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Full Range Scattering Estimates and Their Application to Cloaking

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Abstract

We establish very precise estimates for the time harmonic scattering effects of an inhomogeneity. Our estimates are valid at all frequencies, and are independent of the contents of the inhomogeneity. The involved constants are independent of the frequency. We use these estimates to assess the effectivity of approximate electromagnetic cloaks constructed by so called "mapping techniques".

Contents

1. Introduction

In this paper we study solutions to the (inhomogeneous) Helmholtz equation, that is, the reduced wave equation, in all of $\mathbb{R}^d$, $d = 2, 3$. In particular, we are interested in scattering from an (unknown) inhomogeneity surrounded by an absorbing ("lossy") layer. We establish very precise L^2 estimates for a large class of such scattering solutions. Special emphasis is placed on the case when the incident wave is a plane wave. The novelty of our estimates is threefold: (1) the involved constants are independent of frequency, (2) the estimates apply to all frequencies, and (3) the estimates are completely independent of the material parameters inside the inhomogeneity.

Estimates of the effect of a small inhomogeneity are extremely useful in order to assess the approximate effectiveness of the cloaking technique known as "cloaking by mapping". If one uses the very natural approximation scheme introduced in [8] (for zero frequency, that is, for the steady state conductivity problem) (see also [19] for a similar scheme) then the estimation of the degree of cloaking amounts exactly to the estimation of the effect of the presence of a small inhomogeneity. To obtain a proper estimate of the degree of cloaking (in the sense that it holds irrespective of the object being cloaked) it is important that the estimation of the effect of the small inhomogeneity (on the voltage potential) be independent of its "contents".

For the corresponding approximate "cloaking by mapping" approach to work at any fixed, non-zero frequency, it is necessary to employ an absorbing ("lossy") layer right outside the cloaked area. If such a layer is not present then it is well known that there exists a family of objects that will defy any attempts at cloaking (see [9] for the case of a bounded domain, and [13] or [1] when it comes to the entire space). For a detailed discussion of "lossless" cloaking issues, and the quite complicated resonance phenomena (given a fixed object, and a "resonant" frequency), we refer the reader to [14].

Suppose the incident wave is a plane wave of frequency ω, and let $v_{s,\varepsilon}$ denote the scattered field caused by an inhomogeneity of diameter $\approx \varepsilon$, surrounded by a "lossy" layer of thickness $\approx \varepsilon/2$ with permittivity (or index of refraction) $1 + \frac{i}{\omega\varepsilon\lambda}$, $0 < \lambda < 1$. One of our main results (Theorem 2 of Section 2.3) asserts that

(a) For large frequencies, namely $\omega > 1/\varepsilon$,

$$\frac{1}{\beta}\int_{B_\beta \setminus B_\varepsilon} |v_{s,\varepsilon}|^2 \leqq C\varepsilon^{d-1} \quad \forall\, \beta > \varepsilon.$$

(b) For moderate to small frequencies, namely $0 < \omega \leqq 1/\varepsilon$,

(b1) for $d = 3$,

$$\frac{1}{\beta}\int_{B_\beta \setminus B_\varepsilon} |v_{s,\varepsilon}|^2 \leqq C \max\{1, \lambda^2/(\omega^2\varepsilon^2)\}\varepsilon^2 \quad \forall\, \beta > \varepsilon,$$

(b2) for $d = 2$,

$$\frac{1}{\beta}\int_{B_{2\beta} \setminus B_\beta} |v_{s,\varepsilon}|^2 \leqq C \max\{1, \lambda^2/(\omega^2\varepsilon^2)\}\beta \frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\varepsilon\omega)|^2}, \quad \forall\, \beta > \varepsilon.$$

This result is a follow up to Theorem 1 (Section 2.3) which concerns scattering estimates for the Helmholtz equation with a "general" source in the presence of an appropriate "lossy" layer. Given the fact that (after the rescaling $x \to x/\varepsilon$) the relevant parameter in the Helmholtz equation really is $\omega\varepsilon$ (not ω) it is not surprising that our estimates degenerate as $\omega\varepsilon$ goes to 0. However, it is not a priori clear exactly how sharp they are. To address this point we show that the above estimates are optimal in the following sense: for fixed ε and β there exist scattered fields generated by incidents waves (plane waves for $d = 3$) such that the left-hand sides of $(b1)$ and $(b2)$ are of the same order as right-hand sides of $(b1)$ and $(b2)$ (see Lemmas 7, 8 in the Appendix).

With the extreme choice $\lambda = 0$, using the two dimensional estimates (a) and $(b2)$, we recover the optimal estimates given in Proposition 3 of [7] for the case when the total field vanishes on the boundary of the circular "lossy" layer. This is consistent with the well-known fact that an infinitely "lossy" layer effectively behaves as a sound-soft barrier (see for example [5] or [11]).

For any fixed frequency ω, with ε tending to zero, one will eventually achieve that ω is less than ε^{-1}. Thus the appropriate estimates are $(b1)$ and $(b2)$. These two estimates now assert that the right choice for λ is of magnitude smaller than or equal to ε, in which case the scattering effects (measured in norm) are bounded by $C\varepsilon$ (for $d = 3$) and $C/|\log \varepsilon|$ (for $d = 2$). Such estimates, with $\lambda = \varepsilon$, were obtained in [9] for a bounded domain (see also [13], where the author used a quite different "lossy" layer, for the whole space).

For the proof of the high frequency estimate (a) we use a variant of Morawetz's multiplier technique (see [12]) in which we take into account the effect of the "lossy" layer. The particular way we implement the multipliers is related to the approach taken by Perthame and Vega [18]. For the low frequency case [estimates $(b1)$ and $(b2)$] our proof may be viewed as an extension of the proof found in [13].

We apply our scattering estimates to assess the effectivity of approximate cloaking schemes (Theorem 3 of Section 3). The approximate cloaking schemes we consider are so-called "cloaking by mapping schemes" that include a "lossy" layer, as previously discussed in [9]. The fact that our scattering estimates are very precise in their dependence on frequency makes it possible to estimate the degree of cloaking as a function of frequency. We consider incident waves only in the form of plane waves (although our method can be applied in a much more general setting). From our assessment we may conclude that it is never possible, with one fixed scheme, to obtain cloaking (by mapping) uniformly in frequency. The obstructions to uniform cloaking are related to low frequency "probing" and they are most severe in two dimensions. To be more precise: (1) in three dimensions it is possible to achieve cloaking uniformly in frequency, using a fixed mapping but allowing the amount of absorption (conductivity) in the "lossy" layer to depend on frequency (becoming unbounded as $\omega \to 0$); (2) in two dimensions a prescribed level of cloaking will require both a mapping and an amount of absorption (conductivity) that depend on frequency (as $\omega \to 0$).

An important application of the high frequency scattering estimates proven in this paper is the analysis of approximate, damped cloaking for the full wave equation. An analysis of such space-time approximate cloaking (using these high frequency estimates) may be found in [16], where sharp estimates are derived under the additional assumption that the objects being cloaked have range-bound material parameters.

The approach to cloaking based on change of variables was introduced by Greenleaf–Lassas–Uhlmann [2], Pendry–Schurig–Smith [17], and

LEONARD [10]. Their "transformation optics" schemes use a singular change of coordinates which blows up a point to a cloaked region. Although this approach is excellent in many aspects, it has the defect that one needs to work with a singular structure. This gives difficulties in practice as well as in theory, see for example, [3] and [21]. The reader can find a survey on cloaking in [4]. The approximate cloaking schemes we consider represent a natural regularization of these singular schemes, obtained from a change of variables that tranforms a small ball with a thin "lossy" layer to a unit-size cloaked region, surrounded by a lossy layer (as in [9]).

2. Scattering Estimates

As already mentioned, our analysis is significantly different, depending on whether the frequency ω is smaller than or larger than the reciprocal diameter of the scattering inhomogeneity. We start by considering the case in which ω is larger than the reciprocal diameter.

2.1. The High Frequency Case

2.1.1. Preliminaries In this section we establish two lemmas that are crucial for the proof of our scattering estimates. These lemmas are localized versions of results already derived in [18]. In order to state and prove the two lemmas we shall need some convenient notation. We denote $r = |x|$, and $e_r = x/|x|$. We use v' synonymously with $v_r = \frac{\partial}{\partial r} v$, and define $\nabla_{\partial B_r} v := \nabla v - e_r v_r$, $\operatorname{div}_{\partial B_r} F := \operatorname{div} F - \partial_r(e_r \cdot F)$, where B_r denotes the ball of radius r. $\Re$ signifies the real part of the associated expression, and $\Im$ its imaginary part. We shall repeatedly use that

$$\int_{B_R \setminus B_\alpha} u = \int_\alpha^R \int_{\partial B_1} u(r\sigma)\, \mathrm{d}\sigma\, r^{d-1}\, \mathrm{d}r,$$

and the latter integral we shall often write, for shorthand,

$$\int_\alpha^R \int_{\partial B_1} r^{d-1} u,$$

implicitly implying that we think of the function $u(x) = u(r\sigma)$ as a function of the "two" variables $(r, \sigma) \in \mathbb{R} \times \partial B_1$. Our first lemma establishes a very useful integral identity.

Lemma 1. *Let $d \geqq 2, \omega > 0, 0 < \alpha < \beta < R < \infty$, and let P and Q be two continuous real functions defined on $[\alpha, R]$, with $P, Q \in C^2([\alpha, \beta])$ and $P, Q \in C^2([\beta, R])$. For any $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, u complex valued, we then have the* *identity*

$$
\begin{aligned}
&\Re\left(\int_{B_R\setminus B_\alpha}(P(r)\bar{u}_r+Q(r)\bar{u})(\Delta u+\omega^2 u)\right)\\
&=\omega^2\int_{B_R\setminus B_\alpha}\left(Q(r)-\frac{d-1}{2r}P(r)-\frac{1}{2}P'(r)\right)|u|^2\\
&\quad+\int_{B_R\setminus B_\alpha}\left(\frac{d-1}{2r}P(r)-\frac{1}{2}P'(r)-Q(r)\right)|u_r|^2\\
&\quad+\frac{1}{2}\int_{B_R\setminus B_\alpha}\left(P'(r)+\frac{d-3}{r}P(r)-2Q(r)\right)|\nabla_{\partial B_r}u|^2\\
&\quad+\frac{1}{2}\int_{B_R\setminus B_\alpha\setminus\partial B_\beta}\left(Q''(r)+\frac{d-1}{r}Q'(r)\right)|u|^2\\
&\quad+\frac{1}{2}\int_{\partial B_\beta}(Q'(\beta_+)-Q'(\beta_-))|u|^2+F(\alpha,u)-F(R,u),
\end{aligned}
$$

where F is defined by

$$
\begin{aligned}
F(t,u)=-\frac{\omega^2}{2}\int_{\partial B_t}P(t)|u|^2-\frac{1}{2}\int_{\partial B_t}P(t)|u'|^2+\frac{1}{2}\int_{\partial B_t}Q'(t)|u|^2\\
-\frac{1}{2}\int_{\partial B_t}Q(t)(|u|^2)'+\frac{1}{2}\int_{\partial B_t}P(t)|\nabla_{\partial B_t}u|^2.
\end{aligned}
$$

Proof. We recall that

$$
\Delta u=T_1(u)+T_2(u),
$$

where

$$
\begin{aligned}
T_1(u)&=\frac{1}{r^{d-1}}(r^{d-1}u')' \quad \text{and ,}\\
T_2(u)&=\mathrm{div}_{\partial B_r}(\nabla_{\partial B_r}u)=\frac{1}{r^2}\,\mathrm{div}_{\partial B_1}(\nabla_{\partial B_1}u(r\cdot))=\frac{1}{r^2}\Delta_\sigma u(r\cdot),
\end{aligned}
$$

with $\Delta_\sigma=\mathrm{div}_{\partial B_1}(\nabla_{\partial B_1}\ \cdot\ )$ denoting the Laplace-Beltrami operator on ∂B_1. In the following computations we initially ignore terms contributed from ∂B_R; of course we account for these terms at the very end.

Step 1 We calculate

$$
E_1:=\Re\int_{B_R\setminus B_\alpha}(P(r)\bar{u}_r+Q(r)\bar{u})u.
$$

Since $(|u|^2)'=\bar{u}u'+\bar{u}'u=2\Re(\bar{u}'u)$ this becomes (modulo terms from ∂B_R)

$$
\begin{aligned}
E_1&=\frac{1}{2}\int_\alpha^R\int_{\partial B_1}P(r)r^{d-1}(|u|^2)'+\int_{B_R\setminus B_\alpha}Q(r)|u|^2\\
&=-\frac{1}{2}\int_\alpha^R\int_{\partial B_1}(P(r)r^{d-1})'|u|^2-\frac{1}{2}P(\alpha)\alpha^{d-1}\int_{\partial B_1}|u|^2(\alpha\sigma)+\int_{B_R\setminus B_\alpha}Q(r)|u|^2.
\end{aligned}
$$

A simple computation therefore gives

$$E_1 = \int_{B_R \setminus B_\alpha} \left(Q(r) - \frac{d-1}{2r} P(r) - \frac{1}{2} P'(r) \right) |u|^2 - \frac{1}{2} \int_{\partial B_\alpha} P(\alpha) |u|^2, \qquad (2.1)$$

modulo terms from ∂B_R.

Step 2 We calculate

$$E_2 := \Re \int_{B_R \setminus B_\alpha} P(r) \bar{u}_r T_1(u).$$

This becomes

$$\begin{aligned} E_2 &= \Re \int_{B_R \setminus B_\alpha} P(r) \bar{u}' \left(u'' + \frac{d-1}{r} u' \right) \\ &= \int_{B_R \setminus B_\alpha} \frac{d-1}{r} P(r) |u'|^2 + \frac{1}{2} \int_\alpha^R \int_{\partial B_1} P(r) r^{d-1} (|u'|^2)' \\ &= \int_{B_R \setminus B_\alpha} \frac{d-1}{r} P(r) |u'|^2 - \frac{1}{2} \int_\alpha^R \int_{\partial B_1} (P(r) r^{d-1})' |u'|^2 - \frac{1}{2} P(\alpha) \alpha^{d-1} \\ &\quad \times \int_{\partial B_1} |u'|^2 (\alpha\sigma), \end{aligned}$$

and a simple computation therefore gives

$$E_2 = \int_{B_R \setminus B_\alpha} \left(\frac{d-1}{2r} P(r) - \frac{1}{2} P'(r) \right) |u'|^2 - \frac{1}{2} \int_{\partial B_\alpha} P(\alpha) |u'|^2, \qquad (2.2)$$

modulo terms from ∂B_R.

