Stability of slow blow-up solutions for the critical focussing nonlinear wave equation on $\mathbb{R}^{3+1}$

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Abstract In this brief survey we outline the recent advances on the stability issues of certain finite time type II blow-up solutions for the energy critical focusing wave equation $\square u = -u^5$ in $\mathbb{R}^{3+1}$. Hereafter we use the convention $\square = -\partial_t^2 + \triangle$. The objective of this article is twofold: firstly we describe the construction of singular solutions contained in [19] and [16], and secondly we undertake a detailed analysis of their stability properties enclosed in [12] and [4].

1 Introduction

Despite its naive appearance, the semilinear wave equation

$$\square u = -u^5, \quad u : \mathbb{R}^{1+3}_{t,x} \to \mathbb{R} \tag{1}$$

is an excellent simplistic model since its main features are shared with multiple geometric and physical equations such as critical Wave-Maps and Yang-Mills equations. However, as we shall see, the price to pay to avoid many technical issues is the ubiquity of type I blow-up solutions which constitute the generic blow-up scenario.

Local well-posedness up to the optimal regularity class $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ of the Cauchy problem for equation (1) coupled with initial data was proved by Lindblad and Sogge [21] and it relies on the celebrated Strichartz estimates, see also [25] for a detailed description. Moreover, as a typical trademark for focusing equations, the conserved energy

$$E(u)(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla_{t,x} u|^2 - \frac{1}{6}|u|^6 dx$$

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is not positive definite, making the extensions of local solutions to global one a highly non-trivial question.

In fact, several obstructions to long-time existence of solution of (1) have been uncovered. For instance, Levine [20] demonstrated via a convexity argument that break down in finite time occur for initial data with negative energy. Nonetheless, Levine's argument is indirect, and it does not provide much information about the exact nature of the blow-up. More primitive blow-up solutions can be explicitly constructed by the ODE technique: let $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(x) = 1$ if $|x| \leq 2T$, set the initial data $u_0(0,x) = (\frac{3}{4})^{1/4} T^{-1/2} \phi(x)$, and $u_t(0,x) = (\frac{3}{64})^{1/4} T^{-3/2} \phi(x)$. Then the solution of (1) behaves like the so called *fundamental self-similar solution*

$$u(t,x) = (\frac{3}{4})^{1/4} (T-t)^{-1/2} \tag{2}$$

for $0 < t < T$ and $|x| < T - t$. As this example shows, singularities can arise in finite time even for smooth compactly supported initial data. Observe that for these solutions the critical Sobolev norm diverges as time approaches the maximum time of existence:

$$\limsup_{t \to T} \|\nabla_{t,x} u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \to +\infty. \tag{3}$$

Motivated by such blow-up mechanism it is common to define a blow-up solution u with maximum forward time of existence $T < +\infty$ of *type I* if (3) holds, and of *type II* otherwise, that is if $\|\nabla_{t,x} u(t,\cdot)\|_{L^2}$ remains bounded up to the break down time. The dichotomy between type I and type II blow-up solutions is well understood at this point in time.

Another explicit solution of (1) is the Aubin-Talenti function

$$W(x) = \Big(1 + \frac{|x|^2}{3}\Big)^{-1/2}$$

which is the unique (up to symmetries) positive solution to the associated elliptic equation $\triangle W = -W^5$ and it is the minimizer of the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, see [1] and [26]. Through a remarkable series of works Duyckaerts, Kenig, and Merle [7, 9, 8, 10] provided a complete abstract classification of all possible type II blow-up solutions in finite time in terms of a finite number of rescaled W plus a small radiation term.

Theorem 1 ([10]) *Let u be a radial type II solution of* (1) *which breaks down in finite time T. Then there exist finitely many continuous functions $\lambda_j(t)$, $j = 1, \ldots, k$, with* $\lim_{t\to T}(T-t)\lambda_j(t) = +\infty$, *and*

$$\lim_{t \to T} \Big| \log\Big(\frac{\lambda_i(t)}{\lambda_j(t)}\Big)\Big| = +\infty, \qquad \textit{for } i \neq j,$$

such that

$$u(t,x) = \sum_{j=1}^{k} \pm W_{\lambda_j(t)}(x) + \eta(t,x)$$

and where $(\eta, \partial_t \eta) \in C([0,T], \dot{H}^1 \times L^2)$ *and* $W_\lambda(x) = \lambda^{1/2} W(\lambda x)$.

The extent of the previous result is essential in the progress of understanding type II blow-up solutions. However, due to the nature of the arguments, namely the famous concentration compactness method, the Duyckaerts, Kenig, and Merle program does not demonstrate the existence of all such possible blow-up dynamics. In fact, at the best of author's knowledge it emerges that only finite time blow-up solutions with one bulk term W are known to exist. Moreover, the precise blow-up dynamics is unknown and it does not appear to give any information on the stability of such solutions.

Complementary, an explicit finite time type II blow-up was constructed by Krieger, Schlag and Tataru [19]. The breakthrough [19] consists in establishing the existence of a family of rough blow-up solutions displaying a continuum of blow-up rates slower then the one provided by the self-similar blow-up. In addition, all previously known blow-up solutions become singular along a hypersurface, vice-versa the ones furnished in [19] exhibit a one-point blow-up. In a subsequent work [16], the first two authors extended the range of allowed blow-up speeds up to reach arbitrary close the self-similar blow-up speed.

Other concrete realizations for finite-time type II dynamics where established: Hillairet and Raphaël [11] constructed type II smooth solution for the energy critical semilinear wave equation $\Box u = -u^3$ in $\mathbb{R}^{4+1}$, with the fixed scaling law

$$\lambda(t) = t^{-1} e^{\sqrt{|\log t|}}, \qquad \text{as } t \to 0.$$

The set of initial data leading to such type II blow-up is given by a co-dimension one Lipschitz manifold. Another constructive approach was given in Krieger, Donninger, Huang, and Schlag [13], where the authors provided a finite time blow-up solution of type II with oscillating scaling law, that is of the form $u(t,x) = W_{\lambda(t)}(x) + \eta(t,x)$ where $\lambda(t) = t^{-\nu(t)}$ and $\nu(t) = \nu + \epsilon_0 \frac{sin(\log t)}{\log t}$, with $\nu > 3$ and $|\epsilon_0| \ll 1$ be arbitrary and η a small error.

A deeper study of the stability of such blow-up scenarios has been the subject of a number of recent works. Persuaded by numerical evidence provided by Bizon et al. [3], which suggested that finite time blow-up for (1) are generically of type I, in a sequence of pioneering works Donninger and Schörkhuber [6] and Donninger [5] settled the asymptotic stability of the ODE blow-up solution (2) in the energy norm. On the other hand, Krieger, Nakanishi, and Schlag [15] elucidated that type II solutions are unstable in the energy norm in the following precise sense.

Theorem 2 *([15]) Let* $\lambda(t) \to +\infty$ *as* $t \to T$*, and*

$$u(t,x) = W_{\lambda(t)}(x) + \eta(t,x)$$

be a type II blow up solution on $I \times \mathbb{R}^3$ *for* (1)*, such that*

$$\sup_{t\in I} \|\nabla_{t,x}\eta(t,\cdot)\|_{L^2_x} \leq \delta \ll 1$$

for some sufficiently small $\delta > 0$, where $I = [0,T]$ denotes the maximal life span of the Shatah-Struwe solution u. Also, assume that $t_0 \in I$. Then there exists a co-dimension one Lipschitz manifold Σ in a small neighborhood of the data $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big) \in \Sigma$ in the energy topology $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, such that initial data $(f,g) \in \Sigma$ result in a type II solution, while initial data

$$(f,g) \in B_\delta \backslash \Sigma,$$

where $B_\delta \subset \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is a sufficiently small ball centered at $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big)$, either lead to blow-up in finite time, or solutions scattering to zero, depending on the 'side of Σ' these data are chosen from.

