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Abstract

In certain control settings, information about the reference signals to be tracked or the disturbances affecting a system may be known in advance. This is often the case in motion control, machine-tooling, robotics and, more generally, mechatronic systems. Control systems that exploit early knowledge are called preview-based. The main contributions of this thesis are design methods for two-degree-of-freedom controllers and preview-based feedforward compensators for general closed-loop controllers.

A design method is presented for the preview-based two-degree-of-freedom (RST) controller. Controllers of fixed order which eliminate permanent error are first parameterized. A suitable cost function definition then results in a convex minimization problem. This cost function penalizes both tracking error and excessive command activity.

One contribution of this thesis is an analytic solution to an $\mathcal{H}_2$-norm minimization problem. This solution can be used to determine controller parameters as part of the design method described above. In addition, convex optimization methods are shown to be well-suited to solve the numerical problems that result from the design method, and a convex functional that allows one to design a preview-based controller with specified maximum phase error is presented. This contribution makes it possible to generalize a well-known zero phase error tracking controller.

A design method permitting one to add a preview-based compensator to an existing closed-loop controller is also given. This makes it possible to add preview compensation to a controller, without opening the control loop, so that it exploits preview knowledge.
Version abrégée

Dans certaines situations, des informations concernant la consigne ou les perturbations qui agissent sur un système sont disponibles en avance. Ceci est souvent le cas dans la commande des mouvements, des machines-outils, en robotique et, plus généralement, dans les systèmes mécatroniques. Les régulateurs qui exploitent cette information sont appelés preview-based.

Une des contributions principales de cette thèse est une méthode de synthèse pour les régulateurs avec preview à deux degrés de liberté (RST). Les régulateurs d’ordre fixe qui éliminent les écarts permanents sont d’abord paramétrés. Ensuite, une fonction-coût est définie. Il en résulte un problème de minimisation convexe. Cette fonction-coût pénalise l’écart de poursuite et une activité de commande excessive.

Une autre contribution de cette thèse est une solution analytique à un problème de minimisation $H_2$. Cette solution peut être utilisée pour déterminer les paramètres du régulateur dans le contexte de la méthode décrite ci-dessus. Les méthodes d’optimisation convexes sont bien adaptées pour résoudre les problèmes numériques issus de cette technique de synthèse. Une fonctionnelle convexe qui permet la synthèse de régulateurs avec preview avec une erreur de phase maximale spécifiée a priori est présentée. Cette contribution permet de généraliser le régulateur à erreur de phase nulle décrit dans la littérature.

Une méthode est fournie pour synthétiser un compensateur avec preview pour un régulateur en boucle fermée donné. Il devient alors possible d’exploiter la preview sans altérer le comportement en boucle fermée.
# Contents

1 Introduction ........................................ 1
   1.1 Motivation ...................................... 2
   1.2 State of the art .................................. 3
   1.3 Contributions ................................... 6
   1.4 Organization of the thesis ....................... 6
   1.5 Notation ........................................ 7

2 Preview .............................................. 9
   2.1 Basic principles .................................. 9
   2.2 Interpretations of preview ....................... 12
      2.2.1 Minimization of undershoot .................. 12
      2.2.2 Zero-phase-error tracking .................... 15
      2.2.3 Preview in the regulation setting .......... 16

3 Open-loop preview-based compensation for FIR systems 19
   3.1 Notation .......................................... 19
   3.2 Approximating a pure delay ...................... 20
   3.3 Norms and constraints ........................... 24
      3.3.1 Step response ................................. 24
      3.3.2 Ramp response ................................ 25
      3.3.3 Sinusoidal excitation ......................... 26
   3.4 Problem solutions ................................ 27
      3.4.1 $\ell_2$ case ................................ 28
      3.4.2 $\ell_\infty$ case .............................. 30
   3.5 Example .......................................... 30
   3.6 Remarks ......................................... 31
Chapter 1

Introduction

Given a process to be controlled, the purpose of control theory is to design a control system that generates a plant input, as a function of past external signals and plant measurements, resulting in a closed-loop system satisfying competing criteria such as stability and performance. The controller should be such that these criteria may be met in a robust way, so that limited plant uncertainty and disturbance signals will not prevent the controller from satisfying them.

In classical input-output linear control theory [17, 13, 6], a transfer function is used to characterize time-domain input-output responses, as well as frequency-domain characteristics of a plant or closed-loop system. When a plant is not linear, classical linear control can be used, to a certain degree, to stabilize the plant around a given operating point. Manual and computer-aided design methods exist to design stabilizing controllers which meet certain performance criteria. Frequency-domain results, based on transfer functions, allow one to easily determine stability margins of closed-loop systems by plotting a Nyquist or Bode plot. Thus, robustness properties appear in a natural way in the classical setting. Classical control theory resulted in the advent of the PID controller, which due to its simplicity is used in the great majority of the control systems in use today.

Much recent work is based on the pioneering work of Kalman, who developed the principle concepts of what is now referred to as state space con-
control [14, 13, 6]. In this framework, linear input-output relationships can be written in matrix form, with the appearance of intermediate state-variables. Low-complexity computer algorithms exist which can be used to simulate systems, and design stabilizing controllers by performing pole-placement and state-estimation. The combination of pole-placement, which requires state-variables to be available, and state-estimation, which estimates state-variables by using plant outputs, is a pillar of state-space control.

In practice, both state-space control and input-output control have their strong points, and computer algorithms allow one to easily convert systems between the two frameworks. In addition, robustness measures such as the modulus margin have interpretations in both settings. Therefore, the practicing control engineer generally resorts to both state-space and classical control methods depending on the situation.

In certain control settings, information about the reference signals to be tracked or the perturbations affecting a system may be known in advance. For simplicity, it is assumed that the future knowledge is always known a fixed time in advance, called the *preview*, and control systems that exploit early knowledge are called *preview-based*.

This thesis presents a comprehensive study of how preview information may be used to improve controller performance. The majority of the results presented in this thesis are presented in the input-output setting. In this setting, it is more natural to impose degree constraints than in the state-space setting.

1.1 Motivation

Many results exist in the control literature describing both open and closed-loop optimal control for dynamic systems. When performing open-loop optimization, it is assumed that the reference signal is fully known in advance, which is not always a realistic assumption. Open-loop control lacks robustness, so much effort has been expended on the field of optimal closed-loop control, the goal being to minimize some cost function representing a trade-off between performance and robustness of the compensated system. Unlike
open-loop control, however, most closed-loop control results are based on the assumption that the reference signal is not known in advance.

There are numerous industrial applications, like motion control, machine-tooling, and robotics, where the control signal may be known in advance to some degree. This information is rarely exploited in closed-loop control, although there is great potential to increase system performance by using this information. This failure to exploit early knowledge of reference signals is analogous to the situation that the driver of an automobile is in when there is little or no visibility of the road ahead. When future road conditions are available because of good visibility, the driver can anticipate changes resulting in safer and more enjoyable travel. An automatic control system that is designed to use early knowledge will also be able to anticipate changes and provide better performance than one that only reacts to input changes a-posteriori.

Preview can also be used to greatly improve the tracking for non-minimum phase systems exhibiting undershoot or oscillation. It is shown in this thesis that some ad-hoc methods for dealing with these types of systems can be generalized using the principle of preview.

1.2 State of the art

The idea of preview is not new. For instance, the paper [27] presents a linear quadratic preview-based controller based on finite-time optimization. When the problem length approaches infinity, a time-invariant controller is obtained. In the more recent paper [23], an $H_\infty$ preview problem with perfect state measurement is solved using Hamiltonian methods. The linear quadratic results are obtained as a limiting case. What is interesting to note in this paper is that the differences between the optimal output-feedback preview controller and the standard $H_\infty$ optimal controller occur only in a feedforward term. The state-feedback term is identical to that found in the standard $H_\infty$ controller. This sort of separation of the preview from the closed-loop part of the controller gives credibility to the argument that one may design a closed-loop controller to obtain robustness, and a feedforward controller can be designed separately to improve performance. Although these results are based on state-feedback,
in [16] an output feedback $\mathcal{H}_\infty$ solution is given. In [28], a closed-loop design method based on the algebraic methods of [31] is proposed. The problem is reduced to a model-matching problem in $\ell_1$, $\mathcal{H}_2$, or $\mathcal{H}_\infty$ for which the solution is known. An interesting game theoretic solution to the preview problem is given in [22] where one player represents the control and the other player represents a disturbance. The signals of one of the players are previewed by the other advantaged player—normally the control player.

The above results are very interesting and have relatively wide applicability, but in certain applications using the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm to specify both performance and robustness criteria may not be appropriate. Specifying properties such as monotonicity, settling-time limits, and phase error is impossible in this setting. Some of these results do not permit the incorporation of the command signal into the cost function. Also, these types of design methods may yield very high order controllers. For instance, in [35] a $\mathcal{H}_\infty$ preview controller is designed using spectral factorization methods, but the resulting feedforward controller is of the combined order of the plant and feedback controller.

The preview results above are all closed-loop designs. A number of open-loop preview problems have been solved. In [10], for instance, $\ell_1$, $\ell_2$, and $\ell_\infty$ optimal preview-based feedforward compensators were designed using duality theory to find analytic optimal SISO solutions for a rational feedforward compensator. The command signal, however, is not taken into consideration in the cost function, thus limiting the applicability.

As described above, a kind of separation property exists with respect to preview-based controllers. This must not be confused with the separation principle which allows optimal state-feedback and optimal state estimation to be combined resulting in an optimal output-feedback controller. This preview separation justifies the process of first designing a standard closed-loop controller that provides robustness, and then designing a preview-based feedforward compensator that exploits preview information. This is done in several papers that attempt to improve tracking performance for non-minimum phase systems. The zero-phase error preview compensator of [26] is used in this way. Stable plant dynamics must be canceled by a feedforward compensator,
resulting in a FIR system. Then a design method is applied to determine a
FIR compensator for the remaining FIR system such that, taking preview into
account, the phase error of the compensated plant is zero. These results are
extended in [33]. Unlike the zero-phase compensator of [26], which uses the
same number of samples of preview information as there are non-minimum
phase zeros (sometimes called unstable zeros), this method allows one to ex-
ploit additional preview information, if available. The additional information
is used to reduce the frequency response modulus of the error.

This is also similar to the approach taken in both [12] and [9]. In these
papers, since the non-minimum phase zeros cannot be canceled directly, a
non-causal series expansion of the inverse of the plant zeros is calculated. The
availability of preview information makes it possible to truncate this expansion
and transform it into a causal approximation of the inverse of the non-minimum
phase plant zeros. For the case when there is one non-minimum phase zero, it
is shown in the former paper that this approximation results in a compensated
system which has minimum undershoot. An interesting result given in this
paper is that given enough preview information, a compensator which results
in a monotone step response can always be found for a FIR plant with a pair of
complex non-minimum phase zeros. In [9], it is shown that this approximation
method is equivalent to the solution to a weighted least squares problem in the
frequency domain. The paper [11] presents a means of adapting the monotone
step response results of [12] to the two-degree-of-freedom synthesis approach
described in [20, 2]. This approach is sometimes called the \textit{RST} approach.

The methods of [26, 33, 12, 9] all assume that plant poles and stable plant
zeros are all canceled, and only non-minimum phase plant zeros are taken into
consideration. This makes sense, since the other dynamics can be canceled with
a stable feedforward compensator. However, if one wishes to design a low order
compensator, this may not be the best approach to use. Also, these methods
all perform optimization with the unique goal of reducing tracking error. As
stated above, however, other objectives like reducing command activity may
also be of interest.
1.3 Contributions

This thesis presents a preview-based synthesis approach for the SISO two-degree-of-freedom controller, as well as a preview-based synthesis method permitting a control system designer to adapt an existing closed-loop controller, without changing loop dynamics, so that it can exploit preview. These methods allow the feedforward compensator numerator and denominator degrees to be specified by the compensator designer. The preview-based controller design problem becomes an affine optimization problem in the vector space of rational transfer functions.

An analytical solution of the least-squares type is given for a simple affine $\mathcal{H}_2$ solver which can then be used to find a solution to the preview controller design problem. The solution method is based on vector space methods, and it solves affine problems with multiple competing $\mathcal{H}_2$ criteria.

Also, an approach is presented for solving the preview control design problem using convex optimization methods. This has the advantage of permitting the integration of a variety of optimization criteria tailored to the specific problem. One particular convex functional which appears to have been overlooked by the control community, the maximum phase angle functional, is studied in detail. It is shown that this functional can be used to generalize the zero-phase-error results of [26].

1.4 Organization of the thesis

This thesis is organized as follows. In Chapter 2, the basic principles of preview are introduced, and a few examples that clarify the different interpretations of preview are given. The relationship between preview and delay tracking is clarified. A geometric interpretation of preview is given in Chapter 3, as well as analytic solutions to a basic FIR preview problem. In Chapter 4 a preview-based design algorithm for two-degree-of-freedom controllers is presented, along with a few numerical examples. This algorithm converts a controller design problem into an optimization problem. A zero-placement method is proposed in Chapter 5, suggesting a method that can be used to add preview capability to general closed-loop controllers, without requiring the loop to be opened. An
example clarifying the robustness of this method is given. In Chapter 6, an analytic solution is derived for a particular $\mathcal{H}_2$ problem that is used to find concrete solutions to intermediate problems of Chapters 4 and 5. Chapter 7 presents how convex optimization may also be used to solve these intermediate problems. A particular convex functional, the maximum phase error functional, is studied. In Section 7.7, a more complete example is provided, clarifying the relationship between preview and other optimization criteria. Finally, the results are summarized, and future research directions are pointed out in Chapter 8.

1.5 Notation

A list of symbols appears at the end of this thesis.

As a general rule, capital letters in formulas will represent elements of $\mathbb{R}[q]$. $V$ and $W$, the exceptions to the above rule, will be used to represent rational functions in Chapter 6. To enhance readability, the dependence on $q$ will often not be shown. The meaning should be clear from the context.
1. INTRODUCTION
Chapter 2

Preview

In this chapter, preview will be presented in detail. It will be introduced through a discussion of tracking error, defined as a certain function of the plant output and the reference signal. If the tracking error $e(k)$ is small, the controller will be said, informally, to provide good tracking. In addition to providing good tracking, it may be necessary for a preview controller to meet other criteria, to be discussed in later chapters.

2.1 Basic principles

Assume that a plant $G$ has input $u$ and output $y$, and $y_r$ is the reference signal that the output $y$ is desired to follow. It is assumed that some controller is measuring $y$, and perhaps $y$ as well, and generating an input $u$ to the plant. The details of this controller are not important this early in the discussion.

Let a fixed reference signal $\hat{y}_r(k)$ be given. For simplicity in the presentation, the unit step signal $\hat{y}_r(k) = 1(k)$ will be used as the reference signal in the figures that follow in this chapter.

When using preview it is assumed that the reference signal is known a certain time in advance by the controller. Thus, the true input to the controller, $y_r(k)$ will be generated by shifting forward the fixed reference signal $\hat{y}_r(k)$ by some amount that is known to the controller.

Initially it will be assumed that the reference signal is not provided in
advance, so that $y_r(k) = \hat{y}_r(k)$. If the system has an internal delay of $\gamma_1$ samples, defining $e(k) = y_r(k) - y(k)$ and attempting to follow the reference signal immediately doesn’t make sense because it is an unachievable goal. This is illustrated in Figure 2.1. Shading is used to emphasize which signal difference contributes to the error $e(k)$. This should not necessarily be interpreted as area, because a norm used to quantify the error has not yet been specified.

![Figure 2.1: System with delay, $y_r(k) = \hat{y}_r(k)$, $e(k) = y_r(k) - y(k)$](image)

It is more meaningful to compare a delayed version of the reference signal with the plant output instead (see Figure 2.2(a)). This figure is the same as Figure 2.1, except that the error signal is the difference between the reference signal delayed by $\gamma_1$ samples and the output $y(k)$: $e(k) = q^{-\gamma_1}y_r(k) - y(k)$.

![Figure 2.2: System with delay, $e(k) = q^{-\gamma_1}y_r(k) - y(k)$](image)

In many tracking applications, the reference signal is available in advance.
In this case, it is possible to cancel the plant delay by sending the reference signal exactly $\gamma_1$ samples in advance, by using $y_r(k) = q^\gamma \hat{y}_r(k)$ as seen in Figure 2.2(b).

