Approximate Cloaking via Transformation Optics for Electromagnetic Waves

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This thesis is dedicated to my beloved parents.

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Abstract

Cloaking via transformation optics was introduced by Pendry, Schurig, and Smith (2006) for the Maxwell system and Leonhardt (2006) in the geometric optics setting. They used a singular change of variables which blows up a point into a cloaked region. The same transformation had been used by Greenleaf, Lassas, and Uhlmann (2003) in an inverse context. This singular structure implies difficulties not only in practice but also in analysis. To avoid using the singular structure, regularized schemes have been proposed. One of them was suggested by Kohn, Shen, Vogelius, and Weinstein (2010) for which they used a transformation which blows up a small ball instead of a point into the cloaked region. In this thesis, we study the approximate cloaking via transformation optics for electromagnetic waves in both the time-harmonic regime and time-dependent regime. In the time-harmonic regime, the cloaking device *only* consists of a layer constructed by the mapping technique, no (damping) lossy-layer is required. Due to the fact that no-lossy layer is required, resonance might appear. The analysis is therefore delicate and the phenomena are complex. In particular, we show that the energy can blow up inside the cloaked region in the resonant case and/whereas cloaking is *achieved* in *both* non-resonant and resonant cases. Moreover, the degree of visibility *depends* on the compatibility of the source inside the cloaked region and the system. These facts are new and distinct from known mathematical results in the literature. In the time-dependent regime, the cloaking device also consists of a fixed lossy layer. Our approach is based on estimates on the degree of visibility in the frequency domain for all frequency in which the frequency dependence is explicit. The difficulty and the novelty in the analysis are in the low and high frequency regimes. To this end, we implement the variational technique in low frequency and the multiplier and duality techniques in high frequency domain. The first part of the thesis is inspired by the work of Nguyen (2012) and the second part by the work of Nguyen and Vogelius (2012) on the wave equation.

Résumé

L'invisibilité basée sur la transformation optique a été introduite par Pendry, Schurig et Smith (2006) pour l'équation de Maxwell et Leonhardt (2006) dans l'optique géométrique. Ils ont utilisé un changement singulier de variables qui explose un point dans une région à rendre invisible. La même transformation avait été utilisée par Greenleaf, Lassas et Uhlmann dans un contexte inverse. Cette structure singulière implique des difficultés non seulement dans la pratique mais aussi dans l'analyse. Pour éviter d'utiliser la structure singulière, des régimes régularisés ont été proposés. L'un d'eux a été suggéré par Kohn, Shen, Vogelius, et Weinstein (2010) dans lequel ils ont utilisé une transformation qui fait exploser une petite balle à la place d'un point dans la région à rendre invisible. Dans cette thèse, nous étudions l'invisibilité approximative via la transformation optique des ondes électromagnétiques en régime harmonique et temporel. Dans le régime harmonique, on utilise uniquement une couche construite par la technique de transformation, aucune couche avec perte n'est requise. En raison du fait qu'aucune couche avec perte n'est requise, une résonance peut apparaître. L'analyse est donc délicate et les phénomènes sont complexes. En particulier, nous montrons que l'énergie peut exploser à l'intérieur de la région à rendre invisible dans le cas de résonance et / tandis que l'invisibilité est obtenu à la fois dans les deux cas non-résonance et résonance. De plus, le degré de visibilité dépend de la compatibilité entre la source dans la région à rendre invisible et le système considéré. Ces faits sont nouveaux et distincts des faits connus dans la littérature. Dans le régime temporel, le système est également constitué d'une couche à pertes fixe. Notre approche est basée sur les estimations du degré de visibilité dans le domaine de fréquences pour toutes les fréquences dans lesquelles la dépendance en fréquence est explicite. La difficulté et la nouveauté de l'analyse se situent dans les régimes de basses et hautes fréquences. Pour arriver à cette fin, nous mettons en œuvre la technique de variations en basses fréquences et la technique de multiplicateur et dualité dans le domaine des hautes fréquences. La première partie de la thèse s'inspire du travail de Nguyen (2012) et la seconde partie par les travaux de Nguyen et Vogelius (2012) sur les équations des ondes.

Contents

Introduction

In simple terms, invisibility cloaking (*cloaking*) is to make a target object invisible. In the past decades, it emerges as an interesting topic that appeals many mathematicians and physicists.

In 2006, Pendry, Schurig, and Smith in [49] suggested a cloaking method based on a transformation for the Maxwell system. The method was also introduced in the same year by Leonhardt [27] in the geometric optics setting. They used a singular change of variables which blows up a point into a cloaked region. The same transformation had been used by Greenleaf, Lassas, and Uhlmann to establish the non uniqueness of Calderon's problem in [18]. The singular nature of the cloaks presents various difficulties in practice as well as in theory: (1) they are hard to fabricate and (2) in certain cases the correct definition of the corresponding electromagnetic fields is not obvious. To avoid using the singular structure, various regularized schemes have been proposed. One of them was suggested by Kohn, Shen, Vogelius, and Weinstein in [24] in 2008, in which they used a transformation which blows up a small ball of radius ρ instead of a point into the cloaked region.

In the acoustic context, the approximate cloaking schemes introduced in [24] have been studied extensively in [16, 17, 23, 36, 35, 46, 10, 3, 21, 19]. Both time-harmonic and time regime have been well considered. In the time-harmonic regime, without the lossy (damping) layer, the field inside the cloaked region might depend on the field outside, and resonance might appear and affect the cloaking ability of the cloak, see [35]. With a fixed lossy layer, the cloaking is always achieved and the degree of visibility is known to be ρ in $\mathbb{R}^3$. Approximate cloaking was also investigated for the time-dependent acoustic waves in [45]. Cloaking was shown to be achieved with the same order of visibility as in the time-harmonic case.

There are other ways to achieve cloaking effects, using plasmonic coating [2], active exterior sources [52], complementary media [25, 38], or via localized resonance [33, 29, 37, 39].

The objective of the thesis is to study the cloaking scheme based on the regularized transformations proposed in [24] in the *electromagnetic context*. Similar to the acoustic context, we want to analyze the following aspects of the cloaking:

- In the time-harmonic context, no lossy layer is required. In this context, both non-resonant and resonant cases are considered. We provide the optimal degree of visibility for each case and study the asymptotic behavior of the energy inside the cloaked region as the regularized parameter ρ tends to 0.

- In the time-dependent context, we consider an additional fixed lossy layer. We estimate the cloaking effect by analyzing the Fourier's transform in time of the electromagnetic waves. In doing this, the analysis involves estimating the degree of visibility in the time-harmonic regime where the dependence on the frequency is explicit.

The more rigorous formulation and statements of results will be stated in Chapter 1 and 2. However, to give the reader an idea on the settings and the results without going too much into definitions and assumptions, we try to summarize them below.

The cloaking device via transformation optics
Assume for simplicity that the target cloaked object occupies the region $B_{1/2}$ and is characterized by a pair of permittivity, permeability (ε_O, μ_O). The cloak (cloaking device) occupies the annular region $B_2 \setminus B_{1/2}$. For $0 < \rho < 1$, define

$$F_\rho = \begin{cases} x & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(\dfrac{2-2\rho}{2-\rho} + \dfrac{|x|}{2-\rho}\right)\dfrac{x}{|x|} & \text{in } B_2 \setminus B_\rho, \\ \dfrac{x}{\rho} & \text{in } B_\rho. \end{cases}$$

The medium composed of the target object, the cloak, and the homogeneous medium outside the cloak is described by the triple $(\varepsilon_c, \mu_c, \sigma_c)$ below

$$(\varepsilon_c, \mu_c, \sigma_c) = \begin{cases} (I, I, 0) & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(F_{\rho_*} I, F_{\rho_*} I, 0\right) & \text{in } B_2 \setminus B_1, \\ (I, I, \sigma) & \text{in } B_1 \setminus B_{1/2}, \\ (\varepsilon_O, \mu_O, 0) & \text{in } B_{1/2}. \end{cases} \tag{0.0.1}$$

For a matrix $A \in \mathbb{R}^{3\times 3}$ and for a bi-Lipschitz homeomorphism T, we use following notation is being used:

$$T_* A(y) = \frac{DT(x)A(x)DT^T(x)}{|\det DT(x)|} \text{ with } y = T(x).$$

One may consider $\sigma = 0$ (no lossy layer) or $\sigma = 1$ (fixed lossy layer). A schematic sketch of the cloaking device without the lossy layer is provided in Figure 1.

Cloaking in the time-harmonic regime
With the cloak and the object, in the time-harmonic regime of frequency $\omega > 0$, the electromagnetic field generated by current $J \in [L^2(\mathbb{R}^3)]^3$ is the unique (Silver-Müller) radiating solution

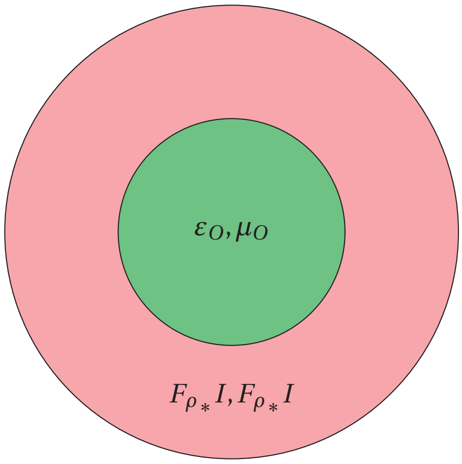

Figure 1: Schematic sketch of the cloaking object (the green part) and cloaking device (the red part) without the lossy layer, in term of permittivity and permeability of the medium.

$(E_c, H_c) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ of the system

$$\begin{cases} \nabla \times E_c = i\omega\mu_c H_c & \text{in } \mathbb{R}^3, \\ \nabla \times H_c = -i\omega\varepsilon_c E_c + J & \text{in } \mathbb{R}^3. \end{cases} \tag{0.0.2}$$

The electromagnetic field generated by $J|_{\mathbb{R}^3\setminus B_1}$ in homogeneous medium is the unique (Silver-Müller) radiating solution $(E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ to the system

$$\begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega E + J|_{\mathbb{R}^3\setminus B_1} & \text{in } \mathbb{R}^3. \end{cases} \tag{0.0.3}$$

Our goal consists of estimating $(E_c, E_c) - (E, H)$ in $\mathbb{R}^3 \setminus B_2$ and thereby confirming the cloaking effect for the proposed system.

Cloaking for electromagnetic waves via transformation optics has been mathematically investigated by several authors. Greenleaf, Kurylev, Lassas, and Uhlmann in [16] and Weder in [55, 56] studied cloaking for the singular scheme mentioned above by considering finite energy solutions. Concerning this approach, the information inside the cloaked region is not seen by observers outside. Approximate cloaking for the Maxwell equations using schemes in the spirit of [24] was considered in [7, 4, 26]. In [4], Ammari et al. investigated cloaking using additional layers inside the transformation cloak. These additional layers depending on the cloaked object were chosen in an appropriate way to cancel first terms in the asymptotic expansion of the polarization tensor to enhance the cloaking property. In [7], Bao, Liu, and Zou studied approximate cloaking using a lossy layer inside the transformation cloak. Their approach is as follows. Taking into account the lossy layer, one easily obtains an estimate for the electric field inside the lossy layer. This estimate depends on the property of the lossy layer and degenerates as the lossy property disappears. They then used the equation of the electric field in the lossy layer to derive estimates for the electric field on the boundary of the lossy region in some negative Sobolev norm. The cloaking estimate can be finally deduced from the integral representation for the electric field. This approach essentially uses the property of the lossy-layer and does not provide an optimal estimate of the degree of visibility in general. For example, when a fixed lossy layer is employed, they showed that the degree of visibility is of the order ρ^2, which is not optimal. In [26], Lassas and Zhou considered the transformation cloak in a symmetric setting, dealt with the non-resonant case (see Definition 1.1.2) and studied the limit of the solutions of the approximate cloaking problem near the cloak interface using separation of variables. Other regularized schemes are considered in [14].

We consider the situation where the cloaking device *only* consists of a layer constructed by the mapping technique and there is no source in that layer. Due to the fact that no-lossy (damping) layer is required, resonance might appear and the analysis is subtle. Our analysis is given in both non-resonant and resonant cases (Definition 1.1.2) and the results can be briefly summarized as follows.

i) In the non-resonant case, cloaking is achieved, and the energy remains finite inside the cloaked region.

ii) In the resonant case, cloaking is also *achieved*. Nevertheless, the degree of invisibility varies and depends on the compatibility (see (1.1.12) and (1.1.17)) of the source with the system. Moreover, the energy inside the cloaked region might explode in the incompatible case. See Theorems 1.1.2 and 1.1.3.

iii) The degree of visibility is of the order ρ^3 for both non-resonant and resonant cases if no source is inside the cloaked region (Theorems 1.1.1 and 1.1.2).

We also investigate the behavior of the field in the *whole* space (Theorems 1.1.1, 1.1.2, and 1.1.3) and establish the optimality of the convergence rate (Section 1.4). Our results are new and distinct from the works mentioned above. First, cloaking takes place even if the energy explodes inside the cloaked region as δ goes to 0. Second, in the resonant case with finite energy inside the cloaked region, the fields inside the cloaked region satisfy a non-local structure. Optimal estimates for the degree of visibility are derived for all cases. In particular, in the case of a fixed lossy layer (non-resonant case), the degree of visibility is of the order ρ^3 instead of ρ^2 as obtained previously . Both non-resonant and resonant cases are analyzed in details without assuming the symmetry of the cloaking setting.

Our approach is different from the ones in the works mentioned. It is based on severals subtle estimates for the effect of small inclusion involving the blow-up structure. Part of the analysis is on Maxwell's equations in the low frequency regime, which is interesting in itself. Our approach in this regime is inspired from [35] where the acoustic setting was considered. Nevertheless, the analysis for the electromagnetic setting is challenging and requires further new ideas due to the non-standard structure coming from the mapping technique and the complexity of electromagnetic structures/phenomena in comparison with acoustic ones. The Helmholtz decomposition and Stokes' theorem are involved in the Maxwell context.

The analysis of the cloaking for time-harmonic Maxwell's equation is presented in Chapter 1.

Cloaking in the time regime
In this regime, we use the time - dependent Maxwell equations. With the cloaking device and the cloaked object, the electromagnetic wave generated by $\mathscr{J}$ with zero data at the time 0 is the unique weak solution $(\mathscr{E}_c, \mathscr{H}_c) \in L^\infty_{\text{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \varepsilon_c \dfrac{\partial \mathscr{E}_c}{\partial t} = \nabla \times \mathscr{H}_c - \mathscr{J} - \sigma_c \mathscr{E}_c & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \mu_c \dfrac{\partial \mathscr{H}_c}{\partial t} = -\nabla \times \mathscr{E}_c & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \mathscr{E}_c(0,) = \mathscr{H}_c(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{0.0.4}$$

In the homogeneous space, the field generated by $\mathscr{J}$ with zero data at the time 0 is the unique weak solution $(\mathscr{E}, \mathscr{H}) \in L^\infty_{\text{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \dfrac{\partial \mathscr{E}}{\partial t} = \nabla \times \mathscr{H} - \mathscr{J} & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \dfrac{\partial \mathscr{H}}{\partial t} = -\nabla \times \mathscr{E} & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \mathscr{E}(0,) = \mathscr{H}(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{0.0.5}$$

Analogous to the time-harmonic regime, we would like to estimate $(\mathscr{E}_c, \mathscr{H}_c) - (\mathscr{E}, \mathscr{H})$ in $\mathbb{R}^3 \setminus B_2$ and thereby confirm the cloaking effect for the proposed system.

Concerning the analysis, we first transform the Maxwell equations in the time domain into a family of the Maxwell equations in the time-harmonic regime by taking the Fourier transform of the solutions with respect to time. After obtaining appropriate estimates on the near-invisibility for the Maxwell equations in the time-harmonic regime, we simply invert the Fourier transform. This idea has its roots in the work of Nguyen and Vogelius in [45] (see also [47]) in the cloaking context and was used to establish the validity of impedance boundary conditions in the time domain in [40]. To implement this idea, the heart matter is to obtain the degree of visibility in which the dependence on frequency is *explicit* and well-controlled. The analysis involves the variational method, the multiplier technique, and the duality methods in different ranges of frequency. An intriguing fact about the Maxwell equations in the time-harmonic regime worthy mentioned is that the multiplier technique does not fit for the cloaking purpose in the very large frequency regime and the dual method is involved instead.

Another key technical point is the proof of the radiating condition for the Fourier transform in time of the weak solutions of general Maxwell equations, a fact which is interesting in itself.

The analysis of the cloaking for time-dependent Maxwell's equations is in Chapter 2.

1 Cloaking for time-harmonic Maxwell's equations

In this chapter, we study the cloaking for the time-harmonic Maxwell equations. We consider the waves at a fixed frequency $\omega > 0$. The chapter uses the materials of the *submitted* version of [41] by H. M. Nguyen and L. Tran.

1.1 Mathematical setting and statement of the main results

In this section, we describe the problem and state the main results for cloaking in the time-harmonic setting. For simplicity of notations, we suppose that the cloak occupies the annular region $B_2 \setminus B_1$ and the cloaked region is the unit ball B_1 in $\mathbb{R}^3$ in which the permittivity and the permeability are given by two 3×3 matrices ε, μ respectively. Here and in what follows, for $r > 0$, let B_r denote the open ball in $\mathbb{R}^3$ centered at the origin and of radius r. Through this chapter, we assume that

$$\varepsilon, \mu \text{ are real, symmetric,} \tag{1.1.1}$$

and uniformly elliptic in B_1, i.e., for a.e. $x \in B_1$ and for some $\Lambda \geq 1$,

$$\frac{1}{\Lambda}|\xi|^2 \leq \langle \varepsilon(x)\xi, \xi\rangle, \langle \mu(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3. \tag{1.1.2}$$

We assume in addition that ε, μ are piecewise C^1 in order to ensure the well-posedness of Maxwell's equations in the frequency domain (via the unique continuation principle). In the spirit of the scheme in [24], the permittivity and permeability of the cloaking region are given by

$$(\varepsilon_c, \mu_c) := (F_{\rho_*} I, F_{\rho_*} I) \text{ in } B_2 \setminus B_1,$$

where $F_\rho : \mathbb{R}^3 \to \mathbb{R}^3$ with $\rho \in (0, 1/2)$ is defined by

$$F_\rho = \begin{cases} x & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(\dfrac{2-2\rho}{2-\rho} + \dfrac{|x|}{2-\rho}\right)\dfrac{x}{|x|} & \text{in } B_2 \setminus B_\rho, \\ \dfrac{x}{\rho} & \text{in } B_\rho. \end{cases}$$

We denote

$$F_0(x) = \lim_{\rho \to 0} F_\rho(x) \text{ for } x \in \mathbb{R}^3.$$

As usual, for a matrix $A \in \mathbb{R}^{3\times 3}$ and for a bi-Lipschitz homeomorphism T, the following notation is used:

$$T_* A(y) = \frac{DT(x)A(x)DT^T(x)}{|\det DT(x)|} \text{ with } y = T(x).$$

Assume that the medium is homogeneous outside the cloak and the cloaked region. In the presence of the cloaked object and the cloaking device, the medium in the whole space $\mathbb{R}^3$ is given by (ε_c, μ_c) which is defined as follows

$$(\varepsilon_c, \mu_c) = \begin{cases} (I, I) & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(F_{\rho *} I, F_{\rho *} I\right) & \text{in } B_2 \setminus B_1, \\ (\varepsilon, \mu) & \text{in } B_1. \end{cases} \tag{1.1.3}$$

With the cloak and the object, in the time-harmonic regime of frequency $\omega > 0$, the electromagnetic field generated by current $J \in [L^2(\mathbb{R}^3)]^3$ is the unique (Silver-Müller) radiating solution $(E_c, H_c) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ of the system

$$\begin{cases} \nabla \times E_c = i\omega\mu_c H_c & \text{in } \mathbb{R}^3, \\ \nabla \times H_c = -i\omega\varepsilon_c E_c + J & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1.4}$$

For an open subset U of $\mathbb{R}^3$, denote

$$H(\text{curl}, U) := \left\{\phi \in [L^2(U)]^3;\ \nabla \times \phi \in [L^2(U)]^3\right\}$$

and

$$H_{\text{loc}}(\text{curl}, U) := \left\{\phi \in [L^2_{\text{loc}}(U)]^3;\ \nabla \times \phi \in [L^2_{\text{loc}}(U)]^3\right\}.$$

Recall that, for $\omega > 0$, a solution $(E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus B_R)]^2$, for some $R > 0$, of the Maxwell equations

$$\begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3 \setminus B_R, \\ \nabla \times H = -i\omega E & \text{in } \mathbb{R}^3 \setminus B_R \end{cases}$$

is called radiating if it satisfies one of the (Silver-Muller) radiation conditions

$$H \times x - |x| E = O(1/|x|) \quad \text{and} \quad E \times x + |x| H = O(1/|x|) \text{ as } |x| \to +\infty. \tag{1.1.5}$$

Here and in what follows, for $\alpha \in \mathbb{R}$, $O(|x|^\alpha)$ denotes a quantity whose norm is bounded by $C|x|^\alpha$ for some constant $C > 0$.

Denote J_{ext} and J_{int} the restriction of J into $\mathbb{R}^3 \setminus B_1$ and B_1 respectively. It is clear that

$$J = \begin{cases} J_{\text{ext}} & \text{in } \mathbb{R}^3 \setminus B_1, \\ J_{\text{int}} & \text{in } B_1. \end{cases} \tag{1.1.6}$$

In the homogeneous medium (without the cloaking device and the cloaked object), the electromagnetic field generated by J_{ext} is the unique (Silver-Müller) radiating solution $(E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ to the system

$$\begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega E + J_{\text{ext}} & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1.7}$$

We next introduce the functional space $\mathcal{N}$ which is related to the notion of resonance and plays a role in our analysis.

Definition 1.1.1. *Let D be a smooth bounded subset of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus D$ is connected. Set*

$$\mathcal{N}(D) := \Big\{ (\boldsymbol{E}, \boldsymbol{H}) \in [H(\text{curl}, D)]^2 : (\boldsymbol{E}, \boldsymbol{H}) \textit{ satisfies the system } (1.1.8) \Big\},$$

where

$$\begin{cases} \nabla \times \boldsymbol{E} = i\omega\mu \boldsymbol{H} & \textit{in } D, \\ \nabla \times \boldsymbol{H} = -i\omega\varepsilon \boldsymbol{E} & \textit{in } D, \\ \nabla \times \boldsymbol{E} \cdot \nu = \nabla \times \boldsymbol{H} \cdot \nu = 0 & \textit{on } \partial D. \end{cases} \tag{1.1.8}$$

In the case $D = B_1$, we simply denote $\mathcal{N}(B_1)$ by $\mathcal{N}$.

The notions of resonance and non-resonance are defined as follows:

Definition 1.1.2. *The cloaking system* (1.1.3) *is said to be non-resonant if $\mathcal{N} = \{(0,0)\}$. Otherwise, the cloaking system* (1.1.3) *is called resonant.*

Our main result in the non-resonance case is the following theorem. The cloaking is always achieved as indicated in (1.1.9). This can be seen by taking $K \subset \mathbb{R}^3 \setminus B_2$ in (1.1.9) and noting that $F_\rho = Id$ in $\mathbb{R}^3 \setminus B_2$. Moreover, the behavior of (E_c, H_c) outside ∂B_1 is also described (see (1.1.9) and (1.1.10)). More precisely, we have

Theorem 1.1.1. *Let $\rho \in (0, 1/2)$, $R_0 > 2$, and let $J \in L^2(\mathbb{R}^3)$ be such that* $\operatorname{supp} J_{\text{ext}} \subset\subset B_{R_0} \setminus B_2$. *Let $(E_c, H_c), (E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ be the radiating solutions of* (1.1.4) *and* (1.1.7) *respectively. Assume that system (1.1.3) is non-resonant. We have, for all $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$,*

$$\|(F_\rho^{-1} * E_c, F_\rho^{-1} * H_c) - (E, H)\|_{H(\text{curl}, K)} \le C\left(\rho^3 \|J_{\text{ext}}\|_{L^2(B_{R_0} \setminus B_2)} + \rho^2 \|J_{\text{int}}\|_{L^2(B_1)}\right), \tag{1.1.9}$$

for some positive constant C depending only on $R_0, \omega, K, \mu, \varepsilon$. Moreover,

$$\lim_{\rho \to 0} (E_c, H_c) = Cl(0, J_{\text{int}}) \text{ in } [H(\text{curl}, B_1)]^2, \tag{1.1.10}$$

where $Cl(0, J_{\text{int}})$ is defined in Definition 1.1.3. The following notation is used

$$F_\rho^{-1} * E_c := (DF_\rho^T E_c) \circ F_\rho \quad \text{and} \quad F_\rho^{-1} * H_c := (DF_\rho^T H_c) \circ F_\rho.$$

Remark 1.1.1. *Note that $F_\rho^{-1} *$ above is different from ${F_\rho^{-1}}_*$ in the definition of μ_c, ε_c.*

The notation $Cl(\cdot, \cdot)$ used in Theorem 1.1.1 is defined as follows.

Definition 1.1.3. *Assume that $\mathcal{N} = \{(0,0)\}$. Let $\theta_1, \theta_2 \in [L^2(B_1)]^3$. Define $Cl(\theta_1, \theta_2) = (E_0, H_0)$ where $(E_0, H_0) \in [H(\text{curl}, B_1)]^2$ is the unique solution to the system*

$$\begin{cases} \nabla \times E_0 = i\omega\mu H_0 + \theta_1 & \text{in } B_1, \\ \nabla \times H_0 = -i\omega\varepsilon E_0 + \theta_2 & \text{in } B_1, \\ \nabla \times E_0 \cdot \nu = \nabla \times H_0 \cdot \nu = 0 & \text{on } \partial B_1. \end{cases} \tag{1.1.11}$$

Remark 1.1.2. The existence and the uniqueness of (E_0, H_0) are established in Lemma 1.3.4.

Remark 1.1.3. *In [56], the conditions*

$$\nabla \times E_0 \cdot \nu|_{\text{int}} = \nabla \times H_0 \cdot \nu|_{\text{int}} = 0$$

are also imposed on the boundary of the cloaked region. This is different from [16], where the following boundary conditions are given

$$E_0 \times \nu|_{\text{int}} = H_0 \times \nu|_{\text{int}} = 0.$$

The novelty of Theorem 1.1.1 relies on the fact that no lossy layer is required. The result holds for a general class of pair (ε, μ). Applying Theorem 1.1.1 to the case where a fixed lossy-layer is used, one obtains that the degree of visibility is of the order ρ^3 which is better than ρ^2 as established previously in [7] for the case $J_{\text{int}} \equiv 0$. In contrast with [7, 4, 14], in Theorem 1.1.1, the estimate of visibility is considered up to the cloaked region and the behavior of the electromagnetic fields are established inside the cloaked region.

We next consider the resonance case. We begin with the compatible case, i.e., (1.1.12) below holds.

Theorem 1.1.2. *Let* $\rho \in (0, 1/2)$, $R_0 > 2$, *and* $J \in [L^2(\mathbb{R}^3)]^3$ *be such that* $\operatorname{supp} J_{\text{ext}} \subset\subset B_{R_0} \setminus B_2$. *Let* $(E_c, H_c), (E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ *be the radiating solutions of* (1.1.4) *and* (1.1.7) *respectively. Assume that system (1.1.3) is resonant and the following compatibility condition* **holds***:*

$$\int_{B_1} J_{\text{int}} \cdot \bar{\boldsymbol{E}} dx = 0 \quad \textit{for all } (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{N}. \tag{1.1.12}$$

We have, for all $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$,

$$\|(F_\rho^{-1} * E_c, F_\rho^{-1} * H_c) - (E, H)\|_{H(\text{curl}, K)} \leq C\Big(\rho^3 \|J_{\text{ext}}\|_{L^2(B_{R_0} \setminus B_2)} + \rho^2 \|J_{\text{int}}\|_{L^2(B_1)}\Big), \tag{1.1.13}$$

for some positive constant C depending only on R_0, ω, K, μ, *and* ε. *Moreover,*

$$\lim_{\rho \to 0} (E_c, H_c) = Cl(0, J_{\text{int}}) \textit{ in } [H(\text{curl}, B_1)]^2, \tag{1.1.14}$$

where $Cl(0, J_{\text{int}})$ *is defined in Definition 1.1.4.*

In Theorem 1.1.2, we use the following notion:

Definition 1.1.4. *Assume that* $\mathcal{N} \neq \{(0,0)\}$. *Let* $\theta_1, \theta_2 \in [L^2(B_1)]^3$ *be such that*

$$\int_{B_1} \big(\theta_2 \cdot \bar{\boldsymbol{E}} - \theta_1 \cdot \bar{\boldsymbol{H}}\big)\, dx = 0 \quad \textit{for all } (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{N}. \tag{1.1.15}$$

Let $(E_0, H_0, E^\perp, H^\perp) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2 \times \mathcal{N}^\perp$ *be the unique solution of the following systems*

$$\begin{cases} \nabla \times E_0 = \nabla \times H_0 = 0 & \textit{in } \mathbb{R}^3 \setminus B_1, \\ \operatorname{div} E_0 = \operatorname{div} H_0 = 0 & \textit{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times E_0 = i\omega\mu H_0 + \theta_1 & \textit{in } B_1, \\ \nabla \times H_0 = -i\omega\varepsilon E_0 + \theta_2 & \textit{in } B_1, \end{cases} \quad \textit{and} \quad \begin{cases} \nabla \times E^\perp = i\omega\mu H^\perp & \textit{in } B_1, \\ \nabla \times H^\perp = -i\omega\varepsilon E^\perp & \textit{in } B_1, \\ \varepsilon E^\perp \cdot \nu = E_0 \cdot \nu|_{\text{ext}} & \textit{on } \partial B_1, \\ \mu H^\perp \cdot \nu = H_0 \cdot \nu|_{\text{ext}} & \textit{on } \partial B_1. \end{cases} \tag{1.1.16}$$

such that

$$|\big(E_0(x), H_0(x)\big)| = O(|x|^{-2}) \textit{ for large } |x|.$$

Denote $Cl(\theta_1, \theta_2)$ *the restriction of* (E_0, H_0) *in* B_1.

Remark 1.1.4. *We note that the definition of* $Cl(0, J_{\text{int}})$ *varies between Definition 1.1.3 and*

Definition 1.1.4 depending on the resonance of the system (1.1.3). *To indicate the limit of* (E_c, H_c) *in* B_1 *and to simplify the set of notations, we use* $Cl(0, J_{\mathrm{int}})$ *for both cases.*

Remark 1.1.5. *In Definition 1.1.4,* (E_0, H_0) *is determined by a non-local structure* (1.1.16). *This is new to our knowledge.*

Here and in what follows, $\mathcal{N}(D)^{\perp}$ denotes the orthogonal space of $\mathcal{N}(D)$ with respect to the standard scalar product in $[L^2(D)]^6$. The uniqueness and the existence of $(E_0, H_0, E^{\perp}, H^{\perp})$ are given in Lemmas 1.3.5 and 1.3.6.