In spite of the universality of type I blow-up for equation (1), with the purpose of study more sophisticated equations at the critical regime where only type II dynamic is present, it is fundamental to investigate further type II blow-up solutions and its stability properties. The stability of solutions constructed in [16] and [19] was analyzed by Krieger [12] where a conditional result requiring two extra co-dimensions was obtained for solutions which blow-up at a rate sufficiently close to the self-similar one. The optimal stability result was achieved by the author and Krieger in [4]. In the second part of this article we outline the proof of the latter results.

To place these results in a proper context, some more discussion on similar results for different equations is in order. As a matter of fact, the work [19] is an occurrence in a triplets of works [19, 17, 18], dedicated to the explicit construction of rough type II singular solutions respectively for semilinear wave equations, for the co-rotational critical wave maps from $\mathbb{R}^{2+1} \to S^2$, and for the critical Yangs-Mills equations in $4+1$ dimensions under the spherically symmetric ansatz. A parallel construction of a smooth finite time type II singular solution with fixed blow-up speed was carried out by Raphaël and Rodnianski [23] for the co-rotational critical wave maps in $2+1$ dimensions with S^2 target, and for the critical $SO(4)$ Yangs-Mills equations in $4+1$ dimensions. Concerning the stability issue, the method employed by Raphaël and Rodnianski implies that their solutions are stable. Furthermore, in a recent breakthrough Krieger and Miao [14] were able to show that the solutions constructed in [17] for the co-rotational critical wave maps are stable in a suitable topology. The corresponding result for the Yang-Mills problem is still open.

2 The construction of slow blow-up solutions

In this section we describe the construction of explicit finite time type II blow-up solutions contained in the works [16] and [19]. We shall be interested exclusively in the case of radial solutions, thus the energy critical focusing semilinear wave equation under radial symmetry can be written as:

$$-u_{tt} + u_{rr} + \frac{2}{r} u_r = -u^5. \tag{4}$$

The goal is to construct a solution $u \in C((0, t_0], H^{1+}) \times C^1((0, t_0], H^+)$ of (4) which blows-up at the space-time origin, and the blow-up is of type II, hence its space-time gradient remains bounded on the interval of existence: $\sup_{t \in [0,t_0]} \|\nabla_{t,x} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} < \infty$. Notice that, due to the time reversibility of the wave equation, we start evolving the dynamics from initial data at time $t_0 > 0$ and solve the wave equation backwards in time until the blow-up time $t = 0$. Here t_0 is a small positive constant that will be defined later.

We state here the results of [19] and [16]. The main difference between them is the lower bound for ν. In [19] the restriction $\nu > 1/2$ was imposed, and in [16] the result was extended to include $\nu > 0$.

Theorem 3 *([19], [16]) Let $\nu > 0$ and $\lambda(t) = t^{-1-\nu}$ the scaling parameter. There exists a class of solutions to equation* (4) *of the form*

$$u_\nu(t, r) = W_{\lambda(t)}(r) + \eta(t, r) =: u_0(t, r) + \eta(t, r)$$

inside the truncated light cone $K = \{(t, r) \in (0, t_0) \times \mathbb{R}^+ : \ t > r\}$. The term u_0 is called bulk term or non-oscillatory elliptic term and it is given by the rescaling of W. The second term η is called oscillatory radiation part and it is composed by two distinct functions: $\eta = \eta^e + \varepsilon$. Here η^e is an non-oscillatory term satisfying $\eta^e \in C^\infty(K)$ and $\mathcal{E}_{loc}(\eta^e)(t) \lesssim (t\lambda(t))^{-2}|\log t|^2$ as $t \to 0$, hence its local energy vanishes as time $t \to 0$. The local energy relative to the origin is defined as

$$\mathcal{E}_{loc}(u)(t) = \int_{|x|<t} \frac{1}{2}|\nabla_{t,x} u|^2 - \frac{1}{6}|u|^6 dx.$$

On the other hand ε is rougher, that is $(\varepsilon(t, \cdot), \varepsilon_t(t, \cdot) \in (H^{\frac{\nu}{2}+1-}(\mathbb{R}^3) \times H^{\frac{\nu}{2}-}(\mathbb{R}^3))$ and $\mathcal{E}_{loc}(\varepsilon)(t) \to 0$ as $t \to 0$. Moreover, outside the light cone we have the bound

$$\int_{|x|\geq t} \frac{1}{2}|\nabla_{t,x} u_\nu|^2 - \frac{1}{6}|u_\nu|^6 dx \leq C < \infty.$$

Notice that the bulk term $u_0 \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ blows-up at the space-time origin. However $u_0(t, \cdot) \notin L^2(\mathbb{R}^3)$, since it does not decay sufficiently fast at space infinity. To obtain a solution in H^{1+} it suffices to multiply u_0 by a bump function $\chi \in C_0^\infty(\mathbb{R})$ which equals one on the ball of radius t_0. In this way, on every fixed positive time slice we clearly have $u_0(t, \cdot)\chi(t) \in H^{1+}(\mathbb{R}^3)$ and $\partial_t(u_0(t, \cdot)\chi(t)) \in H^+(\mathbb{R}^3)$. Notice that $u_0(t, \cdot)\chi(t, \cdot)$ is in $C_0^\infty(\mathbb{R}^3)$ therefore clearly is type II. The rougher part of the solution which gives the overall regularity $C((0, t_0], H^{\frac{\nu}{2}+1-}) \times C^1((0, t_0], H^{\frac{\nu}{2}-})$ is the term η^e. In fact, although it is smooth inside the light cone, namely $\eta^e \in C^\infty(K)$, it reveals a cusp singularity along the boundary ∂K of the light cone, implying that $\eta^e(t, \cdot) \in H^{1+\frac{\nu}{2}-}_{loc}(\mathbb{R}^3)$ and $\partial_t \eta^e(t, \cdot) \in H^{\frac{\nu}{2}-}_{loc}(\mathbb{R}^3)$.

Before outlining the proof of Theorem 3, a final remark on the blow-up speed of u_ν is in order. Clearly, the $L^\infty(\mathbb{R}^3)$ norm of the ODE blow-up solution (2) concentrates, as time approaches the break-down time, at a rate which is proportional to $t^{-1/2}$. Diversely, the blow-up of speed of u_ν solutions is proportional to $t^{-1/2-\nu/2}$. Hence type II solutions blow-up faster that type I and, as ν approaches 0, the different blow-up speeds become comparables. Moreover, by varying the parameter $\nu > 0$, which it is not a priori fixed, we obtain a blow-up solution with prescribed blow-up speed, i.e. a *continuum of blow-up speeds*.

The proof of Theorem 3 is based on a two steps procedure mimicking the strategy of others constructions of type II dynamics contained in [11, 23, 22]. Firstly, one constructs a sequence of approximate solutions which solve (4) up to a small error. This approximation method will not lead to an exact solution by passing to the limit due to the divergence of the coefficients. Hence one needs to terminate the process after finitely many steps. Secondly, one completes the approximate solution to an exact solution via a fixed point argument. In regard to the second step, the argument used to prove Theorem 3 differs drastically from the strategy employed in [11, 23, 22]. In the latter pioneering works the remaining error is controlled via Morawetz and viral type identities, whereas the present proof hinges on a constructive parametrix approach.