Now, let’s return to the case where $y_r(k) = \hat{y}_r(k)$. As was done in Figure 2.2(a), the design of the error signal takes into consideration the system delay $\gamma_1$. However, it may be possible to increase tracking precision (reduce the error), at the cost of the introduction of additional delay $\gamma$, by using an error signal which delays $y_r(k)$ by $\gamma$ additional samples before making the comparison. Through this anticipative behavior, a controller design which minimizes the error signal may result in tracking performance that is unachievable otherwise (see Figure 2.3(a)). In addition, this often results in a smoothing of the control signal. However, these improvements come at the cost of additional delay. This delay, however, is not problematic if the reference signal is sent $\gamma_1 + \gamma$ samples in advance, as is clear in Figure 2.3(b), where $y_r(k) = q^{\gamma_1 + \gamma} \hat{y}_r(k)$.

Thus, if the reference signal is provided early by defining $y_r(k) = q^{\gamma_1 + \gamma} \hat{y}_r(k)$, and the error signal is defined as $e(k) = q^{-\gamma_1 - \gamma} y_r(k) - y(k)$, the goal of designing a preview-based controller can be achieved by designing a controller that makes $e(k)$ small in some sense. This will be discussed extensively in the following chapters.
2.2 Interpretations of preview

Preview has been used successfully in several contexts. In this section, a few different examples of the use of preview are presented.

2.2.1 Minimization of undershoot

Assume that it is desired to design an open-loop FIR compensator \( C(q^{-1}) \) for the single non-minimum phase zero plant \( B(q^{-1}) = 1 - zq^{-1} \), where \( z = 1.2 \) (see Figure 2.4). The reference signal is the unit step \( 1(k) \). It is known that an undershoot will be present. With no compensator, one obtains the response in Figure 2.5 (after scaling for unit DC gain).

\[
\begin{array}{c}
C(q^{-1}) \rightarrow B(q^{-1})
\end{array}
\]

Figure 2.4: Open loop compensation of \( 1 - zq^{-1} \).

A way to reduce the undershoot can be described as follow. Choose the coefficients \( \{c_i\} \) of the polynomial \( C \in \mathbb{R}_{<n-1}[q^{-1}] \) in a way that minimizes the maximal undershoot that results when applying a step input to the system.
C(q^{-1})B(q^{-1})$, subject to the constraint that the DC gain of $C(q^{-1})B(q^{-1})$ be equal to 1. The solution $c_i = z^i(1 - z^{n-1})$ was derived in [12] using linear programming methods. In the case where $n = 11$, the solution shown in

![Graph](image)

(a) Step response of $C(q^{-1})B(q^{-1})$.

(b) Step response of $q^{-10}$.

Figure 2.6: Comparison of compensated system with pure delay

Figure 2.6(a) is found. It is clear that the system reacts slower to the step input, but with much less undershoot than in the uncompensated case. There is no intuitive way to extend this minimax method to the case with general $B$ if, for example, there is no undershoot.

It is shown in [12] that the same compensator can be derived by designing $C$ by performing a series expansion so that $CB$ approximates $q^{-n}$ (see
Comparing the undershoot-minimizing response 2.6(a) to the step response of $q^{-10}$ shown in Figure 2.6(b), this makes sense. Note that the compensator that minimizes undershoot results in a response similar to that of a pure delay.

Thus, when using preview, good tracking behavior may be related to tracking a pure delay, and then canceling this delay with preview. Again, the basic idea is that if the reference signal is sent to the controller $\Delta$ samples in advance, and the controller attempts to track a pure $\Delta$-sample delay, the end result is that the reference signal is tracked without delay, as desired. But since the controller receives the reference signal in advance, it is nevertheless able to anticipate changes to improve tracking precision.

Using series truncation, however, is a somewhat arbitrary method. In [9], however, it is shown to have certain frequency-domain approximation properties. When $B$ is of degree 1, the same compensator can be obtained by designing a preview-optimal compensator that minimizes tracking error using the $\ell_\infty$-norm.

Using series truncation, however, is a somewhat arbitrary method. In [9], however, it is shown to have certain frequency-domain approximation properties. When $B$ is of degree 1, the same compensator can be obtained by designing a preview-optimal compensator that minimizes tracking error using the $\ell_\infty$-norm.

The above methods were derived with the assumption that the plant to be controlled is a FIR system. In the more general case with a discrete-time rational plant $G(q^{-1})$ (Figure 2.8), it is nevertheless possible to use this approach. One must first design a stabilizing feedback controller $K(q^{-1})$, where $F(q^{-1})$ describes the sensor dynamics, and then cancel all poles and the stable zeros of the closed-loop system with a feedforward compensator $\hat{C}(q^{-1})$.

This method results in a FIR transfer function $B(q^{-1})$ from the reference signal to the output that contains only the non-minimum phase plant zeros. One may
then design the compensator $C(q^{-1})$ for $B(q^{-1})$, as before. In practice, it may not be desirable, however, to cancel all closed-loop poles and stable plant zeros. Canceling all closed-loop poles and stable plant zeros results in a compensator of higher order, when in fact certain existing poles and zeros may desirable for the feedforward controller.

Figure 2.8: The tracking problem

### 2.2.2 Zero-phase-error tracking

When the relative degree of a minimum-phase discrete-time plant $G$ is zero, it is possible to invert it exactly to obtain perfect tracking. When the relative degree is positive, this can also be done, but only approximately. When non-minimum phase zeros are present, however, this is no longer possible. Nevertheless, as shown in [26], with the use of preview information it is possible to obtain zero phase error tracking.

As in Section 2.2.1, it is assumed that plant poles and stable plant zeros have been canceled, and only a FIR plant $B(q^{-1})$ remains. If a compensator $B(q)$ is used, the resulting frequency response is

$$B(q)B(q^{-1})_{q=e^{j\omega h}} = B(e^{j\omega h})B(e^{-j\omega h}) = |B_r(\omega) - jB_i(\omega)||B_r(\omega) + jB_i(\omega)|$$

$$= B_r^2(\omega) + B_i^2(\omega)$$

where $B_r(\omega)$ and $B_i(\omega)$ are the real and imaginary parts, respectively, of the frequency response of $B(q^{-1})$. Since this frequency response is real, the phase angle is zero and the phase tracking is perfect. This compensator, however,
will in general not be causal. Thus, it is necessary to divide the compensator by $q^\Delta$, where $\Delta$ is the degree of $B$. After adjusting for unit DC gain, the compensator

$$\frac{B(q)}{q^\Delta B(1)^2}$$

is obtained. By dividing by $q^\Delta$, the phase error is no longer zero. However, if the reference signal is provided $\Delta$ samples early ($y_r = q^\Delta \hat{y}_r$), the phase error will be zero with respect to $\hat{y}_r$. Thus, this is a preview compensator.

### 2.2.3 Preview in the regulation setting

In the regulation setting, one considers the external disturbances that a generalized plant is subject to, the goal being to reject these disturbances by choosing an appropriate controller. The diagram of Figure 2.9 presents a generalized plant $G$ with two inputs $w$ and $u$, and two outputs $z$ and $y$. The signal $w$ represents external plant disturbances, and the signal $y$ is a signal that is available for measurement by a controller $K$. The controller can apply inputs to the plant via the signal $u$, and the goal of the controller is to make the signal $z$ small in some sense.

![Figure 2.9: Generalized plant $G$ with controller $K$](image)

The analogous interpretation of preview, in the regulation setting, is that it consists of information about a perturbation that becomes available to the controller before it affects the plant. This is shown in Figure 2.10, where MUX is a multiplexer that stacks the two input signals into a vector, $\Delta$ is
the number of samples of available preview, and the dashed line represents a new generalized plant. In [16], this interpretation of preview is used to find a solution to the closed-loop $\mathcal{H}_\infty$ preview problem.

Figure 2.10: Generalized plant $\hat{G}$ with preview-based controller $K$
Chapter 3

Open-loop preview-based compensation for FIR systems

In this chapter a geometric method for designing preview-based compensators for simple FIR systems is presented. A generalization of this material is presented in Chapter 5. This material is only presented for completeness, and the methods of Chapters 4 and 5 are recommended in practice for several reasons. First of all, cancellation of plant dynamics is not necessary. Using the methods of Chapter 7 to derive solutions, a much wider range of applicable norms is available. Furthermore, the reference signal is parameterized in a way that makes the methods much more general.

3.1 Notation

The following notation is specific to this chapter: the polynomials

\[ B(q^{-1}) = b_0 + b_1q^{-1} + \ldots + b_mq^{-m} \]

and

\[ C(q^{-1}) = c_0 + c_1q^{-1} + \ldots + c_{n-m-1}q^{-(n-m-1)} \]
will be used, respectively, to represent the zeros of the system which cannot be canceled, and a compensator. \( \phi(q^{-1}) := B(q^{-1})C(q^{-1}) \) will denote the compensated system. To apply these results to general systems (non FIR), the method of Section 2.2.1 can be applied. In that case, \( B(q^{-1}) \) is the polynomial transfer function shown in Figure 2.8.

To simplify calculations, vectors will often be used instead of polynomials. Clearly, \( \mathbb{R}_{<n}[x] \) (the vector space of polynomials in \( x \) of order \( n-1 \) over the field \( \mathbb{R} \)) is isomorphic to \( \mathbb{R}^n \) considered as a vector space. Throughout this chapter any polynomial

\[
A(q^{-1}) = a_0 + a_1q^{-1} + \ldots + a_{n-1}q^{-(n-1)}
\]

will be identified with the vector

\[
A := [a_0 \ a_1 \ \ldots \ a_{n-1}]' \in \mathbb{R}^n.
\]

Thus \( B(q^{-1}), C(q^{-1}), \) and \( \phi(q^{-1}) \) will often be represented respectively by \( B, C, \) and \( \phi, \) vectors in \( \mathbb{R}^{m+1}, \mathbb{R}^{n-m} \) and \( \mathbb{R}^n. \)

The norm of a vector will be written as \( \|x\| \). If \( A \) is a matrix, the semi-norm \( \|x\|_A \) will be defined as \( \|Ax\| \). If the nullspace of \( A \) is empty, \( \|\|_A \) is also a norm. In what follows we will call \( \|x\|_A \) a scaled norm with respect to the reference norm \( \|\| \), and the scaling factor \( A. \)

The Gateaux differential of \( f \) at \( x \) in direction \( \hat{x} \) will be denoted by \( \delta f(x; \hat{x}) \). The unit impulse signal, which otherwise is denoted by \( \delta(k) \), will not be used in this chapter.

## 3.2 Approximating a pure delay

As discussed in Section 2.2.1, the preview problem can be solved by designing a compensator to track a pure delay, and then providing the reference signal in advance to cancel that delay. In this section, a geometric approach to designing a compensator \( C \) for a plant \( B \) is presented. The approach is based on delay tracking. The problem can be formulated as follows:

**Problem 1:** Find the compensator \( C(q^{-1}) \) so that, for a given integer \( \Delta \), the compensated system \( \phi(q^{-1}) = C(q^{-1})B(q^{-1}) \) behaves
3.2. APPROXIMATING A PURE DELAY

like the $\Delta$-sample delay $q^{-\Delta}$, the system that delays its input signal by $\Delta$ samples. $\phi$ is equal to $q^{-\Delta}$. This new definition is used to permit the bold-typed variation of $\phi$ in what follows. It is assumed that $\Delta < n$.

In order to quantify what it means for $\phi(q^{-1})$ to behave like the $\Delta$-sample delay, which will be called $d_{\Delta}(q^{-1})$, a norm that will measure closeness of $\phi(q^{-1})$ and $d_{\Delta}(q^{-1})$ is introduced. To write this problem as a minimum norm problem, it will also be necessary to define the operator

$$T : \mathbb{R}_{<(n-m)}[x] \longrightarrow \mathbb{R}_n[x], \quad TC(q^{-1}) = C(q^{-1})B(q^{-1}).$$

From this definition there is a natural induced operator $T : \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$ which satisfies the following commutative diagram:

Clearly,

$$TC(q^{-1}) = C(q^{-1})B(q^{-1})$$
$$= b_0c_0 + (b_1c_0 + b_0c_1)q^{-1} + \ldots + (b_mc_{n-m-1})q^{-(n-1)}.$$
Hence $T$ can be represented explicitly as:

$$
T = \begin{bmatrix}
    b_0 & 0 & \ldots & 0 & 0 \\
    b_1 & b_0 & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \ddots & 0 & \vdots \\
    b_m & \vdots & b_0 & 0 \\
    0 & b_m & b_1 & b_0 \\
    \vdots & 0 & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \ldots & 0 & b_m
\end{bmatrix} \in \mathbb{R}^{n \times (n-m)}
$$

The above problem may now be solved by finding the compensator $C$ such that $\phi = TC$ is closest to $d_\Delta$ with respect to a certain norm. This is represented in Figure 3.1, where the problem now is to find the point in the linear variety $\text{Im}(T) \cap J$ that is closest to $d_\Delta$. $J$ is a hyperplane which represents the unity DC gain constraint.

The norm chosen for the above minimization problem depends on the tracking goals. In the above example, the goal was to ensure that the system would follow a step reference signal relatively well (with little undershoot). So in that
case it makes sense to choose a norm which results in a compensated system that has a step response that is close to the step response of a delay. In some instances, one might choose a norm such that the ramp response of the system would be close to the ramp response of a delay.

In addition to choosing a norm, certain constraints might be necessary. For example, when following a reference signal in the form of a step, one would clearly like the DC-gain of the compensated signal to be equal to one.

Note that if we consider the simple non-minimum phase system $B(q^{-1}) = 1 - zq^{-1}$ where $z > 1$, we have $b_0 = 1$ and $b_1 = -z$. It is well known that no stable compensator can make this system have a monotone step response. In the framework presented here, this characteristic has a very simple proof, which is derived below.

Lemma 1 If $B(q^{-1}) = 1 - zq^{-1}$ where $z > 1$, a FIR compensator providing a monotone step response does not exist.

**Proof** By linearity and time-invariance, a step response of an LTI system is monotone if and only if the impulse response of that system is non-negative. But the first $n$ elements of the impulse response of $\phi(q^{-1})$ are obviously the elements of the vector $\phi$, and the others are zero. So if there exists some $C$ such that $TC$ is in the first orthant, one obtains a solution with a monotone step response. But for this simple case it is easy to see that $\text{Ker}(T') = [z^{-n} \ldots z^{-1}]'$ so that the image of $T$ is orthogonal to a vector in the first orthant and so no element of the range space can be inside the first orthant, proving that a FIR compensator providing a monotone response does not exist. As long as $z > 0$, a FIR compensator which provides a monotone step response does not exist. As long as $z > 0$, a FIR compensator which provides a monotone step response does not exist.

Note that if $z \leq 0$, $\text{Ker}(T')$ leaves the first orthant, opening up the possibility of a compensated system with monotone step response. When one is not limited to FIR compensators, it is possible to cancel zeros such that $0 < z \leq 1$. The region $z > 1$, however, remains a problem region.
3.3 Norms and constraints

In order to construct a system that approximates a pure delay, it is necessary to define realistic norms and constraints that can be used to solve the problem proposed above.

3.3.1 Step response

Assume that the input generally changes in steps. Given a compensated system \( \phi(q^{-1}) = \phi_0 + \phi_1 q^{-1} + \ldots + \phi_{n-1} q^{-(n-1)} \) at rest, it is simple to see that the step response \( y(k) \) of \( \phi(q^{-1}) \) is

\[
\begin{align*}
y(0) &= \phi_0 \\
y(1) &= \phi_0 + \phi_1 \\
y(2) &= \phi_0 + \phi_1 + \phi_2 \\
& \vdots \\
y(n-1) &= \phi_0 + \ldots + \phi_{n-1} \\
y(n) &= \phi_0 + \ldots + \phi_{n-1} \\
y(n+1) &= \phi_0 + \ldots + \phi_{n-1} \\
& \vdots 
\end{align*}
\]

The step response of \( d_{\Delta}(q^{-1}) \) is

\[
y_{\Delta}(k) = 1(k - \Delta) = \begin{cases} 
0 & k \leq \Delta - 1 \\
1 & k \geq \Delta 
\end{cases}
\]

Define

\[
A_s := \begin{bmatrix} 
1 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
& & \ddots & \vdots \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 
\end{bmatrix} \in \mathbb{R}^{n \times n}
\]

It is not difficult to see that the first \( n \) samples of \( y(k) \) are the elements of the vector \( A_s \phi \), and the first \( n \) samples of \( y_{\Delta}(k) \) are the elements of \( A_s d_{\Delta} \). If \( \phi(q^{-1}) \) is constrained to have unit DC-gain, \( \phi_0 + \ldots + \phi_{n-1} = 1 \), and all of the remaining samples of \( y_{\Delta}(k) \) and \( y(k) \) will be equal.


