.

For the simulations we do not use the implicit scheme, but a semi-implicit version close to it that is both more technical and practical (reyling on a slightly different parametrisation of the approximation manifold with isolated mean, see [11]). With some technical details, the results carry over for that time-stepping scheme too.

In the first numerical experiment, we choose a rank R=6 to ensure the error associated to the rank truncation is negligible. The rates observed in Figure 1a are those predicted by Theorem 3.1, both for the $L^2_{\hat{\mu}}(L^2(D))$ and the SUPG error. Figures 1b and 1c display the errors of DLR approximations computed with R=1,2,3. The error is quasi-optimal with respect to the error obtained when using the optimal rank-R truncation.

References

- [1] G. CERUTI AND C. LUBICH, An unconventional robust integrator for dynamical low-rank approximation, BIT Numerical Mathematics, 62 (2022), pp. 23–44.
- F. FEPPON AND P. F. J. LERMUSIAUX, Dynamically Orthogonal numerical schemes for efficient stochastic advection and lagrangian transport, SIAM Review, 60 (2018), pp. 595– 625.
- [3] S. GIERE, T. ILIESCU, V. JOHN, AND D. WELLS, SUPG reduced order models for convection-dominated convection-diffusion-reaction equations, Computer Methods in Applied Mechanics and Engineering, 289 (2015), pp. 454–474.
- [4] V. John, B. Moreau, and J. Novo, Error analysis of a SUPG-stabilized POD-ROM method for convection-diffusion-reaction equations, Computers & Mathematics with Applications, 122 (2022), pp. 48–60.
- V. John and J. Novo, Error analysis of the supg finite element discretization of evolutionary convection-diffusion-reaction equations, SIAM Journal on Numerical Analysis, 49 (2011), pp. 1149–1176.
- [6] Y. KAZASHI, F. NOBILE, AND E. VIDLIČKOVÁ, Stability properties of a projector-splitting scheme for dynamical low rank approximation of random parabolic equations, Numerische Mathematik, 149 (2021), pp. 973–1024.
- [7] E. Kieri, C. Lubich, and H. Walach, Discretized dynamical low-rank approximation in the presence of small singular values, SIAM Journal on Numerical Analysis, 54 (2016), pp. 1020–1038.

- [8] O. Koch and C. Lubich, *Dynamical low-rank approximation*, SIAM Journal on Matrix Analysis and Applications, 29 (2007), pp. 434–454.
- [9] C. Lubich and I. V. Oseledets, A projector-splitting integrator for dynamical low-rank approximation, BIT Numerical Mathematics, 54 (2014), pp. 171–188.
- [10] E. Musharbash and F. Nobile, Dual dynamically orthogonal approximation of incompressible navier stokes equations with random boundary conditions, Journal of Computational Physics, 354 (2018), pp. 135–162.
- [11] F. Nobile and T. Trigo Trindade, Petrov-Galerkin Dynamical Low Rank Approximation: SUPG stabilisation of advection-dominated problems. In preparation.
- [12] A. Quarteroni and A. Valli, Numerical approximation of partial differential equations, vol. 23, Springer Science & Business Media, 2008.
- [13] T. P. Sapsis and P. F. Lermusiaux, Dynamically orthogonal field equations for continuous stochastic dynamical systems, Physica D: Nonlinear Phenomena, 238 (2009), pp. 2347–2360
- [14] D. B. SZYLD, The many proofs of an identity on the norm of oblique projections, Numerical Algorithms, 42 (2006), pp. 309–323.
- [15] D. Torlo, F. Ballarin, and G. Rozza, Stabilized weighted reduced basis methods for parametrized advection dominated problems with random inputs, SIAM/ASA Journal on Uncertainty Quantification, 6 (2018), pp. 1475–1502.