2.1 The renormalization step

The aim of this step is to iteratively construct a very accurate approximate solution near the singularity depending on two parameters k and ν which has the form

$$u_k(t,r) = W_{\lambda(t)}(r) + \eta_k^e(t,r)$$

where the k-th non-oscillatory term $\eta_k^e(t,r) = \sum_{j=1}^k v_j(t,r)$ is a sum of small corrections and $\lambda(t) = t^{-1-\nu}$. The bulk term $u_0(t,r) = W_{\lambda(t)}(r)$ is very far from being an approximate solution of (4), indeed it produces an error $e_0 = \Box u_0 + (u_0)^5$ which blows up like t^{-2} as $t \to 0$. In [19] the authors adopt the strategy of adding successive corrections functions v_j so that the the error $e_k = \Box u_k + u_k^5$ generated by the approximate solution u_k can be made arbitrary small in a suitable sense by picking k suitable large. More precisely, the corrections v_j are chosen in order to force e_k to go to zero like t^N as $t \to 0$ in the energy norm restricted to a light cone, where N can be made arbitrarily large by taking k large.

The construction consists in a delicate bookkeeping procedure to iteratively reduce the size of the generated error by alternating between amelioration near the spatial origin and improvements near the light cone. The finite sequence of approximate solution u_k is defined recursively. Set $u_0 = W_{\lambda(t)}(r)$, then for $k \geq 1$ the k-th approximation u_k is given in terms of the previous one via the following algorithm: let u_{k-1} be the approximate solution which generates the error $e_{k-1} = \Box u_{k-1} + u_{k-1}^5$, then one updates u_{k-1} by adding a correction, i.e. $u_k = u_{k-1} + v_k = u_0 + v_1 + \cdots + v_k$, thereby the error e_k produced by the improved approximation u_k is smaller than e_{k-1} in a suitable sense. To define the appropriate correction v_k we distinguish between k

even or k odd. The odd corrections are the solutions of the following inhomogeneous second order ODEs:

$$\begin{cases} (\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4(t,r))v_{2k-1}(t,r) = e_{2k-2}(t,r) & \text{in } \mathbb{R}_r^+, \\ v_{2k-1}(t,0) = \partial_r v_{2k-1}(t,0) = 0. \end{cases} \tag{5}$$

The heuristic which leads to such formulation is that when $r \ll t$ we expect the term involving the time derivative in (4) to be negligible. On the other hand, for even corrections we improve the approximate solution near the light cone $r \approx t$, thus we can roughly estimate u_0 by zero and we are led to the $1 + 1$-inhomogeneous hyperbolic equation

$$\begin{cases} (-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)v_{2k}(t,r) = e_{2k-1}(t,r) & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_r^+, \\ v_{2k}(t,0) = \partial_r v_{2k}(t,0) = 0. \end{cases} \tag{6}$$

The Cauchy problem (5) is a standard Sturm-Liouville problem and it is solved via the variation of parameter method. Whereas the hyperbolic character of (6) is controlled by using self-similar coordinate $a = r/t$ and a brillant ansatz on the form of the solution.

2.2 Completion to an exact solution

The main point of this second part of the argument is to perturb around the approximate solution constructed in the previous step, and thus to look for an exact solution of (4) of the form $u_\nu = u_{2k-1} + \varepsilon$. Notice that we stop the approximation algorithm after an odd number of cycles. By imposing that u_ν to be an exact solution we force an equation for ε:

$$(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r)\varepsilon - 5\lambda^2(t)W^4(\lambda(t)r)\varepsilon = e_{2k-1} + \tilde{N}(u_{2k-1}, \varepsilon) \tag{7}$$

where $\tilde{N}(u,\varepsilon) = (\varepsilon + u)^5 - u^5 - 5\varepsilon u_0^4$. To avoid treating a nonlinear hyperbolic equation with time-dependent potential one removes the time dependency of the potential by introducing new coordinates $(t,r) \to (\tau, R)$, where

$$\tau(t) = \int_t^{t_0} \lambda(s)ds + \nu^{-1}t_0^{-\nu} = \nu^{-1}t^{-\nu}, \qquad R(t,r) = \lambda(t)r.$$

The price to pay is that the time derivative ∂_t is transformed into the operator $\lambda(\tau)\partial_\tau + \frac{\lambda'(\tau)}{\lambda(\tau)}R\partial_R$. Let us set $v(\tau,R) = \varepsilon(t(\tau), \lambda^{-1}(\tau)R)$ and $\beta(\tau) = \lambda'(\tau)/\lambda(\tau)$, then equation (7) is transformed into

$$\Big[\Big(\partial_\tau + \beta(\tau)R\partial_R\Big)^2 - \beta(\tau)\Big(\partial_\tau + \beta(\tau)R\partial_R\Big) - \partial_R^2 - \frac{2}{R}\partial_R\Big]v(\tau,R) - 5W^4(R)v(\tau,R) = \lambda^{-2}(\tau)e_{2k-1}(\tau,R) + \lambda^{-2}(\tau)\tilde{N}(u_{2k-1},v)(\tau,R).$$

Subsequently, in order to get rid of the first derivative in the R variable, we consider the function $\tilde{\varepsilon}(\tau,R) = Rv(\tau,R)$, this new function satisfies the equation

$$(\mathcal{D}^2 + \beta(\tau)\mathcal{D} + \mathcal{L})\tilde{\varepsilon}(\tau,R) = f[\tilde{\varepsilon}](\tau,R), \qquad \text{in } \mathbb{R}^+_\tau \times \mathbb{R}^+_R \tag{8}$$

where $\mathcal{D} = \partial_\tau + \beta(\tau)(R\partial_R - 1)$, $\mathcal{L} = -\partial_R^2 - 5W^4(R)$, and

$$f[\tilde{\varepsilon}](\tau,R) = \lambda^{-2}(\tau)\Big(Re_{2k-1} + N(u_{2k-1},\tilde{\varepsilon})\Big)$$

and

$$N(u_{2k-1},\tilde{\varepsilon})(\tau,R) = 5\tilde{\varepsilon}(u_{2k-1}^4 - u_0^4) + R\Big(\frac{\tilde{\varepsilon}}{R} + u_{2k-1}\Big)^5 - Ru_{2k-1}^5 - 5u_{2k-1}^4\tilde{\varepsilon}.$$

To look for a solution of equation (8) a prototypical Fourier transform, namely the *distorted Fourier transform associated to the operator* $\mathcal{L}$, is applied imitating the procedure to convert to the frequencies sides the free wave equation. The spectral properties of the operator $\mathcal{L}$ play a pivotal role and are analyzed in details in [19]. This operator, when restricted to functions on $[0,\infty)$ with Dirichlet condition at $R = 0$, has a simple negative eigenvalue $\xi_d < 0$ (the subscript d referring to *discrete spectrum*), and a corresponding L^2-normalized positive ground state $\phi_d \in L^2(0,\infty) \cap C^\infty([0,\infty))$ decaying exponentially and vanishing at the origin $R = 0$. This mode will cause exponential growth for the linearized evolution $e^{it\sqrt{\mathcal{L}}}$. However, in [19] and [16] the authors avoid this problem by imposing vanishing initial data at $t = 0$ for the function ε, which is equivalent to impose zero data at $\tau = \infty$ for the function $\tilde{\varepsilon}$. In the subsequent works [12] and [4], where no such freedom of imposing zero initial data is acceptable, only a co-dimension one condition will ensure that the forward flow will remains bounded.