\(\phi(q^{-1})\) and \(d_\Delta(q^{-1})\) will have step responses that are close with respect to the reference norm if \(\|A_x\phi - A_xd_\Delta\|\), or \(\|\phi - d_\Delta\|_{A_x}\) is small. \(A_x\) has full rank so \(\|\cdot\|_{A_x}\) is a norm.

### 3.3.2 Ramp response

Assume that the system \(\phi(q^{-1})\) is at rest, and a ramp of slope \(\sigma\), \(r(k) = (k\sigma)1(k)\) = \(\{0, \sigma, 2\sigma, 3\sigma, \ldots\}\), is applied as the input. Since \(y(k) = \phi_0 r(k) + \phi_1 r(k-1) + \ldots + \phi_{n-1} r(k-(n-1))\), one easily finds

\[
\begin{align*}
 y(0) &= \phi_0 r(0) = 0 \\
 y(1) &= \phi_0 r(1) + \phi_1 r(0) = \phi_0 \sigma \\
 y(2) &= (2\phi_0 + \phi_1)\sigma \\
 y(3) &= (3\phi_0 + 2\phi_1 + \phi_2)\sigma \\
 \vdots \\
 y(n-1) &= ((n-1)\phi_0 + (n-2)\phi_1 + \ldots + \phi_{n-2})\sigma \\
 y(n) &= y(n-1) + (\phi_0 + \ldots + \phi_{n-1})\sigma \\
 y(n+1) &= y(n-1) + 2(\phi_0 + \ldots + \phi_{n-1})\sigma \\
 \vdots \\
 y(n-1 + j) &= y(n-1) + j(\phi_0 + \ldots + \phi_{n-1})\sigma
\end{align*}
\]

The ramp response of \(d_\Delta(q^{-1})\) is

\[
y_\Delta(k) = (k - \Delta)(\sigma)1(k - \Delta) = \begin{cases} 0 & k \leq \Delta \\ (k - \Delta)\sigma & k > \Delta \end{cases}
\]

Define the ramp following error \(e_\Delta(k) := y(k) - y_\Delta(k)\). Then we have:

\[
e_\Delta(n-1 + j) - e_\Delta(n + j) = (\phi_0 + \ldots + \phi_{n-1} - 1)\sigma \quad \forall j \geq 0.
\]

Obviously it is necessary that the DC-gain of \(\phi(q^{-1})\) be equal to 1. Otherwise, the ramp-following error will increase with time. Thus, we assume that \(\phi(q^{-1})\) has unit DC-gain, and we derive, for all \(j \geq 0\),

\[
e_\Delta(n-1 + j) = y(n-1) - n\sigma + \Delta \sigma.
\]
3. OPEN-LOOP PREVIEW-BASED COMPENSATION FOR FIR SYSTEMS

The error can be eliminated by setting it to zero:

\[ 0 = e_\Delta (n - 1) = y(n - 1) - n\sigma + \Delta\sigma \]

which implies that

\[ (n - \Delta)\sigma = y(n - 1) = (n\phi_0 + (n - 1)\phi_1 + \ldots + \phi_{n-1})\sigma \]

or in other words:

\[ n - \Delta = n\phi_0 + (n - 1)\phi_1 + \ldots + \phi_{n-1} \]

Thus, with the following constraint

\[ \begin{bmatrix} n & n - 1 & \ldots & 2 & 1 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \phi = \begin{bmatrix} n - \Delta \\ 1 \end{bmatrix} \]

the system output is guaranteed, in steady-state, to follow the ramp with exactly \( \Delta \)-sample delay. Note that this constraint is independent of the slope \( \sigma \).

In addition, note that \( \phi(q^{-1}) \) and \( \Delta(q^{-1}) \) will have transient ramp responses that are close with respect to the reference norm if \( \| A_r \phi - A_r \Delta \| \), or \( \| \phi - \Delta \|_{A_r} \) is small where \( A_r \) is the nonsingular matrix:

\[ A_r := \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 2 & 1 & 0 & \ldots & 0 \\ 3 & 2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ n - 1 & n - 2 & \ldots & 1 & 0 \\ n & n - 1 & \ldots & 2 & 1 \end{bmatrix} \]

3.3.3 Sinusoidal excitation

Assume that it is necessary for the compensated system to follow a sinusoid of angular frequency \( \omega \) with no tracking error. Here the constraint on \( \phi \) that will result in a compensator with zero steady-state tracking error is derived.

An appropriate constraint is:

\[ \phi(q^{-1})|_{e^{j\omega n}} = \Delta(q^{-1})|_{e^{j\omega n}} \]
3.4 Problem solutions

In other words,
\[ \phi_0 + \phi_1 e^{-j\omega h} + \ldots + \phi_{n-1} e^{-j\omega h(n-1)} = e^{-j\omega h\Delta} \]
which can be written as the following linear constraint:
\[ \begin{bmatrix} 1 & e^{-j\omega h} & \ldots & e^{-j\omega h(n-1)} \end{bmatrix} \phi = e^{-j\omega h\Delta} \]

In order to guarantee that a solution with real coefficients is obtained, the same constraint must be applied for negative frequencies:
\[ \phi(q^{-1})|_{e^{-j\omega h}} = q^{-\Delta}|_{e^{-j\omega h}} \]

This results in the constraint:
\[ \begin{bmatrix} 1 & e^{j\omega h} & \ldots & e^{j\omega h(n-1)} \end{bmatrix} \phi = e^{j\omega h\Delta} \]

If these constraint equations are added and subtracted from one another, the following equivalent constraint is obtained:
\[ \begin{bmatrix} 1 & \cos(\omega h) & \ldots & \cos((n-1)\omega h) \\ 0 & \sin(\omega h) & \ldots & \sin((n-1)\omega h) \end{bmatrix} \phi = \begin{bmatrix} \cos(\Delta\omega h) \\ \sin(\Delta\omega h) \end{bmatrix} \]

Compared to the results in [33], this gives lower order compensators, but only cancels the phase and gain error for a specific frequency, as opposed to eliminating the phase error for all frequencies and assigning a weighting function to the gain error.

3.4 Problem solutions

The problem formulation can now be given more precisely.

**Problem 2:** Given \( T \in \mathbb{R}^{n \times (n-m)} \), a scaling factor \( A \in \mathbb{R}^{n \times n} \), a constraint matrix \( W \in \mathbb{R}^{p \times (n-m)} \), a constraint vector \( w \in \mathbb{R}^p \), and the delay factor \( \Delta \in \{1, \ldots, n-1\} \), find a compensator \( C \in \mathbb{R}^{n-m} \) such that \( f(C) := \|TC - d\Delta\|_A \) is minimized subject to the constraint that \( g(C) := WC - w = 0 \).
Note that $A$ is a function of the type of signal we would most closely like to follow. $W$ and $w$ are determined by the constraints such as the DC-gain, ramp following, and sinusoid following constraints. The choice of the reference norm will be discussed later.

What is interesting here is that this problem formulation simply says that a compensator $C$ is desired such that some measure of the error between the output and a pure delay is minimized. Looking back at Chapter 2, it is clear that this is a formulation of a preview-based controller design problem.

3.4.1 $\ell_2$ case

If the $\ell_2$ norm is chosen as the reference norm, an explicit solution may be found for problem 2.

Define

$$J := \{ C \mid WC = w \}$$

$$M^* := \text{Im}(T) \cap J$$

If $M^*$ is not empty, then it must be a subspace. $\|x\|_A = \|Ax\|_2 = \sqrt{x^T A^T A x}$, so the scaled norm is induced by an inner product, and from the classical projection theorem [21], there is a unique solution $C$.

Define $\hat{T} := AT$, and $\hat{d} = Ad_\Delta$, where the delay factor $\Delta$ is understood. The following proposition gives necessary conditions that the unique solution must satisfy:

Lemma 2 If $M^*$ is not empty and $W$ has full rank, then there is a solution $\lambda$ to the equation

$$W(\hat{T}'\hat{T})^{-1}W'\lambda = 2W(\hat{T}'\hat{T})^{-1}\hat{T}'\hat{d} - 2w$$

and the $C$ which solves Problem 2, with the $\ell_2$-norm as reference norm, is given by

$$C = (\hat{T}'\hat{T})^{-1}(\hat{T}'\hat{d} - \frac{1}{2}W'\lambda).$$

Proof We already know that a solution exists. Necessary conditions for the existence of a solution are derived below, and these necessary conditions will
3.4. PROBLEM SOLUTIONS

imply the existence of an appropriate $\lambda$, and provide the coefficients for the optimal compensator. Writing out the norm explicitly, one obtains:

\[ f(C) = (\hat{T}C - \hat{d})(\hat{T}C - \hat{d}) \]
\[ g(C) = WC - w \]

Using the definition of the Gateaux differential

\[ \delta T(C; h) = \lim_{\alpha \to 0} \frac{1}{\alpha}(T(C + \alpha h) - T(C)) \]

it is simple to derive:

\[ \delta f(C; h) = 2h^T(\hat{T}C - \hat{d}) \]
\[ \delta g(C; h) = Wh \]

For the problem to be regular, it is simple to derive that the rows of $W$ must be linearly independent.

Given regularity, the necessary condition for extrema [21] is that $\delta f(C; h) + \delta g(C; h)\lambda = 0 \ \forall h$. This means

\[ 2h^T(\hat{T}C - \hat{d}) + h'W'\lambda = 0 \ \forall h \]

which clearly implies

\[ 2\hat{T}(\hat{T}C - \hat{d}) + W'\lambda = 0 \]

which leads to:

\[ \hat{T}^T C = \hat{T}^T \hat{d} - \frac{1}{2} W' \lambda \]

and therefore

\[ C = (\hat{T}^T \hat{T})^{-1}(\hat{T}^T \hat{d} - \frac{1}{2} W' \lambda) \quad (3.1) \]

The constraint $g(C) = 0$ is then written as:

\[ WC = W(\hat{T}^T \hat{T})^{-1}(\hat{T}^T \hat{d} - \frac{1}{2} W' \lambda) = w \]

which yields

\[ W(\hat{T}^T \hat{T})^{-1} W' \lambda = 2W(\hat{T}^T \hat{T})^{-1} \hat{T}^T \hat{d} - 2w \]

from which $\lambda$ can be found. Substitution of $\lambda$ into Equation 3.1 yields the optimal $C$. \qed
3. OPEN-LOOP PREVIEW-BASED COMPENSATION FOR FIR SYSTEMS

3.4.2 \( \ell_\infty \) case

With the \( \ell_\infty \) norm as the reference norm, the problem can be solved for numerically. The cost function \( f(C) \), being a scaled and shifted norm on a subspace of \( \mathbb{R}^n \), is convex [24, Functional Operations]. If desired, the \( \ell_2 \) norm solution can be used to establish initial conditions for the \( \ell_\infty \) problem.

3.5 Example

The undershoot minimization problem of Section 2.2.1 can be solved approximately by applying Lemma 2. Again, \( B(q^{-1}) = 1 - zq^{-1} \), where \( z = 1.2 \). Instead of seeking to minimize undershoot, it is straightforward to approximate a pure delay by choosing \( A = A_s, W = [1, \ldots, 1]^T \) and \( w = 1 \). Applying Lemma 2, \( \lambda = -0.5892 \) is obtained, with the results in Figure 3.2. The behavior without the compensator is plotted for comparison. The results are suboptimal compared to direct undershoot minimization. This is due to the use of the \( \ell_2 \)-norm.

![Figure 3.2: Step response of \( C(q^{-1})B(q^{-1}) \).](image)

If, instead, one solves Problem 2 numerically, with the \( \ell_\infty \) norm as reference norm, or using methods such as those in [8], the same results as in Section 2.2.1 are obtained.
3.6 Remarks

The $\ell_\infty$ solution tends to give a smoother response than the $\ell_2$ solution. The principal advantage to the $\ell_2$ case is that an analytic solution for any number of non-minimum phase zeros is available. In the case where there is one non-minimum phase zero, the $\ell_\infty$ solution is the same as the solution in [12]. Thus, in this chapter one sees that the problem of designing a compensator that minimizes the maximum undershoot for a simple system with one non-minimum phase zero can be generalized. And the way to generalize the problem is to solve the preview design problem for some error function. This will be discussed further in chapters 4 and 5.

One weakness of the results in this chapter is that certain results must be derived for every reference signal that is used. Also, the results here are only optimal for the FIR plant. If the method is used to design a compensator for a closed-loop plant, the command signal of that plant is not considered at all in the optimization process. Thus, this method, like the methods in [9, 10, 12, 26, 33], may result in excessive command behavior. This is rectified in chapters 4 and 5.
Chapter 4

Two-degree-of-freedom preview-based controller

The RST, or two-degree-of-freedom, model-reference approach has become a popular method for designing simple controllers [2, 18, 20]. A reference model

\[ H_m(q) = \frac{B_m(q)}{A_m(q)} \]

is chosen, and a controller is found so that the closed-loop transfer function is equal to \( H_m(q) \). This simple method is based on classical control theory, and is intuitive for control system designers familiar with the PID approach. Separate tuning of the loop properties and the tracking properties is straightforward, and addition of internal models is intuitive. It is often possible to find an appropriate denominator polynomial \( A_m(q) \) by choosing closed-loop poles that lie within a region inside the unit circle with reasonable damping. Simple optimal approaches that permit one to choose the numerator polynomial \( B_m(q) \) are lacking, so control system designers often simply select an appropriately scaled polynomial consisting of the plant zeros which are unstable or poorly damped so that the closed-loop system is stable with unit D.C. gain. Although this method works for unit step reference signals, it is by no means optimal, and does not work for general reference signals like sinusoidal references or ramps. In this chapter, it is shown how to reduce the controller design method to a simple optimization problem that can be solved using the methods in
Chapters 6 and 7. The designer must still choose the closed-loop poles $A_m$. The closed-loop zeros, $B_m$, are placed to minimize a cost function. Thus, this approach is of the pole-placement type.

The method proposed in this chapter makes it possible to find an optimal controller that minimizes a cost function consisting of a weighted sum of terms penalizing the control action and tracking error resulting from a given reference signal. The optimization is performed over all controllers of fixed degree maintaining fixed closed-loop poles, and eliminating permanent tracking error. Various reference signals may be used, and the relationship between reference signal complexity and controller order is demonstrated. The number of samples of preview information is variable, and can be chosen to be zero to obtain optimal controllers that do not exploit preview.

Although preview has been studied extensively in [33, 12, 26], these methods which place zeros optimally with respect to a FIR system are not appropriate if a controller which cancels all plant poles and stable plant zeros is not desired. This is also true concerning the results in Chapter 3. As mentioned in Section 2.2.1, canceling all plant poles and stable zeros may needlessly increase compensator order. If one chooses not to cancel all plant poles, these methods lose their optimality. In addition, these methods may result in excessive control action. The methods presented in chapters 4 and 5 take into consideration closed-loop poles from the beginning, and since the actuator signal is part of the cost function, excessive control action can be avoided.

In [11] it is demonstrated that it is possible to add preview to a controller in order to achieve better tracking without necessarily separating minimum phase and non-minimum phase dynamics and designing two separate controllers. In [11] it is suggested that by choosing the $T$ polynomial that results in the shortest error response for a deadbeat system, a good response for the non-deadbeat system should occur. The problem to this approach is that control action may be high, and a short error response for a deadbeat system may decay slowly when system poles are present. Optimizing the true closed-loop error signal seems more appropriate.

This chapter is organized as follows: in Section 4.1 the standard two-degree-of-freedom controller is reviewed. In Section 4.2 the tracking error signal and
the set of *admissible* controllers are defined, and the set of all admissible controllers is parameterized. This parameterization can then be used to state a problem in the form of a constrained or unconstrained convex optimization problem that can be solved using the methods of Chapters 6 and 7. These methods will compute controllers which are optimal among all admissible controllers. In Section 4.3, extensions to the basic results are given that allow more flexibility with controller synthesis when preview information is available. Some examples that demonstrate the solution to various control problems are presented in Section 4.4, followed by a section with some concluding remarks.

Although the unit delay operator $q^{-1}$ was used in the previous chapter, in this chapter the advance operator $q$ will be used. When working with the advance operator, causality conditions can and must be specified in terms of polynomial degrees. When working with the delay operator, causality is enforced by the notation.

