Noise-delayed decay for Lévy flights in unstable parabolic potential

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***Abstract*—We obtain the exact formula for the nonlinear relaxation time (NLRT) of anomalous diffusion of a particle in the form of Lévy flights in an unstable parabolic potential. Some peculiarities of the noise-delayed decay phenomenon in such a system compared with Brownian diffusion are discussed. In particular, we show how the NLRT depends on the stability index of the noise and initial position of a particle.**

***Index Terms*—Lévy flights, fully unstable potential, nonlinear relaxation time, noise-delayed decay**

I. Introduction

Recently, interest in the theoretical and experimental investigations of unstable systems has been renewed [1]–[3]. For example, the sensitivity of optomechanical sensors is commonly limited by parametric instability [1]. To suppress this instability in a micron-scale cavity optomechanical system the feedback via optomechanical transduction and electrical gradient force actuation is applied.

Unstable systems have a strong tendency to diverge during a short period of time. This is manifested, for example, in the divergence of all statistical moments of their phase variables. As a result, these characteristics cannot provide exhaustive information about stochastic dynamics of unstable systems. Additional information can be given by studying the evolution of probability distributions [2], [3], as well as by analysis of various temporal characteristics. Moreover, in such systems the constructive role of noise can be manifested. In particular, this role consists in delaying the decay from unstable nonequilbrium states.

The phenomenon of noise-delayed decay of unstable states was well-investigated for the Brownian diffusion [4], [5] (see also [6]) using the theory of first-passage times (FPT) and apparatus of nonlinear relaxation time (NLRT). It was shown that by choosing the optimal value of the noise intensity it is possible to slow down the decay of the initially unstable states of the system. This phenomenon for systems with metastable states is known as noise-enhanced stability (NES) [7]–[13]. However, there are still no studies of the peculiarities of noise-delayed decay phenomenon for anomalous diffusion, in particular for Lévy flights.

In this work we obtain exact results for the NLRT of anomalous diffusion in the form of Lévy flights with an arbitrary Lévy index α in an unstable parabolic potential. A behavior of the NLRT versus the noise intensity parameter σ for different values of Lévy index α and initial positions of a particle are discussed in detail. The effect is compared with the case of overdamped Brownian motion.

II. General relations

The anomalous diffusion in the form of Lévy flights for a particle moving in a potential profile $U(x)$ is described by the following fractional Fokker-Planck equation for the probability density of transitions $P\left(x,t\right|x_0,0)$ [14]–[16]

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[U'\left(x\right)P\right]+\sigma^{\alpha}\frac{\partial^{\alpha}P}{\partial\left|x\right|^{\alpha}}\,, \tag{1}$$

where α is the Lévy index ($0<\alpha<2$) and σ is the parameter characterizing the intensity of Lévy noise. Equation (1) is associated with the following Langevin equation

$$\frac{dx}{dt}=-U'(x)+\xi_{\alpha}(t), \tag{2}$$

where $x(t)$ is the particle coordinate and $\xi_{\alpha}(t)$ is a symmetric white α-stable noise.

The nonlinear relaxation time for a diffusion in a potential profile with a sink is defined as [6]

$$T\left(x_0\right)=\int_0^{\infty}\Pr\left(t,x_0\right)dt, \tag{3}$$

where

$$\Pr\left(t, x_0\right) = \int_{L_1}^{L_2} P\left(x, t \middle| x_0, 0\right) dx \tag{4}$$

represents the probability to find a particle in the interval (L_1, L_2) at the time t, if it starts from the point $x_0 \in (L_1, L_2)$. Substituting (4) in (3) and changing the order of integration we arrive at

$$T\left(x_0\right) = \int_{L_1}^{L_2} Y\left(x, x_0\right) dx, \tag{5}$$

where

$$Y\left(x, x_0\right) = \int_0^{\infty} P\left(x, t \middle| x_0, 0\right) dt. \tag{6}$$

Integrating both parts of (1) with respect to t from 0 to ∞ and taking into account the obvious initial condition $P\left(x, 0\middle| x_0, 0\right) = \delta\left(x - x_0\right)$ and asymptotics (for a potential with a sink) $P\left(x, \infty\middle| x_0, 0\right) = 0$, we obtain the following equation for the function $Y\left(x, x_0\right)$

$$\frac{d}{dx}\left[U'\left(x\right) Y\right] + \sigma^{\alpha} \frac{d^{\alpha} Y}{d\left|x\right|^{\alpha}} = -\delta\left(x - x_0\right). \tag{7}$$

To solve equation (7) it is better to introduce the Fourier transform of the function $Y\left(x, x_0\right)$, i.e.