Let us present below the pivotal result which summarize the main properties of the distorted Fourier transform.

Proposition 1 *([19]) There exists a generalized Fourier basis $\phi(R,\xi)$, $\xi \geq 0$, a eigenstate $\phi_d(R)$, and a spectral measure $\rho(\xi) \in C^\infty((0,\infty))$ with the asymptotic behaviors*

$$\rho(\xi) \sim \begin{cases} \xi^{-\frac{1}{2}}, & \text{if } 0 < \xi \ll 1, \\ \xi^{\frac{1}{2}}, & \text{if } \xi \gg 1, \end{cases}$$

as well as symbol behaviour with respect to differentiation, and such that by defining

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \lim_{b\to+\infty}\int_0^b \phi(R,\xi)f(R)\,dR,$$
$$\hat{f}(\xi_d) = \int_0^\infty \phi_d(R)f(R)\,dR,$$

the map $f \longrightarrow \hat{f}$ is an isometry from L^2_{dR} to $L^2(\{\xi_d\} \cup \mathbb{R}^+, \rho)$, and we have

$$f(R) = \hat{f}(\xi_d)\phi_d(R) + \lim_{\mu\to\infty} \int_0^{\mu} \phi(R,\xi)\hat{f}(\xi)\rho(\xi)\, d\xi,$$

the limits being in the suitable L^2-sense.

The mayor issue in applying the distorted Fourier transform to equation (8) is the term involving $R\partial_R$ contained in the $\mathcal{D}$ operator since $\mathcal{F}(R\partial_R) \neq \xi\partial_\xi \mathcal{F}$. Therefore one defines the *error operator* $\mathcal{K}$ via the equation

$$\mathcal{F}[(R\partial_R - 1)u(\tau, R)](\xi) = \mathcal{A}\hat{u}(\tau,\xi) + \mathcal{K}\hat{u}(\tau,\xi)$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_c \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} \mathcal{K}_{dd} & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_{cc} \end{pmatrix}$$

and $\mathcal{A}_c = -2\xi\partial_\xi - \left(\frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}\right)$. We add the second term in $\mathcal{A}_c$ because later on we shall need the relation $(SM)^{-1}\partial_\tau SM = \mathcal{D}_\tau$ where $\mathcal{D}_\tau = \partial_\tau + \beta(\tau)\mathcal{A}_c$. In other words $\mathcal{K}$ is defined as the solution to the system

$$\begin{cases} \langle (R\partial_R - 1)u(\tau,R), \phi_d\rangle_{L^2_{dR}} = \mathcal{K}_{dd}\langle u(\tau,R),\phi_d\rangle_{L^2_{dR}} + \mathcal{K}_{dc}\hat{u} \\ \mathcal{F}[(R\partial_R - 1)u(\tau,R)] = \mathcal{A}_c\hat{u} + \mathcal{K}_{cd}\langle u(\tau,R),\phi_d\rangle_{L^2_{dR}} + \mathcal{K}_{cc}\hat{u} \end{cases}$$

and we have

$$\mathcal{K}_{dd} = -\frac{1}{2}, \qquad \mathcal{K}_{cd}(\xi) = k_d(\xi),$$

$$\mathcal{K}_{dc}(f) = -\int_0^\infty f(\xi)k_d(\xi)\rho(\xi)d\xi, \qquad \mathcal{K}_{cc}[f](\xi) = \int_0^\infty \frac{F(\xi,\eta)}{\xi - \eta} f(\eta)\rho(\eta)d\eta.$$

where $k_d(\xi)$ is a smooth and rapidly decaying function at $\xi = +\infty$ and the function F is of regularity at least C^2 on $(0,\infty)\times(0,\infty)$, and satisfies further smoothness and decay properties listed in Theorem 5.1 in [19].

We now proceed to transpose equation (8) to the Fourier side. Notice that the time variable remain invariant since we are dealing with Fourier transform in space only. Let us denote the distorted Fourier transform of the unknown function in (8) by $(x_d(\tau), x(\tau,\xi)) = \mathcal{F}(\tilde{\varepsilon})(\tau,\xi)$, that is:

$$x(\tau,\xi) = \int_0^\infty \phi(R,\xi)\tilde{\varepsilon}(\tau,R)\, dR, \qquad x_d(\tau) = \int_0^\infty \phi_d(R)\tilde{\varepsilon}(\tau,R)\, dR.$$

Notice that once the Fourier representation $(x_d(\tau), x(\tau,\xi))$ is known one can easily recover the original function $\tilde{\varepsilon}$ via

$$\tilde{\varepsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)d\xi. \tag{9}$$

Applying the distorted Fourier transform to equation (8) yields to the following system involving one equation for the discrete spectral part and a second equation for the continuous spectral part:

$$(\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi})\mathbf{x}(\tau,\xi) = \mathcal{R}\mathbf{x}(\tau,\xi) + \mathbf{f}(\tau,\xi) \tag{10}$$

where $(\tau,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\mathbf{x}(\tau,\xi) = \big(x_d(\tau), x(\tau,\xi)\big)^T$, and

$$(\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi}) = \begin{pmatrix} \partial_\tau^2 + \beta(\tau)\partial_\tau + \xi_d & 0 \\ 0 & \mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \xi \end{pmatrix}.$$

The inhomogeneous terms on the right-hand-side of (10) are composed of a linear source term

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_{dd} & \mathcal{R}_{dc} \\ \mathcal{R}_{cd} & \mathcal{R}_{cc} \end{pmatrix}$$

where

$$\begin{aligned}
\mathcal{R}_{dd} &= -2\beta(\tau)\mathcal{K}_{dd}\partial_\tau - \beta^2(\tau)\Big(\mathcal{K}_{dd}^2 + \mathcal{K}_{dc}\mathcal{K}_{cd} + \mathcal{K}_{dd} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathcal{K}_{dd}\Big),\\
\mathcal{R}_{dc} &= -2\beta(\tau)\mathcal{K}_{dc}\mathcal{D}_\tau - \beta^2(\tau)\Big(\mathcal{K}_{dd}K_{dc} + \mathcal{K}_{dc}\mathcal{K}_{cc} - \mathcal{K}_{dc}\mathcal{A}_c + \mathcal{K}_{dc} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathcal{K}_{dc}\Big),\\
\mathcal{R}_{cd} &= -2\beta(\tau)\mathcal{K}_{cd}\partial_\tau - \beta^2(\tau)\Big(\mathcal{K}_{cd}\mathcal{K}_{dd} + \mathcal{K}_{dc}\mathcal{K}_{cd} + \mathcal{A}_c\mathcal{K}_{cd} + \mathcal{K}_{cd} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathcal{K}_{cd}\Big),\\
\mathcal{R}_{cc} &= -2\beta(\tau)\mathcal{K}_{cc}\mathcal{D}_\tau - \beta^2(\tau)\Big(\mathcal{K}_{cd}\mathcal{K}_{dc} + \mathcal{K}_{cc}^2 + [\mathcal{A}_c,\mathcal{K}_{cc}] + \mathcal{K}_{cc} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathcal{K}_{cc}\Big),
\end{aligned} \tag{11}$$

plus the nonlinear term (observe that $\tilde{\varepsilon}$ depends on the unknown functions $x_d(\tau), x(\tau,\xi)$ via (9)):

$$\mathbf{f}(\tau,\xi) = \begin{pmatrix} f_d(\tau) \\ f(\tau,\xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle \phi_d, Re_{2k-1} + N(u_{2k-1},\tilde{\varepsilon})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathcal{F}\Big(Re_{2k-1} + N(u_{2k-1},\tilde{\varepsilon})\Big)(\tau,\xi) \end{pmatrix}.$$

We coupled system (10) with initial conditions $\lim_{\tau\to\infty} x_d(\tau) = \partial_\tau x_d(\tau) = 0$, and $\lim_{\tau\to\infty} x(\tau,\xi) = \mathcal{D}_\tau x(\tau,\xi) = 0$.