### 4.1 Standard two-degree-of-freedom controller design

The problem of designing a two-degree-of-freedom pole-placement controller (Figure 4.1) for a strictly proper plant $B(q)/A(q)$ is discussed in this section (see also [2, 20]). The general pole placement controller is of the form

$$R(q)u(k) = T(q)y_r(k) - S(q)y(k)$$

where $k \in \mathbb{Z}$ is the discrete time instant, $y_r(k)$ is the reference signal, $y(k)$ is the plant output, and $u(k)$ is the actuator signal.

![Figure 4.1: Two-degree-of-freedom control system](image)

It is simple to derive that $y(k) = \frac{BT}{AR+BS}y_r(k)$. $B$ may contain stable,
marginally stable, and unstable zeros. As in [2], factorize \( B(q) = B^+(q)B^-(q) \)
so that \( B^+(q) \) is the highest degree monic polynomial with stable, well-damped
zeros. \( R = B^+R_fR_0 \) is also defined, where \( R_f \) is a fixed part of \( R \) that may
be chosen, for example, to contain integrators. This results in
\[
y(k) = \frac{B^-T}{AR_fR_1'} + B^-S y_r(k).
\]
Assume that one would like the closed-loop characteristic polynomial to be
equal to \( A_oA_m \), where the closed-loop modes are specified by the stable poly-
nomial \( A_m \), and \( A_o \) is an observer polynomial that will be canceled by \( T \),
by defining \( T = T'A_o \). Equating denominators, one obtains the Diophantine
equation
\[
AR_fR_1' + B^- S = A_oA_m, \tag{4.1}
\]
an equation in the unknowns \( R_1' \) and \( S \), and
\[
y(k) = \frac{B^- T'}{A_m} y_r(k), \quad u(k) = \frac{AT'}{B^+A_m} y_r(k). \tag{4.2}
\]
By choosing \( R_f, A_o \) and \( A_m \) appropriately it is possible to calibrate the sen-
sitivity function to achieve robustness and disturbance rejection [19, 18, 2].
Adaptive approaches to determining suitable \( R \) and \( S \) polynomials can be
found in [1, 30, 20]. A multi-model design approach is presented in [34].
A solution set \( R_1', S \) to Equation (4.1) such that \( \delta S \leq \delta A + \delta R_f - 1 \) exists\(^1\)
under the assumption that \( A_mA_o \) be divisible by the greatest common divisor
of \( AR_f \) and \( B^- \). This assumption will almost always be satisfied because in
most cases \( AR_f \) and \( B^- \) are coprime. The above parameterization provides
many algebraic solutions to Equation (4.1). However, all of these solutions
may not result in a controller satisfying the causality requirement
\[
\delta R \geq \delta S.
\]
The following lemma indicates under what conditions this will be satisfied.
The causality condition
\[
\delta R \geq \delta T
\]
\(^1\)In fact a solution for \( S \) exists for which equality is achieved. See Theorem 10.3 of [20]
for proof.
4.2 TRACKING GOAL

will be discussed later.

**Lemma 3** If $\delta A_0 \geq 2\delta A - \delta A_m - \delta B^+ + \delta R_f - 1$ then $\delta R \geq \delta S$.

**Proof** From the way that $S$ was chosen,

$$\delta S \leq \delta A + \delta R_f - 1. \quad (4.3)$$

This, combined with the fact that the plant is strictly proper, gives us:

$$\delta (BS) \leq 2\delta A + \delta R_f - 2. \quad (4.4)$$

Equation 4.1, which implies that $AR + BS = A_0A_mB^+$, and the lemma hypothesis on $\delta A_0$ lead to:

$$\delta (AR + BS) = \delta B^+ + \delta A_0 + \delta A_m$$

$$\geq 2\delta A + \delta R_f - 1. \quad (4.5)$$

Inequalities (4.4) and (4.5) imply the following:

$$\delta (AR) > \delta (BS)$$

$$\delta (AR + BS) = \delta (AR)$$

$$\delta (AR) \geq 2\delta A + \delta R_f - 1$$

$$\delta R \geq \delta A + \delta R_f - 1.$$ 

Comparing this to Inequality (4.3), it is clear that $\delta R \geq \delta S$. \qed

**4.2 Tracking goal**

As in Chapter 2, the reference signal $y_r(k) = q^{\gamma_{1,}}y_r(k)$, and error signal

$$e(k) = q^{-\gamma_{1}}y_r(k) - y(k) \quad (4.6)$$

will be used, where $\gamma_1 = \delta A - \delta B$ is the plant delay. For simplicity in what follows, $\Delta$ will be defined to be equal to $\gamma_1 + \gamma$. A parameterization of controllers that eliminate permanent tracking error is given in this chapter, and a cost function making clear the trade-off between tracking performance and
control activity is proposed. With the error signal as given above, the preview tracking problem can be visualized as in Figure 4.2.

Suppose that the reference signal is generated by

\[ y_r(k) = \frac{B_c(q)}{A^+_c(q)A^-_c(q)} \delta(k) \]  

where \( A^+_c \) is composed of stable poles, and \( A^-_c \) is composed of unstable poles. The argument \( q \) has been dropped for notational convenience. Most useful reference signals can be generated in this way by consulting a table of \( Z \)-transforms. A step input can be generated, for example, by defining \( B_c = q \), \( A^+_c = 1 \), and \( A^-_c = q - 1 \). It is also possible to use \( A^+_c \) and \( B_c \) to low-pass filter the reference signal. See [29] for a more complete exposition of this type of two-degree-of-freedom tracking controller. From Equations 4.2, 4.6 and 4.7 the error signal can now be represented in the following way:

\[ e(k) = \left( \frac{A_m - q^\Delta B^- T'}{q^\Delta A_m} \right) \frac{B_c}{A^+_c A^-_c} \delta(k). \]

If a second Diophantine equation is solved,

\[ A^-_c P + q^\Delta B^- T' = A_m, \]  

then \( A^-_c \) will be canceled and the following equations representing the error
and actuator signals are obtained:
\[ e(k) = \frac{PB_c}{q^2 A_m A_c} \delta(k), \quad u(k) = \frac{A T'}{B A_m} y_r(k). \]  

(4.9)

The \( \mathcal{H}_2 \) system norm and \( \ell_2 \) norm of the impulse response are equivalent (see [7]), so one can write \( \| e(k) \|_2^2 = \left\| \frac{PB_c}{q^2 A_m A_c} \right\|_{\mathcal{H}_2}^2 \). Thus, in order to achieve good performance while limiting control action, the cost function \( J \) is defined simply as
\[ J = \alpha_1^2 \left\| \frac{PB_c}{q^2 A_m A_c} \right\|_{\mathcal{H}_2}^2 + \alpha_2^2 \left\| \frac{A T'}{B A_m} \right\|_{\mathcal{H}_2}^2. \]

A controller is called admissible if \( \lim_{k \to \infty} e(k) = 0 \) and \( J \) is finite.

The poles of the transfer functions of Equation (4.9) are inside the unit circle. The final value theorem shows that \( e(k) \) converges to zero. Due to linearity, \( e(k) \) converges to zero exponentially and this controller is therefore admissible. Note that the \( \ell_2 \) norm of \( u(k) \) is not weighted directly, because generally it will not be finite.

Given \( \alpha_1 \) and \( \alpha_2 \), it is possible to achieve a compromise between maintaining low control excitation and high tracking precision by finding a controller which is admissible and such that \( J \) is small.

A solution set \( P_0, T'_0 \) to Equation (4.8) such that \( \delta T'_0 < \delta A_c^+ \) exists under the reasonable assumption that \( q^2 B^- \) and \( A_c^+ \) have no common zeros.

With no limitations on the degrees of \( T' \) and \( P \), it is known that all solutions to Equation (4.8) can be parameterized with respect to the polynomial \( Q \) by
\[ T' = T'_0 - A_c^- Q, \quad P = P_0 + q^2 B^- Q \]  

(4.10)

In our case, however, the degree of \( T' \) may be limited to ensure causality. The lemma below shows that it is nevertheless possible to parameterize all solutions that satisfy a degree condition on one of the dependent variables.

**Lemma 4** Let \( m \in \mathbb{N} \) and polynomials \( A, B, \) and \( C \) such that the greatest common divisor of \( A \) and \( B \) divides \( C \) be given. If the equation \( AX + BY = C \) possesses a solution set \((X, Y)\) such that the degree condition \( \delta Y \leq m \) is satisfied, then all solution sets to this equation that satisfy the degree condition may be parameterized by \((X + BQ, Y - AQ)\), where \( Q \) is allowed to vary over the set of all polynomials such that \( \delta Q \leq m - \delta A \).
Proof

Assume that \( \delta Q \leq m - \delta A \). Then \( \delta(Y - AQ) \leq \max(\delta Y, \delta A + \delta Q) \leq m \). So, \((X + BQ, Y - AQ)\) clearly satisfies the equation \( AX + BY = C \) and the degree condition. Conversely, let \((X_2, Y_2)\) be another solution set with \( Y_2 \) satisfying the degree condition. Then by standard results [31], some polynomial \( Q \) exists such that \( X_2 = X + BQ \) and \( Y_2 = Y - AQ \). Then \( \delta AQ = \delta(Y - Y_2) \leq m \), so \( \delta Q \leq m - \delta A \). □

The relative degree of \( S/R \) has already been determined to be non-negative. However, certain values of the degree of the parameter \( Q \) may result in a negative relative degree of \( T/R \). Application of Lemma 4 shows that if a solution to Equation (4.8) exists such that \( \delta T \leq \delta R \), which is equivalent to \( \delta T' \leq \delta R - \delta A_o \), then all solutions satisfying this degree condition may be obtained through the parameterization of Equation 4.10, where

\[
\delta Q \leq \delta R - \delta A_o - \delta A_c. \tag{4.11}
\]

If such a solution does not exist, it is possible to increase the order of the polynomial \( R \).

It is now possible to write the cost function \( J \) as a function of \( Q \):

\[
J(Q) = \alpha_1 \left\| \frac{(P_0 + q^\Delta B^{-1} Q) B_c}{q^\Delta A_m A_c} \right\|_{H_2}^2 + \alpha_2 \left\| \frac{A(T_0 - A_c^{-1} Q)}{B^+ A_m} \right\|_{H_2}^2. \tag{4.12}
\]

For each \( Q \) the controller is admissible. Therefore, the method of Chapter 6 can be used to find a polynomial \( Q^* \) so that \( J(Q^*) = \inf_Q J(Q) \), solving the controller synthesis problem. If the controller designer wishes to use a more general cost function, the methods of Chapter 7 can also be used.

Notice how this is an unconstrained optimization problem, where in Chapter 3 the problem is constrained. The results of Chapter 3 can be obtained using the methods of this chapter by specifying \( \Delta = n - 1, R = q^{n-m-1}, S = 0, A = q^m, B(q) = B(q^{-1}) * q^m, B^+ = R_f = A_o = 1 \).

Note that the preview \( \Delta \) appears in the definition of the cost function \( J(Q) \). So if all controller polynomials are fixed to be of a certain degree, it
is nevertheless possible to apply the reference signal in advance, and define $\Delta$ appropriately, to take advantage of preview information.

### 4.3 Increasing design freedom with non-causal controller

Note that the degree of $Q$ is a measure of the amount of design freedom that is available to improve performance. More complicated reference signals are reflected by $A_c^-$ having higher degree, which reduces the degree of $Q$ (see Equation 4.11). It is possible to increase the degree of $Q$ by increasing the controller order.

Note that the preview factor $\Delta$ doesn’t affect controller structure in any way. $\Delta$ only results in a time shift of the reference signal that is used in computing the cost function. Choosing a large value of $\Delta$ tunes the optimization procedure so that it attempts to find an appropriate optimal $Q$. But this may not be possible without making changes to the degrees of the controller polynomials.

The papers [33, 12, 26] are based on the principle that by prepending a controller with a FIR filter (which effectively adds zeros to the controller) and using open-loop preview it is possible to improve tracking performance of non-minimum phase systems. These results are encouraging, but do not consider denominator dynamics or actuator behavior. Here a similar structural change by which one can achieve similar results, while also taking into consideration the pole dynamics, is proposed. This is done by adding additional zeros to the controller by increasing the order of the $T$ polynomial so that its degree may be larger than the degree of $R$. Obviously this violates causality, but if additional preview information is available this is not a problem. By slight modification (see Equation 4.11),

$$\delta T \leq \delta R + \kappa,$$

may be ensured by choosing

$$\delta Q \leq \delta R - \delta A_o - \delta A_c^- + \kappa$$
Thus if an additional $\kappa$ samples of preview information are available, the
degrees of $T$ and $Q$ can be increased by $\kappa$, resulting in more design freedom.

The choice of $\kappa$ has no theoretical limitation. As in [12], performance
tends to increase as $\kappa$ increases. However, as $\kappa$ increases, sensitivity to plant
perturbations also increases. This does not result in instability, since only
$R$ and $S$ affect loop properties. But it will affect performance, so there are
practical limits to the choice of $\kappa$. $\gamma$ generally should be smaller or equal to
the order of the controller. This makes sense because a low order controller
cannot behave like a long time delay.

4.4 Examples

In order to find a controller, it is necessary to choose weighting factors $\alpha_1$
and $\alpha_2$, and minimize the cost function 4.12 to find $Q$. Substituting $Q$ into
Equation 4.10, one obtains $T$, completing the design.

This procedure will be shown here in a simple example. An extensive ex-
ample will be given in Chapter 7.7. In this example, the methods in Chapter 6
are used to find a numerical minimum to the cost function 4.12.

The third example demonstrates how to apply the results of Section 4.3.

Example 1 With sampling period $h = 0.2$, the zero-hold discretization of the
simple second order system $1/(s(s - 1)(s + 4))$ gives $A = q^3 - 2.6707q^2 +
2.2195q - 0.5488$, $B^+ = 1$, $B^- = B = 0.0012q^2 + 0.0041q + 0.00086$. $R$ and $S$
are chosen to move the discrete-time open-loop poles from $\{1, 0.4493, 1.2214\}$
to $0.7, 0.7, 0.6 + 0.1i, 0.6 - 0.1i$, using a deadbeat observer polynomial $A_o = q$.
A step input is chosen as the reference signal, with $\gamma = 0$, assuming that no
preview information is available. If Theorem 2 is used to minimize $J(Q)$, and
the actuator signal is not weighted, the following controller, with $Q^* = -19.8$,
is obtained: $R = q^2 - 0.0244997q + 0.0555124$, $S = 81.818208q^2 - 114.59664q +
35.288271$, $T = 19.774054q^2 - 17.264213q$. The signals $y$, $y_c$ and $u$ are shown
in Figure 4.3. From this figure, one sees that the output responds quickly to the
change in reference signal, but the actuator signal $u(k)$ might be too aggressive.
The behavior of the controller which one obtains with $\alpha_1^2 = 15$ and $\alpha_2^2 = 1$
is shown in Figure 4.4. $R$ and $S$ are the same as before, but $Q^* = -10.5$,
Figure 4.3: Optimal step responses and actuator signal $\alpha_1^2 = 1, \alpha_2^2 = 0$

and $T = 10.5q^2 - 8q$. Note that although $Q$ has only one degree of freedom, completely different behavior results from this optimization.

**Example 2** Using the same process as above, assume that it is necessary to track a sinusoidal reference signal. Without preview (Figure 4.5), the initial
tracking error is significant. The order of $R$ is not high enough to provide any design freedom for optimization. Therefore, $\kappa = 2$ is used with two samples of preview information, resulting in considerable tracking improvement (Figure 4.6). Note that the plant output $y$ responds to the sinusoid before the reference.
signal arrives. This demonstrates that the use preview results in anticipatory action, which is possible since the reference signal actually is known and provided to the controller in advance.

Figure 4.5: Optimal sin responses and actuator signal; $\alpha_1^2 = 15, \alpha_2^2 = 10, \gamma = 0, \kappa = 0$
Example 3 With sampling period $h = 1$, the zero-hold discretization of the simple non-minimum phase system $\frac{(s - 0.2)(s - 0.4)^2}{(s - 0.2)}$ gives $B(q) = 1.3607q - 1.6631$, $A(q) = q^2 - 2.9836q + 2.2255$. $R$ and $S$ are chosen to move the discrete-
time open-loop poles to $0.6, 0.5 + 0.1i, 0.5 - 0.1i$, using observer polynomial $A_0 = 1$. A step input is chosen as the reference signal, and assuming that no preview information is available, $\gamma = 0$. Using Theorem 2 with weights $\alpha_1 = \alpha_2 = 1$, the following controller, with $Q^* = 0.2835$, is obtained: $R = q - 3.3281$, $S = -1.4290q + 4.5476$, $T = -0.2835q - 0.0604$. As seen in Figure 4.7, there is a considerable amount of undershoot. It is not possible to reduce the undershoot significantly by varying the $\alpha_i$ weights. Assuming that reducing tracking error is important and 12 samples of preview are available, $\gamma = 0$, and $\kappa = 12$ can be used as described above to get the improved results in Figure 4.8. Here, $Q^*$ is a polynomial of degree 12, $R = q + 3.3281$, $S = -1.4290q + 4.5476$, and $T$ is a polynomial of degree 13. Since $T$ is of higher degree than $R$, the system is not causal, but because of the availability of preview information this causes no problem. In addition, this may seem like high order control, but since $T$ is outside the loop, it behaves like a simple FIR signal prefilter, and doesn’t change loop behavior. In addition, as is seen in Figure 4.8, $T$ actually results in a much smoother actuator signal.