Step 3 We calculate

$$E_3 := \Re \int_{B_R \setminus B_\alpha} Q(r) \bar{u} T_1(u).$$

This becomes

$$\begin{aligned} E_3 &= \Re \int_\alpha^R \int_{\partial B_1} Q(r) \bar{u} (r^{d-1} u')' \\ &= -\Re \int_\alpha^R \int_{\partial B_1} (Q(r)\bar{u})' r^{d-1} u' - Q(\alpha) \alpha^{d-1} \Re \int_{\partial B_1} \bar{u}(\alpha\sigma) u'(\alpha\sigma) \\ &= -\int_{B_R \setminus B_\alpha} Q(r) |u'|^2 - \frac{1}{2} \int_\alpha^R \int_{\partial B_1} Q'(r) r^{d-1} (|u|^2)' - \frac{1}{2} \int_{\partial B_\alpha} Q(\alpha) (|u|^2)', \end{aligned}$$

and a simple computation therefore gives

$$\begin{aligned} E_3 = &-\int_{B_R \setminus B_\alpha} Q(r) |u'|^2 + \frac{1}{2} \int_{B_R \setminus B_\alpha \setminus \partial B_\beta} \left(Q''(r) + \frac{d-1}{r} Q'(r) \right) |u|^2 \\ &+ \frac{1}{2} \int_{\partial B_\beta} (Q'(\beta_+) - Q'(\beta_-)) |u|^2 + \frac{1}{2} \int_{\partial B_\alpha} Q'(\alpha) |u|^2 - \frac{1}{2} \int_{\partial B_\alpha} Q(\alpha) (|u|^2)', \end{aligned} \qquad (2.3)$$

modulo terms from ∂B_R.

Step 4 We calculate

$$E_4 := \Re \int_{B_R \setminus B_\alpha} P(r)\bar{u}_r T_2(u).$$

This becomes

$$\begin{aligned} E_4 &= \Re \int_\alpha^R \int_{\partial B_1} P(r) r^{d-3} \bar{u}' \Delta_\sigma u = -\Re \int_\alpha^\infty \int_{\partial B_1} P(r) r^{d-3} \nabla_\sigma \bar{u}' \nabla_\sigma u \\ &= -\frac{1}{2} \int_\alpha^R \int_{\partial B_1} P(r) r^{d-3} (|\nabla_\sigma u|^2)' \\ &= \frac{1}{2} \int_\alpha^R \int_{\partial B_1} (P(r) r^{d-3})' |\nabla_\sigma u|^2 + \frac{1}{2} P(\alpha)\alpha^{d-3} \int_{\partial B_1} |\nabla_\sigma u|^2(\alpha\sigma), \end{aligned}$$

and a simple computation therefore gives

$$E_4 = \frac{1}{2} \int_{B_R \setminus B_\alpha} \left(P'(r) + \frac{d-3}{r} P(r) \right) |\nabla_{\partial B_r} u|^2 + \frac{1}{2} \int_{\partial B_\alpha} P(\alpha) |\nabla_{\partial B_\alpha} u|^2, \tag{2.4}$$

modulo terms from ∂B_R.

Step 5 We calculate

$$E_5 := \Re \int_{B_R \setminus B_\alpha} Q(r)\bar{u} T_2(u).$$

This becomes

$$E_5 = \Re \int_\alpha^R \int_{\partial B_1} Q(r) r^{d-3} \bar{u} \Delta_\sigma u = -\int_\alpha^R \int_{\partial B_1} Q(r) r^{d-3} |\nabla_\sigma u|^2,$$

and so

$$E_5 = -\int_{B_R \setminus B_\alpha} Q(r) |\nabla_{\partial B_r} u|^2. \tag{2.5}$$

Step 6 We now finally calculate

$$E := \Re \left(\int_{B_R \setminus B_\alpha} (P(r)\bar{u}_r + Q(r)\bar{u})(\Delta u + \omega^2 u) \right).$$

A combination of the identities (2.1)–(2.5) yields

$$\begin{aligned} E &= \omega^2 \int_{B_R \setminus B_\alpha} \left(Q(r) - \frac{d-1}{2r} P(r) - \frac{1}{2} P'(r) \right) |u|^2 \\ &\quad + \int_{B_R \setminus B_\alpha} \left(\frac{d-1}{2r} P(r) - \frac{1}{2} P'(r) \right) |u'|^2 \end{aligned}$$

$$
\begin{aligned}
&-\int_{B_R\setminus B_\alpha} Q(r)|u'|^2 + \frac{1}{2}\int_{B_R\setminus B_\alpha}\left(P'(r) + \frac{d-3}{r}P(r)\right)|\nabla_{\partial B_r}u|^2\\
&-\int_{B_R\setminus B_\alpha} Q(r)|\nabla_{\partial B_r}u|^2 + \frac{1}{2}\int_{B_R\setminus B_\alpha\setminus\partial B_\beta}\left(Q''(r) + \frac{d-1}{r}Q'(r)\right)|u|^2\\
&+\frac{1}{2}\int_{\partial B_\beta}(Q'(\beta_+) - Q'(\beta_-))|u|^2 + F(\alpha,u),
\end{aligned}
$$

modulo terms from ∂B_R. Simplifying the expression on the right-hand side and including terms coming from ∂B_R, we finally arrive at

$$
\begin{aligned}
E = \omega^2 &\int_{B_R\setminus B_\alpha}\left(Q(r) - \frac{d-1}{2r}P(r) - \frac{1}{2}P'(r)\right)|u|^2\\
&+\int_{B_R\setminus B_\alpha}\left(\frac{d-1}{2r}P(r) - \frac{1}{2}P'(r) - Q(r)\right)|u_r|^2\\
&+\frac{1}{2}\int_{B_R\setminus B_\alpha}\left(P'(r) + \frac{d-3}{r}P(r) - 2Q(r)\right)|\nabla_{\partial B_r}u|^2\\
&+\frac{1}{2}\int_{B_R\setminus B_\alpha\setminus\partial B_\beta}\left(Q''(r) + \frac{d-1}{r}Q'(r)\right)|u|^2\\
&+\frac{1}{2}\int_{\partial B_\beta}(Q'(\beta_+) - Q'(\beta_-))|u|^2 + F(\alpha,u) - F(R,u),
\end{aligned}
$$

exactly as asserted in the statement of this lemma. □

With particular choices for the functions P and Q, we may use Lemma 1 to derive the following extremely useful localized energy estimate.

Lemma 2. *Given $\beta > 0$ and $d \geqq 2$, define*

$$
P_*(r) = \begin{cases} \dfrac{2\beta}{d-1} & \text{if } r > \beta,\\[2mm] \dfrac{2r}{d-1} & \text{if } 0 < r < \beta, \end{cases} \qquad \text{and} \quad Q_*(r) = \begin{cases} \dfrac{\beta}{r} & \text{if } r > \beta,\\[2mm] 1 & \text{if } 0 < r < \beta. \end{cases}
$$

For any $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, and any $0 < \alpha < \beta < R < \infty$, $\omega > 0$, we then have

$$
\begin{aligned}
&\Re\left(\int_{B_R\setminus B_\alpha}(P_*(r)\bar{u}_r + Q_*(r)\bar{u})(\Delta u + \omega^2 u)\right)\\
&\quad\leqq -\frac{1}{d-1}\int_{B_\beta\setminus B_\alpha}(|\nabla u|^2 + \omega^2|u|^2) + \frac{\beta(3-d)}{2}\int_{B_R\setminus B_\beta}\frac{|u|^2}{r^3} + F_*(\alpha,u) - F_*(R,u),
\end{aligned}
$$

where F_* is defined as in Lemma 1, with $P = P_*$ and $Q = Q_*$.

Remark 1. The weight functions P_* and Q_* were used by PERTHAME–VEGA [18] (in combination with a limiting absorption argument) to establish high frequency estimates for the Helmholtz equation in all of space. As mentioned earlier, these choices are also in the spirit of MORAWETZ and LUDWIG [12].

Proof. With these particular choices of P and Q the expressions in the right-hand side of the identity in Lemma 1 become

$$Q_*(r) - \frac{d-1}{2r}P_*(r) - \frac{1}{2}P_*'(r) = \begin{cases} 0 & \text{if } r > \beta, \\ -\dfrac{1}{d-1} & \text{if } 0 < r < \beta, \end{cases} \tag{2.6}$$

$$\frac{d-1}{2r}P_*(r) - \frac{1}{2}P_*'(r) - Q_*(r) = \begin{cases} 0 & \text{if } r > \beta, \\ -\dfrac{1}{d-1} & \text{if } 0 < r < \beta, \end{cases} \tag{2.7}$$

$$\frac{1}{2}\left(P_*'(r) + \frac{d-3}{r}P_*(r) - 2Q_*(r)\right) = \begin{cases} -\dfrac{2\beta}{r(d-1)} & \text{if } r > \beta, \\ -\dfrac{1}{d-1} & \text{if } 0 < r < \beta, \end{cases} \tag{2.8}$$

$$Q_*''(r) + \frac{d-1}{r}Q_*'(r) = \begin{cases} \dfrac{\beta(3-d)}{r^3} & \text{if } r > \beta, \\ 0 & \text{if } 0 < r < \beta, \end{cases} \tag{2.9}$$

and

$$Q_*'(\beta_+) - Q_*'(\beta_-) = -\frac{1}{\beta}. \tag{2.10}$$

The desired inequality now follows directly from the identity in Lemma 1 by dropping the two negative terms

$$-\frac{2\beta}{d-1}\int_{B_R\setminus B_\beta}\frac{1}{r}|\nabla_{\partial B_r}u|^2 \quad \text{and} \quad -\frac{1}{2\beta}\int_{\partial B_\beta}|u|^2$$

on the right-hand side. □

2.1.2. Scattering Estimates for the High Frequency Case We are now ready to prove a local H^1 estimate for solutions to a Helmholtz equation that models an inhomogeneity surrounded by an absorbing ("lossy") layer in the high frequency regime. A main feature of this estimate is that its constant is independent of both frequency and the contents of the inhomogeneity.

Proposition 1. *Let $d = 2$ or 3, $0 < \lambda < 1$, and $\omega > \omega_0$, for some fixed, positive ω_0. Let a be a real symmetric matrix valued function and σ be a complex function, both defined on $B_{1/2}$. Suppose a is bounded and uniformly elliptic, and suppose σ satisfies* $0 \leqq \text{ess inf}\,\Im(\sigma) \leqq \text{ess sup}\,\Im(\sigma) < +\infty$, *and* $0 < \text{ess inf}\,\Re(\sigma)$

$\leqq$ ess sup $\Re(\sigma) < +\infty$. *Let* $f \in L^2(\mathbb{R}^d)$ *with* supp $f \subset B_4 \setminus \overline{B_1}$, *and let* $v_\omega \in H^1_{\text{loc}}(\mathbb{R}^d)$ *be the unique solution of*

$$\begin{cases} \operatorname{div}(A\nabla v_\omega) + \omega^2 \Sigma v_\omega = f & \text{in } \mathbb{R}^d, \\ \frac{\partial v_\omega}{\partial r} = i\omega v_\omega + o\left(r^{-\frac{d-1}{2}}\right), & \text{as } r \to \infty, \end{cases} \tag{2.11}$$

with

$$A, \Sigma = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_1, \\ I, 1 + i/(\omega\lambda) & \text{in } B_1 \setminus B_{1/2}, \\ a, \sigma & \text{in } B_{1/2}. \end{cases} \tag{2.12}$$

Then

$$\frac{1}{\beta}\int_{B_\beta \setminus B_1} (|\nabla v_\omega|^2 + \omega^2 |v_\omega|^2) \leqq C \int_{\mathbb{R}^d} |f|^2 \quad \text{for any } \beta > 1. \tag{2.13}$$

The constant C *depends on* ω_0, *but is independent of* $a, \sigma, \omega, \beta, \lambda$, *and* f.

Remark 2. Estimate (2.13) is not true when Σ is a real valued function. The main observation here is that such an estimate holds in the presence of an appropriate "lossy" layer (remember λ lies between 0 and 1). A similar phenomenon, for fixed (non-resonant) frequency, was observed in the work of KOHN–ONOFREI–VOGELIUS –WEINSTEIN [9] and NGUYEN [13].

Proof. In this proof $C = C(\omega_0)$ denotes a constant, which may vary from one place to another, but which is always independent of $a, \sigma, \omega, \beta, \lambda$, and f. To simplify notation we drop the subscript ω from v_ω. We note that since

$$\frac{1}{\beta'}\int_{B_{\beta'} \setminus B_1} (|\nabla v|^2 + \omega^2 |v|^2) \leqq \frac{\beta}{\beta'}\frac{1}{\beta}\int_{B_\beta \setminus B_1} (|\nabla v|^2 + \omega^2 |v|^2) \quad \text{for } 1 < \beta' < \beta,$$

it clearly suffices to prove (2.13) for all β sufficiently large. We consider first the case $d = 3$. Multiplying (2.11) by $\bar{v}$ and integrating the expression obtained on B_R, $R > 1$, we obtain

$$\int_{\partial B_R} v_r \bar{v} - \int_{B_R} \langle A\nabla v, \nabla \bar{v}\rangle + \omega^2 \int_{B_R} \Sigma |v|^2 = \int_{B_R} f\bar{v}.$$

By letting R go to infinity, using the outgoing radiation condition, and considering only the imaginary part of these expressions, we get

$$\omega \limsup_{R\to\infty} \int_{\partial B_R} |v|^2 + \frac{\omega}{\lambda}\int_{B_1 \setminus B_{1/2}} |v|^2 \leqq \int_{\mathbb{R}^d} |f||v|. \tag{2.14}$$

It is easy to see that the lim sup on the left-hand side actually is the limit as R tends to ∞, but that is immaterial here. Since $\Delta v + \omega^2 v + i\frac{\omega}{\lambda} v = 0$ in $B_1 \setminus B_{1/2}$ and

$\omega > \omega_0$, it follows from multiplication of (2.11) by $\phi^2 \bar{v}$ and integration by parts that

$$\int_{B_{8/10} \setminus B_{6/10}} |\nabla v|^2 \leqq C\omega^2 \int_{B_1 \setminus B_{1/2}} |v|^2$$

(the Caccioppoli inequality). Use of (2.14) now gives

$$\int_{B_{8/10} \setminus B_{6/10}} |\nabla v|^2 \leqq C\omega^2 \int_{B_1 \setminus B_{1/2}} |v|^2 \leqq C\lambda\omega \int_{\mathbb{R}^d} |f||v|.$$

Thus there exists $\alpha \in (6/10, 8/10)$ such that

$$\int_{\partial B_\alpha} |\nabla v|^2 + \omega^2 |v|^2 \leqq C\lambda\omega \int_{\mathbb{R}^d} |f||v|, \tag{2.15}$$

and so

$$\frac{\omega}{\lambda} \int_{\partial B_\alpha} |v||v'| \leqq C\omega \int_{\mathbb{R}^d} |f||v|. \tag{2.16}$$