The advantage of system (10) is the crucial observation that it can be solved completely explicitly. In fact, define $(Sf)(\tau,\xi) = f(\tau,\lambda^{-2}(\tau)\xi)$ and $(Mf)(\tau,\xi) = \lambda^{-5/2}(\tau)\rho^{1/2}(\xi)f(\tau,\xi)$, then we have the essential identity

$$(\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \xi) = (SM)^{-1}[(\partial_\tau^2 + \beta(\tau)\partial_\tau + \lambda^{-2}(\tau)\xi)]SM$$

which provides the following parametrix

$$
\begin{aligned}
x_d(\tau) &= \int_{\tau_0}^{\infty} -\frac{1}{2|\xi_d|^{1/2}} e^{-|\xi_d|^{1/2}|\tau-\sigma|}(g_d(\sigma) - \beta(\sigma)\partial_\sigma x_d(\sigma))d\sigma, \\
x(\tau,\xi) &= \int_{\tau}^{\infty} \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\sigma)} \frac{\rho^{1/2}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{1/2}(\xi)} \frac{\sin\left[\lambda(\tau)\xi^{1/2}\int_\tau^\sigma \lambda^{-1}(u)du\right]}{\xi^{1/2}} g\Big(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)d\sigma,
\end{aligned} \tag{12}
$$

where $\mathbf{g} = (g_d, g)^T$ represent respectively the right-hand-side of (10).

A contraction argument allow us to conclude the proof by finding an appropriate solution to (10). The fix point iteration is carried out in a weighted Sobolev type spaces defined by means of the following norms. Let $\alpha \in \mathbb{R}^+$, and a function $\mathbf{u}(\xi) = (u_d, u(\xi))^T$, then define the norm

$$
\|\mathbf{u}\|^2_{L^{2,\alpha}_{d\rho}} = |u_d|^2 + \|u\|^2_{L^{2,\alpha}_{d\rho}} := |u_d|^2 + \int_0^\infty |u(\xi)|^2 \langle\xi\rangle^{2\alpha}\rho(\xi)d\xi.
$$

Notice that $\mathcal{F}$ is an isometry from $H^{2\alpha}_{dR}(\mathbb{R}^+)$ to $L^{2,\alpha}_{d\rho}(\mathbb{R}^+)$. Moreover for every τ-dependent function $\mathbf{f}(\tau,\xi) = (f_d(\tau), f(\tau,\xi))^T$ let us define the norm $\|\mathbf{f}\|_{L^{2,\alpha,N}_\rho} = \sup_{\tau>\tau_0} \tau^N \|\mathbf{f}(\tau,\cdot)\|_{L^{2,\alpha}_{d\rho}}$. Defining $\mathbf{x}$ via the explicit formulas (12), we obtain the linear estimate

$$
\|(\mathbf{x}, \mathcal{D}_\tau \mathbf{x})\|_{L^{2,\alpha+\frac{1}{2},N-2}_{d\rho} \times L^{2,\alpha,N-1}_{d\rho}} \lesssim \frac{1}{N}\|\mathbf{g}\|_{L^{2,\alpha,N}_{d\rho}}.
$$

The small factor N^{-1} is crucial for the fixed point argument to work. A similar estimate holds for the inhomogeneous terms on the right-hand-side of (10). More precisely the map $\mathbf{g}$ satisfies the bound

$$
\|\mathbf{g}\|_{L^{2,\alpha,N}_{d\rho}} \lesssim \|(\mathbf{x}, \mathcal{D}_\tau \mathbf{x})\|_{L^{2,\alpha+\frac{1}{2},N-2}_{d\rho} \times L^{2,\alpha,N-1}_{d\rho}} \tag{13}
$$

and it is locally Lipschitz as a map from $L^{2,\alpha+1/2,N-2}_{d\rho}$ to $L^{2,\alpha,N}_{d\rho}$. Here the smallness of the constant is a consequences of the smallness of the error generated by the approximate solution built in the first part of the argument and the smallness of the time interval $(0, t_0]$ where the construction holds. The lack of smoothness of the approximate solution limits the decay in frequencies, hence the nonlinear estimate (13) holds only for $\nu/4 > \alpha$.

In [19], to control the nonlinear factors enclosed in the $f(\tau,\xi)$ term, precisely to obtain the quintilinear bound

$$
H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3) \subset H^{2\alpha}(\mathbb{R}^3),
$$

the authors relies on a standard application of the Leibniz rule and Sobolev embedding, which holds for $\alpha \geq 1/8$, leading to the lower bound on the blow-up speed: $\nu > 1/2$. The latter restriction was removed in [16] by a more detailed analysis of the first iterate of the exact solution, thus yielding to the full expected range $\nu > 0$.

3 The stability of slow blow-up solutions

In what follows we outline the stability results of type II blow-up solutions u_ν constructed in [19] and [16]. The continuum of blow-up rates proper to u_ν and their limited regularity seem to indicate that these solutions are less stable than their smooth analogs built in [11]. Moreover, taking into consideration the parallel results in the parabolic setting [24], [2] it was commonly assumed that imposing a stability condition will single out a quantized set of allowed blow-up speeds. Although these observations had solid foundations, they were disproved in [12] and [4], making the stability of a family of rough solutions with varying concentration rates a unique feature of hyperbolic equations. In fact, the results [12] and [4] demonstrate that rough solutions u_ν are stable along a co-dimension one Lipschitz manifold of data perturbations in a suitable topology, provided that the blow-up speed is sufficiently close to the self-similar ones, i. e. $\nu > 0$ is sufficiently small. The result is optimal in view of [15], since any type II solution with data close enough to the ground state W can be at best stable for perturbations of the data along a co-dimension one hypersurface in energy space.

The main improvement of [4] over [12] is essentially in the number of co-dimensions imposed on the perturbations. In [12] Krieger showed that type II solutions u_ν are stable under an appropriate co-dimension three condition. Precisely, there exists a co-dimension three Lipschitz hypersurface $\Sigma_0 \subset H^{3/2+}_{rad,loc}(\mathbb{R}^3) \times H^{1/2+}_{rad,loc}(\mathbb{R}^3)$ such that if we take the perturbation of the initial data $(\varepsilon_0, \varepsilon_1) \in \Sigma_0$ small enough, then the solution of the perturbed problem

$$\begin{cases} \Box u = -u^5 \quad \text{in } (0, t_0] \times \mathbb{R}^3 \\ u[t_0] = u_\nu[t_0] + (\varepsilon_0, \varepsilon_1) \end{cases} \tag{14}$$

is a type II blow-up solution of exactly of same type as u_ν. In the subsequent work [4] the extra co-dimensions two condition was removed yielding to the optimal result. The precise statement is given below.