In [12], it is pointed out that the FIR compensator that minimizes undershoot for a single non-minimum phase zero plant is such that the zeros of the compensated system are spaced evenly around the origin. Although the situation presented here is more general, this property still holds approximately (see Figure 3), when the number of samples of preview is significant.
Figure 4.7: Non-minimum phase system with no preview; $\alpha_1^2 = 15, \alpha_2^2 = 15, \gamma = 0, \kappa = 0$
Figure 4.8: Non-minimum phase system with preview; $\alpha_1^2 = 15, \alpha_2^2 = 50, \gamma = 12, \kappa = 12$
4. TWO-DEGREE-OF-FREEDOM PREVIEW-BASED CONTROLLER
Chapter 5

Preview for arbitrary closed-loop systems

The definition of the preview-based error signal, the parameterization of admissible controllers, and the optimization methods which were used in the previous chapter can also be applied to arbitrary closed-loop systems, with some restrictions. This is the subject of this chapter.

A straightforward design method is proposed that allows a designer to use a preview-based optimal compensator in conjunction with an arbitrary SISO feedback controller. The compensator is designed to provide asymptotic tracking of a pre-specified reference signal, closed-loop poles are taken into consideration from the beginning so that the controller is truly optimal, and the actuator signal is included in the cost function so that excessive control action can be avoided.

Again, closed-loop system zeros are placed optimally, based on a cost functional. The zero optimization takes into consideration all closed-loop plant dynamics, and plant poles and stable zeros can be optionally be canceled.

5.1 Tracking goal

A plant and controller, $G$ and $K$, are assumed to be known (see Figure 5.1) so the transfer functions from $y_f$ to $u$, $T_{fu}(q)$ and from $y_f$ to $y$, $T_{fy}(q)$ are avail-
5. PREVIEW FOR ARBITRARY CLOSED-LOOP SYSTEMS

Nothing is assumed about the structure of the controller $K$, other than the availability of the above transfer functions. For example, $K$ can be a one-degree-of-freedom PID or $H_{\infty}$ controller, or a two-degree-of-freedom controller. $T_{fu}(q)$ and $T_{fy}(q)$ are assumed to be stable. Define $T_{fu}(q) = N_{fu}(q)/D_{fu}(q)$, and $T_{fy}(q) = N_{fy}(q)/D_{fy}(q)$. Polynomials in $q$ will be used here, so $K_c(q)$, to be designed, is defined as $K_n(q)/K_d(q)$, where $K_n(q), K_d(q) \in \mathbb{R}[q]$. If $K_d(q) = q^{n-1}$, then $K_c$ will be a standard FIR filter. It may, however, be desired to use some of the available poles in $K_d$ to cancel stable plant zeros. If $n$ is small, only a subset of the plant zeros may be canceled.

![Figure 5.1: Design of general compensator to approximate a pure delay](image)

Suppose that the reference signal is generated, as before, with Equation 4.7. The error signal can now be represented as:

$$ e(k) = \left[ \frac{1}{q^\Delta} - \frac{N_{fy}K_n}{D_{fy}K_d} \right] y_r(k) = \left[ \frac{D_{fy}K_d - q^\Delta N_{fy}K_n}{q^\Delta D_{fy}K_d} \right] \frac{B_c}{A_c^+ A_c^-} \delta(k). $$

If $A_c^-$ and $N_{fy}$ are coprime, then a solution set $(P_0, K_{n0})$ to the Diophantine equation

$$ A_c^- P + q^\Delta N_{fy} K_n = D_{fy} K_d \quad (5.1) $$

exists, and $A_c^-$ will be canceled. Thus, by the final value theorem, permanent error will be eliminated.

If $\delta A_c^- \leq n$, then it is possible to find a pair of solutions $(P_0, K_{n0})$ to Equation 5.1 such that $\delta K_n < n$. Then the set of all solutions $(P, K_n)$ that
eliminate permanent error and satisfy $K_n(q) \in \mathbb{R}_{<n}[q]$ can be parameterized by

$$\left\{ (P, K_n) \mid P = P_0 + q^2 N_{f_y}Q, \ K_n = K_{n0} - A_c^*Q, \ Q \in \mathbb{R}_{<n-\delta A_c}^n [q] \right\}$$

with the following equations representing the error signal $e(k)$ and transfer function $K_c$:

$$e(k) = \frac{(P_0 + q^2 N_{f_y}Q)B_c}{q^2 D_{f_y}K_dA_c^*} \delta(k)$$

(5.2)

$$K_c^Q = \frac{(K_{n0} - A_c^*Q)}{K_d}$$

(5.3)

$T_{\delta,e}^Q$ will be used to denote the transfer function in Equation 5.2, so $e(k) = T_{\delta,e}^Q$. Also, $T_{ru}^Q = K_c^Q T_{fu}$, and $T_{fy}^Q = K_c^Q T_{fy}$, which are the $Q$-dependent transfer functions from $y_r$ to $u$ and $y$, respectively, will be used.

To achieve performance goals, it is natural to search for a polynomial $Q$ that makes $T_{\delta,e}^Q$ small without using too much control action. This may be achieved by minimizing the function

$$J^Q = \alpha_1^2 f_{\delta,e}(T_{\delta,e}^Q) + \alpha_2^2 f_{ru}(T_{ru}^Q)$$

(5.4)

for some functionals $f_{\delta,e}$ and $f_{ru}$ on the set of rational proper stable transfer functions $\mathcal{RH}_\infty$, where $\alpha_1, \alpha_2 \in \mathbb{R}$.

A more general formulation can be made. Given a set of functionals $f_{i,j} : \mathcal{RH}_\infty \rightarrow \mathbb{R}$ with $i \in \{ \delta, r \}$ and $j \in \{ f, u, y, e \}$, corresponding weights $\alpha_{i,j} \in \mathbb{R}$, and affine (in $Q$) transfer functions $T_{i,j}^Q$, a suitable optimization goal is to minimize

$$J(Q) = \sum_{i \in I, j \in J} \alpha_{i,j} f_{i,j}(T_{i,j}^Q).$$

(5.5)

If desired, functionals $g_{i,j} : \mathcal{RH}_\infty \rightarrow \mathbb{R}$ can be defined and one may choose to minimize $J(Q)$, for instance, such that the constraints $g_{i,j}(T_{i,j}^Q) < 0$ are satisfied.

As in Chapter 4, for each $Q$ the controller is admissible. Therefore, the methods of Chapters 6 and 7 can be used to find a polynomial $Q^*$ so that $J(Q^*) = \inf_Q J(Q)$, solving the controller synthesis problem. Again, these results degenerate in a straightforward way to the results of Chapter 3.
5.2 Robust performance

Closed-loop controllers are often designed to be robust so that a small plant perturbation will not result in instability or a significant reduction in performance. Given a particular disturbance or a particular reference signal, a closed-loop controller can be easily designed so that it either follows the reference signal or rejects the disturbance in a robust way. That is, plant perturbations will not prevent the controller from asymptotically following the reference signal or rejecting the disturbance. This is called the internal model principle [32] because it is necessary that the controller contain an internal model describing the disturbance or reference signal in order for this robustness property to hold. If a control system is already robust in this sense, it is reasonable to ask whether the combined system will remain robust if the compensator proposed in this section is used. This question can be answered in the affirmative.

First, assume a reference signal $y_f$ is generated as in Chapter 4:

$$y_f = \frac{B_c}{A_c^* A_c} \delta.$$ 

Let a controller $K$ and a nominal plant $G_n$ be given such that the closed-loop system is stable. Define $y_n$ to be the plant output where $y_f$ is as above. In other words,

$$y_n = T_{G_n} f_y \frac{B_c}{A_c^* A_c} \delta.$$ 

If the plant is perturbed to $G_p$, define $y_p$ analogously:

$$y_p = T_{G_p} f_y \frac{B_c}{A_c^* A_c} \delta.$$ 

Assuming that $K$ provides robust performance, we have

$$\lim_{k \to \infty} y_p(k) - y_n(k) = 0$$

since plant perturbations do not affect asymptotic behavior.

The difference between the nominal and perturbed plant outputs of the system with optimal compensator $K_c$ can be written as:

$$e = T_{G_p} f_y \frac{B_c}{A_c^* A_c} K_c \delta - T_{G_n} f_y \frac{B_c}{A_c^* A_c} K_c \delta$$

(5.6)
\[ K_c(T_{fy} - T_{fy}^G) = \frac{B_c}{A_c} \delta \]  
\[ = K_c(y_p - y_n). \]  

Since \( K_c \) can be written as a FIR filter, \( e(k) \) is a finite weighted sum of shifted signals \( y_p(k) - y_n(k) \). Thus \( e(k) \) converges to zero, and the system with optimal compensator also provides robust performance.

### 5.3 Examples

#### Example 4

The plant \( G = (s + 0.5)/(s(s + 0.2)) \) is controlled, with \( h = 1 \), by an existing observer-based controller containing an internal model of the reference signal. The controller is designed to track a sinusoidal reference signal.

\[
T_{fu} = \frac{0.107q^4 - 0.294q^3 + 0.268q^2 - 0.0812q}{q^4 - 3.06q^3 + 3.59q^2 - 1.91q + 0.385}
\]

\[
T_{fy} = \frac{0.122q^3 - 0.187q^2 + 0.0682q}{q^4 - 3.06q^3 + 3.59q^2 - 1.91q + 0.385}
\]

Without preview (Figure 5.2), the initial tracking error is significant.

The control designer wishes to use preview on this system to attempt to improve tracking, without changing loop dynamics. \( K_d \) is chosen to cancel two zeros at 0.9263 and 0.6026, and \( J^Q = f_{\delta,e}(T^Q_{\delta,e}) \) is used as the cost function where \( f_{\delta,e} \) is the function that returns the \( \ell_\infty \) norm of the impulse response. The use of three samples of preview information \( n = \Delta = 3 \) results in considerable tracking improvement (Figure 5.3) with the compensator \( K_c(q) = (1.1q^2 - 1.7q + 0.7)/(q^2 - 1.5q + 0.56) \). Note that the plant output \( y \) responds to the sinusoid before the reference signal arrives. This is because the use preview results in anticipatory action, which is possible since the reference signal actually is known and provided to the controller in advance. The observer-based controller contains an internal model of the reference signal, and is therefore robust in the presence of plant perturbations. The designer wishes to know if the new system is still robust in the presence of plant perturbations, so the plant is perturbed to \( \hat{G} = 1.5(s + 0.7)/(s(s + 0.3)) \). In Figure 5.4 it is clear that the controller still eliminates permanent error, and although
the performance has diminished it is still better than the performance of the system without preview.
5.4 Remarks

A method has been presented in this chapter that allows simple design of preview-based compensators for existing closed-loop systems. The results of
Figure 5.4: Optimal sin responses and actuator signal; $n = 3$; perturbed plant

Chapter 7 make it possible for the designer to choose the optimization criteria necessary for the application in question, allowing various norms and performance measures on the system transfer functions to serve as constraints or weightings. The convex nature of these functionals, combined with the affine
nature of the parameterization of possible systems, results in a well-posed problem which can easily be solved using the ellipsoid method or the analytic results of Chapter 6.

The use of preview can be helpful in reducing tracking problems due to non-minimum phase zeros, as well as improving tracking performance for minimum phase systems. Unlike the methods presented in [33, 12, 26], the approach presented here takes plant zeros and control action into consideration.

Example 4 shows that the use of a low order compensator with little preview can result in significant performance improvement, while preserving robust performance characteristics of the closed-loop controller.
Chapter 6

Solution to affine $\mathcal{H}_2$ problem

The preview design methods in chapters 4 and 5 are not complete unless methods exist for solving the optimization problems that result from these methods. In this chapter, an analytic solution to a particular $\mathcal{H}_2$ problem is derived.

6.1 Analytic solution

In Chapters 4 and 5, a controller parameterization and cost function were elaborated. In this section, two theorems are presented, the second of which permits one to find the unique controller minimizing the cost functionals 4.12 and 5.5 when choosing the special case of the $\mathcal{H}_2$-norm.

In this chapter, $n$ will refer to the degree of the polynomial $Q$, unlike Chapter 5 where it refers to the degree of the polynomial $K_n$. $\mathcal{RH}_2$ denotes the set of strictly proper stable rational transfer functions with real coefficients. Since $\mathcal{RH}_2$ is a Hilbert space, the inner product can be written as a function of the norm. This easily derived formula is called the polarization identity [37]:

$$\langle X, Y \rangle = \left[ \|X + Y\|_{\mathcal{H}_2}^2 - \|X - Y\|_{\mathcal{H}_2}^2 \right] / 4. \quad (6.1)$$

The norm is easily calculated using standard state-space techniques. It is also
possible to write the inner product as
\[
\sum_{i=0}^{\infty} x_i y_i,
\]
where \( \{x_i\} \) and \( \{y_i\} \) are the impulse responses of the \( X(q) \) and \( Y(q) \). This summation exists since the impulse responses of stable linear systems converge to zero asymptotically. Using this summation to find the approximate inner product appears to be computationally less expensive than the use of the polarization identity and state-space \( \mathcal{H}_2 \) norm algorithms.

**Theorem 1** Given \( n \in \mathbb{Z} \) and stable rational functions \( V(q) \), and \( W(q) \) such that \( V(q) \) and \( q^n W(q) \) are strictly proper, and the impulse response of \( W(q) \) contains at least one non-null element, define
\[
Q = q_0 + q_1 q + \ldots + q_n q^n,
\]
\[
\bar{q} = [q_0, \ldots, q_n]^T,
\]
\[
\Psi(V, W) = \begin{bmatrix}
\langle V, W \rangle \\
\langle V, qW \rangle \\
\vdots \\
\langle V, q^n W \rangle
\end{bmatrix}, \text{ and}
\]
\[
\Phi(W) = \begin{bmatrix}
\langle W, W \rangle & \langle qW, W \rangle & \ldots & \langle q^n W, W \rangle \\
\langle W, qW \rangle & \langle qW, qW \rangle & \ldots & \langle q^n W, qW \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle W, q^n W \rangle & \langle qW, q^n W \rangle & \ldots & \langle q^n W, q^n W \rangle
\end{bmatrix}.
\]

Then the solution \( \bar{q} \) to the regular matrix inversion problem
\[
\Psi(V, W) = \Phi(W) \bar{q}
\]
exists and gives the unique polynomial solution \( Q^* \) of degree \( n \) to the minimization problem:
\[
\min_{Q} \| V(q) - W(q) Q(q) \|_{\mathcal{H}_2}^2.
\]

**Proof** \( V - WQ \) can be written as \( V - q_0 W - q_1 (qW) - q_2 (q^2 W) - \ldots - q_n (q^n W) \). The optimal solution is obtained through a simple projection of \( V \) onto the
linear subspace $\tilde{M} = \text{span}\{W, qW, \ldots, q^nW\}$. $\tilde{M}$ is a finite-dimensional subspace of the Hilbert space $RH_2$, and is therefore closed. By the classical projection theorem [21], an optimal solution $Q^*$ exists, and is unique. The unique minimizing solution $Q^*$ satisfies $(V - Q^*W) \perp \tilde{M}$, resulting in $\langle V, qW \rangle - q_0 \langle W, qW \rangle - q_1 \langle qW, qW \rangle - \ldots - q_n \langle q^nW, qW \rangle = 0 \ \forall i \in \{0, \ldots, n\}$. Each choice of $i$ yields one row of Equation 6.3.