In the incompatible case, we have

Theorem 1.1.3. *Let* $\rho \in (0, 1/2)$, $R_0 > 2$, *and* $J \in [L^2(\mathbb{R}^3)]^3$ *be such that* $\operatorname{supp} J_{\mathrm{ext}} \subset\subset B_{R_0} \setminus B_2$. *Let* $(E_c, H_c), (E, H) \in [H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)]^2$ *be the radiating solutions of* (1.1.4) *and* (1.1.7) *respectively. Assume that system (1.1.3) is resonant and the compatibility condition does* **not** *hold, i.e.,*

$$\int_{B_1} J_{\mathrm{int}} \cdot \bar{\boldsymbol{E}} \, dx \neq 0 \quad \text{for some } (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{N}. \tag{1.1.17}$$

We have, for all $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$,

$$\|(F_\rho^{-1} * E_c, F_\rho^{-1} * H_c) - (E, H)\|_{H(\operatorname{curl}, K)} \leq C\Big(\rho^3 \|J_{\mathrm{ext}}\|_{L^2(B_{R_0} \setminus B_2)} + \rho \|J_{\mathrm{int}}\|_{L^2(B_1)}\Big) \tag{1.1.18}$$

and

$$\liminf_{\rho \to 0} \rho \|\big(E_c, H_c\big)\|_{L^2(B_1)} > 0. \tag{1.1.19}$$

Some comments on Theorems 1.1.2 and 1.1.3 are in order. Theorems 1.1.2 and 1.1.3 imply in particular that cloaking is achieved even in the resonance case. Moreover, without any source in the cloaked region, one can achieve the same degree of visibility as in the non-resonant case considered in Theorem 1.1.1. Nevertheless, the degree of visibility varies and depends on the compatibility of the source inside the cloaked region. More precisely, the rate of the convergence of $(E_c, H_c) - (E, H)$ outside $\bar{B}_1$ in the compatible case is of the order ρ^2 which is better than the incompatible resonant case where an estimate of the order ρ is obtained. The rate of the convergence is optimal and discussed in Section 1.4. By (1.1.19), the energy inside the cloaked region blows up at least with the rate $1/\rho$ as $\rho \to 0$ in the incompatible case.

We now describe briefly the ideas of the proofs of Theorems 1.1.1, 1.1.2 and 1.1.3. Set

$$(\mathcal{E}_\rho, \mathcal{H}_\rho) = (F_\rho^{-1} * E_c, F_\rho^{-1} * H_c) \quad \text{in } \mathbb{R}^3. \tag{1.1.20}$$

It follows from a standard change of variables formula (see, e.g., Lemma 1.2.9) that $(\mathcal{E}_\rho, \mathcal{H}_\rho) \in$

$[H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique (Silver-Müller) radiating solution to

$$\begin{cases} \nabla \times \mathcal{E}_\rho = i\omega\mu_\rho \mathcal{H}_\rho & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H}_\rho = -i\omega\varepsilon_\rho \mathcal{E}_\rho + J_\rho & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1.21}$$

where

$$\big(\varepsilon_\rho, \mu_\rho\big) = \big(F_\rho^{-1}{}_*\varepsilon_c, F_\rho^{-1}{}_*\mu_c\big) = \begin{cases} \big(I, I\big) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ \big(\rho^{-1}\varepsilon(\cdot/\rho), \rho^{-1}\mu(\cdot/\rho)\big) & \text{in } B_\rho, \end{cases} \tag{1.1.22}$$

and

$$J_\rho = \begin{cases} J_{\mathrm{ext}} & \text{in } \mathbb{R}^3 \setminus B_2, \\ \rho^{-2} J_{\mathrm{int}}(\cdot/\rho) & \text{in } B_\rho, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.23}$$

We can then derive Theorems 1.1.1, 1.1.2, and 1.1.3 by studying the difference between $(\mathcal{E}_\rho, \mathcal{H}_\rho)$ and (E, H) in $\mathbb{R}^3 \setminus B_1$ and the behavior of $(\mathcal{E}_\rho, \mathcal{H}_\rho)(\rho\cdot)$ in B_1. It is well-known that when material parameters inside a small inclusion are bounded from below and above by positive constants, the effect of the small inclusion is small (see, e.g., [53, 5]). Without this assumption, the effect of the inclusion might not be small (see, e.g., [24, 36]) unless there is an appropriate lossy-layer, see [7, 4, 14]. In our setting, the boundedness assumption is violated (see (1.1.22)) and no lossy-layer is used. Nevertheless, the effect of the small inclusion is still small due to the special structure induced from (1.1.22).

It is worth noting that System (1.1.11), which involves in the definition of resonance and non-resonance, and the condition of compatibility (1.1.12), appears very naturally in our context. Indeed, note that if (E_c, H_c) is bounded in $[H(\mathrm{curl}, B_1)]^2$, one can check that, up to a subsequence, $(\rho\mathcal{E}_\rho, \rho\mathcal{H}_\rho)(\rho\cdot) = (E_c, H_c)$ converges weakly in $[H(\mathrm{curl}, B_1)]^2$ to (E_0, H_0) which satisfies system (1.1.11) with $(\theta_1, \theta_2) = (0, J)$.

The chapter is organized as follows. In Section 1.2, we establish some basic facts and recall some known results related to Maxwell's equations. These materials will be used in the proofs of Theorems 1.1.1, 1.1.2, and 1.1.3. The proofs of Theorems 1.1.1, 1.1.2, and 1.1.3 are given in Section 1.3. Finally, in Section 1.4, we discuss the optimality of the convergence rate in Theorems 1.1.1, 1.1.2, and 1.1.3.

1.2 Preliminaries

In this section, we establish some basic facts and recall some known results related to Maxwell's equations that will be repeatedly used in the proofs of Theorems 1.1.1, 1.1.2, and 1.1.3. In what follows in this section, D denotes a smooth bounded open subset of $\mathbb{R}^3$ and on its boundary ν

denotes its normal unit vector directed to the exterior. We begin with a variant of the classic Stokes' theorem for an exterior domain.

Lemma 1.2.1. *Assume that* $\mathbb{R}^3 \setminus D$ *is simply connected and let* $u \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus D)$ *be such that*

$$\nabla \times u = 0 \text{ in } \mathbb{R}^3 \setminus D \quad \text{and } |u(x)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.2.1}$$

There exists $\xi \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)$ *such that*

$$\nabla \xi = u \text{ in } \mathbb{R}^3 \setminus D \quad \text{and} \quad |\xi(x)| = O(|x|^{-1}) \text{ for large } |x|. \tag{1.2.2}$$

Proof. By [15, Theorem 2.9], there exists $\eta_n \in H^1(B_n \setminus D)$ for large n such that

$$\nabla \eta_n = u \text{ in } B_n \setminus D \quad \text{and} \quad \int_{\partial B_2} \eta_n = 0.$$

It follows that, for $m > n$ large,

$$\eta_m = \eta_n \text{ in } B_n \setminus D.$$

Let η be the limit of η_n as $n \to +\infty$. Then $\eta \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)$ and

$$\nabla \eta = u \text{ in } \mathbb{R}^3 \setminus D.$$

Fix $x, y \in \mathbb{R}^3$ large enough with $|y| > |x|$ and denote $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. Using (1.2.1), we have, by the fundamental theorem of calculus,

$$|\eta(x) - \eta(y)| \le |\eta(|y|\hat{y}) - \eta(|y|\hat{x})| + |\eta(|y|\hat{x}) - \eta(|x|\hat{x})| \le \frac{C}{|y|} + \int_{|x|}^{|y|} \frac{C}{|r|^2}\, dr \tag{1.2.3}$$

for some positive constant C independent of x and y. It follows that

$$|\eta(x) - \eta(y)| \le \frac{C}{|y|} + \frac{C}{|x|}. \tag{1.2.4}$$

Hence $\lim_{|x| \to \infty} \eta(x)$ exists. Denote this limit by η_∞. By letting $|y| \to +\infty$ in (1.2.4), we obtain

$$|\eta(x) - \eta_\infty| \le \frac{C}{|x|}, \text{ for } |x| \text{ large enough.}$$

The conclusion follows with $\xi = \eta - \eta_\infty$. □

Let U be a smooth open subset of $\mathbb{R}^3$. Denote

$$H(\mathrm{div}, U) := \left\{\phi \in [L^2(U)]^3 : \mathrm{div}\phi \in L^2(U)\right\}.$$

Concerning a free divergent field in a bounded domain, one has the following result which is related to Stokes' theorem, see, e.g., [15, Theorems 3.4 and 3.6].

Lemma 1.2.2. *Assume that D is simply connected and let $u \in H(\mathrm{div}, D)$ be such that*

$$\mathrm{div}\, u = 0 \textit{ in } D \quad \textit{and} \quad \int_{\Gamma_i} u \cdot \nu = 0 \textit{ for all connected component } \Gamma_i \textit{ of } \partial D. \tag{1.2.5}$$

There exists $\phi \in [H^1(D)]^3$ such that

$$\nabla \times \phi = u \textit{ in } D \textit{ and } \mathrm{div}\phi = 0 \textit{ in } D.$$

Assume in addition that $u \cdot \nu = 0$ on ∂D. Then ϕ can be chosen such that

$$\phi \times \nu = 0 \textit{ on } \partial D \quad \textit{and} \quad \int_{\Gamma_i} \phi \cdot \nu = 0 \textit{ for all connected component } \Gamma_i \textit{ of } \partial D.$$

Moreover, such a ϕ is unique and, for some positive constant C,

$$\|\phi\|_{H^1(D)} \leq C \|u\|_{L^2(D)}.$$

The following result is a type of Helmholtz decomposition. It is a variant of [15, Corollary 3.4] where σ is a positive constant.

Lemma 1.2.3. *Assume that D is simply connected and let σ be a 3×3 uniformly elliptic matrix-valued function defined in D. For any $v \in [L^2(D)]^3$, there exist $p \in H^1(D)$ and $\phi \in [H^1(D)]^3$ such that*

$$v = \sigma \nabla p + \nabla \times \phi \textit{ in } D, \quad \mathrm{div}\phi = 0 \textit{ in } D \quad \textit{and} \quad \phi \times \nu = 0 \textit{ on } \partial D. \tag{1.2.6}$$

Moreover,

$$\|p\|_{H^1(D)} + \|\phi\|_{H^1(D)} \leq C \|v\|_{L^2(D)}. \tag{1.2.7}$$

Proof. The proof given here is in the spirit of [15] as follows. By Lax-Milgram's theorem, there exists a unique solution $p \in H^1(D)$ with $\int_D p\, dx = 0$ to the equation

$$\int_D \sigma \nabla p \cdot \nabla q\, dx = \int_D v \cdot \nabla q\, dx \text{ for all } q \in H^1(D).$$

Moreover,

$$\|p\|_{H^1(D)} \leq C \|v\|_{L^2(D)}. \tag{1.2.8}$$

Then

$$\mathrm{div}(v - \sigma \nabla p) = 0 \text{ in } D \quad \text{and} \quad (v - \sigma \nabla p) \cdot \nu = 0 \text{ on } \partial D. \tag{1.2.9}$$

By Lemma 1.2.2, there exists $\phi \in [H^1(D)]^3$ such that

$$\begin{cases} \nabla \times \phi = v - \sigma \nabla p & \text{in } D, \\ \operatorname{div} \phi = 0 & \text{in } D, \\ \phi \times \nu = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \|\phi\|_{H^1(D)} \le C\|v - \sigma \nabla p\|_{L^2(D)}. \tag{1.2.10}$$

Combining (1.2.8), (1.2.9), and (1.2.10), we reach the conclusion for such a pair (p, ϕ). □

We next present two lemmas concerning the uniqueness of the exterior problems for electrostatic settings. They are used in the study of the exterior problems in the low frequency regime, see Lemma 1.3.1. The first one is

Lemma 1.2.4. *Assume that* $\mathbb{R}^3 \setminus D$ *is simply connected. Let* $u \in H_{\rm loc}(\operatorname{curl}, \mathbb{R}^3 \setminus D) \cap H_{\rm loc}(\operatorname{div}, \mathbb{R}^3 \setminus D)$ *be such that*

$$\begin{cases} \nabla \times u = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ u \cdot \nu = 0 & \text{on } \partial D, \end{cases}$$

and

$$|u(x)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.2.11}$$

Then $u = 0$ *in* $\mathbb{R}^3 \setminus D$.

Proof. By Lemma 1.2.1, there exists $\xi \in H^1_{\rm loc}(\mathbb{R}^3 \setminus D)$ such that

$$\nabla \xi = u \text{ in } \mathbb{R}^3 \setminus D \quad \text{and} \quad |\xi(x)| = O(|x|^{-1}) \text{ for large } |x|. \tag{1.2.12}$$

Since $\operatorname{div} u = 0$, we have

$$\Delta \xi = 0 \text{ in } \mathbb{R}^3 \setminus D.$$

Since $\nabla \xi \cdot \nu = u \cdot \nu = 0$ on ∂D, it follows that $\xi = 0$ in $\mathbb{R}^3 \setminus D$, see, e.g., [32, Theorem 2.5.15]. Therefore, $u = 0$. □

The second lemma is

Lemma 1.2.5. *Assume that* $\mathbb{R}^3 \setminus D$ *is simply connected and* $u \in H_{\rm loc}(\operatorname{curl}, \mathbb{R}^3 \setminus D) \cap H_{\rm loc}(\operatorname{div}, \mathbb{R}^3 \setminus$

D) *is such that*

$$\begin{cases} \nabla \times u = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ u \times \nu = 0 & \text{on } \partial D, \end{cases} \qquad \int_{\Gamma_i} u \cdot \nu = 0 \text{ for all connected component } \Gamma_i \text{ of } \partial D,$$

and

$$|u(x)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.2.13}$$

Then $u = 0$ *in* $\mathbb{R}^3 \setminus D$.

Proof. By Lemma 1.2.1, there exists $\xi \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus D)$, such that

$$\nabla \xi = u \text{ in } \mathbb{R}^3 \setminus D \quad \text{and} \quad |\xi(x)| = O(|x|^{-1}) \quad \text{for large } |x|. \tag{1.2.14}$$

There exists $\psi \in [H^1_{\text{loc}}(\mathbb{R}^3 \setminus D)]^3$, such that

$$\nabla \times \psi = u \text{ in } \mathbb{R}^3 \setminus D.$$

Fix $\theta \in C^1(\mathbb{R}^3)$ such that $0 \le \theta \le 1$, $\theta = 1$ in B_1 and $\operatorname{supp} \theta \subset B_2$. For $r > 0$, set $\theta_r(\cdot) = \theta(\cdot / r)$ in $\mathbb{R}^3$. Let $t > s > 0$ be large enough (arbitrary) such that $D \subset\subset B_s$. Since $u \times \nu = 0$ on ∂D, we obtain, by integration by parts, that

$$\int_{\mathbb{R}^3 \setminus D} \nabla \times (\theta_t \psi) \cdot \nabla(\theta_s \bar{\xi})\, dx = -\int_{\partial D} \theta_t \psi \cdot \nabla(\theta_s \bar{\xi}) \times \nu\, ds = -\int_{\partial D} \psi \cdot \bar{u} \times \nu\, ds = 0.$$

Letting $t \to +\infty$, we derive that

$$\int_{\mathbb{R}^3 \setminus D} u \cdot \nabla(\theta_s \bar{\xi})\, dx = 0. \tag{1.2.15}$$

We have

$$\int_{B_{2s} \setminus B_s} |u||\xi||\nabla \theta_s|\, dx \le C|B_{2s} \setminus B_s| s^{-2} s^{-1} s^{-1} \le C s^{-1} \to 0 \text{ as } s \to +\infty. \tag{1.2.16}$$

Using the fact that

$$u \cdot \nabla(\theta_s \bar{\xi}) = u\big(\theta_s \nabla \bar{\xi} + \bar{\xi} \nabla \theta_s\big) = \theta_s |u|^2 + u \bar{\xi} \nabla \theta_s \text{ in } \mathbb{R}^3 \setminus D,$$

and combining (1.2.15) and (1.2.16), we obtain

$$\int_{\mathbb{R}^3 \setminus D} |u|^2\, dx = 0,$$

which yields $u = 0$ in $\mathbb{R}^3 \setminus D$. □

The following result is a consequence of the Stratton - Chu formula.

Lemma 1.2.6. *Let $0 < k \le k_0$. Assume that $D \subset\subset B_1$ and $(E, H) \in \big[H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus D)\big]^2$ is a radiating solution to the Maxwell equations*

$$\begin{cases} \nabla \times E = ikH & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H = -ikE & \text{in } \mathbb{R}^3 \setminus \bar{D}. \end{cases}$$

We have

$$\Big|\big(E(x), H(x)\big)\Big| \le \frac{C}{|x|^2}\big(1 + k|x|\big)\|(E,H)\|_{L^2(B_3\setminus D)} \text{ for } |x| > 3, \tag{1.2.17}$$

for some positive constant C independent of x and k.

Proof. Set

$$G_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \text{ for } x, y \in \mathbb{R}^3, x \neq y.$$

It is known that, see, e.g., [12, Theorem 6.6 and (6.10)], the following variant of the Stratton-Chu formula holds, for $x \in \mathbb{R}^3 \setminus \bar{D}$,

$$E(x) = \nabla_x \times \int_{\partial B_2} \nu(y) \times E(y) G_k(x, y)\, dy \\ + ik \int_{\partial B_2} \nu(y) \times H(y) G_k(x, y)\, dy - \nabla_x \int_{\partial B_2} \nu(y) \cdot E(y) G_k(x, y)\, dy. \tag{1.2.18}$$

Using the facts

$$|\nabla G_k(x, y)| \le \frac{C}{|x|^2}(1 + k|x|) \text{ for } y \in \partial B_2, x \in \mathbb{R}^3 \setminus B_3$$

and, since $\Delta E + k^2 E = 0$ in $\mathbb{R}^3 \setminus D$,

$$\|E\|_{L^\infty(\partial B_2)} \le C\|E\|_{L^2(B_3\setminus D)}, \text{ for some positive constant } C \text{ depending only on } k_0,$$

we derive from (1.2.18) that

$$|E(x)| \le \frac{C}{|x|^2}\big(1 + k|x|\big)\|(E,H)\|_{L^2(B_3\setminus D)} \text{ for } |x| > 3. \tag{1.2.19}$$

Similarly, we obtain

$$|H(x)| \le \frac{C}{|x|^2}\big(1 + k|x|\big)\|(E,H)\|_{L^2(B_3\setminus D)} \text{ for } |x| > 3. \tag{1.2.20}$$

The conclusion now follows from (1.2.19) and (1.2.20). □

We next recall compactness results related to $H(\text{curl}, \cdot)$ and $H(\text{div}, \cdot)$.

Lemma 1.2.7. *Let ϵ be a measurable symmetric uniformly elliptic matrix-valued function defined in D. Assume that one of the following two conditions holds*

i) $(u_n)_{n\in\mathbb{N}} \subset H(\mathrm{curl}, D)$ *is a bounded sequence in* $H(\mathrm{curl}, D)$ *such that*

$$\big(\mathrm{div}(\epsilon u_n)\big)_{n\in\mathbb{N}} \textit{ converges in } H^{-1}(D) \textit{ and } \big(u_n \times \nu\big)_{n\in\mathbb{N}} \textit{ converges in } H^{-1/2}(\partial D).$$

ii) $(u_n)_{n\in\mathbb{N}} \subset H(\mathrm{curl}, D)$ *is a bounded sequence in* $H(\mathrm{curl}, D)$ *such that*

$$\big(\mathrm{div}(\epsilon u_n)\big)_{n\in\mathbb{N}} \textit{ is bounded in } L^2(D) \textit{ and } \big(\epsilon u_n \cdot \nu\big)_{n\in\mathbb{N}} \textit{ converges in } H^{-1/2}(\partial D).$$

There exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ *which converges in* $[L^2(D)]^3$.

The conclusion of Lemma 1.2.7 under condition $i)$ is [36, Lemma 1] and has its roots in [20] and [13]. The conclusion of Lemma 1.2.7 under condition $ii)$ can be obtained in the same way. These compactness results play a similar role as the compact embedding of H^1 into L^2 in the acoustic setting and are basic ingredients in our approach.

In what follows, we denote

$$H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \Big\{\phi \in [H^{-1/2}(\Gamma)]^3;\ \phi\cdot\nu = 0 \text{ and } \mathrm{div}_\Gamma \phi \in H^{-1/2}(\Gamma)\Big\},$$

$$\|\phi\|_{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\phi\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \phi\|_{H^{-1/2}(\Gamma)}.$$

The following trace results related to $H(\mathrm{curl}, \cdot)$ and $H(\mathrm{div}, \cdot)$ are standard, see, e.g., [1, 9, 15].

Lemma 1.2.8. *Set* $\Gamma = \partial D$. *We have*

i)

$$\|v \times \nu\|_{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \le C\|v\|_{H(\mathrm{curl},D)} \textit{ for } v \in H(\mathrm{curl}, D).$$

ii)

$$\|v\cdot\nu\|_{H^{-1/2}(\Gamma)} \le C\|v\|_{H(\mathrm{div},D)} \textit{ for } v \in H(\mathrm{div}, D).$$

Moreover, for any $h \in H^{-1/2}(\mathrm{div}_\Gamma, \partial D)$, *there exists* $\phi \in H(\mathrm{curl}, D)$ *such that*

$$\phi \times \nu = h \textit{ on } \partial D, \textit{ and } \|\phi\|_{H(\mathrm{curl},D)} \le C\|h\|_{H^{-1/2}(\mathrm{div}_\Gamma,\partial D)}.$$

Here C *denotes a positive constant depending only on* D.

We finally recall the following change of variables for the Maxwell equations. It is the basic ingredient for cloaking using transformation optics for electromagnetic fields.

Lemma 1.2.9. *Let D, D' be two open bounded connected subsets of $\mathbb{R}^3$ and $F : D \to D'$ be a bijective map such that $F \in C^1(\bar{D}), F^{-1} \in C^1(\bar{D}')$. Let $\varepsilon, \mu \in [L^\infty(D)]^{3\times 3}$, and $j \in [L^2(D)]^3$. Assume that $(E, H) \in [H(\operatorname{curl}, D)]^2$ is a solution of the Maxwell equations*

$$\begin{cases} \nabla \times E = i\omega\mu H & \text{in } D, \\ \nabla \times H = -i\omega\varepsilon E + j & \text{in } D. \end{cases} \tag{1.2.21}$$

Set, in D',

$$E' := F * E := (DF^{-T} E) \circ F^{-1} \quad \text{and} \quad H' := F * H := (DF^{-T} H) \circ F^{-1}.$$

Then $(E', H') \in [H(\operatorname{curl}, D')]^2$ satisfies

$$\begin{cases} \nabla \times E' = i\omega\mu' H' & \text{in } D', \\ \nabla \times H' = -i\omega\varepsilon' E' + j' & \text{in } D', \end{cases} \tag{1.2.22}$$

where

$$\varepsilon' := F_*\varepsilon := \frac{DF\varepsilon DF^T}{|\det DF|} \circ F^{-1}, \quad \mu' := F_*\mu := \frac{DF\mu DF^T}{|\det DF|} \circ F^{-1}, \quad \text{and} \quad j' := F_* j = \frac{DF j}{|\det DF|} \circ F^{-1}.$$

Remark 1.2.1. It is worth noting the difference of $F*$ in the definition of E' and H', and F_* in the definition of ε', μ', and j'.

1.3 Proofs of the main results

This section is devoted to the proof of Theorems 1.1.1, 1.1.2, and 1.1.3 and is organized as follows. In the first subsection, we establish various results related to $(\mathcal{E}_\rho, \mathcal{H}_\rho)$. The proof of Theorem 1.1.1 is given in the second subsection and the ones of Theorems 1.1.2 and 1.1.3 are given in the third subsection.

1.3.1 Some useful lemmas

In this section, $D \subset B_1$ denotes a smooth open bounded subset of $\mathbb{R}^3$, and ε and μ denote two 3×3 matrices defined in D which are both real, symmetric, and uniformly elliptic in D. We also assume that D and $\mathbb{R}^3 \setminus D$ are simply connected and ε, μ are piecewise C^1. The following lemma provides the stability of the exterior problem in the low frequency regime.

Lemma 1.3.1. *Let $0 < \rho < \rho_0$ and let $(E_\rho, H_\rho) \in [H_{\rm loc}(\operatorname{curl}, \mathbb{R}^3 \setminus D)]^2$ be a radiating solution to* *the system*

$$\begin{cases} \nabla \times E_\rho = i\rho H_\rho & in\, \mathbb{R}^3 \setminus D, \\ \nabla \times H_\rho = -i\rho E_\rho & in\, \mathbb{R}^3 \setminus D. \end{cases} \tag{1.3.1}$$

We have, for $R > 1$,

$$\|(E_\rho, H_\rho)\|_{H(\mathrm{curl}, B_R \setminus D)} \le C_R \Big(\|E_\rho \times \nu\|_{H^{-1/2}(\partial D)} + \|H_\rho \cdot \nu\|_{H^{-1/2}(\partial D)} \Big) \tag{1.3.2}$$

and

$$\|(E_\rho, H_\rho)\|_{H(\mathrm{curl}, B_R \setminus D)} \le C_R \Big(\|E_\rho \times \nu\|_{H^{-1/2}(\partial D)} + \|H_\rho \times \nu\|_{H^{-1/2}(\partial D)} \Big), \tag{1.3.3}$$

for some positive constant C_R depending only on ρ_0, D, and R.

Remark 1.3.1. *A similar estimate to* (1.3.2) *but switching the role of E_ρ and H_ρ also holds true.*

Proof. We begin with the proof of (1.3.2). Since (E_ρ, H_ρ) satisfies (1.3.1), it suffices to prove that

$$\|(E_\rho, H_\rho)\|_{L^2(B_R \setminus D)} \le C_R \Big(\|E_\rho \times \nu\|_{H^{-1/2}(\partial D)} + \|H_\rho \cdot \nu\|_{H^{-1/2}(\partial D)} \Big) \tag{1.3.4}$$

for $R > 3$. Fixing $R > 3$, we prove (1.3.4) by contradiction. Suppose that there exist a sequence $(\rho_n)_{n\in\mathbb{N}} \subset (0, \rho_0)$ and a sequence of radiating solutions $\big((E_n, H_n)\big)_{n\in\mathbb{N}} \subset [H(\mathrm{curl}, \mathbb{R}^3 \setminus D)]^2$ of the system

$$\begin{cases} \nabla \times E_n = i\rho_n H_n & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times H_n = -i\rho_n E_n & \text{in } \mathbb{R}^3 \setminus D, \end{cases} \tag{1.3.5}$$

such that

$$\|(E_n, H_n)\|_{L^2(B_R \setminus D)} = 1 \text{ for } n \in \mathbb{N}, \tag{1.3.6}$$

and

$$\lim_{n \to 0} \Big(\|E_n \times \nu\|_{H^{-1/2}(\partial D)} + \|H_n \cdot \nu\|_{H^{-1/2}(\partial D)} \Big) = 0. \tag{1.3.7}$$

Without loss of generality, one might assume that $\rho_n \to \rho_\infty$ as $n \to \infty$ for some $\rho_\infty \in [0, \rho_0]$. We only consider the case $\rho_\infty = 0$. The case $\rho_\infty > 0$ can be proven similarly. From (1.3.5) and (1.3.6), we have

$$\|(E_n, H_n)\|_{H(\mathrm{curl}, B_R \setminus D)} \le C. \tag{1.3.8}$$

Here and in what follows in this proof, C and C_r denote positive constants independent of n.

Applying Lemma 1.2.6, we have

$$\|(E_n, H_n)\|_{H(\mathrm{curl}, B_r\setminus D)} \le C_r \quad \text{for all } r > 3\,. \tag{1.3.9}$$

Since

$$\Delta E_\rho + \rho^2 E_\rho = \Delta H_\rho + \rho^2 H_\rho = 0 \text{ in } \mathbb{R}^3 \setminus D,$$

it follows from (1.3.9) that, for $r > 3$,

$$\|(E_n, H_n)\|_{H^1(B_{r+1}\setminus B_{r-1})} \le C_r.$$

By the trace theory, we have

$$\|(E_n, H_n)\|_{H^{1/2}(\partial B_r)} \le C_r.$$

Since the embedding of $H^{1/2}(\partial B_r)$ into $H^{-1/2}(\partial B_r)$ is compact, by applying i) of Lemma 1.2.7 to (E_n) and by applying ii) of Lemma 1.2.7 to (H_n), without loss of generality, one might assume that (E_n, H_n) converges in $[L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)]^6$. Moreover, the limit $(E, H) \in [H_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)]^2$ satisfies

$$\begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ H \cdot \nu = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times E = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} E = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ E \times \nu = 0 & \text{on } \partial D. \end{cases} \tag{1.3.10}$$

Applying Lemma 1.2.6 to (E_n, H_n) and letting $n \to +\infty$ ($\rho_n \to 0$), we have

$$|\big(E(x), H(x)\big)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.3.11}$$

On the other hand, since $E_n = -\dfrac{1}{i\rho_n}\nabla \times H_n$ in $\mathbb{R}^3 \setminus D$, we have

$$\int_{\Gamma_i} E_n \cdot \nu = 0 \text{ for all connected component } \Gamma_i \text{ of } \partial D. \tag{1.3.12}$$

Since (E_n) converges to E in $[L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)]^3$ and $\operatorname{div} E_n = \operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus D$, it follows that (E_n) converges to E in $H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3 \setminus D)$. This in turn implies, by (1.3.12),

$$\int_{\Gamma_i} E \cdot \nu = 0 \text{ for all connected component } \Gamma_i \text{ of } \partial D. \tag{1.3.13}$$

Applying Lemma 1.2.4 to H, we derive from (1.3.10) and (1.3.11) that

$$H = 0 \text{ in } \mathbb{R}^3 \setminus D. \tag{1.3.14}$$

Similarly, applying Lemma 1.2.5 to E, from (1.3.10), (1.3.11), and (1.3.13), we obtain

$$E = 0 \text{ in } \mathbb{R}^3 \setminus D.^{1} \tag{1.3.15}$$

From (1.3.6), (1.3.14), and (1.3.15) and the fact that (E_n, H_n) converges to (E, H) in $L^2_{\text{loc}}(\mathbb{R}^3 \setminus D)$, we reach a contradiction. The proof of (1.3.2) is complete.

We next deal with (1.3.3). The proof of (1.3.3) is similar to the one of (1.3.2). However, instead of obtaining (1.3.10) and (1.3.13), we have

$$\begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ H \times \nu = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times E = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} E = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ E \times \nu = 0 & \text{on } \partial D, \end{cases}$$

and

$$\int_\Gamma H \cdot \nu = \int_\Gamma E \cdot \nu = 0 \text{ for all connected component } \Gamma \text{ of } \partial D.$$

By the same arguments, we can derive that $(E, H) = (0, 0)$ in $\mathbb{R}^3$, which also yields a contradiction. The details are left to the reader. □

Remark 1.3.2. We have

$$\operatorname{div}_\Gamma(E_\rho \times \nu) = \nabla \times E_\rho \cdot \nu = i\rho H_\rho \cdot \nu \text{ on } \partial D.$$

It follows that, for $0 < \rho < 1$,

$$\|E_\rho \times \nu\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial D)} \le \|E_\rho \times \nu\|_{H^{-1/2}(\partial D)} + \|H_\rho \cdot \nu\|_{H^{-1/2}(\partial D)} \le \frac{1}{\rho} \|E_\rho \times \nu\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial D)},$$

i.e., the bound in the estimate (1.3.2) is an intermediate quantity between $\|E_\rho \times \nu\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial D)}$ and $\rho^{-1}\|E_\rho \times \nu\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial D)}$.