$$\widetilde{Y}\left(k, x_0\right) = \int_{-\infty}^{\infty} Y\left(x, x_0\right) e^{ikx} dx. \tag{8}$$

For smooth potential profiles $U\left(x\right)$, after the Fourier transform (7) can be written in the following form

$$ik\, U'\left(-i\frac{d}{dk}\right)\widetilde{Y} + \sigma^{\alpha}\left|k\right|^{\alpha}\widetilde{Y} = e^{ikx_0}. \tag{9}$$

Substituting $Y\left(x, x_0\right)$ from the backward Fourier transform of (8) in (5) and changing the order of integration we have for the NLRT

$$T\left(x_0\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{Y}\left(k, x_0\right) \frac{e^{-ikL_1} - e^{-ikL_2}}{ik} dk. \tag{10}$$

One can easily check that after replacing k with $-k$ (9) coincides with the equation for the complex conjugate function $\widetilde{Y}^*\left(k, x_0\right)$, i.e. $\widetilde{Y}\left(-k, x_0\right) = \widetilde{Y}^*\left(k, x_0\right)$. As a result, (10) can be rearranged in a simpler form

$$T\left(x_0\right) = \frac{1}{\pi}\,\mathrm{Re}\left\{\int_0^{\infty} \widetilde{Y}\left(k, x_0\right) \frac{e^{-ikL_1} - e^{-ikL_2}}{ik} dk\right\}, \tag{11}$$

where Re denotes the real part of expression.

Thus, according to (11), it is sufficient to solve equation (9) only in the area of $k > 0$, i.e. the following one

$$U'\left(-i\frac{d}{dk}\right)\widetilde{Y} - i\sigma^{\alpha}k^{\alpha-1}\widetilde{Y} = \frac{e^{ikx_0}}{ik}. \tag{12}$$

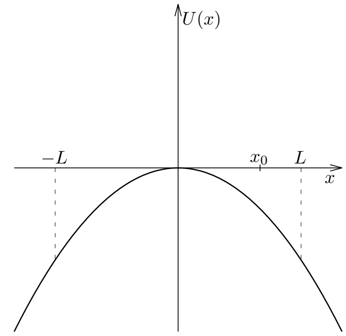

Fig. 1. The unstable parabolic potential.

III. NLRT for a particle in the unstable parabolic potential

Let us solve the problem for the inverse parabolic potential $U(x) = -bx^2/2$, see Fig. 1. Substituting this potential in (12) we arrive at the following first-order differential equation

$$\frac{d\widetilde{Y}}{dk} - \frac{\sigma^{\alpha}k^{\alpha-1}}{b}\widetilde{Y} = -\frac{e^{ikx_0}}{bk}. \tag{13}$$

The general solution of (13) has the form

$$\widetilde{Y}\left(k, x_0\right) = e^{\frac{(\sigma k)^{\alpha}}{\alpha b}}\left(C + \frac{1}{b}\int_k^{\infty} e^{iqx_0 - \frac{(\sigma q)^{\alpha}}{\alpha b}} \frac{dq}{q}\right), \tag{14}$$

where the integral converges. In accordance with the properties of Fourier transform (8): $\widetilde{Y}\left(k, x_0\right) \to 0$ at $k \to \infty$. As a result, we find from (14) the unknown constant: $C = 0$ and, as a consequence,

$$\widetilde{Y}\left(k, x_0\right) = \frac{1}{b} e^{\frac{(\sigma k)^{\alpha}}{\alpha b}} \int_k^{\infty} e^{iqx_0 - \frac{(\sigma q)^{\alpha}}{\alpha b}} \frac{dq}{q}. \tag{15}$$

Substituting (15) in (11) and changing the order of integration we get

$$T\left(x_0\right) = \frac{1}{\pi b}\,\mathrm{Re}\Bigg\{\int_0^{\infty} e^{iqx_0 - \frac{(\sigma q)^{\alpha}}{\alpha b}} \frac{dq}{q} \times \int_0^{q} e^{\frac{(\sigma k)^{\alpha}}{\alpha b}} \frac{e^{-ikL_1} - e^{-ikL_2}}{ik} dk\Bigg\}. \tag{16}$$

If we put in (16) to simplify $L_1 = -L$, $L_2 = L$, we arrive at

$$T\left(x_0\right) = \frac{2}{\pi b}\int_0^{\infty} \frac{\cos qx_0}{q} e^{-\frac{(\sigma q)^{\alpha}}{\alpha b}} dq \int_0^{q} \frac{\sin kL}{k} e^{\frac{(\sigma k)^{\alpha}}{\alpha b}} dk. \tag{17}$$

Equation (17) is the exact result obtained in quadratures for the NLRT of Lévy flights with an arbitrary Lévy index α in the unstable parabolic potential.