To properly enunciate the co-dimension conditions imposed in [12] we have to closely analyse the initial value problem on the Fourier side. We shall seek to construct a solution of (14) by perturbing around the exact solution u_ν, thus we make the following ansatz:

$$u(t, r) = u_\nu(t, r) + \varepsilon(t, r)$$

where $(\varepsilon, \partial_t \varepsilon)$ matches the initial data at time $t = t_0$: $(\varepsilon, \partial_t \varepsilon)\big|_{t=t_0} = (\varepsilon_0, \varepsilon_1)$. In analogy with the argument of the previous section we introduce the renormalized coordinates $(\tau, R) = (\nu^{-1} t^{-\nu}, \lambda(t) r)$, we set $\tilde{\varepsilon} = R\varepsilon$, and we apply the distorted Fourier transform to the equation satisfied by $\tilde{\varepsilon}$. Thus we obtain the following equation in terms of the Fourier variable $\mathbf{x}(\tau, \xi) = \mathcal{F}(\tilde{\varepsilon})(\tau, \xi)$:

$$(\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi})\mathbf{x}(\tau, \xi) = \mathcal{R}\mathbf{x}(\tau, \xi) + \mathbf{f}[\tilde{\varepsilon}](\tau, \xi) \tag{15}$$

where $(\tau, R) \in [\tau_0, \infty) \times \mathbb{R}^+$ and the linear source terms $\mathcal{R}$ are as in (11) and the nonlinear terms are defined by

$$\mathbf{f}[\tilde{\varepsilon}](\tau, \xi) = \begin{pmatrix} f_d[\tilde{\varepsilon}](\tau) \\ f[\tilde{\varepsilon}](\tau, \xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle \phi_d, N(u_\nu, \tilde{\varepsilon})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathcal{F}\left(N(u_\nu, \tilde{\varepsilon})\right)(\tau, \xi) \end{pmatrix}.$$

Instead of coupling the system (15) with vanishing initial data at $\tau = +\infty$ we shall impose initial data at the corresponding initial time $\tau = \tau_0$:

$$\begin{aligned} x_d(\tau_0) &= x_{0d}, & \partial_\tau x_d(\tau_0) &= x_{1d}, \\ x(\tau_0, \xi) &= x_0(\xi), & \mathcal{D}_\tau x(\tau_0, \xi) &= x_1(\xi). \end{aligned} \tag{16}$$

One can compute the initial data on the physical side $(\tilde{\varepsilon}_0, \tilde{\varepsilon}_1)$ in terms of the initial data on the Fourier side $(\mathbf{x}_0, \mathbf{x}_1)$, and vice-versa, via the formulas:

$$\begin{aligned} \mathcal{F}(\tilde{\varepsilon}_0) &= x_0, & -\mathcal{F}\left(\frac{\tilde{\varepsilon}_1}{\lambda}\right) &= x_1 + \beta_\nu(\tau_0)\mathcal{K}_{cc}x_0 + \beta_\nu(\tau_0)\mathcal{K}_{cd}x_{0d}, \\ \langle \phi_d, \tilde{\varepsilon}_0\rangle_{L^2_{dR}} &= x_{0d}, & -\langle \phi_d, \frac{\tilde{\varepsilon}_1}{\lambda}\rangle_{L^2_{dR}} &= x_{1d} + \beta_\nu(\tau_0)\mathcal{K}_{dd}x_{0d} + \beta_\nu(\tau_0)\mathcal{K}_{dc}x_0. \end{aligned} \tag{17}$$

We now present the Theorem contained in [4] which states that the blow-up phenomenon described in Theorem 3 is stable under a suitable co-dimension one class of data perturbations.

Theorem 4 *([4])Assume $0 < \nu \ll 1$, and assume $t_0 = t_0(\nu) > 0$ is sufficiently small, so that the solutions u_ν constructed in [16] and [19] exist on $(0, t_0] \times \mathbb{R}^3$. Let $\delta_1 = \delta_1(\nu) > 0$ be small enough, and let $\mathcal{B}_{\delta_1} \subset \tilde{S} \times \mathbb{R}$ be the δ_1-vicinity of $((0,0),0) \in \tilde{S} \times \mathbb{R}$, where $\tilde{S}$ is the Banach space defined as the completion of $C_0^\infty(0,\infty) \times C_0^\infty(0,\infty)$ with respect to the norm*

$$\|(x_0, x_1)\|_{\tilde{S}} = \|\langle\xi\rangle^{\frac{1}{2}+2\delta_0} \min\{\tau_{0,0}\xi^{\frac{1}{2}}, 1\}^{-1}\xi^{\frac{1}{2}-\delta_0}x_0\|_{L^2_{d\xi}} + \|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1\|_{L^2_{d\xi}}.$$

Then there is a Lipschitz function $\gamma_1 : \mathcal{B}_{\delta_1} \to \mathbb{R}$, such that for any triple $((x_0, x_1), x_{0d}) \in \mathcal{B}_{\delta_1}$, the quadruple

$$((x_0, x_1), (x_{0d}, x_{1d})), \ x_{1d} = \gamma_1(x_0, x_1, x_{0d})$$

determines a data perturbation pair $(\varepsilon_0, \varepsilon_1) \in H^{\frac{3}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3) \times H^{\frac{1}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3)$ via (17)*, and such that the perturbed initial data*

$$u_\nu[t_0] + (\varepsilon_0, \varepsilon_1) \tag{18}$$

lead to a solution $\tilde{u}(t,x)$ on $(0, t_0] \times \mathbb{R}^3$ admitting the description

$$\tilde{u}(t,x) = W_{\tilde{\lambda}(t)}(x) + \epsilon(t,x), \ (\varepsilon(t,\cdot), \varepsilon_t(t,\cdot)) \in H^{1+\frac{\nu}{2}-}_{rad,loc} \times H^{\frac{\nu}{2}-}_{rad,loc}$$

where the parameter $\tilde{\lambda}(t)$ *equals* $\lambda(t)$ *asymptotically*

$$\lim_{t\to 0} \frac{\tilde{\lambda}(t)}{\lambda(t)} = 1.$$

The proof of Theorem 4 builds on the previous work [12] thence let us describe below the main ingredients contained in the latter breakthrough.

3.1 Conditional stability result

The strategy of [12] consists in solving system (15) coupled with (16) iteratively: define the following sequence $\mathbf{x}^{(j)}(\tau,\xi) = \mathbf{x}^{(0)}(\tau,\xi) + \sum_{k=1}^{j} \triangle\mathbf{x}^{(k)}(\tau,\xi)$, where the *zero-th iterate* solves the homogeneous system:

$$\begin{cases} (\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi})\mathbf{x}^{(0)}(\tau,\xi) = 0 \\ (\mathbf{x}^{(0)}, \mathcal{D}_\tau \mathbf{x}^{(0)})|_{\tau=\tau_0} = (\mathbf{x}_0, \mathbf{x}_1) \end{cases}$$

and the the *k-th increment* $\triangle\mathbf{x}^{(k)}$ satisfies the inhomogeneous equation:

$$\begin{cases} (\mathcal{D}_\tau^2 + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi})\triangle\mathbf{x}^{(k)}(\tau,\xi) = \mathcal{R}\triangle\mathbf{x}^{(k-1)}(\tau,\xi) + \triangle\mathbf{f}^{(k-1)}(\tau,\xi) \\ (\triangle\mathbf{x}^{(k)}, \mathcal{D}_\tau \triangle\mathbf{x}^{(k)})|_{\tau=\tau_0} = (\triangle\tilde{\tilde{\mathbf{x}}}_0^{(k)}, \triangle\tilde{\tilde{\mathbf{x}}}_1^{(k)}) \end{cases}$$

where $\triangle\mathbf{f}^{(0)} = \mathbf{f}[\tilde{\varepsilon}^{(0)}]$ and $\triangle\mathbf{f}^{(k-1)} = \mathbf{f}[\tilde{\varepsilon}^{(k-1)}] - \mathbf{f}[\tilde{\varepsilon}^{(k-2)}]$ for $j \geq 2$.