The next lemma gives an estimate for solutions of Maxwell's equations in the low frequency regime, which in turn implies an estimate for the effect of a small inclusion after a change of variables.

Lemma 1.3.2. *Let $0 < \rho < 1/2$, $R > 1/2$, and let $(E_\rho, H_\rho) \in [H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus D)]^2$ be a radiating solution to the system*

$$\begin{cases} \nabla \times E_\rho = i\omega\rho H_\rho & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times H_\rho = -i\omega\rho E_\rho & \text{in } \mathbb{R}^3 \setminus D. \end{cases} \tag{1.3.16}$$

We have

$$\left|\left(E_\rho(x), H_\rho(x)\right)\right| \le C\rho^3 \|(E_\rho, H_\rho)\|_{L^2(B_2 \setminus D)} \quad \text{for } x \in B_{3R/\rho} \setminus B_{2R/\rho},$$

[1]When $\rho_\infty > 0$, instead of being a solution of (1.3.10), (E, H) is the radiating solution of (1.3.1) with $\rho = \rho_\infty$ and $E \times \nu = 0$ on ∂D. This also implies that $(E, H) = (0, 0)$ in $\mathbb{R}^3 \setminus D$.

for some constant C depending only R.

Proof. We only deal with small ρ, since otherwise the conclusion is just a consequence of Stratton-Chu's formula. We have, for $x \in \mathbb{R}^3 \setminus \bar{B}_1$, (see (1.2.18))

$$E_\rho(x) = \int_{\partial B_1} \nabla_x G_k(x,y) \times \big(\nu(y) \times E_\rho(y)\big) dy \\ + i\omega\rho \int_{\partial B_1} \nu(y) \times H_\rho(y) G_k(x,y) dy - \int_{\partial B_1} \nu(y) \cdot E_\rho(y) \nabla_x G_k(x,y) dy, \quad (1.3.17)$$

where $k = \omega\rho$. We claim that

$$\left| \int_{\partial B_1} E_\rho \times \nu \right| \le C\rho \|(E_\rho, H_\rho)\|_{L^2(B_2 \setminus D)}, \quad (1.3.18)$$

and

$$\left| \int_{\partial B_1} H_\rho \times \nu \right| \le C\rho \|(E_\rho, H_\rho)\|_{L^2(B_2 \setminus D)}. \quad (1.3.19)$$

Assuming this, we continue the proof. We have

$$\int_{\partial B_1} \nu \cdot E_\rho \, ds = \frac{1}{i\omega\rho} \int_{\partial B_1} \nu \cdot \nabla \times H_\rho \, ds = 0. \quad (1.3.20)$$

Rewrite (1.3.17) under the form

$$E_\rho(x) = \\ \int_{\partial B_1} \nabla_x G_k(x,0) \times \big(\nu(y) \times E_\rho(y)\big) dy + \int_{\partial B_1} \big(\nabla_x G_k(x,y) - \nabla_x G_k(x,0)\big) \times \big(\nu(y) \times E_\rho(y)\big) dy \\ + i\omega\rho \int_{\partial B_1} \nu(y) \times H_\rho(y) G_k(x,0) dy + i\omega\rho \int_{\partial B_1} \nu(y) \times H_\rho(y) \big(G_k(x,y) - G_k(x,0)\big) dy \\ - \int_{\partial B_1} \nu(y) \cdot E_\rho(y) \nabla_x G_k(x,0) dy - \int_{\partial B_1} \nu(y) \cdot E_\rho(y) \big(\nabla_x G_k(x,y) - \nabla_x G(x,0)\big) dy.$$

Using the facts, for $|x| \in (2R/\rho, 3R/\rho)$ and $y \in \partial B_1$,

$$|G_k(x,y) - G_k(x,0)| \le C\rho^2, \quad |\nabla G_k(x,y) - \nabla G_k(x,0)| \le C\rho^3,$$

and

$$\|(E_\rho, H_\rho)\|_{L^2(\partial B_1)} \le C \|(E_\rho, H_\rho)\|_{L^2(B_2 \setminus D)},$$

we derive from (1.3.18), (1.3.19), and (1.3.20) that

$$|E_\rho(x)| \le C\rho^3 \|(E_\rho, H_\rho)\|_{L^2(B_2 \setminus D)} \text{ for } x \in B_{3R/\rho} \setminus B_{2R/\rho}. \quad (1.3.21)$$

Similarly, we have

$$|H_\rho(x)| \le C\rho^3 \|(E_\rho, H_\rho)\|_{L^2(B_2\setminus D)} \text{ for } x \in B_{3R/\rho} \setminus B_{2R/\rho}. \tag{1.3.22}$$

The conclusion now follows from (1.3.21) and (1.3.22).

It remains to prove Claims (1.3.18) and (1.3.19). We only prove (1.3.18), the proof of (1.3.19) is similar. Let $(\tilde{E}_\rho, \tilde{H}_\rho) \in [H(\mathrm{curl}, B_1)]^2$ be the unique solution to the system

$$\begin{cases} \nabla \times \tilde{E}_\rho = i\omega\rho(1+i)\tilde{H}_\rho & \text{in } B_1, \\ \nabla \times \tilde{H}_\rho = -i\omega\rho(1+i)\tilde{E}_\rho & \text{in } B_1, \\ \tilde{E}_\rho \times \nu = E_\rho \times \nu & \text{on } \partial B_1. \end{cases} \tag{1.3.23}$$

The well-posedness of (1.3.23) follows immediately from Lax-Milgram's theorem. We now prove by contradiction that

$$\|(\tilde{E}_\rho, \tilde{H}_\rho)\|_{L^2(B_1)} \le C\big(\|E_\rho \times \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} + \|H_\rho \cdot \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)}\big). \tag{1.3.24}$$

Assume by contradiction that there exists $(\rho_n)_n \subset (0,1)$ converging to 0, $(E_n, H_n)_n \subset [H(\mathrm{curl}, B_1)]^2$ satisfying

$$\begin{cases} \nabla \times E_n = i\omega\rho_n(1+i)H_n & \text{in } B_1, \\ \nabla \times H_n = -i\omega\rho_n(1+i)E_n & \text{in } B_1, \end{cases} \tag{1.3.25}$$

and that

$$\|(E_n, H_n)\|_{L^2(B_1)} = 1, \text{ for all } n \in \mathbb{N}, \tag{1.3.26}$$

but

$$\|E_n \times \nu\|_{H^{-1/2}(\partial B_1)} + \|H_n \cdot \nu\|_{H^{-1/2}(\partial B_1)} \to 0. \tag{1.3.27}$$

Using Lemma 1.2.7, one can assume that (E_n, H_n) converges to some $(E, H) \in [H(\mathrm{curl}, B_1)]^2$ in $[L^2(B_1)]^6$. It clear from (1.3.25) and (1.3.27) that the limit satisfies

$$\begin{cases} \nabla \times E = 0 & \text{in } B_1, \\ \operatorname{div} E = 0 & \text{in } B_1, \\ E \times \nu = 0 & \text{on } \partial B_1, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times H = 0 & \text{in } B_1, \\ \operatorname{div} H = 0 & \text{in } B_1, \\ H \cdot \nu = 0 & \text{on } \partial B_1, \end{cases}$$

These equations only have zero solutions, thus $(E_n, H_n) \to (0,0)$ in $[L^2(B_1)]^6$. This fact contradicts (1.3.26). We obtain (1.3.24).

It follows that

$$\|(\tilde{E}_\rho, \tilde{H}_\rho)\|_{L^2(B_1)} \le C\|(E_\rho, H_\rho)\|_{L^2(B_2\setminus D)}. \tag{1.3.28}$$

Since

$$\left|\int_{\partial B_1} E_\rho \times \nu\, ds\right| = \left|\int_{\partial B_1} \tilde{E}_\rho \times \nu\, ds\right| = \left|\int_{B_1} \nabla \times \tilde{E}_\rho\, dx\right| = \left|\int_{B_1} \omega\rho(1+i)\tilde{H}_\rho dx\right|,$$

claim (1.3.18) follows from (1.3.28).

The proof is complete. □

The following compactness result is used in the proof of Theorems 1.1.1, 1.1.2, and 1.1.3.

Lemma 1.3.3. *Let* $\big((E_n, H_n)\big)_n$ *be a bounded sequence in* $[H(\mathrm{curl}, D)]^2$ *and let* $\big((\theta_{1,n}, \theta_{2,n})\big)_n$ *be a convergent sequence in* $[L^2(D)]^6$. *Assume that*

$$\begin{cases} \nabla \times E_n = i\mu H_n + \theta_{1,n} & \text{in } D, \\ \nabla \times H_n = -i\varepsilon E_n + \theta_{2,n} & \text{in } D, \end{cases} \tag{1.3.29}$$

and

$$\big((\nabla \times E_n \cdot \nu, \nabla \times H_n \cdot \nu)\big)_n \text{ converges in } [H^{-1/2}(\partial D)]^2. \tag{1.3.30}$$

Then, up to a subsequence, $\big((E_n, H_n)\big)_n$ *converges in* $[H(\mathrm{curl}, D)]^2$.

Remark 1.3.3. A comparison with Lemma 1.3.3 is necessary. The difference between Lemma 1.3.3 and part i) Lemma 1.2.7 is that the sequence $(E_n \times \nu)_n$ or $(H \times \nu)_n$ is not required to be convergent in $H^{-1/2}(\partial D)$. The difference between Lemma 1.3.3 and part ii) Lemma 1.2.7 is that the sequence $\big(\mathrm{div}(\varepsilon E_n)\big)_n$ or $\big(\mathrm{div}(\mu H_n)\big)_n$ is not required to be bounded in $L^2(D)$. Nevertheless, in Lemma 1.3.3, (1.3.29) is assumed.

Proof. It suffices to prove that, up to a subsequence, $\big((E_n, H_n)\big)_n$ converges in $[L^2(D)]^6$. By Lemma 1.2.3, there exist $(q_n)_n \subset H^1(D)$ and $(\phi_n)_n \subset [H^1(D)]^3$ such that, for all n,

$$\varepsilon E_n = \varepsilon\nabla q_n + \nabla \times \phi_n \text{ in D}, \quad \mathrm{div}\phi_n = 0 \text{ in } D, \quad \text{and} \quad \phi_n \times \nu = 0 \text{ on } \partial D. \tag{1.3.31}$$

Moreover, we have

$$\|q_n\|_{H^1(D)} + \|\phi_n\|_{[H^1(D)]^3} \le C\|E_n\|_{L^2(D)} \le C, \tag{1.3.32}$$

for some positive constant C independent of n. From (1.3.32), without loss of generality, one might assume that

$$(q_n)_n \text{ and } (\phi_n)_n \text{ converge in } L^2(D) \text{ and } [L^2(D)]^3 \text{ respectively.} \tag{1.3.33}$$

From (1.3.31) and an integration by parts, we derive that, for all n,

$$\int_D \varepsilon \nabla q_n \cdot \nabla p \, dx = \int_D \varepsilon E_n \cdot \nabla p \, dx \text{ for } p \in H^1(D). \tag{1.3.34}$$

This implies, by (1.3.29), for $m, n \in \mathbb{N}$,

$$\begin{aligned}\int_D \varepsilon \nabla (q_n - q_m) \cdot \nabla (\bar{q}_n - \bar{q}_m) \, dx &= \int_D \varepsilon (E_n - E_m) \cdot \nabla (\bar{q}_n - \bar{q}_m) \, dx, \\ &= i \int_D \Big(\nabla \times \big(H_n - H_m \big) - (\theta_{2,n} - \theta_{2,m}) \Big) \cdot \nabla (\bar{q}_n - \bar{q}_m) \, dx.\end{aligned}$$

An integration by parts yields

$$\begin{aligned}&\int_D \varepsilon \nabla (q_n - q_m) \cdot \nabla (\bar{q}_n - \bar{q}_m) \, dx \\ &\qquad = i \int_{\partial D} \nabla \times \big(H_n - H_m \big) \cdot \nu \; (\bar{q}_n - \bar{q}_m) \, ds - i \int_D (\theta_{2,n} - \theta_{2,m}) \cdot \nabla (\bar{q}_n - \bar{q}_m) \, dx.\end{aligned}$$

By (1.3.30) and the convergence of $(\theta_{1,n}, \theta_{2,n})$ in $[L^2(D)]^6$, the LHS of the above identity converges to 0 as $m, n \to \infty$. Hence, by the ellipticity of ε, $(\nabla q_n)_n$ is a Cauchy sequence and thus converges in $[L^2(D)]^3$. From (1.3.31), we have

$$\int_D \varepsilon^{-1} \nabla \times (\phi_n - \phi_m) \cdot \nabla \times (\bar{\phi}_n - \bar{\phi}_m) \, dx = \int_D \nabla \times (E_n - E_m) \cdot (\bar{\phi}_n - \bar{\phi}_m) \, dx.$$

By the ellipticity of ε and the convergence of (ϕ_n) in $L^2(D)$, we derive that $\big(\nabla \times \phi_n\big)_n$ is a Cauchy sequence in $[L^2(D)]^3$ and thus converges in $[L^2(D)]^3$. Since

$$E_n = \nabla q_n + \varepsilon^{-1} \nabla \times \phi_n,$$

$(E_n)_n$ converges in $[L^2(D)]^3$.

Similarly, up to a subsequence, $(H_n)_n$ converges in $[L^2(D)]^3$. □

Using Lemma 1.3.3 and applying the Fredholm theory, one can prove the well-posedness of (E_0, H_0) in Definitions 1.1.3 and 1.1.4. The first result in this direction is

Lemma 1.3.4. *Let $\theta_1, \theta_2 \in [L^2(D)]^3$. The system*

$$\begin{cases} \nabla \times E = i\mu H + \theta_1 & \text{in } D, \\ \nabla \times H = -i\varepsilon E + \theta_2 & \text{in } D, \\ \nabla \times E \cdot \nu = \nabla \times H \cdot \nu = 0 & \text{on } \partial D, \end{cases} \tag{1.3.35}$$

has a solution (E,H) *in* $[H(\mathrm{curl},D)]^2$ *if and only if*

$$\int_D \theta_2 \cdot \bar{\boldsymbol{E}}\, dx - \int_D \theta_1 \cdot \bar{\boldsymbol{H}}\, dx = 0 \quad \textit{for all } (\boldsymbol{E},\boldsymbol{H}) \in \mathcal{N}(D). \tag{1.3.36}$$

In particular, system (1.3.35) has a unique solution $(E,H) \in \mathcal{N}(D)^{\perp}$ *if and only if (1.3.36) holds.*

Proof. Lemma 1.3.4 is derived from the Fredholm theory. Since ε and μ are uniformly elliptic, by Lemma 1.2.3, there exist $p_1, p_2 \in H^1(D)$ and $\phi_1, \phi_2 \in [H^1(D)]^3$ such that

$$\theta_1 = \mu \nabla p_1 + \nabla \times \phi_1, \quad \theta_2 = \varepsilon \nabla p_2 + \nabla \times \phi_2 \text{ in } D, \tag{1.3.37}$$

and

$$\nabla \times \phi_1 \cdot \nu = \nabla \times \phi_2 \cdot \nu = 0 \text{ on } \partial D. \tag{1.3.38}$$

Set $(E_0, H_0) := (-i\nabla p_2, i\nabla p_1)$ in D. Then $(E_0, H_0) \in [H(\mathrm{curl},D)]^2$ is a solution to

$$\begin{cases} \nabla \times E_0 = i\mu H_0 + \mu \nabla p_1 & \text{in } D, \\ \nabla \times H_0 = -i\varepsilon E_0 + \varepsilon \nabla p_2 & \text{in } D, \\ \nabla \times E_0 \cdot \nu = \nabla \times H_0 \cdot \nu = 0 & \text{on } \partial D. \end{cases} \tag{1.3.39}$$

We have

$$\int_D \varepsilon \nabla p_2 \cdot \bar{\mathbf{E}}\, dx - \int_D \mu \nabla p_1 \cdot \bar{\mathbf{H}}\, dx = 0 \quad \text{for all } (\mathbf{E},\mathbf{H}) \in \mathcal{N}(D). \tag{1.3.40}$$

From (1.3.37), (1.3.38), (1.3.39), and (1.3.40), by considering $(E - E_0, H - H_0)$ instead of (E,H), one might assume that $(\theta_1, \theta_2) \in H(\mathrm{div}, D)$,

$$\mathrm{div}(\theta_1) = \mathrm{div}(\theta_2) = 0 \text{ in } D \quad \text{and} \quad \theta_1 \cdot \nu = \theta_2 \cdot \nu = 0 \text{ on } \partial D. \tag{1.3.41}$$

This is assumed from now on.

Set

$$\mathbb{V} = \Big\{\varphi \in H(\mathrm{curl},D) : \mathrm{div}(\varepsilon\varphi) = 0,\ \varepsilon\varphi \cdot \nu = 0 \text{ on } \partial D,\ \nabla \times \varphi \cdot \nu = 0 \text{ on } \partial D\Big\}.$$

Since ε and μ are real, symmetric and uniformly elliptic, $\mathbb{V}$ is a Hilbert space equipped with the scalar product

$$< E, \varphi >_{\mathbb{V},\mathbb{V}} = \int_D \mu^{-1} \nabla \times E \cdot \nabla \times \bar{\varphi}\, dx + \int_D \varepsilon E \cdot \bar{\varphi}\, dx \quad \text{for } E, \varphi \in \mathbb{V}. \tag{1.3.42}$$

Let $A : \mathbb{V} \to \mathbb{V}$ be defined by

$$< AE, \varphi >_{<\mathbb{V},\mathbb{V}>} = -2 \int_D \varepsilon E \cdot \bar{\varphi} \, dx \text{ for all } \varphi \in \mathbb{V}. \tag{1.3.43}$$

Since ε is symmetric, one can easily check that A is self-adjoint. Since ε and μ are symmetric and uniformly elliptic, by Lemma 1.2.7, A is compact.

Let $g \in \mathbb{V}$ be such that

$$< g, \varphi >_{<\mathbb{V},\mathbb{V}>} = \int_D i\theta_2 \cdot \bar{\varphi} + \int_D \mu^{-1} \theta_1 \cdot \nabla \times \bar{\varphi} \text{ for all } \varphi \in \mathbb{V}. \tag{1.3.44}$$

We claim that

$$\text{system (1.3.35) has a solution in } [H(\operatorname{curl}, D)]^2 \\ \text{if and only if the equation } u + Au = g \text{ in } \mathbb{V} \text{ has a solution in } \mathbb{V} \tag{1.3.45}$$

and

$$(E, H) \text{ is a solution of (1.3.35) if and only if} \\ E + AE = g \text{ in } \mathbb{V} \text{ and } H = -i\mu^{-1}(\nabla \times E - \theta_1). \tag{1.3.46}$$

Assuming this, we continue the proof. By (1.3.45) and the Fredholm theory, see, e.g., [8, Chapter 6], system (1.3.35) has a solution if and only if

$$\langle g, \varphi \rangle_{\mathbb{V},\mathbb{V}} = 0 \text{ for all } \varphi \in \mathbb{V} \text{ such that } \varphi + A\varphi = 0 \text{ in } \mathbb{V}, \tag{1.3.47}$$

since A is self-adjoint. Applying (1.3.46) with $g = \theta_1 = \theta_2 = 0$ and using (1.3.42), (1.3.43), and (1.3.44), we derive that condition (1.3.47) is equivalent to the fact that

$$\int_D \theta_2 \cdot \bar{\mathbf{E}} \, dx - \int_D \theta_1 \cdot \bar{\mathbf{H}} \, dx = 0 \quad \text{for all } (\mathbf{E}, \mathbf{H}) \in \mathcal{N}(D),$$

which is (1.3.36).

The rest of the proof is devoted to establishing Claims (1.3.45) and (1.3.46). Let $(E, H) \in [H(\operatorname{curl}, D)]^2$ be a solution to (1.3.35). From (1.3.41), we derive that $E \in \mathbb{V}$. Fix $\varphi \in \mathbb{V}$. Then $\nabla \times \varphi \cdot \nu = 0$ on ∂D. By Lemma 1.2.2, there exists $\varphi_0 \in [H^1(D)]^3$ such that

$$\nabla \times \varphi_0 = \nabla \times \varphi \text{ in } D, \quad \operatorname{div} \varphi_0 = 0 \text{ in } D, \quad \text{and} \quad \varphi_0 \times \nu = 0 \text{ on } \partial D. \tag{1.3.48}$$

Since $\nabla \times (\varphi_0 - \varphi) = 0$ and D is simply connected, there exists $\xi \in H^1(D)$ such that

$$\varphi_0 - \varphi = \nabla \xi \text{ in } D. \tag{1.3.49}$$

We have, for $\varphi \in \mathbb{V}$,

$$\int_D \mu^{-1} \nabla \times E \cdot \nabla \times \bar{\varphi} \, dx = i \int_D H \cdot \nabla \times \bar{\varphi} + \mu^{-1} \theta_1 \cdot \nabla \times \bar{\varphi} \, dx. \tag{1.3.50}$$

Using (1.3.48) and an integration by parts, we obtain

$$\int_D H \cdot \nabla \times \bar{\varphi} \, dx = \int_D H \cdot \nabla \times \bar{\varphi}_0 \, dx = \int_D \nabla \times H \cdot \bar{\varphi}_0 \, dx. \tag{1.3.51}$$

Using (1.3.49) and the fact $\nabla \times H \cdot \nu = 0$ on ∂D, we also get, by an integration by parts,

$$\int_D \nabla \times H \cdot \bar{\varphi}_0 \, dx = \int_D \nabla \times H \cdot \bar{\varphi} \, dx.$$

This implies, by (1.3.51),

$$\int_D H \cdot \nabla \times \bar{\varphi} \, dx = \int_D \nabla \times H \cdot \bar{\varphi} \, dx. \tag{1.3.52}$$

A combination of (1.3.50) and (1.3.52) yields

$$\int_D \mu^{-1} \nabla \times E \cdot \nabla \times \bar{\varphi} \, dx = i \int_D \nabla \times H \cdot \bar{\varphi} + \mu^{-1} \theta_1 \cdot \nabla \times \bar{\varphi} \, dx. \tag{1.3.53}$$

We derive from (1.3.35) and (1.3.53) that

$$\int_D \mu^{-1} \nabla \times E \cdot \nabla \times \bar{\varphi} \, dx = \int_D \varepsilon E \cdot \bar{\varphi} \, dx + i \int_D \theta_2 \cdot \bar{\varphi} \, dx + \int_D \mu^{-1} \theta_1 \cdot \nabla \times \bar{\varphi} \, dx. \tag{1.3.54}$$

It follows from (1.3.42), (1.3.43), and (1.3.44) that

$$E + AE = g \text{ in } \mathbb{V}.$$

Conversely, assume that there exists $u \in \mathbb{V}$ such that $u + Au = g$. Set

$$E = u \text{ and } H = -i\mu^{-1}(\nabla \times E - \theta_1) \text{ in } D.$$

Using (1.3.54), one can check that (E, H) satisfies the first two equations of (1.3.35). It is clear that $\nabla \times E \cdot \nu = 0$ on ∂D by the definition of $\mathbb{V}$. Since $\nabla \times H = -i\varepsilon E + \theta_2$ in D, $\varepsilon E \cdot \nu = 0$ on ∂D ($E \in \mathbb{V}$), and $\theta_2 \cdot \nu = 0$ on ∂D by (1.3.41), we obtain

$$\nabla \times H \cdot \nu = 0 \text{ on } \partial D.$$

The proof is complete. □

Remark 1.3.4. One of the key points in the proof of Lemma 1.3.4 is the identity

$$\int_D H \cdot \nabla \times \bar{E} \, dx = \int_D \nabla \times H \cdot \bar{E} \, dx,$$

if $E, H \in H(\mathrm{curl}, D)$ is such that $\nabla \times E \cdot \nu = \nabla \times H \cdot \nu = 0$ on ∂D, see (1.3.52). This ensures the variational character of system (1.3.35).

The following lemma yields the uniqueness of (E_0, H_0) in Definition 1.1.4.

Lemma 1.3.5. *Let* $[(E,H),(\tilde{E},\tilde{H})] \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2 \times \mathcal{N}(D)^{\perp}$ *be such that*

$$\begin{cases} \nabla \times E = \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} E = \operatorname{div} H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times E = i\mu H & \text{in } D, \\ \nabla \times H = -i\varepsilon E & \text{in } D, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times \tilde{E} = i\mu \tilde{H} & \text{in } D, \\ \nabla \times \tilde{H} = -i\varepsilon \tilde{E} & \text{in } D, \\ \varepsilon \tilde{E} \cdot \nu = E \cdot \nu|_{\mathrm{ext}} & \text{on } \partial D, \\ \mu \tilde{H} \cdot \nu = H \cdot \nu|_{\mathrm{ext}} & \text{on } \partial D, \end{cases} \tag{1.3.55}$$

and

$$\Big| \big(E(x), H(x)\big) \Big| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.3.56}$$

Then $(E,H) = (0,0)$ *in* $\mathbb{R}^3$ *and* $(\tilde{E},\tilde{H}) = (0,0)$ *in* D.

Proof. Applying Lemma 1.2.1 to $\bar{E}$, there exists a function $\theta \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)$ such that

$$\nabla \theta = \bar{E} \text{ in } \mathbb{R}^3 \setminus D \text{ and } |\theta(x)| = O(|x|^{-1}) \text{ for large } |x|. \tag{1.3.57}$$

For $R > 0$ large, since $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus D$, we have

$$\int_{B_R \setminus D} |E|^2 \, dx = \int_{B_R \setminus D} E \cdot \nabla \theta \, dx = \int_{\partial B_R} (E \cdot \nu) \theta \, ds - \int_{\partial D} (E \cdot \nu)|_{\mathrm{ext}} \theta \, ds.$$

Letting R tend to $+\infty$ and using (1.3.56) and (1.3.57), we obtain

$$\int_{\mathbb{R}^3 \setminus D} |E|^2 \, dx = - \int_{\partial D} (E \cdot \nu)|_{\mathrm{ext}} \theta \, ds. \tag{1.3.58}$$

Extend θ in D so that the extension belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and still denote this extension by θ. We derive from the system of $(\tilde{E},\tilde{H})$ in (1.3.55) that

$$\begin{aligned} - \int_{\partial D} (E \cdot \nu)|_{\mathrm{ext}} \theta \, ds &= - \int_{\partial D} (\varepsilon \tilde{E} \cdot \nu) \theta \, ds = - \int_D \varepsilon \tilde{E} \cdot \nabla \theta \, dx - \int_D \operatorname{div}(\varepsilon \tilde{E}) \theta \, dx \\ &= \int_D -i \nabla \times \tilde{H} \cdot \nabla \theta \, dx = -i \int_{\partial D} \tilde{H} \cdot (\nabla \theta \times \nu) \, ds = -i \int_{\partial D} \tilde{H} \cdot (\bar{E} \times \nu) \, ds. \end{aligned} \tag{1.3.59}$$

Combining (1.3.58) and (1.3.59) yields

$$\int_{\mathbb{R}^3 \setminus D} |E|^2 \, dx = -i \int_{\partial D} \tilde{H} \cdot (\bar{E} \times \nu) \, ds. \tag{1.3.60}$$

Similarly, we have

$$\int_{\mathbb{R}^3\setminus D} |H|^2\,dx = i\int_{\partial D} \tilde{E}\cdot(\bar{H}\times\nu)\,ds. \tag{1.3.61}$$

An integration by parts implies

$$\int_{\partial D} \tilde{H}\cdot(\bar{E}\times\nu)\,ds - \int_{\partial D} \tilde{E}\cdot(\bar{H}\times\nu)\,ds$$
$$= \int_D \nabla\times\tilde{H}\cdot\bar{E}\,dx - \int_D \nabla\times\bar{E}\cdot\tilde{H}\,dx - \int_D \nabla\times\tilde{E}\cdot\bar{H}\,dx + \int_D \nabla\times\bar{H}\cdot\tilde{E}\,dx.$$

Using the equations of (E,H) and $(\tilde{E},\tilde{H})$ in D in (1.3.55), we obtain

$$\int_{\partial D} \tilde{H}\cdot(\bar{E}\times\nu)\,ds - \int_{\partial D} \tilde{E}\cdot(\bar{H}\times\nu)\,ds = 0. \tag{1.3.62}$$

A combination of (1.3.60), (1.3.61), and (1.3.62) yields

$$\int_{\mathbb{R}^3\setminus D} \big(|E|^2+|H|^2\big)dx = 0.$$

We derive that $E = H = 0$ in $\mathbb{R}^3\setminus D$. This implies, by the unique continuation principle see, e.g., [48, Theorem 1],

$$E = H = 0 \text{ in } D$$

and, since $(\tilde{E},\tilde{H})\in\mathcal{N}(D)^{\perp}$,

$$\tilde{E} = \tilde{H} = 0 \text{ in } D.$$

The proof is complete. □

1.3.2 Approximate cloaking in the non-resonant case - Proof of Theorem 1.1.1

The key ingredient in the proof of Theorem 1.1.1 is the following lemma whose proof uses various results in Section 1.2 and Section 1.3.1

Lemma 1.3.6. *Let $0<\rho<\rho_0$, $\theta_\rho = (\theta_{1,\rho},\theta_{2,\rho})\in[L^2(D)]^6$, and $h_\rho = (h_{1,\rho},h_{2,\rho})\in[H^{-1/2}(\mathrm{div}_\Gamma,\partial D)]^2$. Let $(E_\rho,H_\rho)\in[\bigcap_{R>1}H(\mathrm{curl},B_R\setminus\partial D)]^2$ be the unique radiating solution to the system*

$$\begin{cases}\nabla\times E_\rho = i\rho H_\rho & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times H_\rho = -i\rho E_\rho & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times E_\rho = i\mu H_\rho + \theta_{1,\rho} & \text{in } D,\\ \nabla\times H_\rho = -i\varepsilon E_\rho + \theta_{2,\rho} & \text{in } D,\\ [E_\rho\times\nu] = h_{1,\rho},\ [H_\rho\times\nu] = h_{2,\rho} & \text{on } \partial D.\end{cases}$$

Assume that $\mathcal{N}(D) = \{(0,0)\}$. *We have*

$$\|(E_\rho, H_\rho)\|_{L^2(B_5)} \le C\Big(\|\theta_\rho\|_{L^2(D)} + \|h_\rho\|_{H^{-1/2}(\mathrm{div}_\Gamma,\partial D)}\Big), \tag{1.3.63}$$

for some positive constant C *depending only on* ρ_0, ε, μ. *Assume in addition that*

$$\lim_{\rho\to 0}\|h_\rho\|_{H^{-1/2}(\mathrm{div}_\Gamma,\partial D)} = 0 \quad \text{and} \quad \lim_{\rho\to 0}\theta_\rho = \theta \text{ in } [L^2(D)]^6,$$

for some $\theta = (\theta_1,\theta_2) \in [L^2(D)]^6$. *We have*

$$\lim_{\rho\to 0}(E_\rho, H_\rho) = Cl(\theta_1,\theta_2) \text{ in } [H(\mathrm{curl}, D)]^2. \tag{1.3.64}$$

Here and in what follows on ∂D, $[u]$ denotes the jump of u across ∂D for an appropriate (vectorial) function u, i.e., $[u] = u|_{\mathrm{ext}} - u|_{\mathrm{int}}$ on ∂D. Moreover the following notation is used in the thesis

$$\bigcap_{R>1} H(\mathrm{curl}, B_R\setminus\partial D) = \Big\{u:\mathbb{R}^3\to\mathbb{R}^3 \text{ such that } u\in H(\mathrm{curl}, D) \text{ and } u\in H(\mathrm{curl}, B_R\setminus D) \text{ for all } R>1\Big\}$$

Proof. By Lemma 1.2.8, without loss of generality, one might assume that $h_{1,\rho} = h_{2,\rho} = 0$ on ∂D. This is assumed from now on.