An application of Lemma 2 yields

$$\begin{aligned} \frac{1}{2} \int_{B_\beta \setminus B_\alpha} |\nabla v|^2 + \omega^2 |v|^2 \leqq F_*(\alpha, v) - F_*(R, v) + \left| \int_{\mathbb{R}^d} f(r\bar{v}' + \bar{v}) \right| \\ + \frac{\omega}{\lambda} \int_{B_1 \setminus B_{1/2}} |v||v'|, \end{aligned} \tag{2.17}$$

for any $R > \beta > 4$. Recall that

$$F_*(\alpha, v) = -\frac{\omega^2}{2}\alpha \int_{\partial B_\alpha} |v|^2 - \frac{\alpha}{2} \int_{\partial B_\alpha} |v'|^2 - \frac{1}{2} \int_{\partial B_\alpha} (|v|^2)' + \frac{\alpha}{2} \int_{\partial B_\alpha} |\nabla_{\partial B_\alpha} v|^2.$$

Since

$$-\frac{1}{2} \int_{\partial B_\alpha} (|v|^2)' \leqq \int_{\partial B_\alpha} |v||v'| \leqq \frac{\omega_0^2 \alpha}{2} \int_{\partial B_\alpha} |v|^2 + \frac{1}{2\omega_0^2 \alpha} \int_{\partial B_\alpha} |v'|^2,$$

we may conclude

$$\begin{aligned} F_*(\alpha, v) &\leqq \frac{\alpha}{2} \int_{\partial B_\alpha} |\nabla_{\partial B_\alpha} v|^2 + \left(\frac{1}{2\omega_0^2 \alpha} - \frac{\alpha}{2} \right) \int_{\partial B_\alpha} |v'|^2 \\ &\leqq C \int_{\partial B_\alpha} |\nabla v|^2. \end{aligned}$$

It now follows from (2.15) that

$$F_*(\alpha, v) \leqq C\lambda\omega \int_{\mathbb{R}^d} |f||v|. \tag{2.18}$$

We next estimate $F_*(R, v)$ for R large. By definition of F we have

$$\begin{aligned}-F_*(R, v) &= \frac{\beta\omega^2}{2}\int_{\partial B_R} |v|^2 + \frac{\beta}{2}\int_{\partial B_R} |v'|^2 + \frac{\beta}{2}\int_{\partial B_R} \frac{|v|^2}{R^2} \\ &\quad + \frac{\beta}{2}\int_{\partial B_R} \frac{(|v|^2)'}{R} - \frac{\beta}{2}\int_{\partial B_R} |\nabla_{\partial B_R} v|^2 \\ &\leqq \frac{\beta\omega^2}{2}\int_{\partial B_R} |v|^2 + \frac{\beta}{2}\int_{\partial B_R} |v'|^2 + \frac{\beta}{2}\int_{\partial B_R} \frac{|v|^2}{R^2} + \frac{\beta}{2}\int_{\partial B_R} \frac{(|v|^2)'}{R}.\end{aligned}$$

Using the outgoing radiation condition ($v'(x) = i\omega v(x) + o(r^{-1})$ as $r = |x| \to \infty$) and the fact that $v(x) = O(r^{-1})$ as $r \to \infty$, we now obtain

$$\limsup_{R\to\infty} -F_*(R, v) \leqq \beta\omega^2 \limsup_{R\to\infty} \int_{\partial B_R} |v|^2. \tag{2.19}$$

It is easy to see that the lim sups on both sides actually are the limits as R tends to ∞, but that is immaterial here. A combination of (2.17), (2.18), and (2.19) (and use of (2.14) and (2.16)) yields

$$\begin{aligned}\int_{B_\beta\setminus B_\alpha} |\nabla v|^2 + \omega^2|v|^2 \leqq C\Bigg(&\beta\omega\int_{\mathbb{R}^d} |f||v| + \lambda\omega\int_{\mathbb{R}^d} |f||v| + \int_{\mathbb{R}^d} |f||v'| \\ &+ \int_{\mathbb{R}^d} |f||v| + \omega\int_{\mathbb{R}^d} |f||v|\Bigg),\end{aligned}$$

or, after simplification,

$$\int_{B_\beta\setminus B_\alpha} |\nabla v|^2 + \omega^2|v|^2 \leqq C\omega\left(\beta + 1 + \lambda + \frac{1}{\omega}\right)\int_{\mathbb{R}^d} |f||v| + C\int_{\mathbb{R}^d} |f||v'|. \tag{2.20}$$

From the fact that $\omega > 2$, and $0 < \lambda < 1$, it follows that

$$\int_{B_\beta\setminus B_\alpha} |\nabla v|^2 + \omega^2|v|^2 \leqq C\omega\beta\int_{\mathbb{R}^d} |f||v| + C\int_{\mathbb{R}^d} |f||v'|, \quad \text{for any } \beta > 4. \tag{2.21}$$

Since f has support inside $B_4 \setminus B_\alpha$

$$\omega\int_{\mathbb{R}^d} |f||v| + \int_{\mathbb{R}^d} |f||v'| \leqq \frac{c}{2}\int_{B_4\setminus B_\alpha} (|\nabla v|^2 + \omega^2|v|^2) + \frac{1}{c}\int_{\mathbb{R}^d} |f|^2, \tag{2.22}$$

for any $c > 0$. By taking $\beta = 5$ in (2.21) and using (2.22) with c sufficiently small, we now obtain

$$\int_{B_5\setminus B_\alpha} |\nabla v|^2 + \omega^2|v|^2 \leqq C\int_{\mathbb{R}^d} |f|^2,$$

and therefore

$$\omega\int_{\mathbb{R}^d} |f||v| + \int_{\mathbb{R}^d} |f||v'| \leqq C\int_{\mathbb{R}^d} |f|^2. \tag{2.23}$$

A combination of (2.21) and (2.23) yields

$$\frac{1}{\beta}\int_{B_\beta\setminus B_\alpha}(|\nabla v|^2+\omega^2|v|^2)\leqq C\int_{\mathbb{R}^d}|f|^2,\quad \text{for any } \beta>4.$$

This verifies the lemma in the case $d=3$.

The only essential difference in the case $d=2$ (when compared to the case $d=3$) is the presence of the additional positive term

$$\frac{\beta(3-d)}{2}\int_{B_R\setminus B_\beta}\frac{|v|^2}{r^3}=\frac{\beta}{2}\int_{B_R\setminus B_\beta}\frac{|v|^2}{r^3}$$

on the right-hand side of (2.17). We now show that this term can be absorbed by the term $\omega^2\int_{B_\beta\setminus B_\alpha}|v|^2$ for any β sufficiently large. To this end, we note that v has the expansion

$$v(x)=\sum_{k=-\infty}^{\infty}d_k H_k^{(1)}(\omega r)e^{ik\theta},\quad 4<r,$$

where $H_k^{(1)}$ is the first kind Hankel function of order k. It is well-known (see [20]) that

$$r|H_k^{(1)}(r)|^2\leqq r'|H_k^{(1)}(r')|^2\quad \text{for } 0<r'\leqq r\quad \text{for any } k\neq 0,$$

and that

$$r|H_0^{(1)}(r)|^2\leqq Cr'|H_0^{(1)}(r')|^2\quad \text{for }\ 1<r'\leqq r.$$

Consequently

$$\begin{aligned}\int_{\partial B_r}|v|^2&=2\pi\sum_{k=-\infty}^{\infty}|d_k|^2 r|H_k^{(1)}(\omega r)|^2\\ &\leqq C2\pi\sum_{k=-\infty}^{\infty}|d_k|^2 r'|H_k^{(1)}(\omega r')|^2=C\int_{\partial B_{r'}}|v|^2\end{aligned}\tag{2.24}$$

for $4<r'\leqq r$. Based on (2.24) we estimate

$$\beta\int_{\mathbb{R}^2\setminus B_\beta}|v|^2/r^3=\beta\int_\beta^\infty\frac{1}{r^3}\int_{\partial B_r}|v|^2\,\mathrm{d}r\leqq C\frac{1}{\beta}\int_{\partial B_\beta}|v|^2,\tag{2.25}$$

and similarly,

$$\int_{B_\beta\setminus B_4}|v|^2\geqq C^{-1}(\beta-4)\int_{\partial B_\beta}|v|^2,\tag{2.26}$$

for any $\beta>4$. A combination of (2.25) and (2.26) yields

$$\beta\int_{\mathbb{R}^2\setminus B_\beta}|v|^2/r^3\leqq C\frac{1}{\beta}\int_{\partial B_\beta}|v|^2\leqq\frac{C}{\beta(\beta-4)}\int_{B_\beta\setminus B_4}|v|^2,$$

and for β sufficient large (that $C/\beta(\beta-4) < \omega_0^2/2$) this gives

$$\beta \int_{\mathbb{R}^2 \setminus B_\beta} |v|^2/r^3 \leqq \frac{\omega^2}{2} \int_{B_\beta \setminus B_4} |v|^2 \leqq \frac{\omega^2}{2} \int_{B_\beta \setminus B_\alpha} |v|^2,$$

since $\omega > \omega_0$, and $\alpha \in (6/10, 8/10)$. We conclude that the additional term of the right-hand side of (2.17) may be absorbed by (half of) the left-hand side. The rest of the proof of (2.13) for the case $d = 2$ (and β sufficiently large) proceeds exactly as before. □

2.2. The Low Frequency Case

2.2.1. Some Useful Lemmas In this section, we establish some preliminary results that will be used in the proof of Proposition 2, that is, in the proof of our scattering estimates for the low frequency regime. We begin with the following

Lemma 3. *Let $d = 2, 3$, let D be a smooth open subset of $\mathbb{R}^d$ with $D \subset B_1$, and such that $\mathbb{R}^d \setminus \bar{D}$ is connected. Suppose $0 < \omega < \omega_0$ for some sufficiently small $\omega_0 > 0$. For $f \in L^2(\mathbb{R}^d)$, with $\operatorname{supp} f \subset B_4$, and $g \in H^{\frac{1}{2}}(\partial D)$, let $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a solution of*

$$\begin{cases} \Delta v_\omega + \omega^2 v_\omega = f & \text{in } \mathbb{R}^d \setminus \bar{D}, \\ v_\omega = g & \text{on } \partial D, \\ v_\omega \text{ satisfies the outgoing radiation condition.} \end{cases} \tag{2.27}$$

Then

$$\|v_\omega\|_{H^1(B_\beta \setminus D)} \leqq C_\beta (\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for all } \beta \geqq 1, \tag{2.28}$$

for some positive constant $C_\beta = C(\omega_0, \beta, D)$, independent of ω. Furthermore, for all $\beta \geqq 1$ we have

$$\begin{cases} \|v_\omega\|_{L^2(B_\beta \setminus D)} \leqq C\beta^{\frac{1}{2}} (\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 3, \\ \|v_\omega\|_{L^2(B_{2\beta} \setminus B_\beta)} \leqq C\beta (\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) \dfrac{|H_0^{(1)}(\beta\omega)|}{|H_0^{(1)}(\omega)|} & \text{for } d = 2, \end{cases} \tag{2.29}$$

with $C = C(\omega_0, D)$ independent of ω and β. If the data depend on ω (that is, $g = g_\omega$ and $f = f_\omega$) in such a way that $\|f_\omega\|_{L^2(\mathbb{R}^d)} + \|g_\omega\|_{H^{\frac{1}{2}}(\partial D)}$ is bounded, and $f_\omega \to 0$ weakly in $L^2(\mathbb{R}^d)$, $g_\omega \to 0$ in $L^2(\partial D)$ as $\omega \to 0$, then

$$\lim_{\omega \to 0} \|v_\omega\|_{L^2(B_\beta \setminus D)} = 0 \quad \text{for any } \beta \geqq 1. \tag{2.30}$$

Remark 3. Statement (2.28) with $f = 0$ is proved in [13, Lemma 1]. Statements (2.29), (2.30) and the inclusion of a non-trivial f are not found in [13], however, the proof of these "extensions" follow along the lines of the proof of Lemma 1 in [13]. For completeness we give the details here.

Proof of Lemma 3. The proof for the case $d = 3$ is the simplest of the two. It can be obtained by modifying the proof for the case $d = 2$, which we now proceed to give. We recall the following properties of $H_k^{(1)}$, the Hankel function of the first kind of order k, see for instance [20, page 143 and page 446],

$$\lim_{r \to 0} \frac{1}{|\ln r|} H_0^{(1)}(r) = \frac{2}{i\pi}, \quad \lim_{r \to 0} r \frac{\mathrm{d}H_0^{(1)}(r)}{\mathrm{d}r} = -\frac{2}{i\pi}, \tag{2.31}$$

and $r|H_k^{(1)}(r)|^2$, $k \neq 0$, is a monotonically decreasing function on $\mathbb{R}_+$, so that

$$t|H_k^{(1)}(t)|^2 \leqq s|H_k^{(1)}(s)|^2, \quad \text{for all } 0 < s \leqq t, \text{ and any } k \neq 0. \tag{2.32}$$

We first prove by contradiction that

$$\|v_\omega\|_{L^2(B_5 \setminus D)} \leqq C(\|f\|_{L^2(\mathbb{R}^2)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}), \quad 0 < \omega < \omega_0, \tag{2.33}$$

for some positive constant C depending only on ω_0 and D (ω_0 sufficiently small). Suppose this is not true. Then there exist a sequence $\omega_n \to 0_+$ and sequences $f_n \in L^2(\mathbb{R}^2)$, with supp $f_n \subset B_4$, $g_n \in H^{\frac{1}{2}}(\partial D)$ such that

$$\lim_{n \to \infty} \|f_n\|_{L^2(\mathbb{R}^2)} + \|g_n\|_{H^{\frac{1}{2}}(\partial D)} = 0 \quad \text{and} \quad \|v_n\|_{L^2(B_5 \setminus D)} = 1, \tag{2.34}$$

where $v_n \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus D)$ is a solution of

$$\begin{cases} \Delta v_n + \omega_n^2 v_n = f_n & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ v_n = g_n & \text{on } \partial D, \\ v_n \text{ satisfies the outgoing radiation condition.} \end{cases} \tag{2.35}$$

Since $\Delta v_n + \omega_n^2 v_n = 0$ in $\mathbb{R}^2 \setminus \overline{B_4}$, and v_n satisfies the outgoing radiation condition, it follows that v_n can be represented as

$$v_n(x) = \sum_{k=-\infty}^{\infty} a_{k,n} H_k^{(1)}(\omega_n|x|) e^{ik\theta} \quad |x| > 4.$$

We decompose

$$v_n = v_{0,n} + v_{1,n}, \tag{2.36}$$

where

$$v_{0,n} = a_{0,n} H_0^{(1)}(\omega_n|x|) \quad \text{and} \quad v_{1,n} = \sum_{k \neq 0} a_{k,n} H_k^{(1)}(\omega_n|x|) e^{ik\theta}. \tag{2.37}$$