IV. General formula for NLRT in special cases

Let us verify the result given by (17) in the absence of noise $\xi_{\alpha}(t)$. Putting $\sigma = 0$ in (17) we arrive at

$$\begin{aligned} T_{dyn} &= \frac{2}{\pi b}\int_0^{\infty} \frac{\cos\left(qx_0\right)}{q} dq \int_0^{q} \frac{\sin kL}{k} dk \\ &= \frac{2}{\pi b}\int_0^{\infty} \frac{\cos\left(qx_0\right) \mathrm{si}\left(qL\right)}{q} dq, \end{aligned} \tag{18}$$

where $\mathrm{si}(x)$ in the sine integral function. Using the following auxiliary integral $(\alpha > 0, \beta > 0)$

$$\int_0^\infty \frac{\cos(\beta x)\,\mathrm{si}(\alpha x)}{x}\,dx = \begin{cases} (\pi/2)\ln(\alpha/\beta)\,, & \alpha > \beta, \\ 0, & \alpha < \beta \end{cases}$$

and taking into account that $|x_0| < L$, we find the dynamical time in which the particle from its initial position x_0 reaches one of the boundaries $\pm L$, rolling down the potential profile (see Fig. 1)

$$T_{dyn} = \frac{1}{b}\,\ln\frac{L}{|x_0|}\,. \tag{19}$$

On the other hand, the direct integration of (2) with $U(x) = bx^2/2$ and $\xi_\alpha(t) = 0$,

$$\int_{x_0}^{L}\frac{dx}{x} = \int_0^{T_{dyn}} b\,dt$$

gives the same result.

Let us show now that in the case of Lévy index $\alpha = 1$ the NLRT (17) can be written in the form of a single integral. Substituting $\alpha = 1$ in (17) and changing the order of integration we arrive at

$$T(x_0) = \frac{2}{\pi b}\int_0^\infty \frac{\sin kL}{k}\,e^{\sigma k/b}dk \int_k^\infty \frac{\cos(q x_0)}{q}\,e^{\sigma q/b}dq. \tag{20}$$

Differentiating both sides of (20) with respect to the parameter x_0 and calculating the internal integral, we have

$$T'(x_0) = -\frac{2}{\pi}\int_0^\infty \frac{(b x_0 \cos k x_0 + \sigma \sin k x_0)\sin kL}{k\,(\sigma^2 + b^2 x_0^2)}\,dk. \tag{21}$$

Applying Dirichlet

$$\int_0^\infty \frac{\sin \alpha x}{x}\,dx = \frac{\pi}{2}\,\mathrm{sgn}(\alpha)$$

and Frullani formulas

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\left|\frac{b}{a}\right|,$$

where $\mathrm{sgn}(x)$ is the sign function, we obtain from (21)

$$T'(x_0) = \frac{\frac{\sigma}{\pi}\ln\left|\frac{L - x_0}{L + x_0}\right| - b x_0 \mathbb{1}(L - x_0)}{\sigma^2 + b^2 x_0^2}\,, \tag{22}$$

where $\mathbb{1}(x)$ is the step function. According to (17),

$$\lim_{x_0 \to \infty}\langle T(x_0)\rangle = 0.$$

As a consequence, we find from (22) the NLRT for the case $\alpha = 1$

$$T(x_0) = \frac{1}{2b}\ln\frac{\sigma^2 + b^2 L^2}{\sigma^2 + b^2 x_0^2} + \frac{\sigma}{\pi}\int_{x_0}^\infty \ln\left|\frac{L + z}{L - z}\right|\frac{dz}{\sigma^2 + b^2 z^2}\,, \tag{23}$$

where $|x_0| < L$. Obviously, in the case of $\sigma = 0$ (23) coincides with (19).