We first prove (1.3.63) by contradiction. Assume that there exist sequences $(\rho_n)_n \subset (0,\rho_0)$, $\big((E_n, H_n)\big)_n \subset [H_{\mathrm{loc}}(\mathrm{curl},\mathbb{R}^3)]^2$, $\big((\theta_{1,n},\theta_{2,n})\big)_n \subset [L^2(D)]^6$ such that

$$\begin{cases} \nabla\times E_n = i\rho_n H_n & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times H_n = -i\rho_n E_n & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times E_n = i\mu H_n + \theta_{1,n} & \text{in } D,\\ \nabla\times H_n = -i\varepsilon E_n + \theta_{2,n} & \text{in } D, \end{cases} \tag{1.3.65}$$

$$\|(E_n, H_n)\|_{L^2(B_5)} = 1 \text{ for all } n\in\mathbb{N}, \tag{1.3.66}$$

and

$$\lim_{n\to+\infty}\|(\theta_{1,n},\theta_{2,n})\|_{L^2(D)} = 0. \tag{1.3.67}$$

Without loss of generality, one might assume that $\rho_n \to \rho_\infty \in [0,\rho_0]$. We only consider the case $\rho_\infty = 0$. The case $\rho_\infty > 0$ can be proved similarly.

We have

$$\nabla \times E_n \cdot \nu|_{\text{int}} = \nabla \times E_n \cdot \nu|_{\text{ext}} = i\rho_n H_n \cdot \nu|_{\text{ext}} \to 0 \text{ in } H^{-1/2}(\partial D) \text{ as } n \to \infty. \tag{1.3.68}$$

Similarly, we obtain

$$\nabla \times H_n \cdot \nu|_{\text{int}} \to 0 \text{ in } H^{-1/2}(\partial D) \text{ as } n \to \infty. \tag{1.3.69}$$

Applying Lemma 1.3.3 to $\big((E_n, H_n)\big)_n$ in D, without loss of generality, one might assume that

$$\big((E_n, H_n)\big)_n \text{ converges in } [H(\text{curl}, D)]^2 \text{ as } n \to \infty. \tag{1.3.70}$$

Applying i) of Lemma 1.2.8, we derive that

$$\big((E_n \times \nu, H_n \times \nu)\big)_n \text{ converges in } [H^{-1/2}(\text{div}_\Gamma, \partial D)]^2 \text{ as } n \to \infty.$$

It follows from (1.3.66), Lemma 1.2.6, and i) of Lemma 1.2.7 that

$$\big((E_n, H_n)\big)_n \text{ converges in } [L^2_{\text{loc}}(\mathbb{R}^3 \setminus D)]^6 \text{ as } n \to \infty. \tag{1.3.71}$$

Let (E, H) be the limit of (E_n, H_n) in $[L^2_{\text{loc}}(\mathbb{R}^3)]^6$. Then $(E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ and [2]

$$\begin{cases} \nabla \times E = \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \text{div}\, E = \text{div}\, H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times E = i\mu H & \text{in } D, \\ \nabla \times H = -i\varepsilon E & \text{in } D. \end{cases} \tag{1.3.72}$$

We derive from (1.3.68) and (1.3.69) that

$$\nabla \times E \cdot \nu|_{\text{int}} = \nabla \times H \cdot \nu|_{\text{int}} = 0 \text{ on } \partial D. \tag{1.3.73}$$

Applying Lemma 1.2.6, we have

$$|\big(E(x), H(x)\big)| \le \frac{C}{|x|^2} \text{ for } |x| > 3, \tag{1.3.74}$$

for some positive constant C. Combining (1.3.72) and (1.3.73) yields that $(E, H)|_D \in \mathcal{N}(D)$. Since $\mathcal{N}(D) = \{(0, 0)\}$, it follows that $E = H = 0$ in D. Hence

$$E \times \nu = H \times \nu = 0 \text{ on } \partial D. \tag{1.3.75}$$

[2] In the case $\rho_\infty > 0$, the limit (E, H) satisfies the radiating condition and is a solution to Maxwell equations in $\mathbb{R}^3$ with vanished data. It follows that $(E, H) = (0, 0)$, which also gives a contradiction.

We have, for each connected component Γ of ∂D,

$$\int_\Gamma E\cdot \nu|_{\text{ext}} = \lim_{n\to\infty}\int_\Gamma E_n|_{\text{ext}}\cdot \nu = \lim_{n\to\infty}\frac{1}{-i\rho_n}\int_\Gamma (\nabla\times H_n)\cdot\nu|_{\text{ext}} = 0$$

and similarly

$$\int_\Gamma H\cdot \nu|_{\text{ext}} = 0.$$

Using (1.3.72), (1.3.74), and (1.3.75), and applying Lemma 1.2.5 to (E,H) in $\mathbb{R}^3\setminus D$, we obtain

$$E = H = 0 \text{ in } \mathbb{R}^3\setminus D.$$

Thus $E = H = 0$ in $\mathbb{R}^3$, which, by using (1.3.70) and (1.3.71), contradicts (1.3.66). Therefore, (1.3.63) is proved.

We next establish (1.3.64). Fix an arbitrary sequence $(\rho_n)_n$ converging to 0. From (1.3.63), one obtains that

$$\|(E_{\rho_n}, H_{\rho_n})\|_{L^2(B_5)} \le C\Big(\|\theta_{\rho_n}\|_{L^2(D)} + \|h_{\rho_n}\|_{H^{-1/2}(\operatorname{div}_\Gamma,\partial D)}\Big) \le C.$$

Using the same argument as above, one obtains that, up to a subsequence, (E_{ρ_n}, H_{ρ_n}) converges in $[H(\operatorname{curl},\mathbb{R}^3)]^2$ to (E,H), the unique solution of

$$\begin{cases} \nabla\times E = \nabla\times H = 0 & \text{in } \mathbb{R}^3\setminus D,\\ \operatorname{div} E = \operatorname{div} H = 0 & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times E = i\mu H + \theta_1 & \text{in } D,\\ \nabla\times H = -i\varepsilon E + \theta_2 & \text{in } D.\end{cases} \tag{1.3.76}$$

This system implies $\nabla\times E\cdot\nu|_{\text{int}} = \nabla\times H\cdot\nu|_{\text{int}} = 0$ on ∂D. Since $\mathcal{N}(D) = \{(0,0)\}$, we have $(E,H)|_D = Cl(\theta_1,\theta_2)$. Since $(\rho_n)\to 0$ arbitrarily, assertion (1.3.64) follows. The proof is complete. □

We are ready to give the

Proof of Theorem 1.1.1. Let $(E_{1,\rho}, H_{1,\rho}) \in [H_{\text{loc}}(\operatorname{curl},\mathbb{R}^3\setminus B_\rho)]^2$ be the unique radiating solution to the system

$$\begin{cases} \nabla\times E_{1,\rho} = i\omega H_{1,\rho} & \text{in } \mathbb{R}^3\setminus B_\rho,\\ \nabla\times H_{1,\rho} = -i\omega E_{1,\rho} + J_{\text{ext}} & \text{in } \mathbb{R}^3\setminus B_\rho,\\ E_{1,\rho}\times\nu = 0 & \text{on } \partial B_\rho,\end{cases} \tag{1.3.77}$$

extend $(E_{1,\rho}, H_{1,\rho})$ by $(0,0)$ in B_ρ, and still denote this extension by $(E_{1,\rho}, H_{1,\rho})$. Define

$$(E_{2,\rho}, H_{2,\rho}) := (E, H) - (E_{1,\rho}, H_{1,\rho}) \quad \text{and} \quad (E_{3,\rho}, H_{3,\rho}) := (\mathscr{E}_\rho, \mathscr{H}_\rho) - (E_{1,\rho}, H_{1,\rho}) \quad \text{in } \mathbb{R}^3.$$

Then $(E_{2,\rho}, H_{2,\rho}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus B_\rho)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times E_{2,\rho} = i\omega H_{2,\rho} & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ \nabla \times H_{2,\rho} = -i\omega E_{2,\rho} & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ E_{2,\rho} \times \nu = E \times \nu & \text{on } \partial B_\rho, \end{cases}$$

and $(E_{3,\rho}, H_{3,\rho}) \in [\bigcap_{R>1} H(\text{curl}, B_R \setminus \partial B_\rho)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times E_{3,\rho} = i\omega \mu_\rho H_{3,\rho} & \text{in } \mathbb{R}^3 \setminus \partial B_\rho, \\ \nabla \times H_{3,\rho} = -i\omega \varepsilon_\rho E_{3,\rho} + J_\rho \chi_{B_\rho} & \text{in } \mathbb{R}^3 \setminus \partial B_\rho, \\ [E_{3,\rho} \times \nu] = 0, \, [H_{3,\rho} \times \nu] = -H_{1,\rho} \times \nu|_{\text{ext}} & \text{on } \partial B_\rho, \end{cases} \tag{1.3.78}$$

where χ_D denotes the characteristic function of a subset D of $\mathbb{R}^3$. Recall that J_ρ is defined in (1.1.23). Set

$$\tilde{E}_{2,\rho}(x) = E_\rho(\rho x) \text{ and } \tilde{H}_{2,\rho}(x) = H_\rho(\rho x) \text{ for } x \in \mathbb{R}^3 \setminus B_1.$$

Then $(\tilde{E}_{2,\rho}, \tilde{H}_{2,\rho}) \in [H(\text{curl}, \mathbb{R}^3 \setminus B_1)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times \tilde{E}_{2,\rho} = i\omega\rho \tilde{H}_{2,\rho} & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \tilde{H}_{2,\rho} = -i\omega\rho \tilde{E}_{2,\rho} & \text{in } \mathbb{R}^3 \setminus B_1, \\ \tilde{E}_{2,\rho} \times \nu = E(\rho\,\cdot) \times \nu & \text{on } \partial B_1. \end{cases} \tag{1.3.79}$$

By Lemmas 1.3.1 and 1.3.2 (also Remark 1.3.2), we have, for $R > 1/2$ and for $x \in B_{3R} \setminus B_{2R}$,

$$\begin{aligned} \left| \left(\tilde{E}_{2,\rho}\Big(\frac{x}{\rho}\Big), \tilde{H}_{2,\rho}\Big(\frac{x}{\rho}\Big) \right) \right| &\le C\rho^3 \| (\tilde{E}_{2,\rho}, \tilde{H}_{2,\rho}) \|_{L^2(B_2 \setminus B_1)} \\ &\le C\rho^3 (\| E(\rho.) \times \nu \|_{H^{-1/2}(\partial B_1)} + \rho^{-1} \| \operatorname{div}_{\partial B_1} (E(\rho.) \times \nu) \|_{H^{-1/2}(\partial B_1)}) \\ &\le C\rho^3 (\| E(\rho.) \times \nu \|_{H^{-1/2}(\partial B_1)} + \| H(\rho.) \cdot \nu \|_{H^{-1/2}(\partial B_1)}). \end{aligned}$$

Here and in what follows in this proof, C denotes a positive constant depending only on ρ_0, R_0, and R. It follows from the definition of $(\tilde{E}_{2,\rho}, \tilde{H}_{2,\rho})$ that

$$\| (E_{2,\rho}, H_{2,\rho}) \|_{L^2(B_{3R} \setminus B_{2R})} \le C\rho^3 \| J_{\text{ext}} \|_{L^2(\mathbb{R}^3 \setminus B_2)}. \tag{1.3.80}$$

From now on in this proof, for any vector field v, we denote [3]

$$\hat{v}(\cdot) := \rho v(\rho\, \cdot). \tag{1.3.81}$$

We claim that

$$\|\hat{H}_{1,\rho} \times \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} + \|\hat{E}_{1,\rho} \cdot \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} \le C\rho \|J_{\text{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)} \tag{1.3.82}$$

and, for $R > 1/2$,

$$\|(E_{3,\rho}, H_{3,\rho})\|_{L^2(B_{3R} \setminus B_{2R})} \le C\big(\rho^3 \|J_{\text{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)} + \rho^2 \|J_{\text{int}}\|_{L^2(B_1)}\big). \tag{1.3.83}$$

It is clear that (1.1.9) follows from (1.3.80) and (1.3.83). Moreover, by Lemma 1.3.6, assertion (1.1.14) now follows from (1.3.82) and the fact that $(E_c, H_c) = (\hat{E}_{3,\rho}, \hat{H}_{3,\rho})$ in B_1.

It remains to establish (1.3.82) and (1.3.83). It is clear that $(\hat{E}_{3,\rho}, \hat{H}_{3,\rho}) \in [\bigcap_{R>0} H(\text{curl}, B_R \setminus \partial B_1)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times \hat{E}_{3,\rho} = i\omega\rho \hat{H}_{3,\rho} & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{H}_{3,\rho} = -i\omega\rho \hat{E}_{3,\rho} & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{E}_{3,\rho} = i\omega\mu \hat{H}_{3,\rho} & \text{in } B_1, \\ \nabla \times \hat{H}_{3,\rho} = -i\omega\varepsilon \hat{E}_{3,\rho} + J_{\text{int}} & \text{in } B_1, \\ [\hat{E}_{3,\rho} \times \nu] = 0, [\hat{H}_{3,\rho} \times \nu] = -\hat{H}_{1,\rho} \times \nu|_{\text{ext}} & \text{on } \partial B_1. \end{cases} \tag{1.3.84}$$

By Lemma 1.3.6, we have

$$\|(\hat{E}_{3,\rho}, \hat{H}_{3,\rho})\|_{H(\text{curl}, B_5)} \le C\Big(\|J_{\text{int}}\|_{L^2(B_1)} + \|\hat{H}_{1,\rho} \times \nu|_{\text{ext}}\|_{H^{-1/2}(\text{div}_\Gamma, \partial B_1)}\Big). \tag{1.3.85}$$

Applying Lemma 1.3.1 to $(\hat{E}_{2,\rho}, \hat{H}_{2,\rho})$, by (1.3.81), we obtain

$$\|\hat{H}_{2,\rho} \times \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} + \|\hat{E}_{2,\rho} \cdot \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} \le C\rho \|J_{\text{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)}.$$

Since

$$(E_{2,\rho}, H_{2,\rho}) = (E, H) - (E_{1,\rho}, H_{1,\rho}) \text{ in } \mathbb{R}^3 \setminus B_1,$$

it follows that

$$\|\hat{H}_{1,\rho} \times \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} + \|\hat{E}_{1,\rho} \cdot \nu|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} \le C\rho \|J_{\text{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)},$$

which is (1.3.82).

[3] With this notation, one has $(E_c, H_c)(x) = (\hat{E}_\rho, \hat{H}_\rho)$ in B_1. It is worth noting that $\hat{v}(\cdot) \ne v(\rho\, \cdot)$.

Combining (1.3.82) and (1.3.85) yields

$$\|(\hat{E}_{3,\rho}, \hat{H}_{3,\rho})\|_{H(\mathrm{curl}, B_5)} \le C\left(\|J_{\mathrm{int}}\|_{L^2(B_1)} + \rho\|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)}\right). \tag{1.3.86}$$

Applying Lemma 1.3.2, and using (1.3.86), we obtain

$$\left|\left(\hat{E}_{3,\rho}\left(\frac{x}{\rho}\right), \hat{H}_{3,\rho}\left(\frac{x}{\rho}\right)\right)\right| \le C\rho^3\left(\|J_{\mathrm{int}}\|_{L^2(B_1)} + \rho\|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3 \setminus B_2)}\right) \text{ for } x \in B_{3R} \setminus B_{2R}.$$

This implies (1.3.83). The proof is complete. □

1.3.3 Approximate cloaking in the resonant case - Proofs of Theorems 1.1.2 and 1.1.3

The key ingredient in the proof of Theorems 1.1.2 and 1.1.3 is the following variant of Lemma 1.3.6.

Lemma 1.3.7. *Let $0 < \rho < \rho_0$, $\theta_\rho = (\theta_{1,\rho}, \theta_{2,\rho}) \in [L^2(D)]^6$, and $h_\rho = (h_{1,\rho}, h_{2,\rho}) \in [H^{-1/2}(\mathrm{div}_\Gamma, \partial D)]^2$, and let $(E_\rho, H_\rho) \in [\bigcap_{R>1} H(\mathrm{curl}, B_R \setminus \partial D)]^2$ be the unique radiating solution to the system*

$$\begin{cases} \nabla \times E_\rho = i\rho H_\rho & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times H_\rho = -i\rho E_\rho & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times E_\rho = i\mu H_\rho + \theta_{1,\rho} & \text{in } D, \\ \nabla \times H_\rho = -i\varepsilon E_\rho + \theta_{2,\rho} & \text{in } D, \\ [E_\rho \times \nu] = h_{1,\rho}, [H_\rho \times \nu] = h_{2,\rho} & \text{on } \partial D. \end{cases}$$

Assume that $\mathcal{N}(D) \neq \{(0,0)\}$. We have

$$\|(E_\rho, H_\rho)\|_{L^2(B_5)} \le C\Big(\rho^{-1}\|\theta_\rho\|_{L^2(D)} + \|h_\rho\|_{H^{-1/2}(\partial D)} + \rho^{-1}\|\mathrm{div}_\Gamma\, h_\rho\|_{H^{-1/2}(\partial D)}\Big). \tag{1.3.87}$$

Assume in addition that, for all $\rho \in (0, \rho_0)$,

$$\int_D \left(\theta_{2,\rho} \cdot \bar{\boldsymbol{E}} - \theta_{1,\rho} \cdot \bar{\boldsymbol{H}}\right) dx = 0 \text{ for all } (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{N}(D). \tag{1.3.88}$$

Then

$$\|(E_\rho, H_\rho)\|_{L^2(B_5)} \le C\Big(\|\theta_\rho\|_{L^2(D)} + \|h_\rho\|_{H^{-1/2}(\partial D)} + \rho^{-1}\|\mathrm{div}_\Gamma\, h_\rho\|_{H^{-1/2}(\partial D)}\Big). \tag{1.3.89}$$

Here C denotes a positive constant depending only on ρ_0, ε, and μ. Moreover, if

$$\lim_{\rho \to 0}\left(\|h_\rho\|_{H^{-1/2}(\partial D)} + \rho^{-1}\|\mathrm{div}_\Gamma\, h_\rho\|_{H^{-1/2}(\partial D)}\right) = 0 \quad \text{and} \quad \lim_{\rho \to 0} \theta_\rho = \theta \text{ in } [L^2(D)]^6,$$

for some $\theta = (\theta_1, \theta_2) \in [L^2(D)]^6$, *then*

$$\lim_{\rho \to 0} (E_\rho, H_\rho) = Cl(\theta_1, \theta_2) \text{ in } [H(\mathrm{curl}, D)]^2. \tag{1.3.90}$$

Remark 1.3.5. In comparison with (1.3.63) in Lemma 1.3.6, in the resonant case $\mathcal{N}(D) \neq \{(0,0)\}$, estimate (1.3.87) is weaker. Under the compatibility condition (1.3.88), estimate (1.3.89) is stronger than (1.3.87). Note that the term $\| \mathrm{div}_\Gamma h_\rho \|_{H^{-1/2}(\partial D)}$ in (1.3.63) of Lemma 1.3.6 is replaced by $\rho^{-1} \| \mathrm{div}_\Gamma h_\rho \|_{H^{-1/2}(\partial D)}$ in (1.3.89). However, this does not affect the estimate for the degree of visibility in the compatible resonant case (in comparison with the non-resonant case) since in the proof of Theorem 1.2, we apply Lemma 1.3.7 to the situation where $\| h_\rho \|_{H^{-1/2}(\partial D)}$ and $\rho^{-1} \| \mathrm{div}_\Gamma h_\rho \|_{H^{-1/2}(\partial D)}$ are of the same order. It is worth noting that the estimates in Lemma 1.3.7 are somehow sharp because of the optimality of the estimates in Theorems 1.1.2 and 1.1.3; this is discussed in Section 1.4.

Proof. We will give the proof of (1.3.89) and (1.3.90) and explain how to modify the proof of (1.3.89) to obtain (1.3.87).

We prove (1.3.89) by contradiction. Assume that there exist sequences $(\rho_n)_n \subset (0, \rho_0)$, $\big((E_n, H_n)\big)_n \subset [\bigcap_{R>0} H(\mathrm{curl}, B_R \setminus \partial D)]^2$, $(\theta_n)_n = \big((\theta_{1,n}, \theta_{2,n})\big)_n \subset [L^2(D)]^6$ such that (1.3.88) holds for $(\theta_{1,n}, \theta_{2,n})$,

$$\begin{cases} \nabla \times E_n = i\rho_n H_n & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times H_n = -i\rho_n E_n & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times E_n = i\mu H_n + \theta_{1,n} & \text{in } D, \\ \nabla \times H_n = -i\varepsilon E_n + \theta_{2,n} & \text{in } D, \\ [E_n \times \nu] = h_{1,n}, [H_n \times \nu] = h_{2,n} \text{ on } \partial D, \end{cases} \tag{1.3.91}$$

$$\|(E_n, H_n)\|_{L^2(B_5)} = 1 \text{ for all } n \in \mathbb{N}, \tag{1.3.92}$$

and

$$\lim_{n \to +\infty} \Big(\|\theta_n\|_{L^2(D)} + \|h_n\|_{H^{-1/2}(\partial D)} + \rho_n^{-1} \| \mathrm{div}_\Gamma h_n \|_{H^{-1/2}(\partial D)} \Big) = 0. \tag{1.3.93}$$

Without loss of generality, we assume that $\rho_n \to \rho_\infty \in [0, \rho_0]$. We will only consider the case $\rho_\infty = 0$. The proof in the case $\rho_\infty > 0$ follows similarly and is omitted.

Similar to (1.3.68) and (1.3.69), we have, by (1.3.93),

$$\lim_{n \to +\infty} \nabla \times E_n|_{\mathrm{int}} \cdot \nu = 0 \quad \text{and} \quad \lim_{n \to +\infty} \nabla \times H_n \cdot \nu|_{\mathrm{int}} = 0 \text{ in } H^{-1/2}(\partial D). \tag{1.3.94}$$

Applying Lemma 1.3.3 and using (1.3.92), without loss of generality, one might assume that $\big((E_n, H_n)\big)_n$ converges in $[L^2(D)]^6$ and hence also in $[L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus D)]^6$ by applying (1.3.3) of Lemma 1.3.1 and i) of Lemma 1.2.7 to $B_R \setminus D$. Moreover, the limit $(E, H) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ satisfies

$$\begin{cases} \nabla \times E = \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \operatorname{div} E = \operatorname{div} H = 0 & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times E = i\mu H & \text{in } D, \\ \nabla \times H = -i\varepsilon E & \text{in } D, \end{cases} \tag{1.3.95}$$

and, by applying Lemma 1.2.6 and letting $\rho_n \to 0$,

$$|\big(E(x), H(x)\big)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.3.96}$$

Since

$$\int_D \big(\theta_{2,n} \cdot \bar{\mathbf{E}} - \theta_{1,n} \cdot \bar{\mathbf{H}}\big)\, dx = 0 \text{ for all } (\mathbf{E}, \mathbf{H}) \in \mathcal{N}(D),$$

by Lemma 1.3.4, there exists a unique $(E_{1,n}, H_{1,n}) \in \mathcal{N}(D)^{\perp}$ solving

$$\begin{cases} \nabla \times E_{1,n} = i\mu H_{1,n} + \theta_{1,n} & \text{in } D, \\ \nabla \times H_{1,n} = -i\varepsilon E_{1,n} + \theta_{2,n} & \text{in } D, \\ \nabla \times E_{1,n} \cdot \nu = \nabla \times H_{1,n} \cdot \nu = 0 & \text{on } \partial D. \end{cases}$$

Denote by $(E_{2,n}, H_{2,n})$ the projection of $(E_n, H_n) - (E_{1,n}, H_{1,n})$ onto $\mathcal{N}(D)$ and define

$$\tilde{E}_n = \rho_n^{-1}(E_n - E_{1,n} - E_{2,n}) \quad \text{and} \quad \tilde{H}_n = \rho_n^{-1}(H_n - H_{1,n} - H_{2,n}) \quad \text{in } D.$$

Then

$$(\tilde{E}_n, \tilde{H}_n) \in \mathcal{N}(D)^{\perp} \tag{1.3.97}$$

and

$$\begin{cases} \nabla \times \tilde{E}_n = i\mu \tilde{H}_n & \text{in } D, \\ \nabla \times \tilde{H}_n = -i\varepsilon \tilde{E}_n & \text{in } D, \\ \nabla \times \tilde{E}_n \cdot \nu = \rho_n^{-1} \nabla \times E_n \cdot \nu|_{\mathrm{int}} & \text{on } \partial D, \\ \nabla \times \tilde{H}_n \cdot \nu = \rho_n^{-1} \nabla \times H_n \cdot \nu|_{\mathrm{int}} & \text{on } \partial D. \end{cases} \tag{1.3.98}$$

We have

$$\rho_n^{-1}\nabla \times E_n \cdot \nu|_{\rm int} = \rho_n^{-1}\nabla \times E_n \cdot \nu|_{\rm ext} + \rho_n^{-1}\operatorname{div}_\Gamma h_{1,n} = iH_n \cdot \nu|_{\rm ext} + \rho_n^{-1}\operatorname{div}_\Gamma h_{1,n} \text{ on } \partial D.$$

This implies, by (1.3.98),

$$\mu \tilde{H}_n \cdot \nu = H_n \cdot \nu|_{\rm ext} - i\rho_n^{-1}\operatorname{div}_\Gamma h_{1,n} \text{ on } \partial D. \tag{1.3.99}$$

Similarly, we have

$$\varepsilon \tilde{E}_n \cdot \nu = E_n \cdot \nu|_{\rm ext} - i\rho_n^{-1}\operatorname{div}_\Gamma h_{2,n} \text{ on } \partial D. \tag{1.3.100}$$

Using (1.3.93), we derive from (1.3.94), (1.3.99), and (1.3.100) that

$$(\varepsilon\tilde{E}_n \cdot \nu, \mu\tilde{H}_n \cdot \nu) \to (E\cdot\nu|_{\rm ext}, H\cdot\nu|_{\rm ext}) \text{ in } H^{-1/2}(\partial D) \text{ as } n\to\infty. \tag{1.3.101}$$

It follows from Lemma 1.3.8 below that $\Big((\tilde{E}_n, \tilde{H}_n)\Big)_n$ is bounded in $[L^2(D)]^6$. Applying Lemma 1.3.3 to $(\tilde{E}_n, \tilde{H}_n)$, one can assume that

$$(\tilde{E}_n, \tilde{H}_n) \text{ converges to some } (\tilde{E}, \tilde{H}) \in \mathcal{N}(D)^\perp \text{ in } [H(\operatorname{curl}, D)]^2. \tag{1.3.102}$$

Moreover, from (1.3.98) and (1.3.101), we have

$$\begin{cases} \nabla\times\tilde{E} = i\mu\tilde{H} & \text{in } D,\\ \nabla\times\tilde{H} = -i\varepsilon\tilde{E} & \text{in } D,\\ \varepsilon\tilde{E}\cdot\nu = E\cdot\nu|_{\rm ext} & \text{on } \partial D,\\ \mu\tilde{H}\cdot\nu = H\cdot\nu|_{\rm ext} & \text{on } \partial D. \end{cases} \tag{1.3.103}$$

Applying Lemma 1.3.5 to (E, H) defined in $\mathbb{R}^3$ and $(\tilde{E}, \tilde{H})$ defined in D and using (1.3.95), (1.3.96), and (1.3.103), we deduce that $E = H = 0$ in $\mathbb{R}^3$, which contradicts (1.3.92). The proof of (1.3.89) is complete.

We next establish (1.3.90). Fix a sequence (ρ_n) converging to 0. From (1.3.89), one obtains that

$$\|(E_{\rho_n}, H_{\rho_n})\|_{L^2(B_5)} \le C\Big(\|\theta_{\rho_n}\|_{L^2(D)} + \|h_\rho\|_{H^{-1/2}(\partial D)} + \rho_n^{-1}\|\operatorname{div}_\Gamma h_{\rho_n}\|_{H^{-1/2}(\partial D)}\Big) \le C.$$

Define $(\tilde{E}_{\rho_n}, \tilde{H}_{\rho_n})$ in D from (E_{ρ_n}, H_{ρ_n}) as in the definition of $(\tilde{E}_n, \tilde{H}_n)$ from (E_n, H_n). Using the same arguments to obtain (1.3.102), we have

$$(\tilde{E}_{\rho_n}, \tilde{H}_{\rho_n}) \text{ converges to } (\tilde{E}, \tilde{H}) \in \mathcal{N}(D)^\perp \text{ in } [H(\operatorname{curl}, D)]^2. \tag{1.3.104}$$

Up to a subsequence, (E_{ρ_n}, H_{ρ_n}) converges to (E,H) in $\big[H_{\rm loc}(\mathrm{curl},\mathbb{R}^3)\big]^2$ and

$$|\big(E(x),H(x)\big)| = O(|x|^{-2}) \text{ for large } |x|. \tag{1.3.105}$$

Moreover, as in (1.3.103), one can show that (1.1.16) holds. Since the limit is unique, assertion (1.3.90) follows.

We finally show how to modify the proof of (1.3.89) to obtain (1.3.87). The proof is also based on a contradiction argument and is similar to the one of (1.3.89). However, we denote by $(E_{2,n}, H_{2,n})$ the projection of (E_n, H_n) onto $\mathcal{N}$ (note that $E_{1,n}$ and $H_{1,n}$ might not exist in this case)) and define

$$\tilde{E}_n = \rho_n^{-1}(E_n - E_{2,n}) \text{ in } D \quad \text{ and } \quad \tilde{H}_n = \rho_n^{-1}(H_n - H_{2,n}) \text{ in } D.$$

Then

$$\begin{cases} \nabla\times\tilde{E}_n = i\mu\tilde{H}_n + \rho_n^{-1}\theta_{1,n} & \text{in } D,\\ \nabla\times\tilde{H}_n = -i\varepsilon\tilde{E}_n + \rho_n^{-1}\theta_{2,n} & \text{in } D,\\ \nabla\times\tilde{E}_n\cdot\nu = \rho_n^{-1}\nabla\times E_n\cdot\nu|_{\rm int} & \text{on } \partial D,\\ \nabla\times\tilde{H}_n\cdot\nu = \rho_n^{-1}\nabla\times H_n\cdot\nu|_{\rm int} & \text{on } \partial D. \end{cases} \tag{1.3.106}$$

Since $(\rho_n^{-1}\theta_n)_n \to (0,0)$ in $[L^2(D)]^6$, the sequence $\big((\tilde{E}_n,\tilde{H}_n)\big)_n$ converges to $(\tilde{E},\tilde{H})$ in $[L^2(D)]^6$. Similar to the proof of (1.3.89), one also derives that $(E,H) = (0,0)$ in $\mathbb{R}^3$. This yields a contradiction. The proof is complete. □

In the proof of Lemma 1.3.7, we used the following lemma:

Lemma 1.3.8. *Assume that D is simply connected and $(E,H) \in \mathcal{N}(D)^\perp$ satisfies*

$$\nabla\times E = i\mu H \text{ in } D \quad \text{and} \quad \nabla\times H = -i\varepsilon E \text{ in } D. \tag{1.3.107}$$

We have

$$\|(E,H)\|_{H(\mathrm{curl},D)} \le C\|(\varepsilon E\cdot\nu,\, \mu H\cdot\nu)\|_{H^{-1/2}(\partial D)},$$

for some positive constant C depending only on D, ε, μ.