As expected from the presence of a resonance of the operator $\mathcal{L}$, an accurate analysis of the zero-th iterate reveals that this term is fast growing toward $\tau = +\infty$. The growth of its discrete spectral part is easily controlled by imposing a vanishing condition on x_{0d} and x_{1d}. However, the growth of the continuous spectral part is more fundamental and it can be investigated via the explicit homogeneous parametrix:

$$\begin{aligned} x^{(0)}(\tau,\xi) &= S[x_0,x_1](\tau,\xi) \\ &:= \frac{\lambda^{5/2}(\tau)}{\lambda^{5/2}(\tau_0)} \frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\big)}{\rho^{1/2}(\xi)} \cos\Big[\lambda(\tau)\xi^{1/2}\int_{\tau_0}^{\tau} \lambda^{-1}(u)du\Big] x_0\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big) \\ &+ \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\tau_0)} \frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\big)}{\rho^{1/2}(\xi)} \frac{\sin\Big[\lambda(\tau)\xi^{1/2}\int_{\tau_0}^{\tau} \lambda^{-1}(u)du\Big]}{\xi^{1/2}} x_1\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big). \end{aligned}$$

Since $\lambda(\tau) \approx \tau^{1+\nu^{-1}}$, hence $x^{(0)}$ grows polynomially in τ. To control such a growth, the following natural co-dimensions two condition on the initial data (x_0, x_1) is imposed:

$$\begin{aligned} &\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(s)\,ds]\,d\xi = 0, \\ &\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(s)\,ds]\,d\xi = 0. \end{aligned} \tag{19}$$

Albeit such *vanishing relations* do not eliminate completely the growth of $x^{(0)}$ at infinity but only reduces it to linear growth, it is sufficient to run the iteration scheme. In fact, by choosing $\nu \leq 1/3$ and thanks to the decaying factor $\lambda^{-2}(\tau)$ appearing in the nonlinear terms $\triangle\mathbf{f}^{(k-1)}$ one can control them in a relatively straightforward way.

Let us briefly discuss the role of the corrections $(\triangle\tilde{\tilde{x}}_0^{(k)}, \triangle\tilde{\tilde{x}}_1^{(k)})$, which a priori should be both set to zero. At each iterative step, the continuous spectral part of the k-th increments are computed via the two explicit parametrices:

$$\triangle x^{(k)} = I[\mathcal{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}] + S[\triangle\tilde{\tilde{x}}_0^{(k)}, \triangle\tilde{\tilde{x}}_1^{(k)}]$$

where $I[g]$ is the Duhamel parametrix for the inhomogeneous problem with source g and vanishing initial data at $\tau = \tau_0$:

$$I[g] = \int_{\tau_0}^{\tau} \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\sigma)} \frac{\rho^{1/2}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{1/2}(\xi)} \frac{\sin\left[\lambda(\tau)\xi^{1/2}\int_\tau^\sigma \lambda^{-1}(u)du\right]}{\xi^{1/2}} g\Big(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)d\sigma.$$

To control the $\tilde{S}$ norm of the low-frequencies component of the k-th increment $(\triangle x^{(k)}, \mathcal{D}_\tau\triangle x^{(k)})$ one splits $I[\mathcal{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}]$, the inhomogeneous parametrix with vanishing initial data at $\tau = \tau_0$, into $I_{>\tau}[\mathcal{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}]$, an inhomogeneous parametrix with vanishing initial data at $\tau = +\infty$, plus $S[\tilde{\triangle}\tilde{x}_0^{(k)}, \tilde{\triangle}\tilde{x}_1^{(k)}]$, a homogeneous solutions with non-vanishing initial data at $\tau = \tau_0$. Therefore we obtain

$$\triangle x^{(k)} = I_{>\tau}[\mathcal{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}] + S[\tilde{\triangle}\tilde{x}_0^{(k)} + \triangle\tilde{\tilde{x}}_0^{(k)}, \tilde{\triangle}\tilde{x}_1^{(k)} + \triangle\tilde{\tilde{x}}_1^{(k)}].$$

The corrections $\triangle\tilde{\tilde{x}}_{0,1}^{(k)}$ ensure that the small error introduced in the initial data will preserve the vanishing conditions (19), leading to an approximation $\varepsilon^{(j)}$ on the physical side with controlled growth. Therefore to guarantee that the vanishing conditions holds throughout each step one needs to adjust the initial data by adding a small correction.

The final portion of [12] consists in proving that such iteration scheme converges by picking τ_0 sufficiently large. This is achieved via a re-iteration argument of the inhomogeneous parametrix which allows to gain enough smallness and to obtain a convergent series. A similar procedure was employed in [18] and [13]. Once the convergence is established, we obtain a solution of system (15) that fulfills the initial data requirements where $(\mathbf{x}_0, \mathbf{x}_1)$ have been replaced by $(\mathbf{x}_0 + \triangle\mathbf{x}_0, \mathbf{x}_1 + \triangle\mathbf{x}_1)$. The corrections $\triangle\mathbf{x}_{0,1}$ are obtained by summing up all the k-th step corrections $\triangle\tilde{\tilde{\mathbf{x}}}_{0,1}^{(k)}$. Moreover, they are small with respect to the $\tilde{S}$ norm when compared to the original initial data $(\mathbf{x}_0, \mathbf{x}_1)$ and they depend in Lipschitz continuous fashion on $\mathbf{x}_{0,1}$.

3.2 Optimal stability result

The elimination of the extra-vanishing conditions (19) imposed on the perturbation accomplished in [4] is attained in a four steps argument. Firstly, notice that one cannot time translate the solution u_ν without introducing an error of regularity $H^{1+\nu/2-}_{rad,loc}(\mathbb{R}^3)$ on each time-slice, that is too weak since the tollerate regularity of the perturbations is $H^{3/2+}_{rad,loc}(\mathbb{R}^3)$. Therefore a subtle modulation of the scaling law $\lambda(t) = t^{-1-\nu}$ is required. Precisely, one needs to work with a more flexible scaling law depending on two additional parameters γ_1 and γ_2. We stipulate the following ansatz:

$$\lambda^{(\gamma)}(t) = \Big(1 + \gamma_1 \frac{t^N}{\langle t^N \rangle} + \gamma_2 \log t \frac{t^N}{\langle t^N \rangle}\Big) t^{-1-\nu}, \tag{20}$$

here $N \gg 1$ is sufficiently large. Notice that $\lambda^{(\gamma)}$ asymptotically equals to λ as $t \to 0$, and such alteration implies a corresponding adjustment of renormalized coordinates (τ, R): let us introduce

$$\tau^{(\gamma)}(t) = \int_t^{t_0} \lambda^{(\gamma)}(s) ds + \nu^{-1} t_0^{-\nu}, \qquad R^{(\gamma)}(t,r) = \lambda^{(\gamma)}(t) r.$$