Since $\{e^{ik\theta}\}_{k=-\infty}^{\infty}$ are orthogonal in $L^2(\partial B_1)$ and $\|v_n\|_{L^2(B_5 \setminus D)} = 1$, it follows from (2.31), (2.32), and (2.37) that

$$|a_{0,n}| \leqq C/|\ln \omega_n| \tag{2.38}$$

and

$$\int_{\partial B_R} |v_{1,n}|^2 \leqq C \quad \forall R > 9/2. \tag{2.39}$$

In particular it follows that

$$\|v_n\|_{L^2(B_R \setminus D)} \leqq C_R \quad \text{for any } R \geqq 1 \text{ (not just for } R = 5). \tag{2.40}$$

From (2.35)

$$\int_{B_5 \setminus D} |\nabla v_n|^2 - \omega_n^2 \int_{B_5 \setminus D} |v_n|^2 = \int_{\partial B_5} \frac{\partial v_n}{\partial r} \overline{v}_n - \int_{\partial D} \frac{\partial v_n}{\partial \nu} \overline{g}_n - \int_{B_5 \setminus D} f_n \overline{v}_n. \tag{2.41}$$

Since $\Delta v_n + \omega_n^2 v_n = 0$ in $\mathbb{R}^2 \setminus \overline{B_4}$ it follows from elliptic regularity results that

$$\|v_n\|_{L^2(\partial B_5)} + \left\| \frac{\partial v_n}{\partial r} \right\|_{L^2(\partial B_5)} \leqq C \|v_n\|_{L^2(B_6 \setminus B_4)} \leqq C.$$

For the last inequality we have used (2.40). It now follows that

$$\left| \int_{\partial B_5} \frac{\partial v_n}{\partial r} \overline{v}_n \right| \leqq \left\| \frac{\partial v_n}{\partial r} \right\|_{L^2(\partial B_5)} \times \|v_n\|_{L^2(\partial B_5)} \leqq C. \tag{2.42}$$

Since $\Delta v_n + \omega_n^2 v_n = f_n$ in $\mathbb{R}^2 \setminus \bar{D}$ (and $\|v_n\|_{L^2(B_5 \setminus D)} = 1$) a simple variational argument gives that

$$\begin{aligned} \left\| \frac{\partial v_n}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} &\leqq C(\|\nabla v_n\|_{L^2(B_5 \setminus D)} + \|v_n\|_{L^2(B_5 \setminus D)} + \|f_n\|_{L^2(B_5 \setminus D)}) \\ &\leqq C(\|\nabla v_n\|_{L^2(B_5 \setminus D)} + 1), \end{aligned}$$

and so

$$\left| \int_{\partial D} \frac{\partial v_n}{\partial \nu} \overline{g}_n \right| \leqq C(\|\nabla v_n\|_{L^2(B_5 \setminus D)} \|g_n\|_{H^{1/2}(\partial D)} + \|g_n\|_{H^{1/2}(\partial D)}). \tag{2.43}$$

The fact that $\|f_n\|_{L^2(B_5 \setminus D)}$, $\|g_n\|_{H^{1/2}(\partial D)}$, and $\|v_n\|_{L^2(B_5 \setminus D)}$ are bounded, in combination with (2.41), (2.42) and (2.43), now yields that

$$\int_{B_5 \setminus D} |\nabla v_n|^2 \leqq C, \tag{2.44}$$

and so from (2.43)

$$\left| \int_{\partial D} \frac{\partial v_n}{\partial \nu} \bar{g}_n \right| \leqq C. \tag{2.45}$$

This last expression actually tends to zero as $n \to \infty$, but that fact will not be used. Since B_5 could be replaced by any B_R in this last argument, we may (after the extraction of subsequences and the use of a diagonalization argument) assume that $v_n \to v$ weakly in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus D)$ and that $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^2 \setminus D)$. We next prove that $\int_{\mathbb{R}^2\setminus D} |\nabla v|^2 < +\infty$. To that end

$$\int_{B_R\setminus D} |\nabla v|^2 \leqq \liminf_{n\to\infty} \int_{B_R\setminus D} |\nabla v_n|^2, \tag{2.46}$$

for any $R > 1$, and by the equivalent of (2.41) (with 5 replaced by R)

$$\int_{B_R\setminus D} |\nabla v_n|^2 \leqq \omega_n^2 \int_{B_R\setminus D} |v_n|^2 + \int_{\partial B_R} \left|\frac{\partial v_n}{\partial r}\right| |v_n| \\ + \left|\int_{\partial D} \frac{\partial v_n}{\partial \nu} \overline{g}_n\right| + \left|\int_{B_R\setminus D} f_n \overline{v}_n\right|.$$

We claim that

$$\liminf_{n\to\infty} \int_{B_R\setminus D} |\nabla v_n|^2 \leqq \limsup_{n\to\infty} \int_{B_R\setminus D} |\nabla v_n|^2 \leqq C, \tag{2.47}$$

with C independent of $R > 1$. It clearly suffices to prove this for R sufficiently large, say $R > 16$. Due to (2.45) (and the fact that $\omega_n \to 0_+$ and $\|f_n\|_{L^2} \to 0$) it thus suffices to prove that

$$\limsup_{n\to\infty} \int_{\partial B_R} \left|\frac{\partial v_n}{\partial r}\right| |v_n| \leqq C, \tag{2.48}$$

with C independent of $R > 16$. We have

$$\int_{\partial B_R} \left|\frac{\partial v_n}{\partial r}\right| |v_n| \leqq \int_{\partial B_R} \left|\frac{\partial v_{0,n}}{\partial r}\right| |v_n| + \int_{\partial B_R} \left|\frac{\partial v_{1,n}}{\partial r}\right| |v_n|.$$

From (2.31) and (2.38)

$$\limsup_{n\to\infty} \sup_{B_{2R}\setminus B_R} |v_{0,n}| \leqq C \quad \text{and} \quad \limsup_{n\to\infty} \sup_{B_{2R}\setminus B_R} R\left|\frac{\partial v_{0,n}}{\partial r}\right| = 0. \tag{2.49}$$

Very shortly we prove that

$$\limsup_{n\to\infty} \sup_{B_{2R}\setminus B_R} |v_{1,n}| + \limsup_{n\to\infty} \sup_{B_{2R}\setminus B_R} R|\nabla v_{1,n}| \leqq C/\sqrt{R} \quad \forall\, R > 16. \tag{2.50}$$

A combination of (2.49) and (2.50) yields

$$\lim_{n\to\infty} \int_{\partial B_R} \left|\frac{\partial v_{0,n}}{\partial r}\right| |v_n| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \int_{\partial B_R} \left|\frac{\partial v_{1,n}}{\partial r}\right| |v_n| \leqq C \quad \forall\, R > 16,$$

from which (2.48) follows immediately. We now return to the proof of (2.50). For $R > 16$, define $V_{R,n}(x) = v_{1,n}(Rx/4)$. It follows from (2.39) that

$$\int_{B_{10}\setminus B_2} |V_{R,n}|^2 \,\mathrm{d}x \leqq \frac{16}{R^2} \int_{B_{10R/4}\setminus B_{R/2}} |v_{1,n}|^2 \,\mathrm{d}x \leqq C/R. \tag{2.51}$$

On the other hand, $\Delta v_{1,n} + \omega_n^2 v_{1,n} = 0$ for $|x| > 4$, and this implies

$$\Delta V_{R,n} + \frac{\omega_n^2 R^2}{16} V_{R,n} = 0 \quad \text{on } B_{10} \setminus B_2. \tag{2.52}$$

Using the standard theory of elliptic equations (and the fact that $\omega_n \to 0$ as $n \to \infty$) we deduce from (2.51) and (2.52) that

$$\limsup_{n\to\infty} \sup_{B_9\setminus B_3} |V_{R,n}(x)| + \limsup_{n\to\infty} \sup_{B_9\setminus B_3} |\nabla V_{R,n}(x)| \leqq C/\sqrt{R} \quad R > 16.$$

We arrive at (2.50) by a change of variables, and this completes the proof of (2.47). From (2.46) and (2.47) it follows that

$$\int_{\mathbb{R}^2\setminus D} |\nabla v|^2 < +\infty. \tag{2.53}$$

Moreover, (2.34), (2.35), (2.49), (2.50) give that $v \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus D)$ satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{2.54}$$

$$\sup_{\mathbb{R}^2\setminus B_{16}} |v| \leqq C, \tag{2.55}$$

and

$$\int_{B_5\setminus D} |v|^2 = 1.$$

We shall now see that the existence of a solution v with these properties is impossible, which means we have arrived at a contradiction, and therefore may conclude that the estimate (2.33) holds. Fix $\phi \in C^1(\mathbb{R}^2)$ such that $0 \leqq \phi \leqq 1$, $\phi = 1$ if $|x| \leqq 1$ and $\phi = 0$ if $|x| > 2$, and define

$$\phi_R(x) = \phi(x/R).$$

Multiplying the first equation of (2.54) by $\bar{v}\phi_R$ and integrating the expression obtained on $\mathbb{R}^2 \setminus D$, we obtain

$$0 = \int_{\mathbb{R}^2\setminus D} \nabla v \nabla(\bar{v}\phi_R) = \int_{\mathbb{R}^2\setminus D} |\nabla v|^2 \phi_R + \int_{\mathbb{R}^2\setminus D} \bar{v} \nabla v \nabla \phi_R. \tag{2.56}$$

Since $|\nabla \phi_R| \leqq C/R$ and $\operatorname{supp} \nabla \phi_R \subset B_{2R} \setminus B_R$, it follows from (2.55) that

$$\left| \int_{\mathbb{R}^2\setminus D} \bar{v} \nabla v \nabla \phi_R \right| \leqq C \left(\int_{B_{2R}\setminus B_R} |\nabla v|^2 \right)^{\frac{1}{2}} \quad R > 16. \tag{2.57}$$

A combination of (2.53) and (2.57) yields

$$\lim_{R\to\infty} \int_{\mathbb{R}^2\setminus D} \bar{v} \nabla v \nabla \phi_R = 0, \tag{2.58}$$

and from the definition of ϕ_R, and (2.56). we therefore get

$$\int_{\mathbb{R}^2\setminus D} |\nabla v|^2 = \lim_{R\to\infty} \int_{\mathbb{R}^2\setminus B} |\nabla v|^2 \phi_R = -\lim_{R\to\infty} \int_{\mathbb{R}^2\setminus D} \bar{v}\nabla v \nabla \phi_R = 0. \quad (2.59)$$

Since $v = 0$ on ∂D it follows that $v \equiv 0$. This is inconsistent with the fact that $\|v\|_{L^2(B_5\setminus D)} = 1$ (and thus completes the proof of (2.33)).

We next use (2.33) to prove (2.28). We first note that the value 5 is not special, and so in place of (2.33) we might as well have proved

$$\|v_\omega\|_{L^2(B_{\beta+1}\setminus D)} \leqq C_\beta(\|f\|_{L^2(\mathbb{R}^2)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for any } \beta \geqq 1.$$

Since $\Delta v_\omega + \omega^2 v_\omega = 0$ in $B_{\beta+1} \setminus B_4$, with $0 < \omega < \omega_0$, local elliptic regularity theory gives

$$\|v_\omega\|_{H^{\frac{1}{2}}(\partial B_\beta)} \leqq C_\beta(\|f\|_{L^2(\mathbb{R}^2)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for any } \beta \geqq 5.$$

It follows from a standard energy estimate that

$$\|v_\omega\|_{H^1(B_\beta\setminus D)} \leqq C_\beta(\|f\|_{L^2(\mathbb{R}^2)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for any } \beta \geqq 5,$$

(and thus for any $\beta \geqq 1$) as asserted in (2.28). To prove (2.30), we proceed as follows. Suppose ω_n is a sequence, with $\omega_n \to 0$. Since $\|f_{\omega_n}\|_{L^2(\mathbb{R}^2)} + \|g_{\omega_n}\|_{H^{\frac{1}{2}}(\partial D)}$ is bounded, it follows from (2.28) (after extraction of subsequences and a diagonalization argument) that $v_{\omega_n} \to v$ weakly in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus D)$ and $v_{\omega_n} \to v$ in $L^2_{\text{loc}}(\mathbb{R}^2 \setminus D)$ along some subsequence (also referred to as ω_n). Since f_{ω_n} converges to 0 weakly in L^2, and $v_{\omega_n}|_{\partial D} = g_{\omega_n}$ converges to 0 in L^2

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ v = 0 & \text{on } \partial D. \end{cases}$$

We also have (as in (2.53) and (2.55)) that

$$\int_{\mathbb{R}^2\setminus D} |\nabla v|^2 < +\infty \quad \text{and} \quad \sup_{\mathbb{R}^2\setminus B_{16}} |v| < +\infty,$$

and so as before we arrive at $v \equiv 0$. In other words: any sequence $v_{\omega_n}, \omega_n \to 0$, contains a subsequence such that the v_{ω_n} tend to 0 in L^2_{loc}; it immediately follows that $\lim_{\omega\to 0} v_\omega = 0$ in L^2_{loc}.

It remains to prove (2.29). To this end we use (2.28), (2.32) and the decomposition (2.36), noting that since (2.28) is already proven it clearly suffices to verify (2.29) for $\beta > 5$. We have

$$\int_{B_{2\beta}\setminus B_\beta} |v_{0,\omega}|^2 \leqq C\beta^2 |a_{0,\omega}|^2 |H_0^{(1)}(\omega\beta)|^2 \leqq C\beta^2 \frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2} \int_{B_5\setminus B_4} |v_\omega|^2, \tag{2.60}$$

and

$$\int_{B_{2\beta}\setminus B_\beta} |v_{1,\omega}|^2 \leqq 2\pi \sum_{k\neq 0} |a_{k,\omega}|^2 \beta^2 |H_k^{(1)}(\omega\beta)|^2$$
$$\leqq 10\pi\beta \sum_{k\neq 0} |a_{k,\omega}|^2 |H_k^{(1)}(5\omega)|^2 \leqq \beta \int_{B_5\setminus B_4} |v_\omega|^2. \quad (2.61)$$

Here we have used (2.32) to estimate

$$\frac{\beta |H_k^{(1)}(\omega\beta)|^2}{|H_k^{(1)}(5\omega)|^2} = 5\frac{\omega\beta |H_k^{(1)}(\omega\beta)|^2}{5\omega |H_k^{(1)}(5\omega)|^2} \leqq 5 \quad \text{for } \beta \geqq 5, \quad k \neq 0.$$

We also note that

$$\beta \leqq C\beta^2 \frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2} \quad \text{for all } \beta \geqq 5,\ 0 < \omega < \omega_0.$$

By a combination of this inequality with (2.60) and (2.61) we arrive at

$$\int_{B_{2\beta}\setminus B_\beta} |v_\omega|^2 \leqq C\beta^2 \frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2} \int_{B_5\setminus B_4} |v_\omega|^2 \quad \text{for} \quad \beta \geqq 5.$$

Finally, using (2.28) (with $\beta = 5$) we obtain

$$\int_{B_{2\beta}\setminus B_\beta} |v_\omega|^2 \leqq C\beta^2 \frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2} (\|f\|_{L^2(\mathbb{R}^2)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for } \beta \geqq 5.$$

This proves (2.29) (in the case $d = 2$). □

Remark 4. Lemma 3 holds without the smallness assumption on ω_0. In order to verify this, it suffices to establish the estimate (2.33) for ω bounded away from zero and infinity, since the rest of the proof is entirely independent of any smallness assumption on ω_0. This version of (2.33) follows by an argument very similar to the one presented here. Since we shall not need this extension here, we leave the details to the reader.