In order to verify linear independence of the vectors $W, qW, \ldots, q^nW$, define $\tilde{w}_i = \{w_1, w_2, w_3, \ldots\}$ as the impulse response of $q^{n-i}W(q)$. Clearly, $\tilde{w}_i$ is the sequence $\tilde{w}_0$ delayed by $i$ samples, and starting with $i$ zeros. The space of impulse responses of elements of $H_2$ is isomorphic with $H_2$ if the inner product $\langle X, Y \rangle = \sum_{i=1}^{\infty} x_i y_i$, where $\{x_i\}$ and $\{y_i\}$ are the impulse responses of $X(q)$ and $Y(q)$, is used. Assuming that $\tilde{w}_0$ is not identically zero, consider the first non-zero element of $\tilde{w}_0$. It cannot be made zero by any linear combination of $\tilde{w}_i$, $i > 0$. So $\alpha_0 \tilde{w}_0 + \alpha_1 \tilde{w}_1 + \ldots = 0$ implies that $\alpha_0 = 0$. The first non-zero element of $\tilde{w}_1$ cannot be made zero by any linear combination of $\tilde{w}_i$, $i > 1$. So $\alpha_1 = 0$. This argument can be repeated, showing the linear independence of the finite set $\tilde{w}_i$. So the solution $Q^*$ is represented by the unique vector $\tilde{q}$, implying the invertibility of the gram matrix.

The $H_2$ norm of a system $G(q)$ is the same as the norm of the system $G(q)q^{-1}$. This is because the energy of the impulse response of a system is backward-shift invariant. If the impulse response of a system is shifted forward, its norm remains the same until the system becomes non-proper. At this point, it may no longer be possible to calculate the norm using the typical state-space methods. The impulse response of non-causal systems have finite energy which may nevertheless be calculated. Or one may simply calculate the $H_2$ norm by multiplying by $q^{-k}$ for some sufficiently large $k$. Thus, Theorem 1 may be used even when $V(q)$ and $q^nW(q)$ are not strictly proper. Simply multiply the denominators of $V$ and $W$ by $q^k$ for some sufficiently large $k$.

The next theorem provides the solution to a generalization of the problem of Theorem 1. Finding the solution $Q^*$ minimizing Equation (4.12) will be the main application.

**Theorem 2** Given $n, m \in \mathbb{Z}$, $\alpha_j \in \{R \setminus 0\}$ and stable rational functions $V^j(q), W^j(q), j \in \{1, \ldots, m\}$ such that $V^j(q)$ and $q^nW^j(q)$ are strictly proper.
for all $j$, and the impulse response of $W^j(q)$ contains at least one non-null element for some $j$, define $Q$, $q$, $\Psi$, and $\Phi$ as in Theorem 1. Then the solution $\bar{q}$ to the regular matrix inversion problem $\sum_{j=1}^m \alpha_j^2 \Psi(V^j, W^j) = (\sum_{j=1}^m \alpha_j^2 \Phi(W^j))\bar{q}$ exists and gives the unique polynomial solution $Q^*$ of degree $n$ to the minimization problem

$$\min_Q \sum_{j=1}^m \alpha_j^2 \|V^j(q) - W^j(q)Q(q)\|^2_{\mathcal{H}_2}. \quad (6.4)$$

**Proof** Theorem 2, like Theorem 1, is solved using the classical projection theorem. Here, however, the projection theorem is used on the space $H$ which is defined as the $m$-fold cartesian product of $\mathcal{R}H_2$: $H = \mathcal{R}H_2 \times \ldots \times \mathcal{R}H_2$. Using the definition of the inner product, induction, and by the completeness of the cartesian product of complete spaces ([15]), it is easily shown that $H$ is a Hilbert space when equipped with the inner product:

$$\langle (X_1, \ldots, X_m), (Y_1, \ldots, Y_m) \rangle_H = \alpha_1^2(X_1, Y_1) + \ldots + \alpha_m^2(X_m, Y_m).$$

Now, defining $V = (V^1, \ldots, V^m)$, $W = (W^1, \ldots, W^m)$, one obtains

$$\|V - q_0W - \ldots - q_n(q^nW)\|^2_H = \sum_{j=1}^m \alpha_j^2 \|V^j - W^jQ\|^2_{\mathcal{H}_2}.$$ 

Application of the classical projection theorem to minimize the left hand side of this expression results in the solution to 6.4. The equation

$$\sum_{j=1}^m \alpha_j^2 \Psi(V^j, W^j) = (\sum_{j=1}^m \alpha_j^2 \Phi(W^j))\bar{q}$$

is a simple consequence of the orthogonality condition, and as long as one of the $W^j$ has a non-null impulse response, the unicity of $\bar{q}$ can be shown as in Theorem 1. \hfill \Box

In order to apply this theorem to minimize $J(Q)$ (given by Equation (4.12)), simply choose $m = 2$, $V^1 = P_0Bc/(qA_mA_+^c)$, $W^1 = -B^cA_m/(A_mA_+^c)$, $V^2 = (AT_0^*)/(B^+A_m)$, and $W^2 = (AA_+^c)/(B^+A_m)$.
6.2 Remarks

The results of Theorems 1 and 2 are not the same as the classic solution to the $\mathcal{H}_2$ problem. In this case, the parameter $Q$ is a member of the set of fixed degree polynomials $\mathbb{R}_{\leq n}[q]$, whereas $Q$ belongs to the set of rationals $\mathbb{R}(q)$ in the classic $\mathcal{H}_2$ case. In the case presented here, the degree of the resulting solution can be chosen in advance. In the $\mathcal{H}_2$ case, the degree of the solution can not be chosen.

The theorems presented above are analytic in nature, and are naturally preferable to numerical solutions. For problems with large $n$ and $m$, where the transfer functions have impulse responses that die out quickly, calculating the inner products that appear in $\Phi$ and $\Psi$ may be faster if Equation 6.2 is calculated approximately, in lieu of Equation 6.1. For typical problems, however, both solution methods yield results almost instantaneously on modern computers. The impulse responses of the $q^iW$ terms can all be calculated from the impulse response of $q^nW$ by performing backward shifts.
6. SOLUTION TO AFFINE $\mathcal{H}_2$ PROBLEM
Chapter 7

Use of convex optimization

In this chapter, powerful convex optimization-based solution methods are presented that can be applied to find solutions to the problems resulting from the preview design methods in chapters 4 and 5. Unlike the solution methods in Chapter 6, these methods are not analytic. However, unlike gradient-based methods, the convex optimization methods described here provide global optima.

To achieve performance goals, the goal is to find a polynomial $Q$ that minimizes Equation 5.5, repeated here for convenience:

$$J(Q) = \sum_{i \in I, j \in J} a_{i,j} f_{i,j}(T_{i,j}^Q).$$

The functionals $f_{i,j} : \mathcal{RH}_1 \rightarrow \mathbb{R}$ are convex or quasi-convex, and for each polynomial $Q$, the $T_{i,j}^Q$ are transfer functions.

In [5], it is shown that the $H_2$ norm, the $H_\infty$ norm, the step response overshoot, the $\ell_\infty$ norm of the impulse response, the peak gain, and various other useful functionals are convex on $\mathcal{RH}_\infty$. In addition, the settling time functional is quasi-convex on this set. For example, if it is desired for some motion-control application to minimize the maximum absolute error without using a high gain compensator, one natural choice would be $f_{\delta,e}$ as the $\ell_\infty$ norm of the impulse response and $f_{ru}$ as the $H_\infty$ norm or the $H_2$ norm.

In this section, these methods are presented, and it is shown that they are well adapted to solve the problems that appear in Chapters 4 and 5. Also, a
particular functional related to the maximum phase error of preview systems is shown to be convex, and its subgradient is calculated. This is shown to be useful to extend in a natural way the Zero-Phase-Error Tracking Controller of [26].

In this chapter, basic principles of convex analysis are presented, and the ellipsoid method, a numerical convex program solver, is presented.

In what follows, \( n \) will refer to the degree of the polynomial \( Q \), unlike Chapter 5 where it refers to the degree of the polynomial \( K_n \).

### 7.1 Convexity and Subgradients

A linear functional \( \phi : X \rightarrow \mathbb{R} \) is convex if

\[
\phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y), \quad \lambda \in [0, 1].
\]

All functionals in this section are assumed convex. A functional \( \phi^g : X \rightarrow \mathbb{R} \) is a subgradient of \( \phi \) at \( x \) if

\[
\phi(z) \geq \phi(x) + \phi^g(z - x), \quad \forall z.
\]

The function \( \phi^g(x; y) : X \times X \rightarrow \mathbb{R} \) will be used to denote \((\phi^g(x))(y)\). Given a subgradient \( \phi^g(x) \) of \( \phi \) evaluated at the point \( x \), \( \phi^g(x; y) \) will be used to denote this subgradient of \( \phi \) evaluated at the point \( x \) in direction \( y \), analogous to the terminology used with directional derivatives.

Although many functionals that are of interest to control system designers do not possess derivatives, all convex functionals of interest possess subgradients. Although the derivative is unique when it exists, functionals may possess many subgradients. This is not a handicap.

What makes subgradients useful when searching for a minimum is that they make it possible to reduce the search space by cutting it in half. From Equation 7.1 it is clear that given any subgradient \( \phi^g \), if \( \phi^g(z - x) > 0 \) then \( \phi(z) > \phi(x) \). This tells us that it is not worth searching in that half-space for which \( \phi(z) > \phi(x) \). Numerical methods for solving convex programs generally use subgradients in this way, and the ellipsoid method presented in Section A is no exception. If given a point \( x \) and a direction \( z - x \) such that \( \phi^g(z - x) > 0 \),
one can immediately conclude that the objective is increasing in the direction \( z - x \).

**Example 5** On \( \mathbb{R} \), the functional \( \phi(x) = |x| \) is differentiable everywhere except at \( x = 0 \). This functional, however, possesses many subgradients at this point. On Figure 7.1 one sees that at \( x = 0 \), all of the dotted lines lying below \( \phi(x) \) and intersecting \( \phi(x) \) at \( x = 0 \) are subgradients of \( \phi(x) \).

Notice that if given a subgradient at \( x = 0 \) with positive slope and another at \( x = 0 \) with negative slope, one can immediately conclude from Equation 7.1 that the neither the open left half plane nor the open right half plane may contain a minimum.

The following rules facilitate the calculation of subgradients of a given functional. \( \phi \) and \( \phi_i \) are given convex functionals.

- If \( \phi \) is differentiable, it has a unique subgradient, the derivative.
- If \( w \geq 0 \), \( w\phi^g \) is a subgradient of \( w\phi \).
- \[ \sum \phi_i^g(x) \]
is a subgradient of

\[ \left( \sum_{i=1}^{m} \phi_i(x) \right)^{sg} \].

- If \( \phi \) is defined as the supremum of a set of functions \( \phi_n \), a subgradient of \( \phi \) at \( x \) can be found by calculating a subgradient of any \( \phi_n \) which achieves the supremum at \( x \).

### 7.2 Basic principles of convex optimization

Several algorithms exist that permit the computation of the global minimum of a convex functional

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

under convex constraint \( g \leq 0 \) for

\[ g : \mathbb{R}^n \rightarrow \mathbb{R}. \]

The Ellipsoid Algorithm, presented in Chapter A is a particularly simple algorithm for solving this kind of convex program. The functionals presented in Sections 7.3 and 7.4, however, are not functionals on \( \mathbb{R}^n \). In this chapter it is shown how to nevertheless apply this algorithm to solve interesting preview control problems by restricting the optimization to a finite subspaces of \( \mathcal{R}\mathcal{H}_\infty \).

Suppose one is given a convex functional \( f : \mathcal{R}\mathcal{H}_\infty \rightarrow \mathbb{R} \) and a transfer function parameterized by \( Q \), \( T^Q = V - QW \) where \( V, W \in \mathbb{R}(q) \), \( Q \in \mathbb{R}[q] \). In this section it is shown how to find the polynomial \( Q \) that minimizes \( f(T^Q) \) by applying the ellipsoid algorithm directly to find the optimal \( Q \) to any desired accuracy. At the end of this section the general form of this problem is presented, with many applications in control system design.

By writing \( Q = x_0 + x_1q + \ldots + x_nq^n \) it is possible to expand \( T^Q \) as

\[ T^Q = H = H^0 + x_0H^1 + x_1H^2 + \ldots + x_nH^{n+1} \]

(7.2)

where \( H^i \in \mathcal{R}\mathcal{H}_\infty \) \( \forall i \) are easily calculated. The transfer functions parametrized by Equation 7.2 make up a finite dimensional affine subset of \( \mathcal{R}\mathcal{H}_\infty \).
7.2. BASIC PRINCIPLES OF CONVEX OPTIMIZATION

$T^Q$ maps $\mathbb{R}_{\leq n}[q]$ into $\mathcal{RH}_\infty$. Since $\mathbb{R}_{\leq n}[q]$ is isomorphic with $\mathbb{R}^{n+1}$, one can consider $T^Q$ as a function mapping $\mathbb{R}^{n+1}$ into $\mathcal{RH}_\infty$. Consequently, $f(T^Q)$ can be viewed as a function mapping $\mathbb{R}^{n+1}$ into $\mathbb{R}$. This function is easily shown to be a convex functional on the finite dimensional space $\mathbb{R}^{n+1}$, and so the ellipsoid algorithm (see Appendix A) can be applied to minimize them if its subgradient can be calculated.

In [5], it is shown that if $\phi^{sg}$ is a subgradient of $f$ at $T^Q$, then a subgradient of $f(T^Q)$ at the point $T^Q$ is

$$\begin{bmatrix}
\phi^{sg}(H^1) \\
\vdots \\
\phi^{sg}(H^{n+1})
\end{bmatrix}.$$  

Thus, every $Q$ corresponds to a point in $\mathbb{R}^{n+1}$ at which $f(T^Q)$ and $f^{sg}(T^Q)$ can be derived. Application of the ellipsoid algorithm to solve this problem is straightforward. Here an example is given with $n = 0$. In this case, the ellipsoid algorithm degenerates to a very simple interval splitting algorithm.

**Example 6** Suppose it is desired to find $\alpha$ minimizing the convex functional

$$\left\| \frac{z}{z-0.5} + \alpha \frac{z}{z-0.4} \right\|_{\mathcal{H}_\infty}$$

In this simple case, $x_0 = Q = \alpha \in \mathbb{R}$, $H^0 = V = z/(z-0.5)$, $H^1 = -W = z/(z-0.4)$. Assume that the optimal solution is in the interval of length 20 centered at 2, $[-8,12]$. Thus the center of the ellipsoid (interval) is $y_0 = 2$, and the matrix representing the ellipsoid degenerates to the scalar length of the single semi-axis, which is equal to 10.

A subgradient of $f(T^Q)$ at $y_0$ can be found by finding a subgradient of the $\mathcal{H}_\infty$-norm functional at the point $z/(z-0.5) + y_0 z/(z-0.4)$ in the direction $z/(z-0.4)$. From Section 7.4 one can easily calculate that 1.6667 is a subgradient at $y_0 = 2$. This indicates that the functional is undecreasing to the right, and so the optimum must be in the half-interval $[-8,2]$ centered at $y_1 = -3$ and with $A_1 = 5$. $1.6667$ is a subgradient at $y_1$ and so the optimum must be in the half-interval $[-3,2]$ centered at $y_2 = -0.5$ and with $A_2 = 2.5$. With each iteration, the interval is cut in two and $y_i$ converges quickly: $\{-1.75, -1.125, -0.8125, -0.90625, \ldots, -1.095\}$. 
The simple example given above does not do justice to convex optimization methods. The problem only has a single objective with no constraints. In the beginning of this chapter, the more general problem of minimizing tracking error and command signal objectives simultaneously is mentioned. The methods presented above can easily be used to solve this simultaneous convex optimization problem.

Thus, the ellipsoid method can be applied, making it a simple matter to find a polynomial $Q$ that results in asymptotic tracking of the reference signal while keeping the error signal small with respect to $f$ and keeping the transfer function $T^Q_{ru}$ small with respect to $f_{ru}$.

In the case where the $H_2$ norm is used for $f$ and $f_{ru}$, a closed-form solution is presented in Chapter 6.

Since positive scaling, summing, and taking the maximum of convex functions are all operations that preserve convexity, and since the ellipsoid method allows convex constraints to be taken into consideration, it is possible to use a wide variety of criteria when solving this type of problem. The methods in [33, 12, 26] are based on $H_2$ and $\ell_\infty$ optimization criteria. Using convex optimization it is possible to solve all of these problems in a more general way, using a single tool that takes plant poles and control action into consideration. Like in [12], but more generally, it is possible to minimize non-monotone behavior and determine if a compensator providing a monotone step response exists, or find a controller that results in an almost-monotone step response.