Proof. It suffices to prove that

$$\|(E,H)\|_{L^2(D)} \le C\|(\varepsilon E\cdot\nu, \mu H\cdot\nu)\|_{H^{-1/2}(\partial D)}. \tag{1.3.108}$$

The proof is via a standard contradiction argument. Assume that there exists a sequence

$\big((E_n, H_n)\big)_n \subset \mathcal{N}(D)^{\perp}$ such that

$$\nabla \times E_n = i\mu H_n \text{ in } D \quad \text{ and } \quad \nabla \times H_n = -i\varepsilon E_n \text{ in } D, \tag{1.3.109}$$

$$\|(E_n, H_n)\|_{L^2(D)} = 1 \text{ for all } n, \tag{1.3.110}$$

and

$$\big(\varepsilon E_n \cdot \nu, \mu H_n \cdot \nu\big) \to 0 \text{ in } [H^{-1/2}(\partial D)]^2. \tag{1.3.111}$$

Applying Lemma 1.3.3, one might assume that (E_n, H_n) converges to some (E_0, H_0) in $[H(\mathrm{curl}, D)]^2$. Then $(E_0, H_0) \in \mathcal{N}(D)^{\perp}$ and

$$\begin{cases} \nabla \times E_0 = i\mu H_0 & \text{in } D, \\ \nabla \times H_0 = -i\varepsilon E_0 & \text{in } D, \\ \nabla \times E_0 \cdot \nu = \nabla \times H_0 \cdot \nu = 0 & \text{on } \partial D. \end{cases} \tag{1.3.112}$$

It follows that $(E_0, H_0) \in \mathcal{N}(D)^{\perp} \cap \mathcal{N}(D)$. Hence $(E_0, H_0) = (0, 0)$ in D, which contradicts (1.3.110). □

We are ready to give the

Proof of Theorem 1.1.2. In this proof, we use the same notations as in the one of Theorem 1.1.1. Similar to the proof of Theorem 1.1.1, using Lemmas 1.3.1 and 1.3.2, we have, for $R > 1/2$,

$$\|(E_{2,\rho}, H_{2,\rho})\|_{L^2(B_{3R}\setminus B_{2R})} \le C\rho^3 \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)}. \tag{1.3.113}$$

Involving the same method used to prove (1.3.82) and (1.3.83), however, applying (1.3.89) in Lemma 1.3.7 instead of Lemma 1.3.6, we have

$$\|\hat{H}_{1,\rho} \times \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} + \|\hat{E}_{1,\rho} \cdot \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} \le C\rho \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)} \tag{1.3.114}$$

and

$$\|(E_{3,\rho}, H_{3,\rho})\|_{L^2(B_{3R}\setminus B_{2R})} \le C\big(\rho^3 \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)} + \rho^2 \|J_{\mathrm{int}}\|_{L^2(B_1)}\big). \tag{1.3.115}$$

It is clear that (1.1.13) follows from (1.3.113) and (1.3.115). Moreover, by Lemma 1.3.6, assertion (1.1.14) now follows from (1.3.114) and the fact that $(E_c, H_c) = (\hat{E}_{3,\rho}, \hat{H}_{3,\rho})$ in B_1. □

Proof of Theorem 1.1.3. In this proof, we use the same notations as in the one of Theorem 1.1.1. Similar to the proof of Theorem 1.1.1, using Lemmas 1.3.1 and 1.3.2, we have, for $R > 1/2$,

$$\|(E_{2,\rho}, H_{2,\rho})\|_{L^2(B_{3R}\setminus B_{2R})} \le C\rho^3 \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)}. \tag{1.3.116}$$

Using the same method used to prove (1.3.83), however, applying (1.3.87) in Lemma 1.3.7 instead of Lemma 1.3.6, we have

$$\|\hat{H}_{1,\rho} \times \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} + \|\hat{E}_{1,\rho} \cdot \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} \le C\rho \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)} \tag{1.3.117}$$

and

$$\|(E_{3,\rho}, H_{3,\rho})\|_{L^2(B_{3R}\setminus B_{2R})} \le C\big(\rho^3 \|J_{\mathrm{ext}}\|_{L^2(\mathbb{R}^3\setminus B_2)} + \rho \|J_{\mathrm{int}}\|_{L^2(B_1)}\big). \tag{1.3.118}$$

It is clear that (1.1.18) follows from (1.3.116) and (1.3.118).

It remains to prove (1.1.19). Using the linearity of the system and applying Theorem 1.1.2, one can assume that $J_{\mathrm{ext}} = 0$, and $J_{\mathrm{int}} = \mathbf{E}_0$ for some $(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{N} \setminus \{(0,0)\}$. From the definition of $\mathcal{N}$, we have

$$\mathbf{E}_0 \not\equiv 0 \quad \text{and} \quad \mathbf{H}_0 \not\equiv 0 \text{ in } B_1.$$

Note that $(\hat{E}_c, \hat{H}_c) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times \hat{E}_c = i\omega\rho\hat{H}_c & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{H}_c = -i\omega\rho\hat{E}_c & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{E}_c = i\omega\mu\hat{H}_c & \text{in } B_1, \\ \nabla \times \hat{H}_c = -i\omega\varepsilon\hat{E}_c + \mathbf{E}_0 & \text{in } B_1. \end{cases} \tag{1.3.119}$$

We prove (1.1.19) by contradiction. Assume that there exists a sequence $(\rho_n)_n \subset (0, 1/2)$ converging to 0 such that

$$\lim_{n\to\infty} \rho_n \|(E_n, H_n)\|_{L^2(B_1)} = 0, \tag{1.3.120}$$

where $(E_n, H_n) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times E_n = i\omega\rho_n H_n & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times H_n = -i\omega\rho_n E_n & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times E_n = i\omega\mu H_n & \text{in } B_1, \\ \nabla \times H_n = -i\omega\varepsilon E_n + \mathbf{E}_0 & \text{in } B_1. \end{cases} \tag{1.3.121}$$

Applying Lemma 1.2.8 to (E_n, H_n) in B_1 and using (1.3.120) and (1.3.121), we obtain

$$\lim_{n\to\infty} \rho_n \big\|\big(E_n \times \nu, H_n \times \nu\big)\big\|_{H^{-1/2}(\partial B_1)} = 0. \tag{1.3.122}$$

By Lemma 1.3.1, we have

$$\lim_{n\to\infty} \rho_n \big\|\big(E_n, H_n\big)\big\|_{L^2(B_2\setminus B_1)} = 0. \tag{1.3.123}$$

Since $\operatorname{div} E_n = \operatorname{div} H_n = 0$ in $\mathbb{R}^3 \setminus B_1$, we have, by Lemma 1.2.8 and (1.3.123),

$$\lim_{n\to\infty} \rho_n \big\|\big(E_n \cdot \nu, H_n \cdot \nu\big)\big\|_{H^{-1/2}(\partial B_1)} = 0.$$

It follows that

$$\lim_{n\to\infty} \big\|\big(\operatorname{div}_\Gamma (E_n \times \nu), \operatorname{div}_\Gamma (H_n \times \nu)\big)\big\|_{H^{-1/2}(\partial B_1)} = \lim_{n\to\infty} \big\|\big(\nabla \times E_n \cdot \nu, \nabla \times H_n \cdot \nu\big)\big\|_{H^{-1/2}(\partial B_1)} = 0. \tag{1.3.124}$$

Using the fact that $(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{N}$, we derive from (1.3.121) that

$$\int_{B_1} \mu^{-1} \nabla \times \bar{\mathbf{E}}_0 \cdot \nabla \times E_n \, dx - \omega^2 \int_{B_1} \varepsilon \bar{\mathbf{E}}_0 \cdot E_n \, dx = -i\omega \int_{\partial B_1} (\nu \times E_n) \cdot \bar{\mathbf{H}}_0 \, ds,$$

and

$$\int_{B_1} \mu^{-1} \nabla \times E_n \cdot \nabla \times \bar{\mathbf{E}}_0 \, dx - \omega^2 \int_{B_1} \varepsilon E_n \cdot \bar{\mathbf{E}}_0 \, dx = -i\omega \int_{\partial B_1} (\nu \times H_n) \cdot \bar{\mathbf{E}}_0 \, ds + i\omega \int_{B_1} \mathbf{E}_0 \cdot \bar{\mathbf{E}}_0.$$

Considering the imaginary part of the two identities yields

$$\Re \left\{ \int_{\partial B_1} (\nu \times H_n) \cdot \bar{\mathbf{E}}_0 ds + \int_{\partial B_1} (\nu \times E_n) \cdot \bar{\mathbf{H}}_0 ds \right\} = \int_{B_1} |\mathbf{E}_0|^2 > 0. \tag{1.3.125}$$

However, since $\nabla \times \mathbf{H}_0 \cdot \nu = 0$ on ∂B_1, by Lemma 1.2.2, there exists $\mathbf{H} \in H(\operatorname{curl}, B_1)$ such that

$$\nabla \times \mathbf{H}_0 = \nabla \times \mathbf{H} \text{ in } B_1 \quad \text{ and } \quad \mathbf{H} \times \nu = 0 \text{ on } \partial B_1.$$

Since $\nabla \times (\mathbf{H}_0 - \mathbf{H}) = 0$ in B_1, by Lemma 1.2.1, there exists $\xi \in H^1(B_1)$ such that

$$\mathbf{H}_0 - \mathbf{H} = \nabla \xi \text{ in } B_1,$$

and hence

$$\mathbf{H}_0 \times \nu = \nabla \xi \times \nu \text{ on } \partial B_1.$$

We have thus

$$\int_{\partial B_1} (\nu \times E_n) \cdot \bar{\mathbf{H}}_0 \, ds = \int_{\partial B_1} (\nu \times E_n) \cdot \nabla \bar{\xi} \, ds = \int_{\partial B_1} \operatorname{div}_\Gamma (\nu \times E_n) \, \bar{\xi} \, ds \to 0 \text{ as } n \to +\infty, \tag{1.3.126}$$

thanks to (1.3.124). Similarly, we obtain

$$\int_{\partial B_1} (\nu \times H_n) \cdot \bar{\mathbf{E}}_0 \, ds \to 0 \text{ as } n \to +\infty. \tag{1.3.127}$$

Combining (1.3.125), (1.3.126), and (1.3.127), we obtain a contradiction. Hence (1.1.19) holds. The proof is complete. □

1.4 Optimality of the degree of visibility

In this section, we present various settings that justify the optimality of the degree of visibility in Theorems 1.1.1, 1.1.2, and 1.1.3. In what follows in this section, we **assume** that

$$\varepsilon = \mu = I \text{ (the identity matrix) in } B_1. \tag{1.4.1}$$

Let $h_n^{(1)}$ ($n \in \mathbb{N}$) be the spherical Hankel function of first kind of order n and let $j_n,\ y_n$ denote respectively its real and imaginary parts. For $-n \le m \le n, n \in \mathbb{N}$, denote Y_n^m the spherical harmonic function of order n and degree m and set

$$U_n^m(\hat{x}) := \nabla_{\partial B_1} Y_n^m(\hat{x}) \quad \text{ and } \quad V_n^m(\hat{x}) := \hat{x} \times U_n^m(\hat{x}) \text{ for } \hat{x} \in \partial B_1.$$

We recall that $Y_n^m(\hat{x})\hat{x}$, $U_n^m(\hat{x})$, and $V_n^m(\hat{x})$ for $-n \le m \le n, n \in \mathbb{N}$ form an orthonormal basis of $[L^2(\partial B_1)]^3$.

We have

Lemma 1.4.1. *System* (1.1.3) *is non-resonant if and only if* $j_n(\omega) \neq 0$ *for all* $n \ge 1$.

Proof. Assume that $j_n(\omega) = 0$ for some $n \ge 1$. Fix such an n and define, in B_1,

$$\mathbf{E}_0(x) = j_n(\omega r) V_n^0(\hat{x}) \text{ and } \mathbf{H}_0(x) = \frac{n(n+1)}{i\omega r} j_n(\omega r) Y_n^0(\hat{x})\hat{x} + \frac{1}{i\omega r}[j_n(\omega r) + \omega r j_n'(\omega r)] U_n^0(\hat{x}),$$

where $r = |x|$ and $\hat{x} = x/|x|$. Then $(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{N}$. System (1.1.3) is hence resonant. Conversely, assume that $j_n(\omega) \neq 0$ for all $n \in \mathbb{N}$. Using separation of variables (see, e.g., [22, Theorem 2.48]), one can check that if $(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{N}$ then $(\mathbf{E}_0, \mathbf{H}_0) = (0, 0)$ in B_1. □

The following result implies the optimality of (1.1.9) with respect to J_{ext}. For computational ease, instead of considering fields generated by J_{ext}, we deal with fields generated by a plane wave. In what follows, we assume that $0 < \rho < 1/2$. We have

Proposition 1.4.1. *Set* $v(x) := (0,1,0)e^{i\omega x_3}$ *for* $x \in \mathbb{R}^3$. *For* $\omega > 0$ *such that* $j_1(\omega) \neq 0$, *we have*

$$\|E_c\|_{L^2(B_4 \setminus B_2)} \ge C\rho^3,$$

for some positive constant C independent of ρ. Here $(E_c, H_c) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ is uniquely determined by

$$\begin{cases} \nabla \times E = i\omega\mu_c H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega\varepsilon_c E & \text{in } \mathbb{R}^3, \end{cases}$$

where $E = E_c + v$ and $H = H_c + \frac{1}{i\omega}\nabla \times v$ and by the radiation condition. Here (ε_c, μ_c) is defined by (1.1.3) where (ε, μ) is given in (1.4.1).

Proof. Let $\omega > 0$ be such that $j_1(\omega) \neq 0$. Set

$$(\mathscr{E}_\rho, \mathscr{H}_\rho) = (F_\rho^{-1} * E, F_\rho^{-1} * H) \text{ in } \mathbb{R}^3,$$

and define

$$(\mathbf{E}_\rho, \mathbf{H}_\rho) = \begin{cases} (\mathscr{E}_\rho - v, \mathscr{H}_\rho - \frac{1}{i\omega}\nabla \times v) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ (\mathscr{E}_\rho, \mathscr{H}_\rho) & \text{in } B_\rho. \end{cases}$$

Set

$$(\tilde{\mathbf{E}}_\rho, \tilde{\mathbf{H}}_\rho) = (\mathbf{E}_\rho, \mathbf{H}_\rho)(\rho\,\cdot) \quad \text{and} \quad \tilde{v} = v(\rho\,\cdot) \quad \text{in } \mathbb{R}^3.$$

We have

$$\begin{cases} \nabla \times \tilde{\mathbf{E}}_\rho = i\rho\omega\tilde{\mathbf{H}}_\rho & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \tilde{\mathbf{H}}_\rho = -i\rho\omega\tilde{\mathbf{E}}_\rho & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \tilde{\mathbf{E}}_\rho = i\omega\tilde{\mathbf{H}}_\rho & \text{in } B_1, \\ \nabla \times \tilde{\mathbf{H}}_\rho = -i\omega\tilde{\mathbf{E}}_\rho & \text{in } B_1, \\ [\tilde{\mathbf{E}}_\rho \times \nu] = -\tilde{v} \times \nu, \quad [\tilde{\mathbf{H}}_\rho \times \nu] = -\frac{1}{i\rho\omega}(\nabla \times \tilde{v}) \times \nu & \text{on } \partial B_1. \end{cases} \tag{1.4.2}$$

Denote

$$A_{\text{ext}} = \int_{\partial B_1} \tilde{\mathbf{E}}_\rho|_{\text{ext}} \cdot \bar{V}_1^1\, ds \quad \text{and} \quad A_{\text{int}} = \int_{\partial B_1} \tilde{\mathbf{E}}_\rho|_{\text{int}} \cdot \bar{V}_1^1\, ds.$$

Using the transmission condition for $\tilde{\mathbf{E}}_\rho \times \nu$ on ∂B_1 and considering only the component with respect to V_1^1 for $\tilde{\mathbf{E}}_\rho$ (see, e.g., [22, Theorem 2.48]), we have

$$A_{\text{ext}} - A_{\text{int}} = \alpha, \tag{1.4.3}$$

where

$$\alpha = -\int_{\partial B_1} \tilde{v} \cdot \bar{V}_1^1\, ds.$$

Using the transmission condition for $\tilde{\mathbf{H}}_\rho \times \nu$ on ∂B_1 and considering the component with respect to U_1^1 for $\tilde{\mathbf{H}}_\rho$ (see, e.g., [22, Theorem 2.48]), we have

$$a_{\text{ext}}(\omega\rho)A_{\text{ext}} - a_{\text{int}}(\omega)A_{\text{int}} = \beta, \tag{1.4.4}$$

where

$$a_{\text{ext}}(r) = \frac{\left(h_1^{(1)}(r) + r h_1'^{(1)}(r)\right)}{-irh_1^{(1)}(r)}, \qquad a_{\text{int}}(r) = \frac{\left(j_1(r) + r j_1'(r)\right)}{-irj_1(r)}, \quad \text{and} \quad \beta = \alpha\, a_{\text{int}}(\omega\rho).$$

Combining (1.4.3) and (1.4.4) yields

$$A_{\text{ext}} = \frac{\beta - \alpha\, a_{\text{int}}(\omega)}{a_{\text{ext}}(\omega\rho) - a_{\text{int}}(\omega)}. \tag{1.4.5}$$

Since

$$h_1^{(1)}(x) = i\frac{d}{dx}\frac{e^{ix}}{x} = \frac{\sin x - x\cos x}{x^2} + i\frac{x\sin x - \cos x}{x^2}, \text{ for } x \in \mathbb{R}, \tag{1.4.6}$$

we derive that

$$\liminf_{\rho\to 0} \rho^{-1}\left|a_{\text{ext}}(\omega\rho) - a_{\text{int}}(\omega)\right|^{-1} > 0. \tag{1.4.7}$$

Since, by separation of variables, (see, e.g., [22, Theorem 2.48]),

$$\left|\int_{\partial B_1} \tilde{v}\cdot\bar{V}_1^1\, ds\right| = \left|\frac{j_1(\omega\rho)}{j_1(\omega)}\int_{\partial B_1} v\cdot\bar{V}_1^1\, ds\right|,$$

we have

$$C^{-1}\rho \le |\alpha| \le C\rho \tag{1.4.8}$$

for some positive constant C independent of ρ. From (1.4.8) and the fact that

$$|a_{\text{int}}(\omega\rho)| \ge C\rho^{-1},$$

we have

$$\liminf_{\rho\to 0}\left|\beta - \alpha\, a_{\text{int}}(\omega)\right| > 0. \tag{1.4.9}$$

Combining (1.4.7) and (1.4.9) yields

$$\liminf_{\rho\to 0}\rho^{-1}|A_{\text{ext}}| > 0. \tag{1.4.10}$$

Since, again by separation of variables,

$$\int_{\partial B_1} \tilde{\mathbf{E}}_\rho(r\hat{x}) \cdot \bar{V}_1^1(\hat{x})\, d\hat{x} = \frac{h_1^{(1)}(\omega\rho r)}{h_1^{(1)}(\omega\rho)} A_{\text{ext}},$$

and, by Lemma 1.2.9,

$$\tilde{\mathbf{E}}_\rho(x/\rho) = \mathbf{E}_\rho(x) = \mathscr{E}_\rho(x) - v(x) = E_c(x) \text{ for } x \in B_4 \setminus B_2,$$

we obtain the conclusion from (1.4.6) and (1.4.10). □

We next show the optimality of (1.1.9) with respect to J_{int}.

Proposition 1.4.2. *Assume that the system is non-resonant and $J_{\text{ext}} = 0$ in $\mathbb{R}^3 \setminus B_2$. There exists $J_{\text{int}} \in [L^2(B_1)]^3$ such that*

$$\liminf_{\rho\to 0} \rho^{-2} \|H_c\|_{L^2(B_4\setminus B_2)} > 0.$$

Proof. Consider

$$J_{\text{int}}(x) = j_1(\omega r) V_1^1(\hat{x}) \text{ in } B_1, \tag{1.4.11}$$

where $r = |x|$ and $\hat{x} = x/|x|$. Set

$$\mathbf{E}_0 = J_{\text{int}} \text{ and } \mathbf{H}_0 = \frac{1}{i\omega} \nabla \times \mathbf{E}_0 \quad \text{in } B_1.$$

Then

$$\begin{cases} \nabla \times \mathbf{E}_0 = i\omega \mathbf{H}_0 & \text{in } B_1, \\ \nabla \times \mathbf{H}_0 = -i\omega \mathbf{E}_0 & \text{in } B_1. \end{cases} \tag{1.4.12}$$

Define

$$(\hat{\mathbf{E}}_\rho, \hat{\mathbf{H}}_\rho) = \rho(\mathscr{E}_\rho, \mathscr{H}_\rho)(\rho\cdot) \text{ in } \mathbb{R}^3,$$

where $(\mathscr{E}_\rho, \mathscr{H}_\rho)$ is given in (1.1.20). Then

$$\begin{cases} \nabla \times \hat{\mathbf{E}}_\rho = i\rho\omega \hat{\mathbf{H}}_\rho & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{\mathbf{H}}_\rho = -i\rho\omega \hat{\mathbf{E}}_\rho & \text{in } \mathbb{R}^3 \setminus B_1, \\ \nabla \times \hat{\mathbf{E}}_\rho = i\omega \hat{\mathbf{H}}_\rho & \text{in } B_1, \\ \nabla \times \hat{\mathbf{H}}_\rho = -i\omega \hat{\mathbf{E}}_\rho + \mathbf{E}_0 & \text{in } B_1. \end{cases}$$

We have

$$\int_{\partial B_1} (\nu \times \hat{\mathbf{H}}_\rho) \cdot \bar{\mathbf{E}}_0 \, ds - \int_{\partial B_1} (\nu \times \hat{\mathbf{E}}_\rho) \cdot \bar{\mathbf{H}}_0 \, ds = \int_{B_1} |\mathbf{E}_0|^2 > 0. \tag{1.4.13}$$

We claim that

$$\liminf_{\rho \to 0} \left| \int_{\partial B_1} (\nu \times \hat{\mathbf{E}}_\rho) \cdot \bar{\mathbf{H}}_0 \, ds \right| = 0. \tag{1.4.14}$$

Assuming this, we have, from (1.4.13),

$$\liminf_{\rho \to 0} \left| \int_{\partial B_1} (\nu \times \hat{\mathbf{H}}_\rho) \cdot \bar{\mathbf{E}}_0 \, ds \right| > 0.$$

This implies, since $j_1(\omega) \neq 0$ by Lemma 1.4.1, that

$$\liminf_{\rho \to 0} \left| \int_{\partial B_1} \hat{\mathbf{H}}_\rho \bar{U}_1^1 \, ds \right| > 0.$$

On the other hand, by the separation of variables (see, e.g., [22, Theorem 2.48]),

$$\int_{\partial B_1} \hat{\mathbf{H}}_\rho(r\hat{x}) \cdot \bar{U}_1^1(\hat{x}) \, d\hat{x} = \frac{h_1^{(1)}(\omega\rho r) + \omega\rho r h_1'^1(\omega\rho r)}{r\big(h_1^{(1)}(\omega\rho) + \omega\rho h_1'^1(\omega\rho)\big)} \int_{\partial B_1} \hat{\mathbf{H}}_\rho(\hat{x}) \cdot \bar{U}_1^1(\hat{x}) d\hat{x}. \tag{1.4.15}$$

Using the fact

$$\liminf_{\rho \to 0} \rho^{-2} \frac{1}{|h_1^{(1)}(\omega\rho) + \omega\rho h_1'^1(\omega\rho)|} > 0,$$

and taking $r = R/\rho$ with $R \in (2,4)$ in (1.4.15), we obtain

$$\liminf_{\rho \to 0} \rho^{-3} \int_2^4 \left| \int_{\partial B_1} \hat{\mathbf{H}}_\rho(R\hat{x}/\rho) \cdot \bar{U}_1^1(\hat{x}) \, d\hat{x} \right| dR > 0.$$

This implies, since $H_c(R\hat{x}) = \mathscr{H}_\rho(R\hat{x}) = \rho^{-1} \mathbf{H}_\rho(R\hat{x}/\rho)$ for $R \in (2,4)$ and $\hat{x} \in \partial B_1$,

$$\liminf_{\rho \to 0} \rho^{-2} \| H_c \|_{L^2(B_4 \setminus B_2)} > 0,$$

which is the conclusion.

It remains to prove (1.4.14). Since

$$\mathbf{H}_0(x) = \frac{1}{i\omega} \nabla \times \mathbf{E}_0(x) = \frac{2}{i\omega r} j_1(\omega r) Y_1^1(\hat{x}) \hat{x} + \frac{1}{i\omega r} [j_1(\omega r) + \omega r j_1'(\omega r)] U_1^1(\hat{x}) \quad \text{in } B_1, \tag{1.4.16}$$

where $r = |x|$ and $\hat{x} = x/|x|$, using the separation of variables (see, e.g., [22, Theorem 2.48]), we have

$$\liminf_{\rho\to 0}\left|\int_{\partial B_1}(\nu\times\hat{\mathbf{E}}_\rho)\cdot\bar{\mathbf{H}}_0\,d\hat{x}\right|\le C\liminf_{\rho\to 0}\left|\int_{\partial B_1}\hat{\mathbf{E}}_\rho(\hat{x})\cdot\bar{V}_1^1(\hat{x})\,d\hat{x}\right| \tag{1.4.17}$$
$$= C\liminf_{\rho\to 0}\left|\frac{-i\omega\rho}{\sqrt{2}}\int_{\partial B_1}\hat{\mathbf{H}}_\rho(\hat{x})|_{\text{ext}}\cdot(\bar{Y}_1^1(\hat{x})\hat{x})\,d\hat{x}\right|.$$

Since, by Lemma 1.3.6,

$$\|\hat{\mathbf{H}}_\rho\|_{H(\text{curl},B_5)}\le C,$$

we have

$$\liminf_{\rho\to 0}\left|\frac{-i\omega\rho}{\sqrt{2}}\int_{\partial B_1}\hat{\mathbf{H}}_\rho|_{\text{ext}}(\bar{Y}_1^1(\hat{x})\hat{x})\,d\hat{x}\right|=0. \tag{1.4.18}$$

Thus, (1.4.14) follows from (1.4.17) and (1.4.18). □

We finally show the optimality of (1.1.18) in the case where $J_{\text{ext}}\equiv 0$ and J_{int} does not satisfy the compatibility condition.

Proposition 1.4.3. *Assume that $J_{\text{ext}}=0$ in $\mathbb{R}^3\setminus B_2$ and $j_1(\omega)=0$. There exists $J_{\text{int}}\in[L^2(B_1)]^3$ such that*

$$\|E_c\|_{L^2(B_4\setminus B_2)}\ge C\rho,$$

for some positive constant C independent of ρ.

Proof. Define J_{int} by (1.4.11). We use the notations in the proof of Proposition 1.4.2. We have

$$\int_{\partial B_1}(\nu\times\hat{\mathbf{H}}_\rho)\cdot\bar{\mathbf{E}}_0\,ds-\int_{\partial B_1}(\nu\times\hat{\mathbf{E}}_\rho)\cdot\bar{\mathbf{H}}_0\,ds=\int_{B_1}|\mathbf{E}_0|^2>0. \tag{1.4.19}$$

Since $j_1(\omega)=0$, it follows that

$$\int_{\partial B_1}(\nu\times\hat{\mathbf{H}}_\rho)\cdot\bar{\mathbf{E}}_0\,ds=0.$$

We derive from (1.4.19) that [4]

$$\liminf_{\rho\to 0}\left|\int_{\partial B_1}(\nu\times\hat{\mathbf{E}}_\rho)\cdot\bar{\mathbf{H}}_0\,ds\right|>0.$$

This implies, by (1.4.16),

$$\liminf_{\rho\to 0}\left|\int_{\partial B_1}\hat{\mathbf{E}}_\rho(\hat{x})\cdot\bar{V}_1^1(\hat{x})\,d\hat{x}\right|>0. \tag{1.4.20}$$

[4] This is the difference between the resonant and the non-resonant cases.

By the separation of variables (see, e.g., [22, Theorem 2.48]), for $r > 2$, we obtain

$$\int_{\partial B_1} \hat{\mathbf{E}}_\rho(r\hat{x}) \cdot \bar{V}_1^1(\hat{x})\, d\hat{x} = \frac{h_1^{(1)}(\omega\rho r)}{h_1^{(1)}(\omega\rho)} \int_{\partial B_1} \hat{\mathbf{E}}_\rho(\hat{x}) \cdot \bar{V}_1^1(\hat{x})\, d\hat{x}. \tag{1.4.21}$$

Taking $r = R/\rho$ with $R \in (2,4)$ in (1.4.21), since $\lim_{\rho\to 0} \rho^{-2} \left| \frac{h_1^{(1)}(\omega R)}{h_1^{(1)}(\omega\rho)} \right| > 0$, we obtain from (1.4.20) that

$$\liminf_{\rho\to 0} \rho^{-2} \int_2^4 \left| \int_{\partial B_1} \hat{\mathbf{E}}_\rho(R\hat{x}/\rho) \cdot \bar{V}_1^1(\hat{x})\, d\hat{x} \right| dR > 0.$$

This implies

$$\liminf_{\rho\to 0} \rho^{-1} \|E_c\|_{L^2(B_4\setminus B_2)} > 0,$$

which is the conclusion. $\square$

2 Cloaking for time-dependent Maxwell's equations

In this chapter, we study the time-dependent Maxwell equations. It can be considered as the continuation of the previous work in Chapter 1 in that the cloaking in time-harmonic regime is the main ingredient for our method in the time-dependent setting. The chapter uses the material of [42] by H. M. Nguyen and L. Tran.

2.1 Mathematical setting and statement of the main results

Let us now describe the problem in more details. For simplicity, we suppose that the cloaking device occupies the annular region $B_2 \setminus B_{1/2}$ and the cloaked region is the ball $B_{1/2}$ in $\mathbb{R}^3$ in which the permittivity and the permeability are given by two 3×3 matrices ε_O, μ_O respectively. In this chapter, for $r > 0$, we denote B_r the ball centered at the origin and of radius r. Throughout this chapter, we assume that, in $B_{1/2}$,

$$\varepsilon_O,\ \mu_O \text{ are real, symmetric,} \tag{2.1.1}$$

and uniformly elliptic, i.e.,

$$\frac{1}{\Lambda}|\xi|^2 \le \langle \varepsilon_O(x)\xi,\xi\rangle, \langle \mu_O(x)\xi,\xi\rangle \le \Lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^3, \tag{2.1.2}$$

for a.e. $x \in B_{1/2}$ and for some $\Lambda \ge 1$. We also assume ε_O, μ_O are piecewise C^1 to ensure the uniqueness of solutions via the unique continuation principle (see [48, 9] and also [51]).