The estimate (2.29) also leads to the following inequalities.

Lemma 4. *Under the assumptions of Lemma 3, we have*

$$\|v_\omega\|_{H^1(B_\beta\setminus D)} \leqq \begin{cases} C(\omega_0, D)\beta^{\frac{1}{2}}(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 3 \\ C(\omega_0, D)\beta(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 2, \end{cases}$$

for any $\beta \geqq 1$.

Proof. First we prove the corresponding $L^2(B_\beta \setminus D)$ bounds. For this purpose it obviously suffices to consider $d = 2$ (the L^2 estimate for $d = 3$ is already part of (2.29)). Since

$$\frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2} \leqq C \quad \text{for } \beta \geqq 1,\ 0 < \omega < \omega_0,$$

(2.29) implies the estimate

$$\int_{B_{2\beta}\setminus B_\beta} |v_\omega|^2 \leqq C\beta^2(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for } d = 2 \quad \text{and} \quad \beta \geqq 1. \tag{2.62}$$

Let $k_0 \geqq 0$ be chosen so that $2^{-k_0}\beta \geqq 1 > 2^{-k_0-1}\beta$. By summation (of k_0 copies) of the inequality (2.62) and (one copy) of (2.28) we now get

$$\begin{aligned}\int_{B_\beta\setminus D} |v_\omega|^2 &= \sum_{k=0}^{k_0-1} \int_{B_{2^{-k}\beta}\setminus B_{2^{-k-1}\beta}} |v_\omega|^2 + \int_{B_{2^{-k_0}\beta}\setminus D} |v_\omega|^2 \\ &\leqq C\sum_{k=0}^{k_0-1}(2^{-k-1}\beta)^2(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) \\ &\quad + C(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) \\ &\leqq C\beta^2(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}),\end{aligned}$$

or

$$\int_{B_\beta\setminus D} |v_\omega|^2 \leqq C\beta^2(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) \quad \text{for } d = 2 \quad \text{and} \quad \beta \geqq 1. \tag{2.63}$$

This verifies that

$$\|v_\omega\|_{L^2(B_\beta\setminus D)} \leqq \begin{cases} C(\omega_0, D)\beta^{\frac{1}{2}}(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 3 \\ C(\omega_0, D)\beta(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 2, \end{cases} \tag{2.64}$$

for any $\beta \geqq 1$. It remains to prove that

$$\|\nabla v_\omega\|_{L^2(B_\beta\setminus D)} \leqq \begin{cases} C(\omega_0, D)\beta^{\frac{1}{2}}(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 3 \\ C(\omega_0, D)\beta(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 2. \end{cases} \tag{2.65}$$

Let $0 \leqq \phi \leqq 1$ be a cut-off function, with

$$\phi(x) = 1 \text{ for } 1 < |x| < \beta + \frac{1}{4} \quad \text{and} \quad \phi(x) = 0 \text{ near } \partial D \text{ and for } |x| > \beta + \frac{1}{2},$$

and such that $|\nabla\phi(x)| \leqq C$, with C independent of β. Multiplication of the identity $\Delta v_\omega + \omega^2 v_\omega = f$ by $\phi^2 \bar{v}_\omega$ and integration by parts gives

$$\int_{B_{\beta+1}\setminus D} |\nabla v_\omega|^2 \phi^2 = \omega^2 \int_{B_{\beta+1}\setminus D} |v_\omega|^2\phi^2 - 2\int_{B_{\beta+1}\setminus D} \bar{v}_\omega \phi \nabla v_\omega \cdot \nabla\phi$$
$$- \int_{B_{\beta+1}\setminus D} f\bar{v}_\omega \phi^2.$$

By use of the estimate

$$\left|2\int_{B_{\beta+1}\setminus D} \bar{v}_\omega \phi\nabla v_\omega \cdot \nabla\phi\right| \leqq \frac{1}{2}\int_{B_{\beta+1}\setminus D} |\nabla v_\omega|^2\phi^2 + 2\int_{B_{\beta+1}\setminus D} |v_\omega|^2 |\nabla\phi|^2$$

(and the bound on $|\nabla\phi|$), it follows that

$$\frac{1}{2}\int_{B_{\beta+1}\setminus D} |\nabla v_\omega|^2\phi^2 \leqq (\omega^2 + C)\int_{B_{\beta+1}\setminus D} |v_\omega|^2 + \int_{B_{\beta+1}\setminus D} |f|^2.$$

Together with (2.64) this immediately yields

$$\int_{B_\beta\setminus B_1} |\nabla v_\omega|^2 \leqq C\int_{B_{\beta+1}\setminus D} |v_\omega|^2 + \int_{B_{\beta+1}\setminus D} |f|^2 \tag{2.66}$$

$$\leqq \begin{cases} C\beta(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 3 \\ C\beta^2(\|f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{H^{\frac{1}{2}}(\partial D)}) & \text{for } d = 2. \end{cases} \tag{2.67}$$

From (2.28) we already know that

$$\|v_\omega\|_{H^1(B_1\setminus D)} \leqq C(\|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{H^{\frac{1}{2}}(\partial D)}),$$

and so the estimate (2.65) is verified. □

The following simple lemma will also be used in the proof of Proposition 2.

Lemma 5. *Let D be a bounded subset of $\mathbb{R}^d$ with a C^1 boundary. There exists a positive constant C depending only on D such that*

$$\|u\|^2_{L^2(\partial D)} \leqq C\|u\|_{L^2(D)}\|u\|_{H^1(D)}, \quad \forall\, u \in H^1(D).$$

Proof. Assume first that $D = \mathbb{R}^d_+$ and $u \in C^1(\mathbb{R}^d_+)$ with compact support. We have (for real u)

$$|u(x', 0)|^2 = -2\int_0^\infty u(x', x_n)\,\frac{\partial u}{\partial x_n}(x', x_n)\,\mathrm{d}x_n.$$

This implies

$$\|u\|^2_{L^2(\mathbb{R}^d_0)} \leqq C\|u\|_{L^2(\mathbb{R}^d_+)}\|\partial u/\partial x_n\|_{L^2(\mathbb{R}^d_+)}.$$

The proof in the general case follows by application of a standard density argument and use of local charts for ∂D. □

Remark 5. Lemma 5 was proved and used in [5]. Similar inequalities related to the quantities div and curl were introduced in [6].

2.2.2. Scattering Estimates for the Low Frequency Case We are now ready to establish the low frequency analog of Proposition 1.

Proposition 2. *Let $d = 2$ or $3, 0 < \lambda < 1$, and $0 < \omega < \omega_0$, for some sufficiently small $\omega_0 > 0$. Let a be a real symmetric matrix valued function and σ be a complex function, both defined on $B_{1/2}$. Suppose a is bounded and uniformly elliptic, and suppose σ satisfies $0 \leqq \text{ess inf}\, \Im(\sigma) \leqq \text{ess sup}\, \Im(\sigma) < +\infty$, and $0 < \text{ess inf}\, \Re(\sigma) \leqq \text{ess sup}\, \Re(\sigma) < +\infty$. Let $f \in L^2(\mathbb{R}^d)$ with $\text{supp}\, f \subset B_4 \setminus B_1$, and let $v_\omega \in H^1_{\text{loc}}(\mathbb{R}^d)$ be the unique solution of*

$$\begin{cases} \text{div}(A\nabla v_\omega) + \omega^2 \Sigma v_\omega = f & \text{in } \mathbb{R}^d, \\ \frac{\partial v_\omega}{\partial r} = i\omega v_\omega + o(r^{-\frac{d-1}{2}}), & \text{as } r \to \infty, \end{cases}$$

with

$$A, \Sigma = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_1, \\ I, 1 + i/(\omega\lambda) & \text{in } B_1 \setminus B_{1/2}, \\ a, \sigma & \text{in } B_{1/2}. \end{cases}$$

Then, for all $\beta \geqq 1$,

$$\begin{cases} \|v_\omega\|_{L^2(B_\beta \setminus B_1)} \leqq C\beta^{\frac{1}{2}} \max\{1, \lambda/\omega\} \|f\|_{L^2} & \text{for } d = 3, \\ \|v_\omega\|_{L^2(B_{2\beta} \setminus B_\beta)} \leqq C\beta \max\{1, \lambda/\omega\} \|f\|_{L^2} \dfrac{|H_0^{(1)}(\omega\beta)|}{|H_0^{(1)}(\omega)|} & \text{for } d = 2, \end{cases} \tag{2.68}$$

with a constant $C = C(\omega_0)$, independent of $a, \sigma, f, \beta, \omega$ and λ.

Proof. We first prove by contradiction that

$$\|v_\omega\|_{L^2(B_5 \setminus B_1)} \leqq C \max\left\{1, \frac{\lambda}{\omega}\right\} \|f\|_{L^2}, \quad 0 < \omega < \omega_0, \tag{2.69}$$

for ω_0 sufficiently small. Suppose this is not true. Then there exist $\{\omega_n\}$, $\{\lambda_n\}$, and $\{f_n\}$, $\text{supp}\, f_n \subset B_4 \setminus B_1$, such that $\omega_n \to 0_+$, $\max\{\frac{\lambda_n}{\omega_n}, 1\}\|f_n\|_{L^2} \to 0$, as $n \to \infty$, and

$$\|v_n\|_{L^2(B_5 \setminus B_1)} = 1. \tag{2.70}$$

As in (2.40) we conclude that the inequality (2.70) implies that

$$\|v_n\|_{L^2(B_R \setminus B_1)} \leqq C_R \quad \text{for any } R > 1. \tag{2.71}$$

We have, for any $\frac{1}{2} < \alpha < 1$,

$$\int_{B_5 \setminus B_\alpha} |\nabla v_n|^2 - \omega_n^2 \int_{B_5 \setminus B_\alpha} |v_n|^2 - i\frac{\omega_n}{\lambda_n} \int_{B_1 \setminus B_\alpha} |v_n|^2$$
$$= -\int_{B_5} f_n \bar{v}_n + \int_{\partial B_5} v_n' \bar{v}_n - \int_{\partial B_\alpha} v_n' \bar{v}_n. \tag{2.72}$$

Since

$$\Im \int_{\partial B_5} v_n' \overline{v}_n = \lim_{R\to\infty} \Im \int_{\partial B_R} v_n' \overline{v}_n = \lim_{R\to\infty} \omega_n \int_{\partial B_R} |v_n|^2 \geqq 0,$$

and

$$-\Im \int_{\partial B_\alpha} v_n' \overline{v}_n = \Im \left(- \int_{B_\alpha} < A\nabla v_n, \nabla \overline{v}_n > +\omega_n^2 \int_{B_\alpha} \Sigma |v_n|^2 \right) \geqq 0,$$

for any $\alpha > 1/2$, it follows from (2.70), (2.72) and the assumption about $\{f_n\}$ that

$$\int_{B_1\setminus B_\alpha} |v_n|^2 \leqq \frac{\lambda_n}{\omega_n} \int_{\mathbb{R}^d} |f_n||v_n| \to 0 \quad \text{as } n \to \infty.$$

The convergence is uniform in $1/2 < \alpha < 1$, and so

$$\int_{B_1\setminus B_{1/2}} |v_n|^2 \to 0 \quad \text{as } n \to \infty. \tag{2.73}$$

From

$$\int_{B_{8/10}\setminus B_{6/10}} |\nabla v_n|^2 \leqq C \int_{B_1\setminus B_{1/2}} |v_n|^2,$$

(Caccioppoli's inequality) it now follows that

$$\int_{B_{8/10}\setminus B_{6/10}} |v_n||\nabla v_n| \leqq C \int_{B_1\setminus B_{1/2}} |v_n|^2 \to 0.$$

As a consequence, for some $\alpha_n \in (6/10, 8/10)$

$$\int_{\partial B_{\alpha_n}} |v_n||v_n'| \leqq C \int_{B_1\setminus B_{1/2}} |v_n|^2 \to 0.$$

Due to (2.71) and elliptic regularity,

$$\left| \int_{\partial B_5} v_n' \bar{v}_n \right| \leqq C.$$

Considering the real part of (2.72) (with $\alpha = \alpha_n$) and using the assumptions on f_n and v_n, and (2.73) we therefore obtain

$$\int_{B_5\setminus B_{\alpha_n}} |\nabla v_n|^2 \leqq C,$$

and so

$$\int_{B_5\setminus B_{8/10}} |\nabla v_n|^2 \leqq C.$$

On the other hand, from (2.73), as n goes to infinity,

$$\int_{B_1\setminus B_{8/10}} |v_n|^2 \leqq \int_{B_1\setminus B_{1/2}} |v_n|^2 \to 0.$$

An application of Lemma 5 gives

$$\|v_n\|^2_{L^2(\partial B_1)} \leqq C\|v_n\|_{L^2(B_1\setminus B_{8/10})}\|v_n\|_{H^1(B_1\setminus B_{8/10})} \to 0 \quad \text{as } n\to\infty.$$

Since $\|f_n\|_{L^2(\mathbb{R}^d)} \to 0$, Lemma 3 (with $D = B_1$) now yields

$$\lim_{n\to\infty} \|v_n\|_{L^2(B_5\setminus B_1)} = 0.$$

This is an obvious contradiction to the fact that $\|v_n\|_{L^2(B_5\setminus B_1)} = 1$, and so we may conclude that (2.69) holds. It is clear that the value 5 plays no particular role in the above proof, in other words, we have established the analog of (2.69) with the left-hand side $\|v_\omega\|_{L^2(B_\beta\setminus B_1)}$ and a constant C_β, that depends on β (for any $\beta \geqq 1$). The proof of the estimates (2.68) now follows from (a slightly modified version of) Lemma 3. Indeed, elliptic regularity and (2.69) gives

$$\|v_\omega\|_{H^{1/2}(\partial B_{9/2})} \leqq C\|v_\omega\|_{L^2(B_5\setminus B_1)} \leqq C\max\left\{1, \frac{\lambda}{\omega}\right\}\|f\|_{L^2}, \quad 0<\omega<\omega_0,$$

and a slight modification of Lemma 3 (with B_1 replaced by B_5, $D = B_{9/2}$, $f = 0$, and $g = v_\omega|_{\partial B_{9/2}}$) now yields

$$\begin{cases} \|v_\omega\|_{L^2(B_\beta\setminus B_{9/2})} \leqq C\beta^{\frac{1}{2}}\max\left\{1, \dfrac{\lambda}{\omega}\right\}\|f\|_{L^2} & \text{for } d=3,\\ \|v_\omega\|_{L^2(B_{2\beta}\setminus B_\beta)} \leqq C\beta\max\left\{1, \dfrac{\lambda}{\omega}\right\}\|f\|_{L^2}\dfrac{|H_0^{(1)}(\beta\omega)|}{|H_0^{(1)}(\omega)|} & \text{for } d=2, \end{cases}$$

with $C = C(\omega_0)$ independent of ω and $\beta \geqq 5$. A combination of these estimates with (2.69) immediately leads to (2.68). $\square$

The same approach that was used to derive Lemma 4 from Lemma 3 may also be applied to Proposition 2, to arrive at the following estimates.