What makes the application of finite dimensional convex optimization algorithms possible for the solution of these problems on the infinite-dimensional space $RH_\infty$ is that we are actually dealing with a finite subspace of $RH_\infty$ through Equation 7.3. The design method presented in [5], allows one to solve certain controller design problems that place poles and zeros optimally. To achieve that goal it is necessary to approximate $RH_\infty$ with a finite subspace. In order to find suboptimal solutions that are close to the optimum, the dimension of the approximating subspace may become large. Determining the distance of a given solution to the optimum is not very straightforward. The methods presented here require much less computational power than the ones in [5], yield solutions arbitrarily close to the optimum in a straightforward way,
7.3 Overshoot functional

Here is an example of a functional often used in control design, the overshoot functional (see [5]). It is assumed that the plant has unit DC gain.

\[ \phi(H) = \sup_{t \geq 0} s^H(t) - 1 \]

where \( s^H \) is the step response of \( H \). We wish to find subgradient of \( \phi \) at \( H_0 \).

Define

\[ \phi^{\text{step},t}(H) = s^H(t). \]

Then

\[ \phi(H) = \sup_{t \geq 0} \phi^{\text{step},t}(H) - 1. \]

Now \( \phi \) can be written as a supremum of functions, and one can use the supremum rule for finding a subgradient. One must simply find a subgradient of one of the functions which achieves the supremum.

If \( t_0 \) is a time at which the step response of \( H_0 \) is maximum, then \( \phi^{\text{step},t_0} - 1 \) achieves the supremum. But \( \phi^{\text{step},t_0} - 1 \) is linear, so its derivative is a subgradient of \( \phi \) at \( H_0 \):

\[ \delta \phi^{\text{step},t_0}(H_0; \delta H) = \lim_{\delta \to 0} \frac{1}{\delta} \left( \phi^{\text{step},t_0}(H_0 + \delta H) - \phi^{\text{step},t_0}(H_0) \right) \]
\[ = \lim_{\delta \to 0} \frac{1}{\delta} \phi^{\text{step},t_0}(H_0) + \delta \phi^{\text{step},t_0}(\delta H_0) - \phi^{\text{step},t_0}(H_0) \]
\[ = \phi^{\text{step},t_0}(\delta H) \]

A subgradient at \( H_0 \) of \( \phi \) is \( \phi^{\text{step},t_0} \). From Figure 7.2 it is clear that

\[ \phi(H_0 + \delta H) - \phi(H_0) \geq \delta \phi^{\text{step},t_0}(\delta H) > 0. \]

At the point \( H_0 \), adding \( \delta H \) for any positive \( \delta \) causes the objective (overshoot) to increase. The peculiar nature of subgradients that makes them different from derivatives is that this does not indicate that by subtracting \( \delta H \) the objective will decrease for every \( \delta \). From Figure 7.2 it is clear that the objective may decrease for small \( \delta \), but for large \( \delta \) it will increase without bound.
7. USE OF CONVEX OPTIMIZATION

7.4 $\mathcal{H}_\infty$-norm

Here is an example of another functional often used in control design, the $\mathcal{H}_\infty$-norm. Define $\phi(H) : \mathcal{RH}_\infty \to \mathbb{R}$ as $\phi(H) = \|H\|_{\mathcal{H}_\infty}$.

In [5] it is shown that $\phi(H)$ is a convex functional of $H$ and its subgradient at $H_0$ in direction $H$ is equal to

$$\phi^s(H_0; H) = \frac{1}{\phi(H_0)} \Re \left( \frac{H_0(e^{j\omega_0})}{H(e^{j\omega_0})} \right)$$

where $\omega_0$ is the frequency where the frequency response magnitude of $H$ is greatest. Here a proof different than the one in [5] will be provided. This proof is more general and will be useful in a later section to evaluate the subgradient for the maximum phase error functional. First, a preliminary lemma that the author did not find in print in this exact form, but that is equivalent to Theorem 23.9 of [24].

Lemma 5 Given the linear transformation $h : X \to Y$ and the convex func-
tional $g : Y \to \mathbb{R}$, the composition $f(x) = g(h(x))$ is convex, and

$$f^{sg}(x_0; y_0) := g^{sg}(h(x_0); h(y_0))$$

is a subgradient of $f$.

**Proof** The convexity of $f$ is obvious and well-known. The linearity of $f^{sg}(x_0)$ is also obvious. Given $x_0, x_1 \in X$,

$$f(x_1) = g(h(x_1)) \geq g(h(x_0)) + g^{sg}(h(x_0); h(x_1) - h(x_0))$$

$$f(x_0) + f^{sg}(x_0; x_1 - x_0)$$

by convexity of $g$ and linearity of $h$.

If $h^\omega : \mathcal{RH}_\infty \to \mathbb{C}$ is defined so that $h^\omega(H) = H(e^{j\omega h})$ is the complex frequency response of $H$ at frequency $\omega$, it is easy to verify that $h^\omega$ is a linear transformation.

The functional $g(z) = |z|$ on $\mathbb{C}$ is convex and differentiable, and so its subgradient at $z$ in the direction $\hat{z}$ is equal to its directional derivative:

$$g^{sg}(z; \hat{z}) = \frac{1}{|\hat{z}|} \Re(\hat{z} \hat{z})$$

Given a transfer function $H \in \mathcal{RH}_\infty$, $f^\omega(H) := g(h^\omega(H))$ is the absolute value of the frequency response at frequency $\omega$. By Lemma 5, the subgradient of $f^\omega$ is:

$$(f^\omega)^{sg}(H_0; H) = g^{sg}(h^\omega(H_0); h^\omega(H))$$

$$= \frac{1}{|H_0(e^{j\omega h})|} \Re \left( \frac{H_0(e^{j\omega h})H(e^{j\omega h})}{\overline{H_0(e^{j\omega h})}H(e^{j\omega h})} \right)$$

Since $\phi(H) = \sup_{0 \leq \omega \leq \omega_N} f^\omega(H)$, where $\omega_N$ is the Nyquist frequency, one obtains

$$\phi^{sg}(H_0; H) = \frac{1}{\phi(H_0)} \Re \left( \frac{H_0(e^{j\omega h})H(e^{j\omega h})}{\overline{H_0(e^{j\omega h})}H(e^{j\omega h})} \right)$$

from the supremum rule of Section 7.1.
7.5 Maximum phase error functional

The classes of functionals that are linear or differentiable are very small compared to the class of convex functionals. In addition, under very mild conditions the existence of subgradients has been proved (see [24, 25]). Thus the power of convex optimization lies in its applicability to a very large class of problems. One functional that is particularly well-suited to preview-related problems is presented below.

The functional $\psi(z)$ returning the phase angle in radians of a complex number $z \in \mathbb{C}$ is not linear. It is not convex, either. Given two complex numbers, $c_1$ with an angle between 90 and 180 degrees, and $c_2$ with an angle between -90 and -180 degrees, the average of $c_1$ and $c_2$ could have an angle larger than the average angle of $c_1$ and $c_2$, violating convexity. However, if $c_1$ and $c_2$ are restricted to lie in the right half plane, this cannot occur, and $\psi(z)$ can be shown to be convex.

**Theorem 3** The functional $\psi(z)$ is convex in the right half plane, and

$$\psi^{sg}(z; \hat{z}) = \Im(\hat{z}/z).$$

**Proof** $\psi(z) = \Im(\ln z)$ is not only convex, but it is differentiable in the right half plane. Its gateaux differential can be calculated to be

$$\delta \psi(z; \hat{z}) = \lim_{\delta \to 0} \Im \left( \frac{\ln(z + \delta \hat{z}) - \ln(z)}{\delta} \right) = \Im \left( \lim_{\delta \to 0} \frac{d\ln(z)}{dz} \hat{z} \right) = \Im(\hat{z}/z)$$

When the gateaux differential evaluated at $z$ in direction $\hat{z}$ exists, it is equal to a subgradient of a convex functional evaluated at $z$ in the direction $\hat{z}$. Since $\psi(z)$ is convex in the right half plane, the result follows immediately.

By writing $z = x + iy$ and $\hat{z} = \hat{x} + i\hat{y}$, and identifying $\mathbb{C}$ with $\mathbb{R}^2$, this subgradient can be written explicitly as a linear functional on $\mathbb{R}^2$.

$$\delta \psi(z; \hat{z}) = \Im(\hat{z}/z) = \Im\left( \frac{\hat{x} + i\hat{y}}{x + iy} \right)$$
7.5. MAXIMUM PHASE ERROR FUNCTIONAL

\[ \Re(\dot{x} + \ddot{y} + i(\dot{y}x - y\dot{x})) \]

\[ = \frac{\dot{y}x - y\dot{x}}{x^2 + y^2} \]

\[ = \left[ \begin{array}{c} -y \\ x \\ \dot{x} \end{array} \right] \cdot \left[ \begin{array}{c} x \\ y \end{array} \right] \]

This has a simple interpretation. If, for example, \( z \) is in the first quadrant, \( \psi^g(z) \) points up and to the left, in the direction of greatest increasing angle.

Given a non-negative weighting function \( w(\omega) \), the functional

\[ \phi(H) = \max_{\omega=0} w(\omega) \psi(H(e^{j\omega})) \]

on the domain \( \mathcal{RH}_\infty \) of rational proper stable transfer functions is also not a convex function of \( H \). However, if \( H(e^{j\omega}) \) is restricted to lie in the right half plane, it is convex. This is shown below, and a subgradient of \( \phi \) is derived. This functional can be used to generalize in a natural way the Zero-Phase-Error Tracking Controller of [26], or the controller of [33].

In [26] preview is used to design a compensator that has zero phase error. In many applications, it may be desirable to have small phase error, but requiring the phase error to be zero on the entire frequency range is quite restrictive and may make it difficult to meet other requirements. Here it is shown how to calculate a subgradient of \( \phi(H) \) in order to find controllers with small, but not necessarily zero, phase error in certain frequency ranges.

In Section 7.4 it is shown how to find a subgradient for the \( \mathcal{H}_\infty \)-norm functional, which is the maximum of the absolute value of the frequency response over a given range of frequencies. In order to compute a subgradient of this norm the absolute value functional \( g(z) \) was composed with the transformation \( h^w \) between a rational transfer function and its complex frequency response. Here, using the same \( h^w \) as in Section 7.4 and defining \( \phi^w(H) := w(\omega)\phi(h^w(H)) \), Lemma 5 and the positive weight rule of Section 7.1 can be used to show that

\[ (\phi^w)^g(H_0, H) = w(\omega)\phi^g(h^w(H_0), h^w(H)) = w(\omega)\Re(H(e^{j\omega}))/H_0(e^{j\omega})) \]
Now, by applying the supremum rule of Section 7.1, one can show that the functional
\[ \phi(H) = \max_{\omega=0} \phi^\omega(H) \]
has subgradient
\[ \phi^{\ast \omega}(H_0; H) = w(\omega_0) \Re (H(e^{j\omega_0 h})/H_0(e^{j\omega_0 h})) \]
where \( \omega_0 \) is the frequency for which \( w(\omega) \Re (H(e^{j\omega h})/H_0(e^{j\omega h})) \) is maximized.

In order to apply the above result, it is necessary to constrain the solution to be such that for all frequencies with non-null weight, the frequency response angle lies in the right half plane. The functional \( \rho(z) = -\Re\{z\} \) is obviously convex, because it is linear. Clearly, \( \rho^{\ast \omega}(z, \bar{z}) = -\Re\{\bar{z}\} \). The composition with \( h^\omega(H) \) is linear, and so the supremum rule can again be used to determine that
\[ (\rho \circ h^\omega)(H) = \max_{\omega=0} w(\omega) \rho(h^\omega(H)) \]
has subgradient
\[ (\rho \circ h^\omega)^{\ast \omega}(H_0; H) = w(\omega_0) \rho(h^\omega(H_0)) \]
where \( \omega_0 \) is the frequency for which \( -w(\omega) \Re \{H(e^{j\omega h})\} \) is maximized.

This functional can be used as a constraint, over an arbitrary frequency range, to guarantee that the phase angle of the transfer function of interest will be between \( -\pi \) and \( \pi \), the range over which the phase angle functional is convex.

The above results allow one to find transfer functions which have a phase error close to zero. In our case, however, what we are interested in is finding transfer functions so that the phase error is close to zero when preview is accounted for. This can be done either by multiplying the transfer function by \( q^\Delta \), or by multiplying the FFT’s of the appropriate impulse responses by \( e^{j\omega h} \) (see Section 7.8) in the subgradient calculation routines.

The above results can be used to find numerical solutions equivalent to the results in [26].

Example 7 Given a FIR plant \( 1 - 2q^{-1} \), the zero-phase-error filter in [26] is \( -2 + q^{-1} \). This same result can be found within the more general framework
of Chapter 5 by choosing $T_{fy} = (q - 2)/q$, $T_{fu} = 1$, and $\Delta = 1$, and then minimizing the maximum angle functional of $T_{fy}$ with weight $w(\omega) = 1 \forall \omega$, while simultaneously constraining the frequency response to lie in the right half plane for all $\omega$.

![Figure 7.3: Step responses of zero-phase-error compensated system](image)

Notice in Figure 7.3 that the step response of the compensated FIR plant has significant undershoot and overshoot. This is partly a consequence of the very strict zero-phase-error requirement, and if such strict phase requirements are not necessary it is possible to reduce the overshoot and undershoot while still keeping the phase error small. For instance, the maximum phase error functional presented above, combined with preview, can be used as follows.

By choosing $T_{fy} = (q - 2)/q$, $T_{fu} = 1$, and $\Delta = 1$, and then minimizing the $\ell_\infty$-norm of the step response error signal, while constraining maximum angle functional of $T_{fy}$ to be less than ten degrees, while simultaneously constraining the frequency response to lie in the right half plane for all $\omega$, the bode plot (taking preview into consideration) and step response shown in Figure 7.4 are obtained. The filter obtained is $-1.56 + 0.56q^{-1}$. With the use of the weighting factor $w(\omega)$, it is possible to improve the step response even further by limiting the phase error only in the frequency ranges for which small phase error is
This example shows the usefulness of the maximum phase error functional. For simplicity, the FIR case is considered here, and the results are compared critical.
7.6 Generalized Preview

As presented, the concept of preview has been shown to be equivalent to following a pure delay. In practice, however, it is not always clear what constitutes a pure delay. For instance, when following a step input one might measure the difference between the system output and the step output of a pure delay, and use the \( \ell_\infty \)-norm as a criteria for optimization. Although this may work well in certain cases, it is not necessary the best criteria to use. This is because there are many situations where this measure might be relatively high even though qualitatively the system behaves in a reasonable way. By insisting that the \( \ell_\infty \)-norm of the difference between the plant output and a delayed unit step be small, it may be difficult or impossible to meet other requirements. For this reason, it may be judicious to generalize the meaning of preview.

The undershoot and overshoot functionals for a given reference input are convex and have easily calculated subgradients. In addition, these functionals remain convex when the undershoot or overshoot is calculated relative to an arbitrary function of time. Thus, the undershoot of the plant output, relative to a lower barrier, can be used to keep the plant output above, or almost above, this lower barrier. The overshoot of the plant output, relative to an upper barrier, can also be used in the same way.

By adding barriers below and above (Figure 7.5) instead of simply calculating the \( \ell_\infty \)-norm of the difference between the plant output and the reference signal delayed by \( \Delta \), the optimization algorithm is given much more freedom to find a solution that has an acceptable step-response error. This additional freedom can then be put to better use, minimizing simultaneously some other criterion.

The barriers of Figure 7.5 can be parameterized so that the inflection points of the two barriers are separated in time by a width \( \Omega \), and have average time \( \Delta \).
7.7 Example

In this chapter an example demonstrating the trade-offs which occur when designing a preview-based controller.

Example 8 The plant

\[
\frac{(q + 1)(q - 1.5 \exp^{j\pi/16})(q - 1.5 \exp^{-j\pi/16})}{(q - 1)^2q^2}
\]

is similar to the discretization of a double integrator, but it has two additional complex non-minimum-phase zeros. The sampling time is \( h = 1 \), and 12 samples of preview information are available. \( R \) and \( S \) have been chosen to place the closed-loop poles at \( \{0.6 + 0.1i, 0.6 - 0.1i, 0.5 + 0.1i, 0.5 - 0.1i, 0.7, 0.7\} \), resulting in \( R = q^3 - 1.73q^2 + 1.34q - 0.13, S = 0.13q^3 - 0.14q^2 + 0.02q \). It is necessary to design the \( T \) polynomial so that the system asymptotically tracks a step input. The design should take command signal magnitude into account to avoid saturation which occurs at \( |u| = 0.025 \).