Let $\rho \in (0,1)$ and let $F_\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F_\rho(x) = \begin{cases} x & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(\dfrac{2-2\rho}{2-\rho} + \dfrac{|x|}{2-\rho}\right)\dfrac{x}{|x|} & \text{in } B_2 \setminus B_\rho, \\ \dfrac{x}{\rho} & \text{in } B_\rho. \end{cases}$$

The cloaking device in $B_2 \setminus B_{1/2}$ constructed via transformation optics technique is characterized by the triple of permittivity, permeability, and conductivity and contains two layers. The first one in $B_2 \setminus B_1$ coming from the transformation technique using the map F_ρ is

$$(F_{\rho *} I, F_{\rho *} I, 0)$$

and the second one in $B_1 \setminus B_{1/2}$ which is a fixed lossy layer is

$$(I, I, 1).$$

Here and in what follows, for a diffeomorphism F and a matrix-valued function A, one denotes

$$F_* A := \frac{DFADF^T}{|\det DF|} \circ F^{-1}. \tag{2.1.3}$$

Remark 2.1.1. Different fixed lossy-layer can be used but for the simplicity of notations and to avoid several unnecessary technical points, the triple $(I, I, 1)$ is considered.

Assume that the medium is homogeneous outside the cloaking device and the cloaked region. In the presence of the cloaked object and the cloaking device, the medium in the whole space $\mathbb{R}^3$ is described by the triple $(\varepsilon_c, \mu_c, \sigma_c)$ given by

$$(\varepsilon_c, \mu_c, \sigma_c) = \begin{cases} (I, I, 0) & \text{in } \mathbb{R}^3 \setminus B_2, \\ (F_{\rho *} I, F_{\rho *} I, 0) & \text{in } B_2 \setminus B_1, \\ (I, I, 1) & \text{in } B_1 \setminus B_{1/2}, \\ (\varepsilon_O, \mu_O, 0) & \text{in } B_{1/2}. \end{cases} \tag{2.1.4}$$

Let $\mathcal{J}$ represent a charge density. We assume that

$$\mathcal{J} \in L^1([0,\infty); [L^2(\mathbb{R}^3)]^3) \text{ with } \operatorname{supp} \mathcal{J} \subset [0, T] \times (B_{R_0} \setminus B_2), \text{ for some } T > 0, R_0 > 2 \tag{2.1.5}$$

and

$$\operatorname{div} \mathcal{J} = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}^3. \tag{2.1.6}$$

With the cloaking device and the cloaked object, the electromagnetic wave generated by $\mathcal{J}$ with zero data at the time 0 is the unique weak solution $(\mathcal{E}_c, \mathcal{H}_c) \in L^\infty_{\text{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \varepsilon_c \dfrac{\partial \mathscr{E}_c}{\partial t} = \nabla \times \mathscr{H}_c - \mathscr{J} - \sigma_c \mathscr{E}_c & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mu_c \dfrac{\partial \mathscr{H}_c}{\partial t} = -\nabla \times \mathscr{E}_c & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathscr{E}_c(0,) = \mathscr{H}_c(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1.7}$$

In the homogeneous space, the field generated by $\mathscr{J}$ with zero data at the time 0 is the unique weak solution $(\mathscr{E}, \mathscr{H}) \in L^\infty_{\text{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \dfrac{\partial \mathscr{E}}{\partial t} = \nabla \times \mathscr{H} - \mathscr{J} & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \dfrac{\partial \mathscr{H}}{\partial t} = -\nabla \times \mathscr{E} & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathscr{E}(0,) = \mathscr{H}(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1.8}$$

The meaning of weak solutions, in a slightly more general context, is as follows

Definition 2.1.1. *Let ε, μ, $\in [L^\infty(\mathbb{R}^3)]^{3\times 3}$, σ_m, $\sigma_e \in L^\infty(\mathbb{R}^3)$ be such that ε and μ are real, symmetric, and uniformly elliptic in $\mathbb{R}^3$, and σ_m and σ_e are real and nonnegative in $\mathbb{R}^3$, and let $f_e, f_m \in L^1_{\text{loc}}([0,\infty); [L^2(\mathbb{R}^3)]^3)$. A pair $(\mathscr{E}, \mathscr{H}) \in L^\infty_{\text{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ is called a weak solution of*

$$\begin{cases} \varepsilon \dfrac{\partial \mathscr{E}}{\partial t} = \nabla \times \mathscr{H} - \sigma_e \mathscr{E} + f_m & \textit{in } (0, +\infty) \times \mathbb{R}^3, \\ \mu \dfrac{\partial \mathscr{H}}{\partial t} = -\nabla \times \mathscr{E} - \sigma_m \mathscr{H} + f_e & \textit{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathscr{E}(0,) = 0; \mathscr{H}(0,) = 0 & \textit{in } \mathbb{R}^3, \end{cases} \tag{2.1.9}$$

if

$$\begin{cases} \dfrac{d}{dt} \langle \varepsilon \mathscr{E}(t,.), E \rangle + \langle \sigma_e \mathscr{E}(t,.), E \rangle - \langle \mathscr{H}(t,.), \nabla \times E \rangle = \langle f_m(t,.), E \rangle, \\ \dfrac{d}{dt} \langle \mu \mathscr{H}(t,.), H \rangle + \langle \sigma_m \mathscr{H}(t,.), H \rangle + \langle \mathscr{E}(t,.), \nabla \times H \rangle = \langle f_e(t,.), H \rangle, \end{cases} \quad \textit{for } t > 0, \tag{2.1.10}$$

for all $(E, H) \in [H(\text{curl}, \mathbb{R}^3)]^2$, and

$$\mathscr{E}(0,.) = \mathscr{H}(0,.) = 0 \textit{ in } \mathbb{R}^3. \tag{2.1.11}$$

Some comments on Definition 2.1.1 are in order. System (2.1.10) is understood in the distributional sense. Initial condition (2.1.11) is understood as

$$\langle \varepsilon \mathscr{E}(0,.), E \rangle = \langle \mu \mathscr{H}(0,.), H \rangle = 0 \quad \text{for all } (E, H) \in [H(\text{curl}, \mathbb{R}^3)]^2. \tag{2.1.12}$$

From (2.1.10), one can check that

$$\langle \varepsilon \mathscr{E}(t,.), E\rangle, \langle \mu \mathscr{H}(t,.), H\rangle \in W^{1,1}_{\mathrm{loc}}([0,+\infty)). \tag{2.1.13}$$

This in turn ensures the trace sense in (2.1.12).

Concerning the well-posedness of (2.1.9), we have, see, e.g., [46, Theorem 3.1],

Proposition 2.1.1. *Let $f_e, f_m \in L^1_{\mathrm{loc}}([0,\infty); [L^2(\mathbb{R}^3)]^3)$. There exists a unique weak solution $(\mathscr{E}, \mathscr{H}) \in L^\infty_{\mathrm{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ of (2.1.9). Moreover, for each $T > 0$, the following estimates hold*

$$\int_{\mathbb{R}^3} |\mathscr{E}(t,x)|^2 + |\mathscr{H}(t,x)|^2 \, dx \le C \left(\int_0^t \left\| \big(f_e(s,.), f_m(s,.)\big) \right\|_{L^2(\mathbb{R}^3)} ds \right)^2 \quad \text{for } t \in [0,T], \tag{2.1.14}$$

for some positive constant C depending only on the ellipticity of ε and μ.

Remark 2.1.2. We emphasize here that the constant C in Proposition 2.1.1 is independent of T. This fact is later used in the proof of the radiating condition. In [46], the authors considered dispersive materials and also dealt with Maxwell equations which are non-local in time. However, this version suffices for our analysis.

We are ready to state the main result of the chapter which is proved in Section 2.3.

Theorem 2.1.1. *Let $\rho \in (0,1)$ and let $(\mathscr{E}_c, \mathscr{H}_c), (\mathscr{E}, \mathscr{H}) \in L^\infty_{\mathrm{loc}}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ be the unique solutions to systems (2.1.7) and (2.1.8) respectively. Assume that $\mathscr{J} \in C^\infty\big((0,+\infty); \mathbb{R}^3)\big)$ satisfying (2.1.5) and (2.1.6). Then, for $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$,*

$$\|(\mathscr{E}_c, \mathscr{H}_c) - (\mathscr{E}, \mathscr{H})\|_{L^\infty((0,T); L^2(K))} \le C\rho^3 \|\mathscr{J}\|_{H^{11}((0,\infty); [L^2(\mathbb{R}^3)]^3)}, \tag{2.1.15}$$

for some positive constant C depending only on K, R_0, and T.

Remark 2.1.3. Assertion (2.1.15) is optimal since it gives the same degree of visibility as in the frequency domain in [41] where the optimality is established.

Remark 2.1.4. Estimate (2.1.15) requires $\mathscr{J}$ is regular. The condition on the regularity of $\mathscr{J}$ is not optimal and the optimality would be studied elsewhere.

Our approach is inspired by the work of Nguyen and Vogelius in [45] (see also [47, 40]) where they study approximate cloaking for the acoustic setting in the time domain. The main idea can be briefly described as follows. We first transform the Maxwell equations in the time domain into a family of the Maxwell equations in the harmonic regime by taking the Fourier transform of solutions with respect to time. After obtaining the appropriate degree of near-invisibility for the Maxwell equations in the time harmonic regime, where the dependence on frequency is *explicit*, we simply invert the Fourier transform. The analysis in the frequency domain ω (in Section 2.2) can be divided into three steps which deal with low and moderate $(0 < \omega < 1)$, moderate and high ($1 < \omega < 1/\rho$), high and very high frequency ($\omega > 1/\rho$) regimes. The analysis in the low and moderate frequency regime (in Section 2.2.1) is based on a variational approach. In comparison with [41], one needs, in addition, to derive estimate for small frequency in which the dependence on the frequency is explicit. In the moderate and high frequency regime, to obtain appropriate estimates, on one hand, we use the multiplier technique for a lossy region. The test functions are inspired from the scalar case due to Morawetz (see [31]). Nevertheless, there is a significant difference between the scalar case and the Maxwell vectorial case. It is known in the scalar case that one can control the normal derivative of a solution to the exterior Helmholtz equation in homogeneous medium by its value on the boundary of a convex bounded subset of $\mathbb{R}^3$. However, in contrast with the scalar case, one cannot either use tangential components of the electromagnetic fields to control the normal component in the same Sobolev norms and conversely. This fact can be seen from the explicit solutions outside a unit ball of Maxwell equations (see, e.g., [22, Theorem 2.50]). This is the reason for which we use the multiplier technique for a lossy region. This point again reveals the distinct structure of Maxwell equations in the time harmonic regime in comparison with the one of the Helmholtz equations. The analysis in the moderate and high frequency regime is given in Section 2.2.2. The analysis in the high and very high frequency regime in Section 2.2.3 is based on the duality method inspired from [28]. The proof of Theorem 2.1.1 based on the frequency analysis is given in Section 2.3. A key technical point to make use of the analysis in the frequency domain is the establishment of the radiating condition for the Fourier transform with respect to time of the solutions of Maxwell equations. The rigorous proof on the radiating condition in a general setting is new to our knowledge and is interesting in itself.

The chapter is organized as follows. Section 2.2 is devoted to the estimates for Maxwell's equations in frequency domain. Section 2.3 gives the proof of Theorem 2.1.1. The assertion on the radiating condition is also stated and proved there.

2.2 Frequency analysis

In this section, we provide estimates to assess the degree of visibility in the frequency domain. We first recall some notations. Let U be a smooth open subset of $\mathbb{R}^3$. We denote

$$H(\mathrm{curl}, U) := \Big\{\phi \in [L^2(U)]^3 : \nabla \times \phi \in [L^2(U)]^3\Big\},$$

$$H(\mathrm{div}, U) := \Big\{\phi \in [L^2(U)]^3 : \mathrm{div}\phi \in L^2(U)\Big\}.$$

We also use the notations $H_{\mathrm{loc}}(\mathrm{curl}, U)$ and $H_{\mathrm{loc}}(\mathrm{div}, U)$ with the usual convention.

Given $\mathbb{J} \in [L^2(\mathbb{R}^3)]^3$ with compact support, let $(\mathbb{E}, \mathbb{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ and $(\mathbb{E}_\rho, \mathbb{H}_\rho) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$

be the corresponding unique radiating solutions of the following systems

$$\begin{cases} \nabla \times \mathbb{E} = i\omega \mathbb{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbb{H} = -i\omega \mathbb{E} + \mathbb{J} & \text{in } \mathbb{R}^3, \end{cases} \tag{2.2.1}$$

and

$$\begin{cases} \nabla \times \mathbb{E}_\rho = i\omega \mu_\rho \mathbb{H}_\rho & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbb{H}_\rho = -i\omega \varepsilon_\rho \mathbb{E}_\rho + \sigma_\rho \mathbb{E}_\rho + \mathbb{J} & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2.2}$$

Here

$$(\varepsilon_\rho, \mu_\rho, \sigma_\rho) = \begin{cases} (I, I, 0) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ (\rho^{-1} I, \rho^{-1} I, \rho^{-1} I) & \text{in } B_\rho \setminus B_{\rho/2}, \\ (F_\rho^{-1}{}_* \varepsilon_O, F_\rho^{-1}{}_* \mu_O, 0) & \text{in } B_{\rho/2} \end{cases} \tag{2.2.3}$$

Recall that, for $\omega > 0$, a solution $(E, H) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus B_R)]^2$, for some $R > 0$, of the Maxwell equations

$$\begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3 \setminus B_R, \\ \nabla \times H = -i\omega E & \text{in } \mathbb{R}^3 \setminus B_R \end{cases}$$

is called radiating if it satisfies one of the (Silver-Muller) radiation conditions

$$H \times x - |x| E = O(1/|x|) \quad \text{and} \quad E \times x + |x| H = O(1/|x|) \text{ as } |x| \to +\infty. \tag{2.2.4}$$

Here and in what follows, for $\alpha \in \mathbb{R}$, $O(|x|^\alpha)$ denotes a quantity whose norm is bounded by $C|x|^\alpha$ for some constant $C > 0$.

Throughout this section, we assume

$$\mathrm{div}\, \mathbb{J} = 0 \quad \text{and} \quad \mathrm{supp}\, \mathbb{J} \subset B_{R_0} \setminus B_2, \tag{2.2.5}$$

for some $R_0 > 2$. One sees later (in Section 2.3) that if $(\hat{\mathscr{E}}_c, \hat{\mathscr{H}}_c)$ and $(\hat{\mathscr{E}}, \hat{\mathscr{H}})$ are the corresponding Fourier transform with respect to t of $(\mathscr{E}_c, \mathscr{H}_c)$ and $(\mathscr{E}, \mathscr{H})$ and if one defines $(\hat{\mathscr{E}}_\rho, \hat{\mathscr{H}}_\rho) = (DF_\rho^T \hat{\mathscr{E}}_c, DF_\rho^T \hat{\mathscr{H}}_c) \circ F_\rho$ in $\mathbb{R}^3$ then $(\hat{\mathscr{E}}, \hat{\mathscr{H}})$ and $(\hat{\mathscr{E}}_\rho, \hat{\mathscr{H}}_\rho)$ satisfy (2.2.1) and (2.2.2) respectively (for some $\mathbb{J}$). This is the motivation for the introduction of $(\mathbb{E}, \mathbb{H})$ and $(\mathbb{E}_\rho, \mathbb{H}_\rho)$.

The goal of this section is to derive estimates for $(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})$ in which the dependence on the frequency ω and ρ is explicit. More precisely, we establish the following three results

Proposition 2.2.1. *Let* $0 < \rho < \rho_0$ *and* $0 < \omega < \omega_0$. *We have*

$$\|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \rho^3 \omega^{-1} \|\mathbb{J}\|_{L^2(\mathbb{R}^3)}, \tag{2.2.6}$$

for some positive constant C_R *depending only on* R_0, R, ω_0, *and* ρ_0.

Proposition 2.2.2. *Let* $0 < \rho < \rho_0$ *and* $0 < \omega_0 \le \omega \le \omega_1 \rho^{-1}$ *and assume that* ρ_0 *is small enough and* ω_0 *is large enough. We have, for* $R > 2$,

$$\|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \omega^3 \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)}, \tag{2.2.7}$$

for some positive constant C_R *depending only on* R, R_0, ω_0, *and* ω_1.

Proposition 2.2.3. *Let* $0 < \rho < 1$, $\omega_1 > 0$, *and* $\omega > \omega_1 \rho^{-1}$. *We have, for* $R > 2$,

$$\|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \omega^{17/2} \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)}, \tag{2.2.8}$$

for some positive constant C_R *depending only on* R_0, R, *and* ω_1.

To motivate the analysis in this section, we define

$$(\mathbf{E}_\rho, \mathbf{H}_\rho) = \begin{cases} (\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H}) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ (\mathbb{E}_\rho, \mathbb{H}_\rho) & \text{in } B_\rho, \end{cases} \tag{2.2.9}$$

and set

$$(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) = (\mathbf{E}_\rho, \mathbf{H}_\rho)(\rho\, \cdot) \text{ in } \mathbb{R}^3. \tag{2.2.10}$$

As in the previous chapter, the following notation is used in the thesis

$$\bigcap_{R>1} H(\text{curl}, B_R \backslash \partial D) = \Big\{ u : \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } u \in H(\text{curl}, D) \text{ and } u \in H(\text{curl}, B_R \backslash D) \text{ for all } R > 1 \Big\}$$

Then $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) \in [L^2_{\text{loc}}(\mathbb{R}^3)]^6$ with $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) \in \cap_{R>1} H(\text{curl}, B_R \setminus \partial B_1)$ is the unique radiating solution of

$$\begin{cases} \nabla \times \widetilde{\mathbf{E}}_\rho = i\omega \tilde{\mu}_\rho \widetilde{\mathbf{H}}_\rho & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ \nabla \times \widetilde{\mathbf{H}}_\rho = -i\omega \tilde{\varepsilon}_\rho \widetilde{\mathbf{E}}_\rho + \tilde{\sigma}_\rho \widetilde{\mathbf{E}}_\rho & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ [\widetilde{\mathbf{E}}_\rho \times \nu] = -\mathbb{E}(\rho\, \cdot) \times \nu & \text{on } \partial B_1, \\ [\widetilde{\mathbf{H}}_\rho \times \nu] = -\mathbb{H}(\rho\, \cdot) \times \nu & \text{on } \partial B_1, \end{cases} \tag{2.2.11}$$

where

$$(\tilde{\varepsilon}_\rho, \tilde{\mu}_\rho, \tilde{\sigma}_\rho) := \begin{cases} (\rho I, \rho I, 0) & \text{in } \mathbb{R}^3 \setminus B_1, \\ (I, I, 1) & \text{in } B_1 \setminus B_{1/2}, \\ (\varepsilon_O, \mu_O, 0) & \text{in } B_{1/2}. \end{cases} \tag{2.2.12}$$

Here and in what follows for a bounded smooth subset D of $\mathbb{R}^3$, we denote $[u] := u|_{\text{ext}} - u|_{\text{int}}$ on ∂D for an appropriate (vectorial) function u. We will study (2.2.11) and using this to derive estimates for $(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})$ in the following three subsections.

2.2.1 Low and moderate frequency analysis - Proof of Proposition 2.2.1

This section is devoted to the proof of Proposition 2.2.1 and contains two subsections. In the first subsection, we present several useful lemmas and the proof of Proposition 2.2.1 is given in the second subsection.

Some useful lemmas

In this subsection, Lemma 2.2.1, 2.2.2 and 2.2.3 are basic results that will be used several times later on. Lemma 2.2.5 is the main result of this subsection. The setting of this lemma resembles that of Proposition 2.2.1. Lemma 2.2.4 is an intermediate result, which will be used in the proof of Lemma 2.2.5.

We first recall the following known result which is the basic ingredient for the variational approach.

Lemma 2.2.1. *Let ϵ be a measurable, symmetric, uniformly elliptic, matrix-valued function defined in D. Assume that one of the following two conditions holds*

i) $(u_n)_{n\in\mathbb{N}} \subset H(\text{curl}, D)$ *is a bounded sequence in* $H(\text{curl}, D)$ *such that*

$$\big(\text{div}(\epsilon u_n)\big)_{n\in\mathbb{N}} \text{ converges in } H^{-1}(D) \text{ and } \big(u_n \times \nu\big)_{n\in\mathbb{N}} \text{ converges in } H^{-1/2}(\partial D).$$

ii) $(u_n)_{n\in\mathbb{N}} \subset H(\text{curl}, D)$ *is a bounded sequence in* $H(\text{curl}, D)$ *such that*

$$\big(\text{div}(\epsilon u_n)\big)_{n\in\mathbb{N}} \text{ converges in } L^2(D) \text{ and } \big((\epsilon u_n) \cdot \nu\big)_{n\in\mathbb{N}} \text{ converges in } H^{-1/2}(\partial D).$$

There exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ which converges in $[L^2(D)]^3$.

The conclusion of Lemma 2.2.1 under condition $i)$ is [36, Lemma 1] and has its roots in [20, 13, 54]. The conclusion of Lemma 2.2.1 under condition $ii)$ can be obtained in the same way.

We have

Lemma 2.2.2. *Let $0 < \omega < \omega_0$ and D be a simply connected, bounded, open subset of $\mathbb{R}^3$ of class C^1 and denote $\Gamma = \partial D$. Let $h \in H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $E \in H(\operatorname{curl}, D)$. We have*

$$\Big|\int_\Gamma \bar{E} \cdot h\, ds\Big| \le C\Big(\omega \|E\|_{L^2(D)} + \|\nabla \times E\|_{L^2(D)}\Big)\left(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1}\|\operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)}\right), \tag{2.2.13}$$

for some positive constant C depending only on D and ω_0.

Here and in what follows, $\bar{u}$ denotes the complex conjugate of u.

Proof. Let $(E^0, H^0) \in [H(\operatorname{curl}, D)]^2$ be the unique solution to

$$\begin{cases} \nabla \times E^0 = i\omega(1+i)H^0 & \text{in } D, \\ \nabla \times H^0 = -i\omega(1+i)E^0 & \text{in } D, \\ E^0 \times \nu = h & \text{on } \Gamma. \end{cases}$$

We prove by contradiction that

$$\|(E^0, H^0)\|_{L^2(D)} \le C\left(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1}\|\operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)}\right) \tag{2.2.14}$$

for some positive constant C depending only on ω_0. Assume that there exist sequences $((E_n, H_n)) \subset [H(\operatorname{curl}, D)]^2$, $(\omega_n) \subset (0, \omega_0)$ and $(h_n) \subset H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ such that

$$\|(E_n, H_n)\| = 1 \text{ for all } n, \tag{2.2.15}$$

$$\|h_n\|_{H^{-1/2}(\Gamma)} + \omega_n^{-1}\|\operatorname{div}_\Gamma h_n\|_{H^{-1/2}(\Gamma)} \text{ converges to } 0, \tag{2.2.16}$$

and

$$\begin{cases} \nabla \times E_n = i\omega_n(1+i)H_n & \text{in } D, \\ \nabla \times H_n = -i\omega_n(1+i)E_n & \text{in } D, \\ E_n \times \nu = h_n & \text{in } \Gamma. \end{cases} \tag{2.2.17}$$

Without loss of generality, one can assume that $\omega_n \to \omega^*$. Applying Lemma 2.2.1, one might assume that (E_n, H_n) converges to some $(E, H) \in [L^2(D)]^6$. We only consider the case $\omega_* = 0$, the case where $\omega_* > 0$ is standard. Then

$$\begin{cases} \nabla \times E = 0 & \text{in } D, \\ \operatorname{div} E = 0 & \text{in } D, \\ E \times \nu = 0 & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times H = 0 & \text{in } D, \\ \operatorname{div} H = 0 & \text{in } D, \\ H \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

We also have, for each connected component Γ_j of Γ,

$$\int_{\Gamma_j} E \cdot \nu \, ds = \lim_{n \to \infty} \int_{\Gamma_j} E_n \cdot \nu \, ds = \lim_{n \to \infty} \Big[\frac{1}{-i\omega_n(1+i)} \int_{\Gamma_j} (\nabla \times H_n) \cdot \nu \, ds \Big] = 0.$$

Since D is simply connected, it follows (see, e.g., [15, Theorems 2.9 and 3.1]) that $E = \nabla \times \xi_E$ and $H = \nabla \xi_H$ for some $\xi_E, \xi_H \in H^1(D)$. We derive from the systems of E and H that

$$\int_D |\nabla \times \xi_E|^2 \, dx = 0 \quad \text{and} \quad \int_D |\nabla \xi_H|^2 \, dx = 0.$$

This yields that $E = H = 0$ in D. We have a contradiction. Therefore, (2.2.14) is proved.

We have

$$\begin{aligned} \int_\Gamma \bar{E} \cdot h \, ds = \int_\Gamma \bar{E} \cdot (E^0 \times \nu) \, ds &= \int_D (\nabla \times \bar{E}) \cdot E^0 \, dx - \int_D \bar{E} \cdot (\nabla \times E^0) \, dx \text{ (integration by parts)} \\ &= \int_D (\nabla \times \bar{E}) \cdot E^0 \, dx - i\omega(1+i) \int_D \bar{E} \cdot H^0 \, dx. \end{aligned}$$

It follows from Hölder's inequality and (2.2.14) that

$$\begin{aligned} \Big| \int_\Gamma \bar{E} \cdot h \, ds \Big| &\le \Big(\omega \|E\|_{L^2(D)} + \|\nabla \times E\|_{L^2(D)} \Big) \|(E^0, H^0)\|_{L^2(D)} \\ &\le C \Big(\omega \|E\|_{L^2(D)} + \|\nabla \times E\|_{L^2(D)} \Big) \big(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1} \| \operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)} \big), \end{aligned}$$

which is (2.2.13). □

The following simple result is used in our analysis.

Lemma 2.2.3. *Let D be a C^1 bounded open subset of $\mathbb{R}^3$ and denote $\Gamma = \partial D$. Let $h \in H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $u \in H(\operatorname{curl}, D)$. We have*

$$\Big| \int_\Gamma \bar{u} \cdot h \Big| \le C \|u\|_{H(\operatorname{curl}, D)} \|h\|_{H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)}. \tag{2.2.18}$$

for some positive constant C independent of h, and u.

Proof. The result is standard. For the convenience of the reader, we present the proof. By the trace theory, see, e.g., [1], there exists $\phi \in H(\operatorname{curl}, D)$ such that

$$\phi \times \nu = h \text{ on } \Gamma \quad \text{and} \quad \|\phi\|_{H(\operatorname{curl},D)} \le C\|h\|_{H^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}$$

for some positive constant C depending only on D. Then, by integration by parts, we have

$$\int_\Gamma \bar{u}\cdot h = \int_\Gamma \bar{u}\cdot(\phi\times\nu) = \int_D \nabla\times\bar{u}\cdot\phi - \int_D \bar{u}\cdot\nabla\times\phi.$$

The conclusion follows by Hölder's inequality. □

We next present an estimate for the exterior domain in the small and moderate frequency regime.

Lemma 2.2.4. *Let $R_0 > 2$, $0 < k < k_0$ and $D \subset B_1$ be a smooth open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus D$ is connected. Let $(f_1, f_2) \in [H(\operatorname{div}, \mathbb{R}^3 \setminus D)]^2$ with support in $B_{R_0} \setminus D$ and assume that $(E, H) \in [\cap_{R>1} H(\operatorname{curl}, B_R \setminus D)]^2$ is a radiating solution of*

$$\begin{cases} \nabla\times E = ikH + f_1 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla\times H = -ikE + f_2 & \text{in } \mathbb{R}^3\setminus\bar{D}. \end{cases} \tag{2.2.19}$$

We have, for $R > 2$,

$$\|(E,H)\|_{L^2(B_R\setminus D)} \le C_R\Big(\|(E\times\nu, H\times\nu)\|_{H^{-1/2}(\partial B_1)} + \|(f_1,f_2)\|_{L^2} + k^{-1}\|(\operatorname{div} f_1, \operatorname{div} f_2)\|_{L^2}\Big), \tag{2.2.20}$$

for some positive constant C_R depending only on D, k_0, R_0, and R.

Proof. By the Stratton-Chu formula, we have, for $x \in \mathbb{R}^3 \setminus \bar{B}_1$,

$$\begin{aligned} E(x) = &\int_{\partial B_{R_0+1/2}} \nabla_x G_k(x,y)\times\big(\nu(y)\times E(y)\big)dy \\ &+ ik\int_{\partial B_{R_0+1/2}} \nu(y)\times H(y) G_k(x,y)dy - \int_{\partial B_{R_0+1/2}} \nu(y)\cdot E(y)\nabla_x G_k(x,y)dy, \end{aligned}$$

and

$$\begin{aligned} H(x) = &\int_{\partial B_{R_0+1/2}} \nabla_x G_k(x,y)\times\big(\nu(y)\times E(y)\big)dy \\ &+ ik\int_{\partial B_{R_0+1/2}} \nu(y)\times H(y) G_k(x,y)dy - \int_{\partial B_{R_0+1/2}} \nu(y)\cdot E(y)\nabla_x G_k(x,y)dy, \end{aligned}$$

where

$$G_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \text{ for } x \ne y. \tag{2.2.21}$$

It follows that, for $R > R_0 + 1$,

$$\|(E,H)\|_{L^2(B_R\setminus D)} \le C_R \|(E,H)\|_{L^2(B_{R_0+1}\setminus D)}. \tag{2.2.22}$$

It hence suffices to prove (2.2.20) for $R = R_0 + 1$ by contradiction. Assume that there exist sequences $(k_n) \subset (0, k_0)$, $\big((f_{1,n}, f_{2,n})\big) \subset L^2(\mathbb{R}^3 \setminus D)$ with support in $B_{R_0} \setminus D$, and $\big((E_n, H_n)\big) \subset [\cap_{R>1} H(\operatorname{curl}, B_R \setminus D)]^2$ such that $\|(E_n, H_n)\|_{L^2(B_{R_0+1}\setminus D)} = 1$,

$$\lim_{n\to+\infty}\Big(\|(E_n \times \nu, H_n \times \nu)\|_{H^{-1/2}(\partial D)} + \|(f_{1,n}, f_{2,n})\|_{L^2} + k_n^{-1}\|(\operatorname{div} f_{1,n}, \operatorname{div} f_{2,n})\|_{L^2}\Big) = 0,$$

and

$$\begin{cases} \nabla \times E_n = i k_n H_n + f_{1,n} & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H_n = -i k_n E_n + f_{2,n} & \text{in } \mathbb{R}^3 \setminus \bar{D}. \end{cases} \tag{2.2.23}$$

Without loss of generality, one might assume that $k_n \to k_*$ as $n \to +\infty$. Using Lemma 2.2.1 and (2.2.22), one can assume that (E_n, H_n) converges to (E, H) in $L^2(B_R \setminus D)$. We first consider the case $k_* = 0$. We have

$$\begin{cases} \nabla \times E = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ E \times \nu = 0 & \text{on } \partial D, \end{cases} \qquad \begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ H \times \nu = 0 & \text{on } \partial D, \end{cases} \tag{2.2.24}$$

$$\operatorname{div} E = 0 \text{ in } \mathbb{R}^3 \setminus \bar{D} \quad \operatorname{div} H = 0 \text{ in } \mathbb{R}^3 \setminus \bar{D}, \tag{2.2.25}$$

and

$$|E(x)| = O(|x|^{-2}) \text{ and } \quad |H(x)| = O(|x|^{-2}) \text{ for large } x. \tag{2.2.26}$$

Assertion (2.2.26) can be derived again from the Stratton-Chu formula using the fact $\lim_{n\to+\infty} k_n = 0$. It follows from (2.2.24) and (2.2.26) that, see, e.g., [41, Lemma 3.1] (see also [15, Chapter I]), there exist $\varphi_E, \varphi_H \in H^1_{\rm loc}(\mathbb{R}^3 \setminus D)$ such that

$$E(x) = \nabla \varphi_E(x) \quad \text{ and } \quad H(x) = \nabla \varphi_H(x),$$

and

$$|\varphi_E(x)| = O(|x|^{-1}) \text{ and } \quad |\varphi_H(x)| = O(|x|^{-1}) \text{ for large } x.$$

From (2.2.25) and the fact $E \times \nu = H \times \nu = 0$ on ∂D, we derive that

$$\int_{\mathbb{R}^3\setminus D} \langle \nabla \varphi_E, \nabla \varphi_E \rangle = \int_{\mathbb{R}^3\setminus D} \langle \nabla \varphi_H, \nabla \varphi_H \rangle = 0.$$

This yields

$$E = \nabla \varphi_E = 0 \quad \text{and} \quad H = \nabla \varphi_H = 0 \text{ in } \mathbb{R}^3 \setminus D.$$

We have a contradiction with the fact $\|(E_n, H_n)\|_{L^2(B_{R_0+1} \setminus D)} = 1$.