Corollary 1. *Under the assumptions of Proposition 2, we have*

$$\|v_\omega\|_{H^1(B_\beta\setminus B_1)} \leqq \begin{cases} C(\omega_0)\beta^{\frac{1}{2}}\max\{1,\lambda/\omega\}\|f\|_{L^2} & \text{for } d=3,\\ C(\omega_0)\beta\max\{1,\lambda/\omega\}\|f\|_{L^2} & \text{for } d=2. \end{cases}$$

2.3. Uniform Scattering Estimates

By a combination of the Propositions 1 and 2 we arrive at our main scattering result.

Theorem 1. *Let $d = 2$ or 3, $0<\lambda<1$, and $0<\omega$. Let a be a real symmetric matrix valued function and σ be a complex function, both defined on $B_{1/2}$. Suppose a is bounded and uniformly elliptic, and suppose σ satisfies $0 \leqq \operatorname{ess\,inf}\Im(\sigma) \leqq \operatorname{ess\,sup}\Im(\sigma) < +\infty$, and $0 < \operatorname{ess\,inf}\Re(\sigma) \leqq \operatorname{ess\,sup}\Re(\sigma) < +\infty$. Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_4\setminus B_1$, and let $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of*

$$\begin{cases} \operatorname{div}(A\nabla v_\omega) + \omega^2\Sigma v_\omega = f & \text{in } \mathbb{R}^d,\\ \frac{\partial v_\omega}{\partial r} = i\omega v_\omega + o\left(r^{-\frac{d-1}{2}}\right), & \text{as } r\to\infty, \end{cases}$$

with

$$A, \Sigma = \begin{cases} I, 1 & in \quad \mathbb{R}^d \setminus B_1, \\ I, 1 + i/(\omega\lambda) & in \quad B_1 \setminus B_{1/2}, \\ a, \sigma & in \quad B_{1/2}. \end{cases}$$

For any $\omega_0 > 0$ there exists a constant C such that

(a) *For $\omega > \omega_0$,*

$$\frac{1}{\beta}\int_{B_\beta \setminus B_1} |v_\omega|^2 \leqq \frac{C}{\omega^2}\int_{\mathbb{R}^d} |f|^2 \quad for \quad all \beta > 1.$$

(b) *For $0 < \omega \leqq \omega_0$, and $d = 3$,*

$$\frac{1}{\beta}\int_{B_\beta \setminus B_1} |v_\omega|^2 \leqq C \max\{1, \lambda^2/\omega^2\}\int_{\mathbb{R}^d} |f|^2 \quad for \quad all\ \beta > 1.$$

For $0 < \omega \leqq \omega_0$, and $d = 2$,

$$\frac{1}{\beta}\int_{B_{2\beta} \setminus B_\beta} |v_\omega|^2 \leqq C \max\{1, \lambda^2/\omega^2\}\beta\int_{\mathbb{R}^d} |f|^2 \frac{|H_0^{(1)}(\omega\beta)|^2}{|H_0^{(1)}(\omega)|^2}, \quad for \quad all\ \beta > 1.$$

The constant C depends on ω_0, but is independent of $a, \sigma, f, \beta, \omega$ and λ.

Remark 6. The low frequency estimates in (b) are weaker than the high frequency estimates in (a) due to the presence of the term involving λ/ω. However, the estimates in (b) are optimal in this regard. We shall discuss the optimality of this part of the estimates in the appendix (see also Remark 9).

Remark 7. A direct combination of the propositions 1 and 2 yields Theorem 1 with the proviso that $\omega_0 > 0$ be sufficiently small. However, note that the estimates in (b) are equivalent to the estimate in (a) for ω bounded away from 0 and infinity. The theorem therefore remains valid if we increase the separator ω_0 between the cases (a) and (b), and so it holds with any fixed separator, as formulated above. For the the remainder of this paper we make the selection $\omega_0 = 1$.

Since the results of Proposition 1 and Corollary 1 pertain to the H^1 norm, we can include derivatives in our estimates. The use of Corollary 1 also eliminates the fraction involving Hankel functions in the low frequency, $d = 2$, case.

Corollary 2. *Under the assumptions of Theorem 1, we have*

$$\frac{1}{\beta}\int_{B_\beta \setminus B_1} (\omega^2|v_\omega|^2 + |\nabla v_\omega|^2) \leqq C\|f\|_{L^2}^2 \quad \omega > 1,$$

and

$$\frac{1}{\beta}\int_{B_\beta \setminus B_1} (|v_\omega|^2 + |\nabla v_\omega|^2) \leqq \begin{cases} C \max\{1, \lambda^2/\omega^2\}\|f\|_{L^2}^2 & for\ d = 3, \\ C\beta \max\{1, \lambda^2/\omega^2\}\|f\|_{L^2}^2 & for\ d = 2, \end{cases} \quad 0 < \omega \leqq 1.$$

From Theorem 1 we may deduce very precise estimates for the scattering effect of an arbitrary object surrounded by a "lossy" layer in the case when the incident wave is a plane wave.

Corollary 3. *Let $d = 2$, or 3 and $\omega > 0$. Suppose a is a real symmetric matrix valued function which is bounded and uniformly elliptic. Suppose $\sigma \in L^\infty(B_{1/2})$ is a complex function with $0 \leqq \text{ess inf}\,\Im(\sigma) \leqq \text{ess sup}\,\Im(\sigma) < +\infty, 0 < \text{ess inf}\,\Re(\sigma) \leqq \text{ess sup}\,\Re(\sigma) < +\infty$, and suppose $0 < \lambda < 1$. Define*

$$A = \begin{cases} I & \text{if } x \in \mathbb{R}^d \setminus B_{1/2}, \\ a(x) & \text{otherwise}, \end{cases} \quad \text{and} \quad \Sigma = \begin{cases} 1 & \text{if } x \in \mathbb{R}^d \setminus B_1, \\ 1 + \dfrac{i}{\omega\lambda} & \text{if } x \in B_1 \setminus B_{1/2}, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

Given $\eta \in \mathbb{R}^d$, with $|\eta| = 1$, let $\mathbf{v}_\omega$ be the solution of

$$\operatorname{div}(A\nabla\mathbf{v}_\omega) + \omega^2\Sigma\mathbf{v}_\omega = 0, \quad \text{in } \mathbb{R}^d,$$

of the form $\mathbf{v}_\omega = v_s + e^{i\omega x\cdot\eta}$, with $v_s \in H^1_{\text{loc}}(\mathbb{R}^d)$, the scattered wave, satisfying the outgoing radiation condition: $\frac{\partial v_s}{\partial r} = i\omega v_s + o\left(r^{-\frac{d-1}{2}}\right)$ as $r \to \infty$. Then

(a) *For $\omega > 1$,*

$$\frac{1}{\beta}\int_{B_\beta\setminus B_1} |v_s|^2 \leqq C \quad \text{for all } \beta > 1.$$

(b) *For $0 < \omega \leqq 1$, and $d = 3$,*

$$\frac{1}{\beta}\int_{B_\beta\setminus B_1} |v_s|^2 \leqq C\max\{1, \lambda^2/\omega^2\} \quad \text{for all } \beta > 1.$$

For $0 < \omega \leqq 1$, and $d = 2$,

$$\frac{1}{\beta}\int_{B_{2\beta}\setminus B_\beta} |v_s|^2 \leqq C\max\{1, \lambda^2/\omega^2\}\beta\frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\omega)|^2}.$$

The constant C is independent of $\omega, \beta, \lambda, \eta, a$ and σ.

Remark 8. By a slight variation of the following proof of Corollary 3 (using Corollary 2 in place of Theorem 1) we may also show that

$$\frac{1}{\beta}\int_{B_\beta\setminus B_1} (\omega^2|v_s|^2 + |\nabla v_s|^2) \leqq C\omega^2 \quad \forall\,\omega > 1,$$

and

$$\frac{1}{\beta}\int_{B_\beta\setminus B_1} (|v_s|^2 + |\nabla v_s|^2) \leqq \begin{cases} C\max\{1, \lambda^2/\omega^2\} & \text{for } d = 3, \\ C\beta\max\{1, \lambda^2/\omega^2\} & \text{for } d = 2, \end{cases} \quad \forall\, 0 < \omega \leqq 1.$$

Proof of Corollary 3. We introduce

$$v = v_s(x) + e^{i\omega\eta\cdot x}\psi(x),$$

where $\psi \in C^\infty(\mathbb{R}^d)$ is a cut-off function with $\psi = 1$ for $x \in B_2$ and $\psi = 0$ for $x \in \mathbb{R}^d \setminus B_3$. The function v is in $H^1_{\text{loc}}(\mathbb{R}^d)$, it satisfies the outgoing radiation condition and

$$\operatorname{div}(A\nabla v) + \omega^2 \Sigma v = f. \tag{2.74}$$

Here the source f is given by

$$f = 2i\omega e^{i\omega\eta\cdot x}\eta \cdot \nabla\psi + e^{i\omega\eta\cdot x}\Delta\psi.$$

An application of Theorem 1 yields the desired estimates. □

Remark 9. The low frequency estimates in (b) of Corollary 3 are significantly weaker than the high frequency estimates in (a) due to the presence of the term λ/ω. As ω approaches 0 these estimates allow for scattered fields (from incident plane waves) whose L^2 norms become unbounded on bounded sets. In the appendix we show that this does indeed occur for $d = 3$, we also show that the L^2 norm (on $B_4 \setminus B_1$) is bounded from below by λ/ω (see Lemma 7). For $d = 2$ the situation is a little bit more complicated: in the appendix we show that there exist locally bounded incident waves for which the $L^2(B_4 \setminus B_1)$ norm of the scattered field is bounded from below by λ/ω, however, the incident waves we exhibit are not plane (see Lemma 8).

From the previous result we obtain (by rescaling) the following result, which provides an estimate of the scattered field, $v_{s,\varepsilon}(x)$, caused by an incident plane wave "hitting" a diametrically small object surrounded by a thin "lossy" layer.

Theorem 2. *Let $d = 2$ or 3, $0 < \varepsilon < 1$, $0 < \lambda < 1$, $\omega > 0$, and $\eta \in \mathbb{R}^d$ with $|\eta| = 1$. Let $\mathbf{v}_\varepsilon(x) = v_{s,\varepsilon}(x) + e^{i\omega x\cdot\eta}$ be the solution of*

$$\operatorname{div}(A_\varepsilon \nabla \mathbf{v}_\varepsilon) + \omega^2 \Sigma_\varepsilon \mathbf{v}_\varepsilon = 0, \quad \text{in } \mathbb{R}^d,$$

where $v_{s,\varepsilon} \in H^1_{\text{loc}}(\mathbb{R}^d)$, the scattered field, satisfies the outgoing radiation condition: $\frac{\partial v_{s,\varepsilon}}{\partial r} = i\omega v_{s,\varepsilon} + o(r^{-\frac{d-1}{2}})$ as $r \to \infty$. Here the coefficients A_ε and Σ_ε are given by

$$A_\varepsilon = \begin{cases} I & \text{if } x \in \mathbb{R}^3 \setminus B_{\varepsilon/2}, \\ a_\varepsilon(x) & \text{otherwise,} \end{cases} \quad \text{and} \quad \Sigma_\varepsilon = \begin{cases} 1 & \text{if } \ x \in \mathbb{R}^3 \setminus B_\varepsilon, \\ 1 + \dfrac{i}{\omega\varepsilon\lambda} & \text{if } \ x \in B_\varepsilon \setminus B_{\varepsilon/2}, \\ \sigma_\varepsilon(x) & \text{otherwise.} \end{cases}$$

a_ε is a real symmetric matrix valued function, that is bounded and uniformly elliptic in $B_{\varepsilon/2}$; $\sigma_\varepsilon \in L^\infty(B_{\varepsilon/2})$ is a complex function with $0 \leqq \operatorname{ess\,inf} \Im(\sigma_\varepsilon) \leqq \operatorname{ess\,sup} \Im(\sigma_\varepsilon) < +\infty$, and $0 < \operatorname{ess\,inf} \Re(\sigma_\varepsilon) \leqq \operatorname{ess\,sup} \Re(\sigma_\varepsilon) < +\infty$. Then

(a) *for* $\omega > 1/\varepsilon$,

$$\frac{1}{\beta}\int_{B_\beta\setminus B_\varepsilon} |v_{s,\varepsilon}|^2 \leqq C\varepsilon^{d-1} \quad \textit{for all } \beta > \varepsilon.$$

(b) *For* $0 < \omega \leqq 1/\varepsilon$, *and* $d = 3$,

$$\frac{1}{\beta}\int_{B_\beta\setminus B_\varepsilon} |v_{s,\varepsilon}|^2 \leqq C\max\{1, \lambda^2/(\omega^2\varepsilon^2)\}\varepsilon^2 \quad \textit{for all } \beta > \varepsilon.$$

For $0 < \omega \leqq 1/\varepsilon$, *and* $d = 2$,

$$\frac{1}{\beta}\int_{B_{2\beta}\setminus B_\beta} |v_{s,\varepsilon}|^2 \leqq C\max\{1, \lambda^2/(\omega^2\varepsilon^2)\}\beta\frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\varepsilon\omega)|^2} \quad \textit{for all } \beta > \varepsilon.$$

Most importantly: the constant C *is independent of* ε, ω, β, λ, η, a_ε *and* σ_ε.

3. Applications to Cloaking

It is by now fairly well-known that estimates of the scattering effect of small inhomogeneities are very related to estimates of the efficiency of approximate cloaks obtained by so-called mapping techniques (see for instance [8,9,15], or [13]). This is especially true for estimates that are uniform with respect to the "contents" of the inhomogeneity. Based on Theorem 2, we shall now, in this spirit, derive efficiency estimates that are also explicit in their frequency dependence. Let us first recall the following basic fact on which our (approximate) change-of-variable-based cloaking schemes rely. The proof of this fact is quite elementary and left to the reader.

Lemma 6. *Let* $d \geqq 2$, *let* A *be a real symmetric matrix valued* L^∞ *function, and let* Σ *be a complex* L^∞ *function defined on* $\mathbb{R}^d$. *Suppose* $F : \mathbb{R}^d \to \mathbb{R}^d$ *is Lipschitz, surjective, and invertible, with* $F(x) = x$ *on* $\mathbb{R}^d \setminus B_2$, *and* $\det DF > c > 0$ *almost everywhere* $x \in \mathbb{R}^d$. *Then* $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *is a (distributional) solution of*

$$\operatorname{div}(A\nabla u) + \omega^2\Sigma u = f \quad \textit{in } \mathbb{R}^d$$

if and only if $v := u \circ F^{-1} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *is a solution of*

$$\operatorname{div}(F_*A\nabla v) + \omega^2 F_*\Sigma v = f_* \quad \textit{in } \mathbb{R}^d.$$

Here

$$F_*A(y) = \frac{DF(x)A(x)DF^T(x)}{\det DF(x)}, \quad F_*\Sigma(y) = \frac{\Sigma(x)}{\det DF(x)}, \quad f_*(y) = \frac{f(x)}{\det DF(x)},$$

with $x = F^{-1}(y)$. *Note that* $u = v$ *outside* B_2.