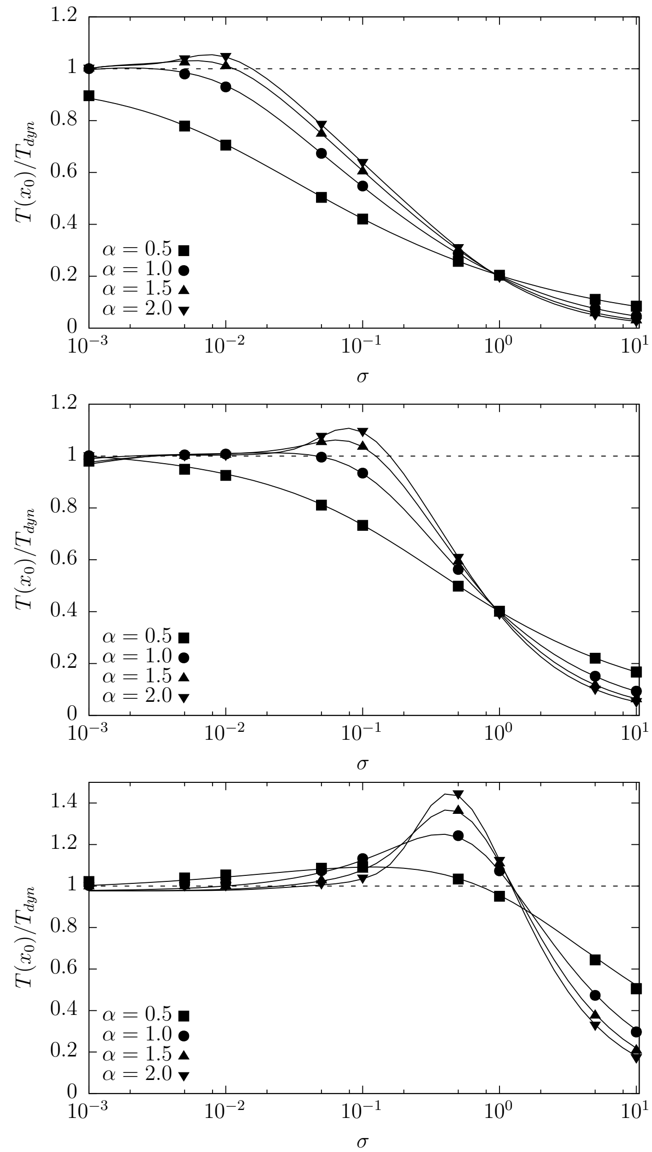

Fig. 2. Normalized nonlinear relaxation time as a function of the noise intensity parameter σ for $L = 1$, different values of Lévy index α and various initial positions of a particle $x_0 = 0.01$ (top panel), $x_0 = 0.1$ (middle panel) and $x_0 = 0.5$ (bottom panel). The case $\alpha = 2$ corresponds to usual Brownian motion. Error bars, which are standard deviations of the mean, are within the symbol size.

V. Discussion

The plot in semi-log scale of the normalized nonlinear relaxation time $T(x_0)/T_{dyn}$ as a function of the scale parameter σ is depicted in Fig. 2, where points correspond to the Monte Carlo simulation of the Langevin equation (2) while solid lines present numerical integration of the exact formula (17). As seen from Fig. 2, Monte Carlo simulations nicely corroborate exact results. Subsequent rows correspond to various values of x_0: $x_0 = 0.01$ (top panel), $x_0 = 0.1$ (middle panel) and $x_0 = 0.5$ (bottom panel).

As it can be seen in Fig. 2, for the fixed interval half-width L the NLRT $T(x_0)$ depends both on the stability index

α and the initial condition x_0. Moreover, it can be a non-monotonous function of the scale parameter σ. Middle and bottom panel of Fig. 2 demonstrate that analogously like in the noise enhanced stability [8] the increasing noise can extend the lifetime of unstable states, i.e. it results in the noise delayed decay (NDD) phenomenon. In such a case, on the one hand, the scale parameter is large enough that particle can surmount the potential barrier. On the other hand, the noise is weak enough thus majority of escapes take place into the direction of the deterministic force. Therefore, the delay in the decay is caused by excursions into the direction of the origin, which slow down the escape rate.

As follows from Fig. 2, the region of NDD (area of increasing NLRT compared with the dynamical decay time T_{dyn}) is most clearly visible at initial position of a particle x_0 close to the boundary $L = 1$ (bottom panel). At the same time, the NDD phenomenon for the noise with Lévy index $\alpha < 1$ can be observed only for very small values of the noise intensity parameter σ. It should be noted also that the effect is great for Brownian diffusion ($\alpha = 2$) than for anomalous one.

VI. Conclusions

Using analytical considerations we have derived the closed formula for the nonlinear relaxation time of Lévy flights in fully unstable potential. In the case of Cauchy noise excitation, i.e. Lévy noise with the stability index $\alpha = 1$, the general formula for the NLRT can be significantly simplified.

In case of unstable potentials, the NLRT is sensitive both to the initial position and noise parameters. In particular, it can be a non-monotonous function of the scale parameter σ. Therefore, for a fixed initial position and the value of the Lévy index it is possible to find such a value of the scale parameter which maximally slows down the escape process, i.e. it extends the lifetime of the fully unstable state.

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