We next consider the case $k_* > 0$. In this case, we have (E, H) satisfies the radiating condition and

$$\begin{cases} \nabla \times E = i k_* H & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H = -i k_* E & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ E \times \nu = H \times \nu = 0 & \text{on } \partial D. \end{cases} \tag{2.2.27}$$

One also reaches that $(E, H) = (0, 0)$ in $\mathbb{R}^3 \setminus D$ and obtains a contradiction. □

In the same spirit, we have

Lemma 2.2.5. *Let $0 < \rho < \rho_0$, $0 < \omega < \omega_0$, $1/2 < r < 1$, and $R_0 > 2$. Let $h = (h_1, h_2) \in [H^{-1/2}(\mathrm{div}_{\partial B_1}, \partial B_1)]^2$. Assume that $(E, H) \in [L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus B_r)]^6$ with $(E, H) \in [\cap_{R>1} H(\mathrm{curl}, (B_R \setminus B_r) \setminus \partial B_1)]^2$ is a radiating solution of*

$$\begin{cases} \nabla \times E = i\omega \tilde{\mu}_\rho H & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ \nabla \times H = -i\omega \tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ [E \times \nu] = h_1, [H \times \nu] = h_2 & \text{on } \partial B_1. \end{cases} \tag{2.2.28}$$

We have, for $R > 2$,

$$\|(E, H)\|_{L^2(B_R \setminus B_r)} \le C_R \Big(\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1, h_2)\|_{H^{-1/2}(\partial B_r)} \\ + \omega^{-1} \|(\mathrm{div}_{\partial B_1} h_1, \mathrm{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big), \tag{2.2.29}$$

for some positive constant C_R independent of (h_1, h_2), (f_1, f_2), ρ, and ω.

Proof. As argued in the proof of Lemma 2.2.4, using Stratton-Chu's formulas, it suffices to prove

$$\|(E, H)\|_{L^2(B_2 \setminus B_r)} \le C_R \Big(\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1, h_2)\|_{H^{-1/2}(\partial B_r)} \\ + \omega^{-1} \|(\mathrm{div}_{\partial B_1} h_1, \mathrm{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big), \tag{2.2.30}$$

by contradiction. Assume that there exist sequences $(\omega_n) \subset (0, \omega_0)$, $\big((f_{1,n}, f_{2,n})\big) \subset L^2(\mathbb{R}^3 \setminus B_r)$

with support in $B_1 \setminus B_r$, and $\big((E_n, H_n)\big) \subset [\cap_{R>1} H(\operatorname{curl}, B_R \setminus D)]^2$ such that

$$\|(E_n, H_n)\|_{L^2(B_2 \setminus B_r)} = 1, \tag{2.2.31}$$

$$\lim_{n\to+\infty} \Big(\|(E_n \times \nu, H_n \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_{1,n}, h_{2,n})\|_{H^{-1/2}(\partial B_r)} \\ + \omega_n^{-1} \|(\operatorname{div}_{\partial B_1} h_{1,n}, \operatorname{div}_{\partial B_1} h_{2,n})\|_{H^{-1/2}(\partial B_1)} \Big) = 0, \tag{2.2.32}$$

and

$$\begin{cases} \nabla \times E_n = i\omega_n \tilde{\mu}_{\rho_n} H_n & \text{in } \mathbb{R}^3 \setminus \bar{B}_r, \\ \nabla \times H_n = -i\omega_n \tilde{\varepsilon}_{\rho_n} E_n + \tilde{\sigma}_{\rho_n} E_n & \text{in } \mathbb{R}^3 \setminus \bar{B}_r, \\ [E_n \times \nu] = h_{1,n}, [H_n \times \nu] = h_{2,n} & \text{on } \partial B_1. \end{cases} \tag{2.2.33}$$

Without loss of generality, one might assume that $\omega_n \to \omega_*$ and $\rho_n \to \rho_*$ as $n \to +\infty$. We first consider the case $\rho_* = 0$. Since, as $n \to +\infty$,

$$(-i\omega_n + 1) E_n \cdot \nu|_{int} = -i\omega_n \rho_n E_n \cdot \nu|_{ext} - \operatorname{div}_{\partial B_1} h_{2,n} \to 0 \text{ in } H^{-1/2}(\partial B_1)$$

and

$$H_n \cdot \nu|_{int} = \rho_n H_n \cdot \nu|_{ext} - (i\omega_n)^{-1} \operatorname{div}_{\partial B_1} h_{1,n} \to 0 \text{ in } H^{-1/2}(\partial B_1),$$

using (2.2.32) and applying Lemma 2.2.1, one can assume that (E_n, H_n) converges to (E, H) in $L^2(B_1 \setminus B_r)$. Moreover,

$$\begin{cases} \nabla \times E = i\omega_* H & \text{in } B_1 \setminus \bar{B}_r, \\ \nabla \times H = -i\omega_* E + E & \text{in } B_1 \setminus \bar{B}_r, \\ \operatorname{div} E = \operatorname{div} H = 0 & \text{in } B_1 \setminus \bar{B}_r, \\ E \times \nu = H \times \nu = 0 & \text{on } \partial B_r, \\ E \cdot \nu = H \cdot \nu = 0 & \text{on } \partial B_1. \end{cases} \tag{2.2.34}$$

As in (2.2.45) below, it is clear that $E_n \to 0$ in $[L^2(B_2 \setminus B_1)]^3$. It follows that $(E, H) = (0, 0)$ in $B_1 \setminus B_r$. We derive that

$$\lim_{n\to+\infty} \|(E_n, H_n)\|_{L^2(B_1 \setminus B_r)} = 0 \tag{2.2.35}$$

and, by [20, Lemma A1],

$$\lim_{n\to+\infty} \|(E_n \times \nu, H_n \times \nu)|_{int}\|_{H^{-1/2}(\partial B_1)} = 0.$$

This yields

$$\lim_{n\to+\infty} \|(E_n \times \nu, H_n \times \nu)|_{ext}\|_{H^{-1/2}(B_2\setminus B_1)} = 0. \tag{2.2.36}$$

This in turn implies, by Lemma 2.2.4, that

$$\lim_{n\to+\infty} \|(E_n, H_n)\|_{L^2(B_2\setminus B_1)} = 0. \tag{2.2.37}$$

Combining (2.2.31), (2.2.35), and (2.2.37), we obtain a contradiction.

We next consider the case $\rho_* > 0$. The proof in this case is similar to the one in Lemma 2.2.4 and omitted (see also [36, Lemma 4] for the case $\omega_* > 0$). □

Remark 2.2.1. The proof gives the following slightly sharper estimate

$$\begin{aligned}\|(E,H)\|_{L^2(B_R\setminus B_r)} \le C_R\Big(&\|(E\times\nu, H\times\nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1,h_2)\|_{H^{-1/2}(\partial B_r)} \\ &+ \|\omega^{-1}(\mathrm{div}_{\partial B_1} h_1, \mathrm{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)}\Big).\end{aligned} \tag{2.2.38}$$

We are ready to give the main result of this section

Lemma 2.2.6. *Let $0<\rho<\rho_0$ and $0<\omega<\omega_0$, and let $h_1, h_2 \in H^{-1/2}(\mathrm{div}_{\partial B_1}, \partial B_1)$. Let $(E_\rho, H_\rho) \in [\cap_{R>1} H(\mathrm{curl}, B_R\setminus\partial B_1)]^2$ be the unique radiating solution of*

$$\begin{cases} \nabla\times E = i\omega\tilde{\mu}_\rho H & \text{in } \mathbb{R}^3\setminus\partial B_1, \\ \nabla\times H = -i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3\setminus B_1, \\ [E\times\nu] = h_1,\ [H\times\nu] = h_2 & \text{on } \partial B_1. \end{cases} \tag{2.2.39}$$

We have

$$\|(E,H)\|_{L^2(B_2\setminus B_{2/3})} \le C\omega^{-1}\Big(\|(h_1,h_2)\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|(\mathrm{div}_{\partial B_1} h_1, \mathrm{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)}\Big),$$

for some positive constant C depending only on ρ_0 and ω_0.

Proof. Multiplying the first equation of (2.2.39) by $\tilde{\mu}_\rho^{-1}\nabla\times\bar{E}$ and integrating over $B_R\setminus\partial B_1$, we have, for $R>1$,

$$\begin{aligned}\int_{B_R\setminus\partial B_1} \tilde{\mu}_\rho^{-1}\nabla\times E\cdot\nabla\times\bar{E}\,dx &= i\omega\int_{B_R\setminus\partial B_1} H\cdot\nabla\times\bar{E}\,dx \\ = i\omega\int_{B_R\setminus\partial B_1}(-i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E)\cdot\bar{E}\,dx &+ i\omega\int_{\partial B_R}(H\times\nu)\cdot\bar{E}\,dx \\ &- i\omega\int_{\partial B_1}(H\times\nu)|_{\mathrm{ext}}\cdot\bar{E}|_{\mathrm{ext}} - (H\times\nu)|_{\mathrm{int}}\cdot\bar{E}|_{\mathrm{int}}.\end{aligned}$$

Using the definition of $\tilde{\sigma}_\rho$ and considering the imaginary part, we have

$$\int_{B_1\setminus B_{1/2}} |E|^2\, dx = \Re\left(\int_{\partial B_1} h_2\cdot \bar{E}|_{\text{ext}}\, dx - \bar{h}_1\cdot H|_{\text{int}}\, dx\right) - \Re\int_{\partial B_R} (H\times\nu)\cdot \bar{E}\, dx. \tag{2.2.40}$$

Letting $R\to\infty$ and using the radiation condition, we derive from (2.2.40) that

$$\begin{aligned}\int_{B_1\setminus B_{1/2}} |E|^2\, dx &\le \left|\int_{\partial B_1} h_2\cdot \bar{E}|_{\text{ext}} - \bar{h}_1 H|_{\text{int}}\, ds\right| \\ &\le \left|\int_{\partial B_1} h_2\cdot \bar{E}|_{\text{ext}}\right| + \left|\int_{\partial B_1} \bar{h}_1\cdot H|_{\text{ext}}\, ds\right| + \left|\int_{\partial B_1} (\bar{h}_1\times\nu)\cdot h_2\, ds\right|.\end{aligned} \tag{2.2.41}$$

Applying Lemma 2.2.2 with $D = B_2\setminus B_1$, we have

$$\left|\int_{\partial B_1} h_2\cdot \bar{E}|_{\text{ext}}\, ds\right| \le C\omega\|(E,H)\|_{L^2(B_2\setminus B_1)}\Big(\|h_2\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|\operatorname{div}_\Gamma h_2\|_{H^{-1/2}(\partial B_1)}\Big) \tag{2.2.42}$$

and

$$\left|\int_{\partial B_1} h_1\cdot \bar{H}|_{\text{ext}}\, ds\right| \le C\omega\|(E,H)\|_{L^2(B_2\setminus B_1)}\Big(\|h_1\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|\operatorname{div}_\Gamma h_1\|_{H^{-1/2}(\partial B_1)}\Big). \tag{2.2.43}$$

Applying Lemma 2.2.3, we obtain

$$\left|\int_{\partial B_1} (\bar{h}_1\times\nu)\cdot h_2\, ds\right| \le C\|(h_1,h_2)\|^2_{H^{-1/2}(\operatorname{div}_{\partial B_1},\partial B_1)}. \tag{2.2.44}$$

Denote

$$M = \|(h_1,h_2)\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|(\operatorname{div}_\Gamma h_1, \operatorname{div}_\Gamma h_2)\|_{H^{-1/2}(\partial B_1)}.$$

Combining (2.2.41), (2.2.42), (2.2.43) and (2.2.44) yields

$$\int_{B_1\setminus B_{1/2}} |E|^2\, dx \le C\Big(\omega M\|(E,H)\|_{L^2(B_2\setminus B_1)} + M^2\Big). \tag{2.2.45}$$

From the equations of (E,H) in $B_1\setminus B_{1/2}$, we have

$$\Delta E + \omega^2 E - i\omega E = 0 \text{ in } B_1\setminus B_{1/2}.$$

It follows from (2.2.45) that

$$\|E\|^2_{L^2(\partial B_{2/3})} + \|\nabla E\|^2_{L^2(\partial B_{2/3})} \le C\Big(\omega M\|(E,H)\|_{L^2(B_2\setminus B_1)} + M^2\Big), \tag{2.2.46}$$

which yields

$$\|(E,H)\|^2_{L^2(\partial B_{2/3})} \le C\Big(\omega^{-1} M\|(E,H)\|_{L^2(B_2\setminus B_1)} + \omega^{-2}M^2\Big). \tag{2.2.47}$$

Using (2.2.47) and applying Lemma 2.2.5 with $r = 2/3$, we derive that

$$\|(E,H)\|^2_{L^2(B_R\setminus B_{2/3})} \le C\Big(\omega^{-1}M\|(E,H)\|_{L^2(B_2\setminus B_1)} + \omega^{-2}M^2\Big),$$

and the conclusion follows. □

We end this subsection with

Lemma 2.2.7. *Let* $0<\rho<1$ *and* $\rho\omega<k_0$, *and let* $D\subset B_1$ *be a smooth, open subset of* $\mathbb{R}^3$. *Assume that* $(E,H)\in[\cap_{R>2}H(\mathrm{curl},B_R\setminus D)]^2$ *is a radiating solution to the system*

$$\begin{cases} \nabla\times E = i\omega\rho H & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times H = -i\omega\rho E & \text{in } \mathbb{R}^3\setminus D.\end{cases}$$

We have, for $R\ge 1$ *and* $x\in B_{3R/\rho}\setminus B_{2R/\rho}$,

$$|E(x)| \le C_R\rho^3\Big((\omega^2+1)\|E\|_{L^2(B_2\setminus D)} + (\omega+1)\omega\|H\|_{L^2(B_2\setminus D)}\Big),$$

for some positive constant C *depending only on* k_0 *and* R.

Proof. By Stratton-Chu's formula, we have, for $x\in\mathbb{R}^3\setminus\bar{B}_1$,

$$E(x) = \int_{\partial B_1}\nabla_x G_k(x,y)\times\big(\nu(y)\times E(y)\big)dy + i\omega\rho\int_{\partial B_1}\nu(y)\times H(y)G_k(x,y)dy - \int_{\partial B_1}\nu(y)\cdot E(y)\nabla_x G_k(x,y)dy, \quad (2.2.48)$$

where $k=\omega\rho$ and G_k is given in (2.2.21).

Let $(\widetilde{E},\widetilde{H})\in[H(\mathrm{curl},B_1)]^2$ be the unique solution to the system

$$\begin{cases} \nabla\times\widetilde{E} = i\omega\rho(1+i)\widetilde{H} & \text{in } B_1,\\ \nabla\times\widetilde{H} = -i\omega\rho(1+i)\widetilde{E} & \text{in } B_1,\\ \widetilde{E}\times\nu = E\times\nu & \text{on } \partial B_1.\end{cases} \quad (2.2.49)$$

By a contradiction argument, see, e.g., [41] (see also the proof of Lemma 2.2.6), we obtain

$$\|(\widetilde{E},\widetilde{H})\|_{L^2(B_1)} \le C\|E\times\nu_{\mathrm{ext}}, H\cdot\nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)}. \quad (2.2.50)$$

Since

$$\left|\int_{\partial B_1}E\times\nu\,ds\right| = \left|\int_{\partial B_1}\widetilde{E}\times\nu\,ds\right| = \left|\int_{B_1}\nabla\times\widetilde{E}\,dx\right| = \left|\int_{B_1}\omega\rho(1+i)\widetilde{H}dx\right|,$$

we obtain

$$\left|\int_{\partial B_1} E \times \nu \, ds\right| \le C\omega\rho \|(E,H)\|_{L^2(B_2\setminus D)}. \tag{2.2.51}$$

Similarly, we have

$$\left|\int_{\partial B_1} H \times \nu \, ds\right| \le C\omega\rho \|(E,H)\|_{L^2(B_2\setminus D)}. \tag{2.2.52}$$

One has

$$\int_{\partial B_1} \nu \cdot E \, ds = \frac{1}{i\omega\rho}\int_{\partial B_1} \nu \cdot \nabla \times H \, ds = 0. \tag{2.2.53}$$

Rewrite (2.2.48) under the form

$$\begin{aligned}
&E(x) = \\
&\int_{\partial B_1} \nabla_x G_k(x,0) \times \big(\nu(y) \times E(y)\big) dy + \int_{\partial B_1} \big(\nabla_x G_k(x,y) - \nabla_x G_k(x,0)\big) \times \big(\nu(y) \times E(y)\big) dy \\
&+ ik \int_{\partial B_1} \nu(y) \times H(y) G_k(x,0) \, dy + ik \int_{\partial B_1} \nu(y) \times H(y)\big(G_k(x,y) - G_k(x,0)\big) dy \\
&- \int_{\partial B_1} \nu(y) \cdot E(y) \nabla_x G_k(x,0) \, dy - \int_{\partial B_1} \nu(y) \cdot E(y)\big(\nabla_x G_k(x,y) - \nabla_x G(x,0)\big) dy.
\end{aligned}$$

Using the facts, for $|x| \in (2R/\rho, 3R/\rho)$ and $y \in \partial B_1$,

$$|G_k(x,y) - G_k(x,0)| \le C(1+\omega)\rho^2, \quad |\nabla G_k(x,y) - \nabla G_k(x,0)| \le C(1+\omega^2)\rho^3,$$

$$\|E\|_{L^2(\partial B_1)} \le C\|E\|_{L^2(B_2\setminus D)} \quad \text{and} \quad \|H\|_{L^2(\partial B_1)} \le C\|H\|_{L^2(B_2\setminus D)},$$

we derive the conclusion from (2.2.51), (2.2.52), and (2.2.53). □

Proof of Proposition 2.2.1

Applying Lemma 2.2.6 to $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)$, we have

$$\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2\setminus B_1)} \le C\omega^{-1}\|\big(\mathbb{E}(\rho\,.), \mathbb{H}(\rho\,.)\big)\|_{L^2(\partial B_1)}. \tag{2.2.54}$$

Since $\operatorname{div} \mathbb{J} = 0$, we have

$$\Delta \mathbb{E} + \omega^2 \mathbb{E} = -i\omega \mathbb{J} \text{ in } \mathbb{R}^3.$$

It follows that, for $x \in B_2$,

$$\mathbb{E}(x) = -i\omega \int_{\mathbb{R}^3} \mathbb{J}(y) G_\omega(x,y) \, dy \quad \text{and} \quad \mathbb{H}(x) = -\nabla_x \times \int_{\mathbb{R}^3} \mathbb{J}(y) G_\omega(x,y) \, dy. \tag{2.2.55}$$

This yields

$$\|\mathbb{E}(\rho.),\mathbb{H}(\rho.)\|_{C(\partial B_1)} \le C\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.2.56}$$

From (2.2.54) and (2.2.56), we obtain

$$\|(\widetilde{\mathbf{E}}_\rho,\widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2\setminus B_1)} \le C\omega^{-1}\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.2.57}$$

Applying Lemma 2.2.7 to $(\widetilde{\mathbf{E}}_\rho,\widetilde{\mathbf{H}}_\rho)$, we have, for $x\in B_{3r/\rho}\setminus B_{2r/\rho}$,

$$\left|\left(\widetilde{\mathbf{E}}_\rho(x),\widetilde{\mathbf{H}}_\rho(x)\right)\right| \le C_r\omega^{-1}\rho^3\|\mathbb{J}\|_{L^2(\mathbb{R}^3)} \text{ for } r>1/2,$$

Since $(\mathbb{E}_\rho,\mathbb{H}_\rho)-(\mathbb{E},\mathbb{H}) = (\widetilde{\mathbf{E}}_\rho,\widetilde{\mathbf{H}}_\rho)(\rho^{-1}\cdot)$ in $\mathbb{R}^3\setminus B_2$, the conclusion follows. □

2.2.2 Moderate and high frequency analysis - Proof of Proposition 2.2.2

This section contains two subsections. In the first subsection, we present several lemmas used in the proof of Proposition 2.2.2. The proof of Proposition 2.2.2 is given in the second subsection. The main objective of the first subsection is Lemma 2.2.9 which is analogous to Lemma 2.2.6 in the low frequency regime. To this end, we use a priori estimate in a bounded domain (in high frequency) in Corollary 2.2.1, which is obtained from Lemma 2.2.8.

Some useful lemmas

We begin with the following lemma that provide a priori estimate for the Maxwell equations in high frequency. The method of multiplication is used.

Lemma 2.2.8. *Let $\omega>\omega_0$, and let Ω be a* **convex** *bounded subset of $\mathbb{R}^3$ of class C^1. Let $j\in H(\operatorname{div},\Omega)$ and let $u\in H(\operatorname{curl},\Omega)\cap H(\operatorname{div},\Omega)$ be such that*

$$\nabla\times\nabla\times u-\omega^2 u = j \text{ in } \Omega, \tag{2.2.58}$$

and $u\cdot\nu$, $(\nabla\times u)\cdot\nu\in L^2(\partial\Omega)$. Then

$$\begin{aligned}&\|(\omega u\times\nu,(\nabla\times u)\times\nu)\|_{L^2(\partial\Omega)}\\ &\qquad\le C\Big(\|(\omega u,\nabla\times u)\|_{L^2(\Omega)}+\|(\omega u\cdot\nu,(\nabla\times u)\cdot\nu)\|_{L^2(\partial\Omega)}+\|j\|_{L^2(\Omega)}+\omega^{-1}\|\operatorname{div} j\|_{L^2(\Omega)}\Big),\end{aligned} \tag{2.2.59}$$

for some positive constant C depending only on Ω and ω_0.

Proof. The analysis is based on the multiplier technique. We first consider $\operatorname{div} j=0$. Multiplying (2.2.58) by $(\nabla \times \bar{u}) \times x$ and integrating over Ω, we obtain

$$\int_\Omega j \cdot (\nabla \times \bar{u}) \times x \, dx = \int_\Omega \nabla \times (\nabla \times u) \cdot (\nabla \times \bar{u}) \times x \, dx - \omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx. \quad (2.2.60)$$

Set

$$I_1 := -\omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx, \quad \text{and} \quad I_2 := \int_\Omega \nabla \times (\nabla \times u) \cdot (\nabla \times \bar{u}) \times x \, dx.$$

We have

$$\begin{aligned} I_1 &= -\omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx = \omega^2 \int_\Omega (\nabla \times \bar{u}) \cdot (u \times x) \, dx \\ &= \omega^2 \int_\Omega \bar{u} \cdot \nabla \times (u \times x) \, dx - \omega^2 \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \quad \text{(by integration by parts)}. \end{aligned}$$

Recall that, for all $v \in [H^1(\Omega)]^3$,

$$\nabla \times (v \times x) = -x \times (\nabla \times v) + v + \nabla(v \cdot x) - x \operatorname{div} v \quad \text{in } \Omega. \quad (2.2.61)$$

Using (2.2.61), and the fact $\operatorname{div} u = \operatorname{div} j = 0$ in Ω, we derive that

$$\begin{aligned} I_1 &= -\omega^2 \int_\Omega \bar{u} \cdot \left[x \times (\nabla \times u)\right] dx + \omega^2 \int_\Omega |u|^2 \, dx \\ &\qquad + \omega^2 \int_\Omega \bar{u} \cdot \nabla(u \cdot x) \, dx - \omega^2 \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \\ &= -\overline{I_1} + \omega^2 \left[\int_\Omega |u|^2 \, dx + \int_{\partial\Omega} (\bar{u} \cdot \nu)(u \cdot x) - \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \right]. \end{aligned}$$

This implies

$$\Re I_1 = \frac{\omega^2}{2} \left(\int_\Omega |u|^2 \, dx + \int_{\partial\Omega} (\bar{u} \cdot \nu)(u \cdot x) - \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \right). \quad (2.2.62)$$

Similarly, we have

$$\Re I_2 = \frac{1}{2} \left(\int_\Omega |\nabla \times u|^2 \, dx + \int_{\partial\Omega} ((\nabla \times \bar{u}) \cdot \nu)((\nabla \times u) \cdot x) - \int_{\partial\Omega} ((\nabla \times \bar{u}) \times \nu) \cdot ((\nabla \times u) \times x) \, ds \right). \quad (2.2.63)$$

Combining (2.2.60), (2.2.62), and (2.2.63) yields

$$\begin{aligned} &\int_\Omega \omega^2 |u|^2 + |\nabla \times u|^2 \, dx - \int_{\partial\Omega} \omega^2 (\bar{u} \times \nu) \cdot (u \times x) + ((\nabla \times \bar{u}) \times \nu) \cdot ((\nabla \times u) \times x) \, ds \\ &\quad + \int_{\partial\Omega} \omega^2 (\bar{u} \cdot \nu)(u \cdot x) + ((\nabla \times \bar{u}) \cdot \nu)((\nabla \times u) \cdot x) \, ds = \Re \left\{ \int_\Omega j \cdot (\nabla \times \bar{u}) \times x \, dx \right\}. \quad (2.2.64) \end{aligned}$$

Using Schwarz's inequality for the RHS, this implies (2.2.59) in the case $\operatorname{div} j = 0$ in Ω.

We next consider arbitrary $\operatorname{div} j$. Let $\phi \in H^1_0(\Omega)$ be the unique solution of

$$\Delta\phi = \operatorname{div} j \quad \text{in } \Omega,$$

It is clear that

$$\|\phi\|_{H^1(\Omega)} \le C\|j\|_{L^2(\Omega)} \tag{2.2.65}$$

and

$$\|\nabla\phi \times \nu\|_{L^2(\partial\Omega)} \le C\|\phi\|_{H^2(\Omega)} \le C\|\operatorname{div} j\|_{L^2(\Omega)}, \tag{2.2.66}$$

for some positive constant C depending only on Ω. Set

$$\tilde{u} = u - \omega^{-2}\nabla\phi \text{ in } \Omega. \tag{2.2.67}$$

We have

$$\nabla \times \nabla \times \tilde{u} - \omega^2\tilde{u} = j - \nabla\phi \text{ in } \Omega.$$

Since $\operatorname{div}(j - \nabla\phi) = 0$ in Ω, applying the previous case to $\tilde{u}$, we obtain the conclusions from (2.2.65), (2.2.66) and (2.2.67). □

As a consequence of Lemma 2.2.8, we can prove

Corollary 2.2.1. *Let $\omega > \omega_0$. Let $j \in H(\operatorname{div}, B_1 \setminus B_{3/4})$ and let $(E, H) \in [H(\operatorname{curl}, B_1 \setminus B_{3/4})]^2$ be such that $E \cdot \nu, H \cdot \nu \in [L^2(\partial B_1)]^3$. Assume that*

$$\begin{cases} \nabla \times E = i\omega H & \text{in } B_1 \setminus B_{3/4}, \\ \nabla \times H = -i\omega E + j & \text{in } B_1 \setminus B_{3/4}. \end{cases} \quad \text{and} \quad \operatorname{div} j = 0 \text{ in } B_1 \setminus B_{3/4}.$$

We have

$$\|(E \times \nu, H \times \nu)\|_{L^2(\partial B_1)} \le C\Big(\|(E, H)\|_{L^2(B_1 \setminus B_{3/4})} + \|(E \cdot \nu, H \cdot \nu)\|_{L^2(\partial B_1)} + \|j\|_{L^2(B_1 \setminus B_{3/4})}\Big),$$

for some positive constant C depending only on ω_0.

Proof. Let $0 \le \phi \le 1$ be a smooth function in B_1 such that $\phi(x) = 0$ in $B_{4/5}$, and $\phi(x) = 1$ in $B_1 \setminus B_{5/6}$. Extend u and j by 0 in $B_{3/4}$ and set $u = \phi E$ in B_1. Then

$$\nabla \times \nabla \times u - \omega^2 u = i\omega\phi j + \nabla \times (\nabla\phi \times E) + \nabla\phi \times (\nabla \times E) \text{ in } B_1. \tag{2.2.68}$$

Since $\Delta E + \omega^2 E = i\omega j$ in $B_1 \setminus B_{3/4}$, we have

$$\|\nabla E\|_{L^2(B_{5/6}\setminus B_{4/5})} \le C\omega\Big(\|E\|_{L^2(B_1\setminus B_{3/4})} + \|j\|_{L^2(B_1\setminus B_{3/4})}\Big). \tag{2.2.69}$$

Applying Lemma 2.2.8 and using (2.2.68) and (2.2.69) one obtains the conclusion. □

The main result of this section is the following lemma which is a variant of Lemma 2.2.6 in the case $\omega_0 < \omega < \omega_1 \rho^{-1}$.

Lemma 2.2.9. *Let $0 < \rho < \rho_0$ and $0 < \omega_0 < \omega < \omega_1/\rho$. Suppose that $h_1, h_2 \in L^2(\mathrm{div}_\Gamma, \partial B_1)$ and let $(E, H) \in [\cap_{R>1} H(\mathrm{curl}, B_R \setminus \partial B_1)]^2$ be the unique radiating solution to the system*

$$\begin{cases} \nabla \times E = i\omega\tilde{\mu}_\rho H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3, \\ [E\times\nu] = h_1,\ [H\times\nu] = h_2 & \text{on } \partial B_1. \end{cases} \tag{2.2.70}$$

If ρ_0 is small enough and ω_0 is large enough, we have that

$$\|(E\times\nu, H\times\nu)_{\mathrm{int}}\|_{L^2(\partial B_1)} \le C\Big(\|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega^{-1}\|(\mathrm{div}_{\partial B_1} h_1, \mathrm{div}_{\partial B_1} h_2)\|_{L^2(\partial B_1)}\Big), \tag{2.2.71}$$

for some positive constant C depending only on ω_0, ω_1, and ρ_0.