Let F_ε, $0 < \varepsilon < 1$, denote the particular continuous, radial Lipschitz mapping $\mathbb{R}^d \to \mathbb{R}^d$ given by

$$F_\varepsilon = \begin{cases} x & \text{if } x \in \mathbb{R}^d \setminus B_2, \\ \left(\dfrac{2-2\varepsilon}{2-\varepsilon} + \dfrac{|x|}{2-\varepsilon}\right)\dfrac{x}{|x|} & \text{if } x \in B_2 \setminus B_\varepsilon, \\ \dfrac{x}{\varepsilon} & \text{if } x \in B_\varepsilon. \end{cases} \tag{3.1}$$

We notice that F_ε transforms B_2 and B_ε into B_2 and B_1, respectively, with $F_\varepsilon = \mathrm{id}$ outside B_2.

The following theorem provides estimates of the degree of near invisibility achieved by

$$\text{the approximate cloak} = \begin{cases} (F_\varepsilon)_*I,\ (F_\varepsilon)_*1 & \text{in } B_2 \setminus B_1, \\ (F_\varepsilon)_*I,\ (F_\varepsilon)_*\left(1 + \dfrac{i}{\omega\varepsilon\lambda}\right) & \text{in } B_1 \setminus B_{1/2}, \end{cases}$$

where the dependence on frequency is explicit. These estimates are optimal in their dependence on ε and ω (as explained in the appendix).

Theorem 3. *Let $d = 2$, or 3 and $\omega > 0$. Suppose a is a real symmetric matrix valued function which is bounded and uniformly elliptic, suppose $\sigma \in L^\infty(B_{1/2})$ is a complex function with $0 \leqq \operatorname{ess\,inf} \Im(\sigma) \leqq \operatorname{ess\,sup} \Im(\sigma) < +\infty$, and $0 < \operatorname{ess\,inf} \Re(\sigma) \leqq \operatorname{ess\,sup} \Re(\sigma) < +\infty$. Define, for $0 < \varepsilon < 1$, and $0 < \lambda < 1$,*

$$A^c_\varepsilon, \Sigma^c_\varepsilon = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_2, \\ (F_\varepsilon)_*I,\ (F_\varepsilon)_*1 & \text{in } B_2 \setminus B_1, \\ (F_\varepsilon)_*I,\ (F_\varepsilon)_*\left(1 + \dfrac{i}{\omega\varepsilon\lambda}\right) & \text{in } B_1 \setminus B_{1/2}, \\ a(x), \sigma(x) & \text{in } B_{1/2}. \end{cases}$$

Given $\eta \in \mathbb{R}^d$, with $|\eta| = 1$, let $\mathbf{u}_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of

$$\operatorname{div}(A^c_\varepsilon \nabla \mathbf{u}_\omega) + \omega^2 \Sigma^c_\varepsilon \mathbf{u}_\omega = 0, \quad \text{in } \mathbb{R}^d,$$

of the form $\mathbf{u}_\omega = u_s + e^{i\omega x\cdot\eta}$, with $u_s \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, the scattered wave, satisfying the outgoing radiation condition: $\frac{\partial u_s}{\partial r} = i\omega u_s + o(r^{-\frac{d-1}{2}})$ as $r \to \infty$. Then

(a) *For $\omega > 1/\varepsilon$,*

$$\frac{1}{\beta}\int_{B_\beta \setminus B_2} |u_s|^2 \leqq C\varepsilon^{d-1} \quad \forall \beta > 2.$$

(b) *For $0 < \omega \leqq 1/\varepsilon$, and $d = 3$,*

$$\frac{1}{\beta}\int_{B_\beta \setminus B_2} |u_s|^2 \leqq C \max\{1, \lambda^2/(\omega^2\varepsilon^2)\}\varepsilon^2 \quad \forall \beta > 2.$$

For $0 < \omega \leqq 1/\varepsilon$*, and* $d = 2$*,*

$$\frac{1}{\beta}\int_{B_{2\beta}\setminus B_\beta} |u_s|^2 \leqq C\max\{1, \lambda^2/(\varepsilon^2\omega^2)\}\beta\frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\varepsilon\omega)|^2} \quad \forall\beta > 2.$$

Most importantly: the constant C *is independent of* $a, \sigma, \omega, \varepsilon, \lambda, \beta$*, and* η*.*

Proof. In the following we drop the subscript ω from the solution $\mathbf{u}_\omega$. Set $\mathbf{u}_\varepsilon = \mathbf{u}\circ F_\varepsilon$ (so that $\mathbf{u}_\varepsilon(x) = \mathbf{u}(x)$ for $|x| > 2$) and define $u_{s,\varepsilon}(x) = \mathbf{u}_\varepsilon(x) - e^{i\omega x\cdot\eta}$ (so that $u_{s,\varepsilon}(x) = u_s(x)$ for $|x| > 2$). Then, by Lemma 6,

$$\operatorname{div}(\tilde{A}_\varepsilon\nabla\mathbf{u}_\varepsilon) + \omega^2\tilde{\Sigma}_\varepsilon\mathbf{u}_\varepsilon = 0,$$

and $\mathbf{u}_\varepsilon(x) = u_{s,\varepsilon}(x) + e^{i\omega x\cdot\eta}$, with $u_{s,\varepsilon} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying the outgoing radiation condition. Here

$$\tilde{A}_\varepsilon, \tilde{\Sigma}_\varepsilon = (F_\varepsilon^{-1})_* A_\varepsilon^c, (F_\varepsilon^{-1})_*\Sigma_\varepsilon^c = \begin{cases} I, 1 & \text{in } \mathbb{R}^d\setminus B_\varepsilon,\\ I, 1 + \dfrac{i}{\omega\varepsilon\lambda} & \text{in } B_\varepsilon\setminus B_{\varepsilon/2},\\ \varepsilon^{2-d}a(x/\varepsilon), \varepsilon^{-d}\sigma(x/\varepsilon) & \text{in } B_{\varepsilon/2}.\end{cases}$$

According to Theorem 2 we have

(a) For $\omega > 1/\varepsilon$,

$$\frac{1}{\beta}\int_{B_\beta\setminus B_\varepsilon} |u_{s,\varepsilon}|^2 \leqq C\varepsilon^{d-1} \quad \text{for all } \beta > \varepsilon.$$

(b) For $0 < \omega \leqq 1/\varepsilon$, and $d = 3$,

$$\frac{1}{\beta}\int_{B_\beta\setminus B_\varepsilon} |u_{s,\varepsilon}|^2 \leqq C\max\{1, \lambda^2/(\varepsilon^2\omega^2)\}\varepsilon^2 \quad \text{for all } \beta > \varepsilon,$$

for $0 < \omega \leqq 1/\varepsilon$, and $d = 2$,

$$\frac{1}{\beta}\int_{B_{2\beta}\setminus B_\beta} |u_{s,\varepsilon}|^2 \leqq C\max\{1, \lambda^2/(\varepsilon^2\omega^2)\}\beta\frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\varepsilon\omega)|^2} \quad \text{for all } \beta > \varepsilon,$$

The constant C is independent of $\varepsilon, \omega, \beta, \lambda, \eta, a$ and σ. Since $u_{s,\varepsilon}(x) = u_s(x)$ for $|x| > 2$, the conclusion follows. □

Remark 10. Even though it might not appear so practically important, it is fairly simple to generalize the "approximate cloaking" results proven in Theorem 3 to the case when B_2, B_1 and $B_{1/2}$ are replaced by $2D$, D and $\frac{1}{2}D$, where D is a bounded, smooth, convex set containing 0. For a general D, the associated map F_ε will of course be more complicated, and no longer radial.

Remark 11. If we take the size of the scattered wave as a measure of approximate invisibility, then Theorem 3 gives a very precise estimate of the degree of "approximate invisibility" associated with

$$\textit{the approximate cloak} = \begin{cases} (F_\varepsilon)_* I,\ (F_\varepsilon)_* 1 & \textit{in } B_2 \setminus B_1, \\ (F_\varepsilon)_* I,\ (F_\varepsilon)_* \left(1 + \dfrac{i}{\omega\varepsilon\lambda}\right) & \textit{in } B_1 \setminus B_{1/2}. \end{cases}$$

For $\omega > 1/\varepsilon$ this ("norm-squared") estimate is $O(\varepsilon^{d-1})$, uniformly in $0 < \lambda < 1$. For $0 < \omega \leqq 1/\varepsilon$, the situation is a little bit different. If we select $\lambda = \omega\varepsilon$ then Theorem 3, in the case $d = 3$, asserts that

$$\frac{1}{\beta} \int_{B_\beta \setminus B_2} |u_s|^2 \leqq C\varepsilon^2 \quad \forall \beta > 2.$$

In other words it guarantees the same degree of "approximate invisibility" as for $\omega > 1/\varepsilon$. For $d = 2$ and $0 < \omega \leqq 1/\varepsilon$ the (best) choice, $\lambda = \omega\varepsilon$, gives

$$\frac{1}{\beta} \int_{B_{2\beta} \setminus B_\beta} |v_{s,\varepsilon}|^2 \leqq C\beta \frac{|H_0^{(1)}(\beta\omega)|^2}{|H_0^{(1)}(\varepsilon\omega)|^2}.$$

It is easy to see that if $\omega = \varepsilon^\gamma$, for some $\gamma > 0$, then the right-hand side is bounded from below by $c_0 > 0$ (independently of ε and $\beta > 2$) and so we have an estimate that predicts very poor "approximate invisibility".

4. Appendix: Two Optimality Results

The purpose of this appendix is to prove two optimality results related to the estimates in (b) of Theorems 1, 2 and 3. These results are a natural extension of those presented in [13] to show that a "lossy" layer is necessary for an approximate invisibility that is independent of the contents of the cloaked region. The coefficients of the Helmholtz equation are now defined as follows

$$A, \Sigma = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_1, \\ I, 1 + \frac{i}{\omega\lambda} & \text{in } B_1 \setminus B_{1/2}, \\ I, q^2/\omega^2 & \text{in } B_{1/2}, \end{cases} \tag{4.1}$$

with $0 < \omega < 1, 0 < \lambda < 1$, and $q \in \mathbb{R}$. $u_s \in H^1_{\text{loc}}(\mathbb{R}^d)$ is the "outgoing" scattered field corresponding to the incident field u_{inc}, that is, u_s satisfies the outgoing radiation condition and $u := u_s + u_{inc}$ is a solution of

$$\operatorname{div}(A\nabla u) + \omega^2 \Sigma u = 0 \quad \text{in } \mathbb{R}^d. \tag{4.2}$$

Lemma 7. *Suppose $d = 3$. There exist positive constants δ_0, c and q, such that for any $0 < \omega < 1, 0 < \lambda < 1$, with $0 < \omega/\lambda < \delta_0$,*

$$\|u_s\|_{L^2(B_4 \setminus B_1)} \geqq \frac{c\lambda}{\omega}. \tag{4.3}$$

Here u_s is the outgoing scattered field corresponding to (4.2) *with an incoming plane wave $u_{inc} = e^{i\omega\eta\cdot x}$, $\eta \in \mathbb{R}^3$, $|\eta| = 1$. The constant c is independent of ω, λ and η.*

Proof. It is well known that the plane wave $u_{inc}(x) = e^{i\omega\eta\cdot x}$ has the Jacobi-Anger expansion

$$e^{i\omega\eta\cdot x} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(\omega|x|) P_n(\cos\theta),$$

where j_n is the spherical Bessel function of order n, P_n is the n'th Legendre polynomial, and θ denotes the angle between x and the direction η. Since this expansion is orthogonal in $L^2(\sin\theta d\theta)$, and the same is true for the corresponding expansion of the solution u_s, it suffices to prove the estimate (4.3) for a single mode. In other words, it suffices consider an incident wave of the form

$$\tilde{u}_{\text{inc}} = j_0(\omega|x|),$$

the mode corresponding to $n = 0$. Let ν be in the first quadrant of the complex plan, such that $\nu^2 = \omega^2 + i\omega/\lambda$. With this we have

$$\begin{cases} u_s = \alpha h_0(\omega|x|) & \text{for } |x| > 1, \\ u_t = \gamma_1 j_0(\nu|x|) + \gamma_2 h_0(\nu|x|) & \text{for } 1/2 < |x| < 1, \\ u_t = \beta j_0(q|x|) & \text{for } |x| < 1/2, \end{cases}$$

where $u_t := u_s + \tilde{u}_{inc}$ in B_1, and $h_0 = h_0^{(1)}$ denotes the (first kind) spherical Hankel function of order 0. Due to the transmission conditions on the boundary of B_1 and $B_{1/2}$,

$$\begin{cases} u_s + \tilde{u}_{inc} = u_t & \text{at } |x| = 1, \\ \dfrac{\partial u_s}{\partial r} + \dfrac{\partial \tilde{u}_{inc}}{\partial r} = \dfrac{\partial u_t}{\partial r} & \text{at } |x| = 1, \\ u_t|_+ = u_t|_- & \text{at } |x| = 1/2, \\ \left.\dfrac{\partial u_t}{\partial r}\right|_+ = \left.\dfrac{\partial u_t}{\partial r}\right|_- & \text{at } |x| = 1/2, \end{cases}$$

and so

$$\begin{cases} \alpha h_0(\omega) + j_0(\omega) = \gamma_1 j_0(\nu) + \gamma_2 h_0(\nu), \\ \alpha\omega h_0'(\omega) + \omega j_0'(\omega) = \gamma_1 \nu j_0'(\nu) + \gamma_2 \nu h_0'(\nu), \\ \gamma_1 j_0(\nu/2) + \gamma_2 h_0(\nu/2) = \beta j_0(q/2), \\ \gamma_1 \nu j_0'(\nu/2) + \gamma_2 \nu h_0'(\nu/2) = \beta q j_0'(q/2). \end{cases} \tag{4.4}$$

From the last two equations of (4.4) it follows that

$$\gamma_2 = B\gamma_1, \tag{4.5}$$

where

$$B = -\frac{j_0(\nu/2)qj_0'(q/2) - \nu j_0'(\nu/2)j_0(q/2)}{h_0(\nu/2)qj_0'(q/2) - \nu h_0'(\nu/2)j_0(q/2)}.$$