A similar iterative procedure that gave rise to the approximate solutions in [19] and [16] can be applied for the more general scaling law (20) to build approximate solutions of the form

$$u^{(\gamma)}_{app}(t,r) = W_{\lambda^{\gamma}(t)}(r) + \sum_{l=1}^{2k-1} v_l(t,r) + \sum_{a=1,2} v_{smooth,a}(t,r) + v(t,r)$$

which solves $\Box u^{(\gamma)}_{app} + (u^{(\gamma)}_{app})^5 = e^{(\gamma)}_{app}$ and where the error satisfies

$$\begin{aligned} e^{(\gamma)}_{app} = (|\gamma_1| + |\gamma_2|)\Big[&O\Big(\log t \frac{\lambda^{1/2} R}{(t\lambda)^{k_0+4}} (1 + (1-a)^{1/2+\nu/2})\Big) \\ &+ O\Big(\log t \frac{\lambda^{1/2} R^{-1}}{(t\lambda)^{k_0+2}} (1 + (1-a)^{1/2+\nu/2})\Big)\Big]. \end{aligned}$$

The main novelty is that we perturb around $W_{\lambda^{(\gamma)}}$ as opposed to W_λ, which when inserted into the equation (4), generates additional error terms. We isolate the terms of the error which depend on $\gamma_{1,2}$ from the part which do not depend on $\gamma_{1,2}$. The former error terms are treated by adding a finite number of corrections v_l following the iterative scheme in [19] and [16]. On the other hand, the latter error terms are decimated by the two corrections $v_{smooth,a}$ which have better regularity property than the previous corrections. The final correction v is introduced to further improve the overall regularity to the error term.

Next, in the modulation step, one shows how to tune the parameters γ_1 and γ_2 such that a comparable procedure from [12] can be applied. Precisely, our point of departure is a singular type II solution constructed in the previous papers [19] and

[16] which has the form $u_\nu = u_{2k-1} + \varepsilon$. Denote the associated initial data on the $t = t_0$ time slice by $(\varepsilon_1, \varepsilon_2)$ and consider $(\mathbf{x}_0, \mathbf{x}_1)$ the corresponding initial data at $\tau = \tau_0$ on the distorted Fourier side (with respect to R) computed via the relations (17). The point is that the initial data $(\mathbf{x}_0, \mathbf{x}_1)$ do not satisfy the vanishing conditions (19) with respect to scaling law λ anymore, thence we can not directly apply the argument of [12] as outlined in the previous section. To circumvent this impasse, we shall seek to complete the approximation $u^{(\gamma)}_{app}$ to an exact solution to the critical focusing wave equation (4) by introducing the function $\bar{\varepsilon}$:

$$u = u^{(\gamma)}_{app} + \bar{\varepsilon}. \tag{21}$$

Denote $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) = \bar{\varepsilon}[t_0]$ the associated initial data of the new perturbation on the $t = t_0$ time slice and consider $(\mathbf{x}^{(\gamma)}_0, \mathbf{x}^{(\gamma)}_1)$ the corresponding initial data at $\tau = \tau_0$ on the distorted Fourier side with respect to $R^{(\gamma)}$. We impose the following relations on the $t = t_0$ time slice

$$\bar{\varepsilon}_0 = \chi_{r \lesssim t_0}\left[W_\lambda(r) - W_{\lambda^{(\gamma)}}(r) - v_{smooth,1,2} - v\right] + \varepsilon_0,$$

as well as

$$\bar{\varepsilon}_1 = \chi_{r \lesssim t_0}\left[\partial_t\left[W_\lambda(r) - W_{\lambda^{(\gamma)}}(r)\right] - \partial_t v_{smooth,1,2} - \partial_t v\right] + \varepsilon_1.$$

Then one proves that there exists a unique choice of the parameters $\gamma_{1,2}$ such that the corresponding vanishing conditions (19) for $(x^{(\gamma)}_0, x^{(\gamma)}_1)$ with respect to the scaling law (20) are satisfied.

Subsequently, we plug the ansatz (21) into (4) to find a corresponding equation for the perturbation $\bar{\varepsilon}$. Proceeding as in the previous section we solve such equation by passing to the distorted Fourier side with respect to $R^{(\gamma)}$. Let us denote the distorted Fourier transform of $\bar{\varepsilon}$ by $\mathbf{x}^{(\gamma)}$, then we obtain the corresponding transport equation on the distorted Fourier side:

$$(\mathcal{D}^2_\tau + \beta(\tau)\mathcal{D}_\tau + \boldsymbol{\xi})\mathbf{x}^{(\gamma)}(\tau, \xi) = \mathcal{R}\mathbf{x}^{(\gamma)}(\tau, \xi) + \mathbf{f}[\bar{\varepsilon}^{(\gamma)}](\tau, \xi) \tag{22}$$

where the linear source terms $\mathcal{R}$ are as in (11) and the nonlinear terms are defined by

$$\mathbf{f}[\bar{\varepsilon}^{(\gamma)}](\tau, \xi) = \begin{pmatrix} f_d[\tilde{\varepsilon}^{(\gamma)}](\tau) \\ f[\bar{\varepsilon}^{(\gamma)}](\tau, \xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle \phi_d, R^{(\gamma)} e^{(\gamma)}_{app} + N(u^{(\gamma)}_{app}, \tilde{\varepsilon}^{(\gamma)})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathcal{F}\left(R^{(\gamma)} e^{(\gamma)}_{app} + N(u^{(\gamma)}_{app}, \tilde{\varepsilon}^{(\gamma)})\right)(\tau, \xi) \end{pmatrix}.$$

The system (22) is coupled with initial data $(\mathbf{x}^{(\gamma)}_0, \mathbf{x}^{(\gamma)}_1)$ which satisfy the vanishing relations (19) with respect to the scaling law (20). Thus we can apply a similar iterative scheme as in [12] to show that there exist corrections $\triangle\mathbf{x}^{(\gamma)}_{0,1}$ such that a solution $\mathbf{x}^{(\gamma)}$ to (22) with perturbed initial data $(\mathbf{x}^{(\gamma)}_0 + \triangle\mathbf{x}^{(\gamma)}_0, \mathbf{x}^{(\gamma)}_1 + \triangle\mathbf{x}^{(\gamma)}_1)$ exists.

The last step consists to estimate the error induced by the small correction terms $\triangle\mathbf{x}^{(\gamma)}_{0,1}$ which have been introduced in the iterative scheme in terms of the original variable R. Hence we analyse $\triangle\varepsilon_{0,1}^{(\gamma)}$ the inverse distorted Fourier transform of $\triangle\mathbf{x}_{0,1}^{(\gamma)}$, with respect to the variable $R^{(\gamma)}$, and we prove that such errors are small when compared to the initial data perturbation. To show smallness one needs to compute the Fourier transform of $\triangle\varepsilon_{0,1}^{(\gamma)}$ with respect to the original variable R yielding to corrections denoted $\triangle\mathbf{x}_{0,1}$, and prove that the latter corrections are small in the $\tilde{S}$ norm when compared to the original initial data $\mathbf{x}_{0,1}$.

Finally, the investigation of the Lipschitz dependence of the corrections $\triangle\mathbf{x}_{0,1}^{(\gamma)}$ with respect to the original data $\mathbf{x}_{0,1}^{(\gamma)}$ is carried out in details in [4] by carefully analyzing the dependence of the error $e_{app}^{(\gamma)}$ from the parameters $\gamma_{1,2}$.

Acknowledgement

The author would like to thank Joachim Krieger and Luigi Forcella for providing insightful comments and numerous corrections.

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