Proof. Applying Corollary 2.2.1, we have

$$\|(E\times\nu|_{\mathrm{int}}, H\times\nu|_{\mathrm{int}})\|^2_{L^2(\partial B_1)} \le C\Big(\|(E,H)\|^2_{L^2(B_1\setminus B_{3/4})} + \|(E\cdot\nu, H\cdot\nu)|_{\mathrm{int}}\|^2_{L^2(\partial B_1)}\Big). \tag{2.2.72}$$

One has, see, e.g., [13],

$$\|(E\cdot\nu, H\cdot\nu)|_{ext}\|^2_{L^2(\partial B_1)} \le C\Big(\|(E\times\nu, H\times\nu)|_{ext}\|^2_{L^2(\partial B_1)} + \|(E,H)\|_{L^2(B_2\setminus B_1)} + \|(E,H)\|_{L^2(\partial B_2)}\Big).$$

for some $C = C_{\omega_1} > 0$. By Lemma 2.2.4, we obtain

$$\|(E\cdot\nu, H\cdot\nu)|_{ext}\|^2_{L^2(\partial B_1)} \le C\|(E\times\nu, H\times\nu)|_{ext}\|^2_{L^2(\partial B_1)} \tag{2.2.73}$$

Since

$$\big(1-(i\omega)^{-1}\big)E\cdot\nu|_{int} = \rho E\cdot\nu|_{ext} + \frac{1}{i\omega}\mathrm{div}_{\partial B_1} h_2 \quad \text{and} \quad H\cdot\nu|_{int} = \rho H\cdot\nu|_{ext} + \frac{1}{i\omega}\mathrm{div}_{\partial B_1} h_1$$

we derive from (2.2.73) that

$$\|(E\cdot\nu, H\cdot\nu)|_{\mathrm{int}}\|^2_{L^2(\partial B_1)} \le C\Big(\rho^2\|(E\times\nu, H\times\nu)|_{ext}\|^2_{L^2(\partial B_1)} + \omega^{-2}\|\mathrm{div}_{\partial B_1}(h_1, h_2)\|^2_{L^2(\partial B_1)}\Big).$$

From the transmission conditions on ∂B_1, we deduce that

$$\begin{aligned}&\|(E\cdot\nu, H\cdot\nu)|_{\mathrm{int}}\|^2_{L^2(\partial B_1)}\\&\quad\leq C\Big(\rho^2\|(E\times\nu, H\times\nu)|_{\mathrm{int}}\|^2_{L^2(\partial B_1)}+\rho^2\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega^{-2}\|\operatorname{div}_{\partial B_1}(h_1,h_2)\|^2_{L^2(\partial B_1)}\Big).\end{aligned}\tag{2.2.74}$$

On the other hand, as in (2.2.41), we have

$$\begin{aligned}\int_{B_1\setminus B_{1/2}}|E|^2\,dx&\leq\left|\int_{\partial B_1}h_2\cdot\bar{E}|_{ext}-\bar{h}_1H|_{int}\,ds\right|\\&\leq C\Big(\omega_0\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega_0^{-1}\|(E\times\nu,H\times\nu)|_{\mathrm{ext}}\|^2_{L^2(\partial B_1)}\Big).\end{aligned}\tag{2.2.75}$$

Since $\Delta E+\omega^2E-i\omega E=0$ in $B_1\setminus B_{1/2}$, it follows that

$$\int_{B_{3/4}\setminus B_{2/3}}|E|^2+\omega^{-2}|\nabla E|^2\,dx\leq C\Big(\omega_0\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega_0^{-1}\|(E\times\nu,H\times\nu)|_{\mathrm{ext}}\|^2_{L^2(\partial B_1)}\Big).\tag{2.2.76}$$

An Integration by parts yields, for $2/3<r<3/4$, that

$$\begin{aligned}&\omega^2\int_{B_1\setminus B_r}|H|^2\,dx-\omega^2\int_{B_1\setminus B_r}|E|^2\,dx\\&\qquad=\Re\Big\{i\omega\int_{\partial B_1}\bar{E}|_{\mathrm{int}}(H\times\nu|_{\mathrm{int}})\,ds-i\omega\int_{\partial B_r}\bar{E}|_{\mathrm{int}}(H\times\nu|_{\mathrm{int}})\,ds\Big\}.\end{aligned}\tag{2.2.77}$$

Combining (2.2.75), (2.2.76) and (2.2.77) yields

$$\int_{B_1\setminus B_{3/4}}|E|^2+|H|^2\,dx\leq C\Big(\omega_0\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega_0^{-1}\|(E\times\nu,H\times\nu)|_{int}\|^2_{L^2(\partial B_1)}\Big).\tag{2.2.78}$$

From (2.2.72), (2.2.74) and (2.2.78), one obtains that, for ρ small enough,

$$\begin{aligned}&\|(E\times\nu|_{\mathrm{int}},H\times\nu|_{\mathrm{int}})\|^2_{L^2(\partial B_1)}\\&\quad\leq C\Big(\omega_0\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega_0^{-1}\|(E\times\nu,H\times\nu)|_{\mathrm{int}}\|^2_{L^2(\partial B_1)}+\omega^{-2}\|\operatorname{div}_{\partial B_1}(h_1,h_2)\|^2_{L^2(B_1)}\Big).\end{aligned}\tag{2.2.79}$$

This implies

$$\|(E\times\nu|_{\mathrm{int}},H\times\nu|_{\mathrm{int}})\|^2_{L^2(\partial B_1)}\leq C\Big(\|(h_1,h_2)\|^2_{L^2(\partial B_1)}+\omega^{-2}\|\operatorname{div}_{\partial B_1}(h_1,h_2)\|^2_{L^2(B_1)}\Big),\tag{2.2.80}$$

for ω_0 large enough and ρ small enough. □

Proof of Proposition 2.2.2.

Apply Lemma 2.2.9, we have

$$\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2\setminus B_1)} \le C\|\big(\mathbb{E}(\rho\,\cdot), \mathbb{H}(\rho\,\cdot)\big)\|_{L^2(\partial B_1)}. \tag{2.2.81}$$

Since $\omega > \omega_0$ large, by (2.2.55), one has

$$\|\mathbb{E}(\rho.), \mathbb{H}(\rho.)\|_{L^2(\partial B_1)} + \omega^{-1}\| \operatorname{div}_{\partial B_1}(\mathbb{E}(\rho.)\times \nu, \operatorname{div}_{\partial B_1}\mathbb{H}(\rho.)\times\nu)\|_{L^2(\partial B_1)} \le C\omega\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.2.82}$$

Applying Lemmas 2.2.9 and 2.2.4, we obtain

$$\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2\setminus B_1)} \le C\omega\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}.$$

The conclusion now follows from Lemma 2.2.7 in the case $\omega\rho < 1$ and from Lemma 2.2.12 in the case $\omega\rho > 1$. □

2.2.3 High and very high frequency analysis - Proof of Proposition 2.2.3

This section contains two subsections. In the first subsection, we present several lemmas used in the proof of Proposition 2.2.3. The proof of Proposition 2.2.3 is given in the second subsection.

Some useful lemmas

We begin this section with a trace-type result for Maxwell's equations in a bounded domain. The analysis is based on the Aubin–Nitsche duality argument, see e.g., [11, Lemma 4.8] (or dual method, see, e.g., [28]). In this subsection, D denotes an open smooth bounded subset of $\mathbb{R}^3$.

Lemma 2.2.10. *Let $\omega > \omega_0 > 0$ and $f \in H(\operatorname{div}, D)$. Assume that $(E, H) \in [H(\operatorname{curl}, D)]^2$ satisfies the equations*

$$\begin{cases} \nabla\times E = i\omega H & \text{in } D,\\ \nabla\times H = -i\omega E + f & \text{in } D. \end{cases} \tag{2.2.83}$$

Then

$$\|E\|_{H^{-1/2}(\partial D)} + \omega\|H\times\nu\|_{H^{-3/2}(\partial D)} \le C\Big(\omega^2\|E\|_{L^2(D)} + \omega\|f\|_{L^2(D)} + \omega^{-1}\|\operatorname{div} f\|_{L^2(D)}\Big),$$

for some positive constant C depending only on D and ω_0.

Remark 2.2.2. It is crucial to our analysis that the constant C is independent of ω.

Proof. We have, from (2.2.83),

$$\Delta E+\omega^2 E=\nabla(\operatorname{div} E)-\nabla\times(\nabla\times E)+\omega^2 E=\frac{1}{i\omega}\nabla(\operatorname{div} f)-i\omega f \text{ in } D. \tag{2.2.84}$$

Fix $\phi\in[H^{1/2}(\partial D)]^3$ (arbitrary). By [15, Theorem 1.6], there exists $\xi\in[H^2(D)]^3$ such that

$$\xi=0 \text{ on } \partial D,\quad \frac{\partial\xi}{\partial\nu}=\phi \text{ on } \partial D, \tag{2.2.85}$$

and

$$\|\xi\|_{H^2(D)}\leq C\|\phi\|_{H^{1/2}(\partial D)}. \tag{2.2.86}$$

Here and in what follows, C denotes a positive constant depending only on D and ω_0. Multiplying (2.2.84) by ξ and integrating by parts, we obtain

$$\int_D(\Delta\xi+\omega^2\xi)E-\int_{\partial D}E\phi=\int_D(\Delta E+\omega^2 E)\xi=\int_D-\frac{1}{i\omega}\operatorname{div} f\operatorname{div}\xi-i\omega f\xi \tag{2.2.87}$$

We derive from (2.2.86) that

$$\left|\int_{\partial D}E\phi\,ds\right|\leq C\Big(\omega^2\|E\|_{L^2(D)}+\omega\|f\|_{L^2(D)}+\omega^{-1}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\phi\|_{H^{1/2}(\partial D)},$$

which implies, since ϕ is arbitrary,

$$\|E\|_{H^{-1/2}(\partial D)}\leq C\Big(\omega^2\|E\|_{L^2(D)}+\omega\|f\|_{L^2(D)}+\omega^{-1}\|\operatorname{div} f\|_{L^2(D)}\Big). \tag{2.2.88}$$

It remains to prove

$$\|H\times\nu\|_{H^{-3/2}(\partial D)}\leq C\Big(\omega\|E\|_{L^2(D)}+\|f\|_{L^2(D)}+\omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big). \tag{2.2.89}$$

Fix $\varphi\in H^{3/2}(\partial D)$ (arbitrary), consider an extension of φ in D such that its $H^2(D)$-norm is bounded by $C\|\varphi\|_{H^{3/2}(\partial D)}$, and still denote this extension by φ. Such an extension exists by the trace theory, see, e.g., [15, Theorem 1.6]. We have

$$\int_{\partial D}H\times\nu\cdot\varphi\,ds=\int_D\Big(\nabla\times\varphi\cdot H-\nabla\times H\cdot\varphi\Big)\,dx. \tag{2.2.90}$$

Since

$$\begin{aligned}\left|\int_D\nabla\times\varphi\cdot H\,dx\right|&=\omega^{-1}\left|\int_D\nabla\times\varphi\cdot\nabla\times E\,dx\right|\\&=\omega^{-1}\left|\int_D\nabla\times\nabla\times\varphi\cdot E\,dx+\int_{\partial D}E\cdot((\nabla\times\varphi)\times\nu)\,ds\right|,\end{aligned}$$

and $\nabla \times H = i\omega E + f$, it follows from (2.2.88) that

$$\Big|\int_D \nabla \times \varphi \cdot H\,dx\Big| \le C\Big(\omega\|E\|_{L^2(D)} + \|f\|_{L^2(D)} + \omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)} \tag{2.2.91}$$

and

$$\Big|\int_D \nabla \times H \cdot \varphi\,dx\Big| \le C\Big(\omega\|E\|_{L^2(D)} + \|f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)}. \tag{2.2.92}$$

Combining (2.2.90), (2.2.91), and (2.2.92) yields

$$\Big|\int_{\partial D} H \times \nu \cdot \varphi\,ds\Big| \le C\Big(\omega\|E\|_{L^2(D)} + \|f\|_{L^2(D)} + \omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)}.$$

Since φ is arbitrary, assertion (2.2.89) follows. The proof is complete. □

Using Lemma 2.2.10, we establish

Lemma 2.2.11. *Let $\omega_1 > 0$, $0 < \rho < 1$, and assume that $\omega\rho > \omega_1$. Given $h_1, h_2 \in H^{3/2}(\operatorname{div}_\Gamma, \partial B_1)$, let $(E, H) \in [\cap_{R>1} H(\operatorname{curl}, B_R \setminus \partial B_1)]^2$ be the unique radiating solution of*

$$\begin{cases} \nabla \times E = i\omega\tilde{\mu}_\rho H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3, \\ [E \times \nu] = h_1; [H \times \nu] = h_2 & \text{on } \partial B_1, \end{cases} \tag{2.2.93}$$

where $(\tilde{\varepsilon}_\rho, \tilde{\mu}_\rho, \tilde{\sigma}_\rho)$ is defined in (2.2.12). We have

$$\|E \times \nu|_{\text{int}}\|_{H^{-1/2}(\partial B_1)} + \omega\|H \times \nu|_{\text{int}}\|_{H^{-3/2}(\partial B_1)} \le C\Big(\omega^4\|h_2\|_{H^{1/2}(\partial B_1)} + \omega^3\|h_1\|_{H^{3/2}(\partial B_1)}\Big),$$

for some positive constant C depending only on ω_1.

Proof. As in (2.2.41), we have

$$\int_{B_1 \setminus B_{1/2}} |E|^2\,dx \le \Big|\int_{\partial B_1} h_2 \cdot \bar{E}|_{\text{ext}} - \bar{h}_1 H|_{\text{int}}\,ds\Big|.$$

This implies

$$\int_{B_1 \setminus B_{1/2}} |E|^2\,dx \le \|h_2\|_{H^{1/2}(\partial B_1)}\|E|_{\text{int}}\|_{H^{-1/2}(\partial B_1)} + \|h_1\|_{H^{3/2}(\partial B_1)}\|H \times \nu|_{\text{int}}\|_{H^{-3/2}(\partial B_1)} + \|h_2\|^2_{L^2(\partial B_1)} \tag{2.2.94}$$

Applying Lemma 2.2.10 to (E, H) with $f = E$ in $B_1 \setminus B_{1/2}$, we have

$$\|E|_{\text{int}}\|_{H^{-1/2}(\partial B_1)} + \omega\|H \times \nu\|_{H^{-3/2}(\partial B_1)} \le C\omega^2\|E\|_{L^2(B_1 \setminus B_{1/2})}.$$

It follows from (2.2.94) that

$$\|E\|_{L^2(B_1\setminus B_{1/2})} \le C\Big(\omega^2\|h_2\|_{H^{1/2}(\partial B_1)} + \omega\|h_1\|_{H^{3/2}(\partial B_1)}\Big). \tag{2.2.95}$$

Applying Lemma 2.2.10 to (E,H) with $f = E$ in $B_1 \setminus B_{1/2}$ again, one has

$$\|E\times\nu|_{\rm int}\|_{H^{-1/2}(\partial B_1)} + \omega\|H\times\nu|_{\rm int}\|_{H^{-3/2}(\partial B_1)} \le C\Big(\omega^4\|h_2\|_{H^{1/2}(\partial B_1)} + \omega^3\|h_1\|_{H^{3/2}(\partial B_1)}\Big)$$

Using the transmission condition at ∂B_1, one reaches the conclusion. □

We end this subsection by a simple consequence of Stratton-Chu's formula.

Lemma 2.2.12. *Let $0<\rho<1$, $\omega>0$ with $\omega\rho>\omega_1$, and $D\subset B_1$. Assume that $(E,H)\in \big[H_{\rm loc}(\operatorname{curl},\mathbb{R}^3\setminus D)\big]^2$ is a radiating solution to the Maxwell equations*

$$\begin{cases} \nabla\times E = i\omega\rho H & in\ \mathbb{R}^3\setminus\bar{D},\\ \nabla\times H = -i\omega\rho E & in\ \mathbb{R}^3\setminus\bar{D}. \end{cases}$$

We have

$$|E(x)| \le \frac{C|\omega\rho|^{3/2}}{|x|}\|E\times\nu\|_{H^{-1/2}(\partial D)} + \frac{C|\omega\rho|^{5/2}}{|x|}\|H\times\nu\|_{H^{-3/2}(\partial D)}\ \textit{for}\ x\in B_{3/\rho}\setminus B_{1/\rho}, \tag{2.2.96}$$

for some positive constant C independent of x, ω, and ρ.

Proof of Proposition 2.2.3.

Apply Lemma 2.2.9, we have

$$\begin{aligned}\|\widetilde{\mathbf{E}}_\rho\times\nu\|_{H^{-1/2}(B_2\setminus B_1)} &+ \omega\|\widetilde{\mathbf{H}}_\rho\times\nu\|_{H^{-3/2}(B_2\setminus B_1)}\\ &\le C\omega^3\|\mathbb{E}(\rho\,\cdot)\times\nu\|_{H^{3/2}(\partial B_1)} + C\omega^4\|\mathbb{H}(\rho\,\cdot)\times\nu\|_{H^{1/2}(\partial B_1)}.\end{aligned} \tag{2.2.97}$$

Since $\omega>\omega_0$ large, by (2.2.55), one has

$$\omega^3\|\mathbb{E}(\rho\,\cdot)\times\nu\|_{H^{3/2}(\partial B_1)} + \omega^4\|\mathbb{H}(\rho\,\cdot)\times\nu\|_{H^{1/2}(\partial B_1)} \le C\omega^6\rho^{1/2}\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.2.98}$$

Applying Lemma 2.2.12, we derive from (2.2.97) and (2.2.98) that

$$\|\widetilde{\mathbf{E}}_\rho\|_{L^2(B_3\setminus B_{1/2})} \le C\omega^{15/2}\rho^3\|\mathbb{J}\|_{L^2(\mathbb{R}^3)},$$

which yields

$$\|\widetilde{\mathbf{H}}_\rho\|_{L^2(B_2\setminus B_1)} \le C\omega^{17/2}\rho^3\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}.$$

The proof is complete. □

2.3 Proof of Theorem 2.1.1

To implement the analysis in the frequency domain, let us introduce the notation of Fourier transform with respect to t:

$$\hat{u}(\omega, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, x) e^{i\omega t}\, dt, \tag{2.3.1}$$

for an appropriate function $u \in L^\infty_{\text{loc}}([0, +\infty), L^2(\mathbb{R}^3))$; here we extend u by 0 for $t < 0$.

The starting point of the frequency analysis is based on the following result:

Proposition 2.3.1. *Let* $f_e, f_m \in L^2([0,\infty); [L^2(\mathbb{R}^3)]^3) \cap L^1([0,\infty); [L^2(\mathbb{R}^3)]^3)$. *Assume that* $(\mathscr{E}, \mathscr{H}) \in L^\infty_{\text{loc}}([0, +\infty), [L^2(\mathbb{R}^3)]^6)$ *be the unique weak solution of* (2.1.9). *Assume that there exists* $R_0 > 0$ *such that* $\operatorname{supp} f_e(t, \cdot)$, $\operatorname{supp} f_m(t, \cdot)$, $\operatorname{supp} \sigma_e$, $\operatorname{supp} \sigma_m \subset B_{R_0}$. *Then, for almost every* $\omega > 0$, $(\hat{\mathscr{E}}, \hat{\mathscr{H}})(\omega, .) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ *is the unique radiating solution to the system*

$$\begin{cases} \nabla \times \hat{\mathscr{E}}(\omega, .) = i\omega\mu\hat{\mathscr{H}}(\omega, .) - \sigma_m \hat{\mathscr{H}}(\omega, \cdot) + \hat{f}_e(\omega, \cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{\mathscr{H}}(\omega, .) = -i\omega\varepsilon\hat{\mathscr{E}}(\omega, .) + \sigma_e \hat{\mathscr{E}}(\omega, .) - \hat{f}_m(\omega, .) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.3.2}$$

Proof. Let $(\mathscr{E}_\delta, \mathscr{H}_\delta) \in L^\infty_{\text{loc}}([0, \infty), [L^2(\mathbb{R}^3)]^6)$ be the unique weak solution to

$$\begin{cases} \varepsilon \dfrac{\partial \mathscr{E}_\delta}{\partial t} = \nabla \times \mathscr{H}_\delta - \sigma_e \mathscr{E}_\delta - \delta \mathscr{E}_\delta + f_m & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mu \dfrac{\partial \mathscr{H}_\delta}{\partial t} = -\nabla \times \mathscr{E}_\delta - \sigma_m \mathscr{H}_\delta - \delta \mathscr{H}_\delta + f_e & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathscr{E}_\delta(0,) = 0; \mathscr{H}_\delta(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.3.3}$$

By the standard Galerkin approach, one can prove that

$$\delta \int_0^{+\infty} \int_{\mathbb{R}^3} |\mathscr{E}_\delta(s, x)|^2 + |\mathscr{H}_\delta(s, x)|^2 \, dx\, ds \le C \|(f_e, f_m)\|^2_{L^2(\mathbb{R}_+, \mathbb{R}^3)}. \tag{2.3.4}$$

for some positive constant independent of δ and (f_e, f_m). Hence $\mathscr{E}_\delta, \mathscr{H}_\delta \in L^2((0,\infty); [L^2(\mathbb{R}^3)]^3)$, and thus $\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta \in L^2((0,\infty); [L^2(\mathbb{R}^3)]^3)$ by Parserval's theorem. It follows, for a.e. $\omega > 0$, that $(\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta) \in H(\text{curl}, \mathbb{R}^3)$ is the unique solution to

$$\begin{cases} \nabla \times \hat{\mathscr{E}}_\delta(\omega, .) = i\omega\mu\hat{\mathscr{H}}_\delta(\omega, .) - (\sigma_m + \delta)\hat{\mathscr{H}}_\delta(\omega, .) + \hat{f}_e(\omega, \cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{\mathscr{H}}_\delta(\omega, .) = -i\omega\varepsilon\hat{\mathscr{E}}_\delta(\omega, .) + (\sigma_e + \delta)\hat{\mathscr{E}}_\delta(\omega, .) - \hat{f}_m(\omega, .) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.3.5}$$

For $0 < \omega_1 < \omega < \omega_2 < \infty$, one can check that the solution of (2.3.5) satisfies

$$\|(\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta)(\omega, .)\|_{H(\text{curl}, B_R)} \le C \|(\hat{f}_e, \hat{f}_m)(\omega, .)\|_{L^2(\mathbb{R}^3)} \le C \|(f_e, f_m)\|_{L^1((0,\infty), L^2(\mathbb{R}^3))}. \tag{2.3.6}$$

for some positive constant C depending only on ε, μ, R, ω_1, and ω_2. Letting $\delta \to 0$ and using the limiting absorption principle, see e.g., [36, (2.28) and the following paragraph], one derives that

$$(\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta)(\omega,) \rightharpoonup (\mathscr{E}_0, \mathscr{H}_0)(\omega,.) \text{ weakly in } [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2 \text{ as } \delta \to 0, \tag{2.3.7}$$

where $(\mathscr{E}_0, \mathscr{H}_0)(\omega,.) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ is the unique radiating solution to the system

$$\begin{cases} \nabla \times \mathscr{E}_0(\omega,.) = i\omega\mu\mathscr{H}_0(\omega,.) - \sigma_m \mathscr{H}_0 + \hat{f}_e(\omega,\cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \mathscr{H}_0(\omega,.) = -i\omega\varepsilon\mathscr{E}_0(\omega,.) + \sigma_e \mathscr{E}_0(\omega,.) - \hat{f}_m(\omega,.) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.3.8}$$

From (2.3.6) and (2.3.7), we have

$$(\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta) \to (\mathscr{E}_0, \mathscr{H}_0) \text{ in distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3 \text{ as } \delta \to 0. \tag{2.3.9}$$

We claim that

$$(\hat{\mathscr{E}}_\delta, \hat{\mathscr{H}}_\delta) \to (\hat{\mathscr{E}}, \hat{\mathscr{H}}) \text{ in distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3. \tag{2.3.10}$$

and the conclusion follows from (2.3.9) and (2.3.10).

It remains to prove (2.3.10). Let $\phi \in [C_c^\infty\big((0,\infty) \times \mathbb{R}^3)\big]^3$. We have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} (\hat{\mathscr{E}}_\delta(\omega,x) - \hat{\mathscr{E}}(\omega,x))\bar{\phi}(\omega,x)\, dx d\omega = \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\mathscr{E}_\delta(t,x) - \mathscr{E}(t,x))\bar{\check{\phi}}(t,x)\, dx dt. \tag{2.3.11}$$

We have, by applying Proposition 2.1.1 to $(\mathscr{E}_\delta - \mathscr{E}, \mathscr{H}_\delta - \mathscr{H})$,

$$\|\mathscr{E}_\delta(t,.) - \mathscr{E}(t,.)\|_{L^2(\mathbb{R}^3)} \le C\delta \int_0^t \|(\mathscr{E}(s,.), \mathscr{H}(s,.))\|_{L^2(\mathbb{R}^3)}\, ds \text{ for } t > 0,$$

and, by applying Proposition 2.1.1 for $(\mathscr{E}, \mathscr{H})$,

$$\|(\mathscr{E}(s,.), \mathscr{H}(s,.))\|_{L^2(\mathbb{R}^3)} \le C\|(f_e, f_m)\|_{L^1((0,\infty),[L^2(\mathbb{R}^3)]^6)} \text{ for } t > 0.$$

It follows that

$$\|\mathscr{E}_\delta(t,.) - \mathscr{E}(t,.)\|_{L^2(\mathbb{R}^3)} \le C\delta t. \tag{2.3.12}$$

From (2.3.12), we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} (\mathscr{E}_\delta(t,x) - \mathscr{E}(t,x))\bar{\check{\phi}}(t,x)\, dx dt \le C\delta \int_{\mathbb{R}} t\|\check{\phi}(t,.)\|_{L^2(\mathbb{R}^3)}\, dt. \tag{2.3.13}$$

From (2.3.13) and the fast decay property of $\check{\phi}$, we derive that

$$\hat{\mathscr{E}}_\delta \to \hat{\mathscr{E}} \text{ in distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3.$$

Similarly, one can prove that

$$\hat{\mathscr{H}}_\delta \to \hat{\mathscr{H}} \text{ in distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3.$$

The proof is complete. $\square$

We are ready to give

Proof of Theorem 2.1.1. Fix $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$ and $T > 0$. Using the fact that $\hat{\mathscr{E}}_c(-k,x) = \overline{\hat{\mathscr{E}}_c}(k,x)$ and $\hat{\mathscr{E}}(-k,x) = \overline{\hat{\mathscr{E}}}(k,x)$ for $k > 0$, one has, for $0 < t < T$,

$$\|\mathscr{E}_c(t,\cdot) - \mathscr{E}(t,\cdot)\|_{L^2(K)} \le \int_0^T \|\partial_t \mathscr{E}_c(t,\cdot) - \partial_t \mathscr{E}(t,\cdot)\|_{L^2(K)} \le T \int_0^\infty \omega \|\hat{\mathscr{E}}_c(\omega,\cdot) - \hat{\mathscr{E}}(\omega,\cdot)\|_{L^2(K)}\, d\omega. \tag{2.3.14}$$

We have, by Proposition 2.2.1,

$$\int_0^1 \omega \|\hat{\mathscr{E}}_c(\omega,.) - \hat{\mathscr{E}}(\omega,.)\|_{L^2(K)}\, d\omega \le C \int_0^1 \rho^3 \|\hat{\mathscr{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)}\, d\omega \le C\rho^3 \|\mathscr{J}\|^2_{L^2(\mathbb{R};L^2(\mathbb{R}^3))}, \tag{2.3.15}$$

by Proposition 2.2.2 (here for simplicity of notations we assume that $\omega_0 = 1$),

$$\int_1^{1/\rho} \omega \|\hat{\mathscr{E}}_c(\omega,.) - \hat{\mathscr{E}}(\omega,.)\|_{L^2(K)}\, d\omega \le C\rho^3 \int_1^{1/\rho} \omega^4 \|\hat{\mathscr{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)}\, d\omega, \tag{2.3.16}$$

and, by Proposition 2.2.3,

$$\int_{1/\rho}^{+\infty} \omega \|\hat{\mathscr{E}}_c(\omega,.) - \hat{\mathscr{E}}(\omega,.)\|_{L^2(K)}\, d\omega \le C\rho^3 \int_{\frac{1}{\rho}}^{+\infty} \omega^{19/2} \|\hat{\mathscr{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)}\, d\omega, \tag{2.3.17}$$

A combination of (2.3.16), and (2.3.17) yields

$$\begin{aligned}\int_1^{\infty} \omega \|\hat{\mathscr{E}}_c(\omega,.) - \hat{\mathscr{E}}(\omega,.)\|_{L^2(K)}\, d\omega &\le C\rho^3 \int_1^{+\infty} \frac{1}{\omega} \|\widehat{\partial_t^{(11)} \mathscr{J}}(\omega,\cdot)\|_{L^2(\mathbb{R}^3))}\, d\omega \\ &\le C\rho^3 \|\mathscr{J}\|_{H^{11}(\mathbb{R},L^2(\mathbb{R}^3))}\end{aligned} \tag{2.3.18}$$

We derive from (2.3.14), (2.3.15) and (2.3.18) that, for $0 < t < T$,

$$\|\mathscr{E}_c(t,\cdot) - \mathscr{E}(t,\cdot)\|_{L^2(K)} \le CT\rho^3 \|\mathscr{J}\|_{H^{11}(\mathbb{R},L^2(\mathbb{R}^3))}.$$

The proof is complete. $\square$

Conclusion

The approximate cloaking for electromagnetic waves is achieved through the transformation optics method in both the time-harmonic and time-dependent regime. In the time-harmonic regime, using *only* a layer constructed by the mapping technique, the energy may blow up inside the cloaked region in the resonant case and/whereas cloaking is always *achieved*. Moreover, the degree of visibility *varies* among ρ, ρ^2 and ρ^3 depending on the resonance or non-resonance of the system and the compatibility of the source inside the cloaked region. These facts are new and distinct from known mathematical results in the literature.

With a fixed lossy layer, estimates on the degree of visibility in the frequency domain for all frequency are established. We implement the variational technique in low frequency and the multiplier and duality techniques in high frequency domain. The frequency dependence is explicitly provided for different frequency ranges. In turn, using these estimates, we show that cloaking is achieved with the degree of visibility ρ^3 in the time-dependent regime.

Using *only* the layer constructed by the mapping technique, it is natural to expect that cloaking is also achieved for the time-dependent Maxwell equations. However, in this case, one may not have good control of the frequency dependence. In turn, the use of Fourier's transform to imply cloaking effect in time domain is not obvious. This problem is closely related to the cloaking without lossy layer for the scalar wave equation, which has not been studied. These questions are interesting and can be the subject of researches in the future.

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- Vietnamese

Publications and conference talks

- H. M. Nguyen and L. Tran, *Approximate Cloaking for Electromagnetic Waves via Transformation Optics: cloaking vs infinite energy*, submitted, 2018
- H. M. Nguyen and L. Tran, *Approximate Cloaking for time-dependent Maxwell's Equations via Transformation Optics*, in preparation, 2018
- L. Tran, *Phenomena in Cloaking using Transformation Optics, Resonance and Non-Resonance*, Atlanta, Jan. 2017, Special Section, Joint Mathematical Meeting