We recall that

$$h_0(t) = \frac{e^{it}}{it}, \quad \text{and} \quad j_0(t) = \frac{\sin t}{t}, \tag{4.6}$$

and as a consequence

$$\frac{th_0'(t)}{h_0(t)} = -1 + it, \tag{4.7}$$

and

$$h_0(\nu/2)qj_0'(q/2) - \nu h_0'(\nu/2)j_0(q/2) = h_0(\nu/2)j_0(q/2)\left(q\frac{j_0'(q/2)}{j_0(q/2)} + 2 - i\nu\right).$$

Now choose q such that $\frac{qj_0'(q/2)}{j_0(q/2)} = -2$ (there exist many such q). Then

$$h_0(\nu/2)qj_0'(q/2) - \nu h_0'(\nu/2)j_0(q/2) = -i\nu h_0(\nu/2)j_0(q/2).$$

On the other hand, it follows from (4.6), with this choice of q, that

$$\begin{aligned} j_0(\nu/2)qj_0'(q/2) - \nu j_0'(\nu/2)j_0(q/2) &= \left[\frac{qj_0'(q/2)}{j_0(q/2)} - \frac{\nu j_0'(\nu/2)}{j_0(\nu/2)}\right] j_0(q/2)j_0(\nu/2) \\ &= [-2 + O(|\nu|^2)]j_0(q/2)j_0(\nu/2). \end{aligned}$$

Thus

$$\frac{1}{B} = -e^{i\nu/2}[1 + O(|\nu|^2)]. \tag{4.8}$$

We next calculate α from the first two equations of (4.4). Set

$$\tilde{\gamma}_2 = \gamma_2\left(1 + \frac{\gamma_1}{\gamma_2}\frac{j_0(\nu)}{h_0(\nu)}\right).$$

Due to (4.8),

$$\tilde{\gamma}_2 = \gamma_2(1 - ie^{-i\nu/2}[1 + O(|\nu|^2)]\sin(\nu)) = \gamma_2[1 - i\nu + O(|\nu|^2)], \tag{4.9}$$

and due to (4.5), (4.6), and (4.8),

$$\begin{aligned} 1 + \frac{\gamma_1}{\gamma_2}\frac{j_0'(\nu)}{h_0'(\nu)} &= 1 + e^{i\nu/2}[1 + O(|\nu|^2)]\frac{\sin(\nu) - \nu\cos(\nu)}{(i+\nu)e^{i\nu}} \\ &= 1 + O(|\nu|^3). \end{aligned} \tag{4.10}$$

A combination of (4.9) and (4.10) yields

$$\gamma_2\left(1 + \frac{\gamma_1}{\gamma_2}\frac{j_0'(\nu)}{h_0'(\nu)}\right) = \tilde{\gamma}_2[1 + i\nu + O(|\nu|^2)][1 + O(|\nu|^3)] = \tilde{\gamma}_2[1 + i\nu + O(|\nu|^2)].$$

The first two equations of (4.4) can therefore be written

$$\begin{cases} \alpha h_0(\omega) + j_0(\omega) = \tilde{\gamma}_2 h_0(\nu), \\ \alpha\omega h_0'(\omega) + \omega j_0'(\omega) = \tilde{\gamma}_2[1 + i\nu + O(|\nu|^2)]\nu h_0'(\nu), \end{cases}$$

which implies

$$\alpha = -\frac{j_0(\omega)[1 + i\nu + O(|\nu|^2)]\nu h_0'(\nu) - \omega j_0'(\omega) h_0(\nu)}{h_0(\omega)[1 + i\nu + O(|\nu|^2)]\nu h_0'(\nu) - \omega h_0'(\omega) h_0(\nu)}. \tag{4.11}$$

Using (4.6) and (4.7) we easily calculate

$$\begin{aligned} & j_0(\omega)[1 + i\nu + O(|\nu|^2)]\nu h_0'(\nu) - \omega j_0'(\omega) h_0(\nu) \\ & \quad = -h_0(\nu)(j_0(\omega)[1 + i\nu + O(|\nu|^2)](1 - i\nu) + \omega j_0'(\omega)) \\ & \quad = -h_0(\nu)[1 + O(|\nu|^2) + O(\omega^2)]. \end{aligned} \tag{4.12}$$

Similarly, we calculate

$$\begin{aligned} & h_0(\omega)[1 + i\nu + O(|\nu|^2)]\nu h_0'(\nu) - \omega h_0'(\omega) h_0(\nu) \\ & \quad = h_0(\omega) h_0(\nu)([1 + i\nu + O(|\nu|^2)](-1 + i\nu) + 1 - i\omega) \\ & \quad = h_0(\omega) h_0(\nu)(-i\omega + O(|\nu|^2)). \end{aligned} \tag{4.13}$$

A combination of (4.11), (4.12), (4.13) yields

$$\alpha = \frac{1 + O(|\nu|^2) + O(\omega^2)}{h_0(\omega)(-i\omega + O(|\nu|^2))}.$$

Since

$$|\nu|^2 = \frac{\omega}{\lambda}(1 + O(\omega^2)) \leqq C\frac{\omega}{\lambda}, \quad \text{and} \quad \omega \leqq \frac{\omega}{\lambda},$$

(remember: $0 < \lambda < 1$ and $0 < \omega/\lambda < \delta_0$ implies that $\omega < \omega/\lambda < \delta_0$) it follows that there exists a positive constant c, independent of ω and λ (and η) such that

$$|\alpha| \geqq \left| \frac{c\lambda}{h_0(\omega)\omega} \right| \tag{4.14}$$

for $0 < \omega/\lambda < \delta_0$ (provided δ_0 is sufficiently small). From (4.14) it follows immediately that

$$\|u_s\|_{L^2(B_4 \setminus B_1)} \geqq \frac{c\lambda}{\omega},$$

and this completes the proof of Lemma 7. □

We note that the corresponding choice $u_{inc} = J_0(\omega|x|)$ does not lead to a lower bound of the order λ/ω for dimension $d = 2$, and indeed, in this case we do not know if such a bound holds for the scattered field created by an incoming plane wave. We are, however, able to establish this lower bound for different incident fields that satisfy

$$\|u_{inc}\|_{L^\infty(K)} \leqq C_K,$$

uniformly in $0 < \omega < 1$, on any compact set $K \subset \mathbb{R}^2$.

Lemma 8. *Suppose $d = 2$, and let u_s denote the scattered field corresponding to the incident wave $u_{inc}(x) = J_2(\omega|x|)e^{2i\theta}/|J_2(\omega)|$. Here J_2 denotes the Bessel function of order 2. There exist positive constants δ_0, c and q [of (4.1)] such that for any $0 < \omega < 1$, $0 < \lambda < 1$, with $0 < \omega/\lambda < \delta_0$,*

$$\|u_s\|_{L^2(B_4 \setminus B_1)} \geqq \frac{c\lambda}{\omega}. \tag{4.15}$$

The constant c is independent of ω and λ.

Proof. Note that for $0 < \omega$ sufficiently small, $J_2(\omega)$ does not vanish, and so u_{inc} is well defined. Let $\tilde{u}_{inc}$ denote the incoming wave

$$\tilde{u}_{inc}(x) = J_2(\omega|x|)e^{2i\theta},$$

and let $\tilde{u}_s$ denote the corresponding scattered field. As in the previous proof, let ν be in the first quadrant of the complex plan, such that $\nu^2 = \omega^2 + i\omega/\lambda$. We then have

$$\begin{cases} \tilde{u}_s = \alpha H_2(\omega|x|)e^{2i\theta} & \text{for } |x| > 1, \\ \tilde{u}_t = \gamma_1 J_2(\nu|x|)e^{2i\theta} + \gamma_2 H_2(\nu|x|)e^{2i\theta} & \text{for } 1/2 < |x| < 1, \\ \tilde{u}_t = \beta J_2(q|x|)e^{2i\theta} & \text{for } |x| < 1/2, \end{cases}$$

with $\tilde{u}_t := \tilde{u}_s + \tilde{u}_i$ in B_1. Here $H_2 = H_2^{(1)}$ denotes the Hankel function (of the first kind) of order 2. Due to the transmission conditions on the boundary of B_1 and $B_{1/2}$,

$$\begin{cases} \alpha H_2(\omega) + J_2(\omega) = \gamma_1 J_2(\nu) + \gamma_2 H_2(\nu), \\ \alpha\omega H_2'(\omega) + \omega J_2'(\omega) = \gamma_1 \nu J_2'(\nu) + \gamma_2 \nu H_2'(\nu), \\ \gamma_1 J_2(\nu/2) + \gamma_2 H_2(\nu/2) = \beta J_2(q/2), \\ \gamma_1 \nu J_2'(\nu/2) + \gamma_2 \nu H_2'(\nu/2) = \beta q J_2'(q/2). \end{cases} \tag{4.16}$$

We recall that

$$\begin{cases} J_2(t) = \dfrac{t^2}{8} + O(t^4), & H_2(t) = -\dfrac{4i}{\pi t^2} + O(1), \\ J_2'(t) = \dfrac{t}{4} + O(t^3), & H_2'(t) = \dfrac{8i}{\pi t^3} + O(t^{-1}), \\ tJ_2'(t) = \dfrac{t^2}{4} + O(t^4), & tH_2'(t) = \dfrac{8i}{\pi t^2} + O(1). \end{cases} \tag{4.17}$$

From the last two equations of (4.16), we have

$$\gamma_2 = B\gamma_1,$$

where

$$B = -\frac{J_2(\nu/2)qJ_2'(q/2) - \nu J_2'(\nu/2)J_2(q/2)}{H_2(\nu/2)qJ_2'(q/2) - \nu H_2'(\nu/2)J_2(q/2)}. \tag{4.18}$$

Since

$$H_2(\nu/2)qJ_2'(q/2) - \nu H_2'(\nu/2)J_2(q/2) = \left(\frac{qJ_2'(q/2)}{J_2(q/2)} - \frac{\nu H_2'(\nu/2)}{H_2(\nu/2)}\right) J_2(q/2) \\ \times H_2(\nu/2),$$

and

$$\frac{\nu H_2'(\nu/2)}{H_2(\nu/2)} = 2\left[\frac{8i}{\pi(\nu/2)^2} + O(1)\right] \Big/ \left[-\frac{4i}{\pi(\nu/2)^2} + O(1)\right] = -4 + O(|\nu|^2),$$

it follows that

$$H_2(\nu/2)qJ_2'(q/2) - \nu H_2'(\nu/2)J_2(q/2) = \left(\frac{qJ_2'(q/2)}{J_2(q/2)} + 4 + O(|\nu|^2)\right) J_2(q/2) \\ \times H_2(\nu/2).$$

By choosing q such that $\frac{qJ_2'(q/2)}{J_2(q/2)} = -4$ (there exist many such q) we obtain

$$H_2(\nu/2)qJ_2'(q/2) - \nu H_2'(\nu/2)J_2(q/2) = O(|\nu|^2)J_2(q/2)H_2(\nu/2). \quad (4.19)$$

With this choice of q, we also have

$$\begin{aligned} J_2(\nu/2)qJ_2'(q/2) - \nu J_2'(\nu/2)J_2(q/2) &= \left(\frac{qJ_2'(q/2)}{J_2(q/2)} - \frac{\nu J_2'(\nu/2)}{J_2(\nu/2)}\right) J_2(\nu/2)J_2(q/2) \\ &= [-8 + O(|\nu|^2)]J_2(\nu/2)J_2(q/2). \quad (4.20) \end{aligned}$$

A combination of (4.18), (4.19), and (4.20) [and use of (4.17)] now gives

$$\frac{1}{B} = O(1/|\nu|^2).$$

Set

$$\tilde{\gamma}_2 = \gamma_2\left(1 + \frac{\gamma_1}{\gamma_2}\frac{J_2(\nu)}{H_2(\nu)}\right).$$

Then

$$\tilde{\gamma}_2 = \gamma_2[1 + O(|\nu|^2)],$$

and

$$\gamma_2\left(1 + \frac{\gamma_1}{\gamma_2}\frac{J_2'(\nu)}{H_2'(\nu)}\right) = \tilde{\gamma}_2[1 + O(|\nu|^2)].$$

Hence the first two equations of (4.16) can be written as follows

$$\begin{cases} \alpha H_2(\omega) + J_2(\omega) = \tilde{\gamma}_2 H_2(\nu), \\ \alpha\omega H_2'(\omega) + \omega J_2'(\omega) = \tilde{\gamma}_2[1 + O(|\nu|^2)]\nu H_2'(\nu), \end{cases}$$

and this implies

$$\alpha = -\frac{J_2(\omega)[1 + O(|\nu|^2)]\nu H_2'(\nu) - \omega J_2'(\omega) H_2(\nu)}{H_2(\omega)[1 + O(|\nu|^2)]\nu H_2'(\nu) - \omega H_2'(\omega) H_2(\nu)}. \tag{4.21}$$

Based on (4.17) we easily calculate

$$\begin{aligned} &J_2(\omega)[1 + O(|\nu|^2)]\nu H_2'(\nu) - \omega J_2'(\omega) H_2(\nu) \\ &\quad = \left([1 + O(|\nu|^2)]\frac{\nu H_2'(\nu)}{H_2(\nu)} - \frac{\omega J_2'(\omega)}{J_2(\omega)}\right) J_2(\omega) H_2(\nu) \\ &\quad = -4(1 + O(|\nu|^2 + \omega^2)) J_2(\omega) H_2(\nu), \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &H_2(\omega)[1 + O(|\nu|^2)]\nu H_2'(\nu) - \omega H_2'(\omega) H_2(\nu) \\ &\quad = \left([1 + O(|\nu|^2)]\frac{\nu H_2'(\nu)}{H_2(\nu)} - \frac{\omega H_2'(\omega)}{H_2(\omega)}\right) H_2(\omega) H_2(\nu) \\ &\quad = O(|\nu|^2 + \omega^2) H_2(\omega) H_2(\nu). \end{aligned} \tag{4.23}$$

Since

$$|\nu|^2 = \frac{\omega}{\lambda}(1 + O(\omega^2)) \leqq C\frac{\omega}{\lambda}, \quad \text{and} \quad \omega^2 \leqq \delta_0 \frac{\omega}{\lambda},$$

a combination of (4.21), (4.22), and (4.23) yields

$$|\alpha| \geqq \frac{|J_2(\omega)|}{O(\omega/\lambda)|H_2(\omega)|} \geqq c\frac{\lambda}{\omega}\frac{|J_2(\omega)|}{|H_2(\omega)|},$$

for some positive constant c. This immediately implies that

$$\begin{aligned} \|u_s\|_{L^2(B_4\setminus B_1)} &= \frac{1}{|J_2(\omega)|}\|\tilde{u}_s\|_{L^2(B_4\setminus B_1)} \\ &= \frac{|\alpha|}{|J_2(\omega)|}\|H_2(\omega|x|)\|_{L^2(B_4\setminus B_1)} \\ &\geqq c\frac{|\alpha||H_2(\omega)|}{|J_2(\omega)|} \geqq c\frac{\lambda}{\omega}, \end{aligned}$$

which completes the proof of Lemma 8. □

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