If \( T = 0.0065q^3 \) is chosen simply so that the compensated system has unit DC-gain, the system exhibits undershoot and oscillation (see \( y_{ref} \) and \( u_{ref} \).
in Figure 7.6). According to [12], it is possible to find a FIR compensator of degree 15 such that a monotone step response occurs. Instead of pursuing this path, it is possible to search for a shorter compensator which results in approximate monotonicity.

Figure 7.6: Almost monotone step responses, $\Delta = 12$, $\kappa = 4$
This can be done by searching for a compensator such that the impulse response of $T_{ry}^Q$ is almost non-negative. The appropriate functional is the impulse-response undershoot functional. In order to achieve a response that appears completely monotone, it is necessary to increase the number of zeros in $T$ by $\kappa = 4$. Finding $Q$ to minimize the impulse-response undershoot of $T_{ry}^Q$, the results in Figure 7.6 are obtained. In this case, $R$ and $S$ are as above, and $T = 0.0063q - 0.0051q^2 + 0.0064q^3 - 0.0052q^4 + 0.006q^5 - 0.004q^6 + 0.0022q^7$. Since $\delta T > \delta R$, this system is non-causal. Here preview has not been used at all in the derivation of this controller. The only consideration made while designing this controller was that the output be monotone. Therefore, the system can be made causal by simply adding 4 steps of delay. With a simple factorization we obtain a standard causal RST controller with $R$ and $S$ as above, and $T = 0.1241q + 1.1221q^2 + q^3$, preceded by the FIR filter $F = 0.0056q^{-4} - 0.0101q^{-3} + 0.0108q^{-2} - 0.0064q^{-1} + 0.0022$.

There are 12 samples of preview available, and the apparent delay exhibited by the monotone step response is longer than this. So the controller above designed for monotone step response is not a very good preview controller, in the sense that the tracking error for $\Delta = 12$ is quite large, according to most measures. If it is necessary that the tracking error be small, and a certain amount of undershoot is tolerable, then a preview tracking controller with $\Delta = 12$ may be designed. Since undershoot occurs with this plant, it makes more sense to use generalized preview with barrier functions rather than attempting to follow the step response of $q^{-\Delta}$ closely. Barrier functions can be designed so as to not penalize small amounts of early oscillatory behavior.

In Figure 7.7 is the step response of a preview-based controller, with $\kappa = 3$, designed with the sole objective that the step-response plant output deviate little from the region between the lower and upper barriers. This is obtained by defining a cost function which is the sum of two cost functionals; the first being the undershoot of the difference between the plant output and the lower barrier, and the second being the overshoot of the difference between the plant output and the upper barrier. In this case, for every response that lies between the two barriers, the cost is zero. Thus, there is a non-unique solution to this problem. The solution appearing in Figure 7.7 is one of the solutions, and
what is noticeable is that there is lots of chatter between the barriers. This is a poor solution because the problem is poorly formulated. There are a number of methods for finding a better solution. One of these is to simply require that the second derivative of the plant output be small. Adding the constraint that
the $\ell_\infty$ norm of the second derivative of $u$, $u_{tt}$, be less than 0.025, a smooth response is obtained. In Figure 7.8 the result is shown, with the additional constraint, meant to avoid command saturation, $|u| < 0.025$. The response is considerably faster than that of the original system, with a command signal with a lower peak value. $R$ and $S$ are unchanged, and $T = 0.0023q - 0.016q^2 + 0.0231q^3 - 0.0055q^4 - 0.0226q^5 + 0.025q^6$. Here, $\kappa = 3$ has been chosen in a somewhat arbitrary way. The larger $\kappa$ is, the better the performance tends to become. However, large $\kappa$ results in a $T$ polynomial of high degree. Generally, this is not a severe problem, since $T$ can again be factored so that this controller is equivalent to an RST controller with $\delta R = \delta S = \delta T = 3$, preceded by a FIR filter of degree 3. Thus, the controller inside the loop remains of low order.

If we wish to do without the FIR filter completely, we can specify $\kappa = 0$, and see what is obtained. In this case, we obtain the results in Figure 7.9. $y$ is almost identical to $y_{ref}$, but $u$ satisfies the additional $\ell_\infty$ constraint, unlike $u_{ref}$. In this case, $T = 0.006q - 0.0126q^2 + 0.0131q^3$, and so we obtain a simple RST controller of degree 3.

If, on the other hand, we choose $\kappa = 11$, additional controller freedom is available to satisfy more stringent saturation requirements. In the response of Figure 7.10(b) one sees that $y$ is almost identical to $y_{ref}$. $u$, however, has an $\ell_\infty$ norm of 0.17, almost half the peak value of $u_{ref}$. The cost of this improvement lies in the degree of the polynomial $T$. The controller, nevertheless, can be simplified to one where $\delta R = \delta S = \delta T = 3$, preceded by a FIR filter of degree 11.

The power of the convex approach is that many different types of criteria may be used, separately or combined. To generalize [26], it is possible to reduce phase error of preview systems, without necessarily reducing it to zero. For instance, if a controller is designed using the same approach as the approach used to design the system of Figure 7.8, without $\ell_\infty$ constraints on $u$, the preview-adjusted phase (the phase of $q^{\Delta} * T_u$) varies by ±50 degrees between 0 and 0.1 Hz. Without noticeably affecting the response, this phase variation can be reduced to ±30 degrees, by using the convex functional of Section 7.5 as a constraint. Using the compensator of [26], however, would completely change the system transient response.
7.8 Remarks

Any implementation of the above theory must make use of a convex solver such as the ellipsoid algorithm or a splitting plane algorithm. These algorithms must
have access to subroutines that calculate the values and subgradients for the various functionals of interest. These calculations, due to equation 7.3, require much scaling and addition of rational transfer functions.

In practice it is more computationally feasible to use impulse responses of
the transfer functions instead of the transfer functions themselves. All of the important calculations necessary to implement the convex optimization algorithms for control applications can be written to use impulse responses instead of transfer functions, providing accuracy good enough for general purpose use.
When impulse responses are used instead of transfer functions, addition and scaling is usually performed much faster as a simple scaled vector addition, or *saxpy*.

When impulse responses are used instead of transfer functions, the frequency response selection of Sections 7.4 and 7.5 can be performed approximately with a Fast Fourier Transform (FFT).

A number of optimization criteria are available when resorting to convex optimization algorithms. The algorithm which should be used is application dependent. When minimizing time-domain errors, the $\ell_1$-norm generally provides good results. It is also possible to minimize overshoots and undershoots, settling time, and other standard measures of performance. Some norms, such as the $\mathcal{H}_\infty$-norm and the $\ell_\infty$-norm, often do not have unique minima. In these cases, it is usually worthwhile to add a two-norm performance measure to the cost function to benefit from this non-uniqueness and improve some other objective simultaneously. Frequency-domain criteria generally can be obtained with the judicious use of the $\mathcal{H}_\infty$-norm. Also, the maximum phase error functional derived above can be combined with preview.
Chapter 8

Conclusions

The ad-hoc methods of [26, 33, 12] are important in their own right because they provide theoretical results which point the control system designer in certain directions, providing information about what can be achieved, and why. However, the multi-objective approaches such as those in Chapters 6 and 7 can incorporate these results, are more flexible and should be used when designing a real controller. Attempting to minimize a single objective, such as tracking error, often provides unsatisfactory results. With the computing power available on modern desktop computers there are few reasons not to use multi-objective design methods. The convex optimization methods make it feasible to use cost functions that are well adapted to most imaginable physical design goals.

It appears that many preview design problems, [26, 33, 12, 10, 9], can be solved by combining preview with convex optimization methods instead of resorting to the analytic results contained in those papers. Doing so makes it possible to combine these results with others, giving the system designer flexibility to make trade-offs between various competing objectives.

The methods in this thesis are meant to solve the preview design problem using feedforward control. As mentioned in the introduction, a type of separation property justifies this approach. The closed-loop system may be designed using a number of methods which are out of the scope of this thesis, and the results provided here complement most closed-loop design methods.
The methods in this thesis concentrate on zero placement. It is possible to place poles as well as zeros to improve performance. The \( H_\infty \) preview methods do this. It might be possible to use LMI design methods to also place poles and zeros simultaneously. This could be a subject of future research. These methods, like the \( H_\infty \) methods, are nevertheless limited in the sense that objectives such as settling time and maximum phase error can not be integrated into these frameworks. The very general methods proposed in [5] would be a good starting point to solve the complete preview design problem. One weakness to this method is that the control designer must choose a finite dimensional approximation of \( H_\infty \), and find the optimal controller in that space. The optimal controller is only found as \( n \) approaches infinity, so computational complexity may be high in practice. And the effect of order reduction on preview behavior would have to be studied.

The full controller design methods tend to result in high-order controllers. As the preview time \( \Delta \) increases, the order of these controllers would also increase. It seems that it would be important to study the limitations of controller order reduction in the high-\( \Delta \) case. Loss of stability is not a problem in the framework presented here because essentially we are choosing the coefficients of a reference signal prefilter which does not affect closed-loop stability properties. The full-controller design methods give the designer little flexibility with respect to choosing controller order.

These results have applications in most settings where a small delay is permitted, so that the reference signal is available in advance. Many motion control and mechatronic applications fit this description. Since preview adds delay to the controller, preview controllers must obviously not be used as inner loops in cascade control schemes.

The degrees of all polynomials in the methods presented here can be specified by the system designer. This makes it possible to choose the amount of preview that is necessary, and not more. This is important because using excessive preview may result in poor behavior. Strictly speaking, robustness is preserved since the preview controller has stable poles and is outside the control loop. When using large amounts of preview, however, any plant modeling errors may result in greatly increased transient tracking error. Thus, there are
practical limits to the choice of $\kappa$ and $\gamma$. It would be interesting to study the sensitivity of transient behavior to plant uncertainty when using large amounts of preview.

Although this thesis has concentrated on using feedforward controllers to improve performance when preview knowledge of the reference signal is available, it is also possible to use preview to improve disturbance rejection when early disturbance knowledge is available. When raw materials are transformed continuously in a manufacturing process, for example, certain properties can be measured shortly before processing to reduce disturbances caused by property variations. For instance, in an aluminum foil milling process, measurement of the raw metal thickness can be done shortly before the metal reaches the rollers. Deviation from the average thickness is generally treated as a disturbance. Preview-based regulation is a very natural approach to reject this type of disturbance. When using the generalized plant approach, both regulation preview and tracking preview can be studied using the same framework (Section 2.2.3). To benefit from low-order control, regulation preview in the RST framework could be studied. In this situation, preview would influence the choice of $R$ and $S$. Whether this can be done effectively, without reducing controller effectiveness with respect to other disturbances, should be studied.
Appendix A

Ellipsoid Algorithm

Solving convex programs on $\mathbb{R}^n$ is simple when subgradients are available for the objective functional and the constraint functional. This appendix is only meant to be a quick introduction to the methods. More details and references are found in [5, 36, 3]. When programs can be written as LMI’s, they can be solved even more efficiently [4]. This, however, requires a rewriting of the problem. For the class of problems presented in this thesis, the speed of the ellipsoid algorithm is largely sufficient on modern computers, easy to understand and code, and easy to use. This algorithm is presented below.

Assume that a convex functional $f : \mathbb{R}^n \to \mathbb{R}$ possessing subgradients is given, and one wishes to find $y \in \mathbb{R}^n$ minimizing $f(y)$. It is not assumed that the gradient exists or is known.

An ellipsoid in $\mathbb{R}^n$ can be characterized by its center $y$ and a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Assume that the solution $y^*$ to the minimization problem is contained in the ellipsoid described by $y_0, A_0$. For real problems this is not a severe handicap—one may simply start with a very large ellipse that is sure to contain the solution. Now use the following algorithm:

1. Find a subgradient of $f$ at $y_k$ to eliminate half of ellipsoid from consideration. The half that can be eliminated is simply the set of $y$ in the half space such that $f^*(y - y_k) > 0$.

2. Find the smallest ellipsoid $E_{k+1}$ which contains the feasible half of $E_k$.
It is easily shown that $E_{k+1}$ exists and is smaller in volume than $E_k$. In addition, the ratio of the two volumes is a function only of $n$. Let $y_{k+1}$ be center of $E_{k+1}$. See [5].

3. Increment $k$.

4. If error within tolerance, exit. Otherwise, go to step 1.

Although this algorithm is very simple to program, it is not the most efficient convex optimization algorithm. Cutting plane algorithms presented in [36] are much more efficient. This algorithm is easily programmed because data storage is only needed to keep track of the center of the ellipse $y_k$ and the matrix $A_k$ describing the ellipse. Given $y_k$, $A_k$, and a subgradient of $f(y_k)$, it is trivial to calculate $y_{k+1}$ and $A_{k+1}$ (see [5]).

If the functionals are not convex, but pseudo-convex, this algorithm will still converge. The exit condition will, however, be more complex.

When a convex constraint $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is present, trivial modifications to the algorithm are necessary. If $g(y_k) \leq 0$, so $y_k$ is feasible, calculate $E_{k+1}$ in exactly the same way as above. If $g(y_k) > 0$, it is necessary to calculate $y_{k+1}$ and $E_{k+1}$ by simply using the subgradient of $g$ instead of the subgradient of $f$. 
List of Symbols

1(k) Unit step function
⟨X, Y⟩ Inner product of X and Y
αi Scalar weighting factor
Δ Total preview (γ1 + γ)
δf(x; ̇x) Gateaux differential of f at x in direction ̇x
δP Degree of a polynomial P ∈ ℝ[q]
δ(k) Unit impulse function
γ Additional delay (Δ − γ1)
γ1 Plant delay
κ Number of zeros added to controller to increase design freedom
ϕ(q−1) Compensated FIR system B(q−1)C(q−1)
ϕ∗g Subgradient of functional ϕ
× Cartesian product
A Denominator of plant P
A⁺ Stable denominator factor of reference signal generator
A− Unstable denominator factor of reference signal generator
Am Closed-loop pole placement polynomial
Am Denominator of reference model
Ao Observer polynomial
B(q−1) Chapter 3: FIR plant
B Numerator of plant P
B⁺ Highest degree monic polynomial consisting of minimum-phase well-damped zeros of B(q)
B⁻  \( B(q)/B^+(q) \) (non-minimum phase or poorly-damped part of \( B(q) \))

\( B_c \)  Numerator of reference signal generator

\( B_m \)  Numerator of reference model

\( \mathbb{C} \)  Set of all complex numbers

\( C(q^{-1}) \)  Chapter 3: FIR compensator

\( d_\Delta(q^{-1}) \)  \( \Delta \)-sample delay, \( q^{-\Delta} \)

\( D_{ab} \)  Denominator of \( T_{ab} \)

\( e(k) \)  Tracking error

\( \text{FIR} \)  Finite Impulse Response

\( h \)  Sampling period

\( H_m \)  Reference model

\( J \)  Cost function

\( K \)  General controller

\( k \)  Discrete time instant, \( k \in \mathbb{Z} \)

\( \mathbb{N} \)  Set of all positive integers

\( N_{ab} \)  Numerator of \( T_{ab} \)

\( P \)  Plant

\( \text{PID} \)  Proportional-Integral-Derivative controller

\( q \)  forward shift operator

\( q^{-1} \)  backward shift operator

\( \mathbb{R} \)  Set of all real numbers

\( \mathbb{R}[q] \)  Set of polynomials in \( q \) over the field \( \mathbb{R} \)

\( \mathbb{R}(q) \)  Set of rationals in \( q \) over the field \( \mathbb{R} \)

\( \mathbb{R}_{\leq n}[q] \)  Set of polynomials \( \{P(q) \in \mathbb{R}[q] \mid s.t. \delta P < n\} \)

\( \mathbb{R}_{n}[q] \)  \( \mathbb{R}_{\leq(n+1)}[q] \)

\( R \)  Pole polynomial of RST controller

\( \text{RST} \)  Type of two-degree-of-freedom controller

\( R_f \)  Fixed part of \( R \) containing, for instance, integrators

\( S \)  Feedback polynomial of RST controller

\( \text{SISO} \)  Single input single output

\( T_{ab} \)  Transfer function from signal \( a \) to signal \( b \)

\( T \)  Feedforward polynomial of RST controller
$u$  Plant input (command)

$y$  Plant output

$y_r$  Reference signal (may be $\Delta$-delayed version of $y_r$)

$\hat{y}_r(k)$  Fixed reference signal

$\mathbb{Z}$  Set of all integers
A. ELLIPSOID ALGORITHM
Bibliography


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