OPTIMAL CONSTANT IN A NEW ESTIMATE FOR THE DEGREE

By

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Abstract. In this paper, we prove the estimate

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N, \mathbb{S}^N),$$

for every $\delta \in (0, \ell_N)$, where $C = C(N)$ is a positive constant depending only on N and $\ell_N = \sqrt{2 + 2/(N+1)}$. We show that the constant ℓ_N in this estimate is optimal. We also present a class of maps from $\mathbb{S}^N$ into $\mathbb{S}^N$, strictly larger than $C(\mathbb{S}^N, \mathbb{S}^N)$, on which we can define the notion of degree and for which the previous inequality still holds.

1 Introduction

In [3], the authors proved that

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N, \mathbb{S}^N), \tag{1.1}$$

for each $0 < \delta < \sqrt{2}$, where $C = C(\delta, N)$ is a positive constant depending only on N and δ.

Here $|x|$ denotes Euclidean norm of x for $x \in \mathbb{R}^{N+1}$.

Estimate (1.1) was initially suggested by J. Bourgain, H. Brezis, and P. Mironescu in [1]. This was proved in [2] in the case $N = 1$ and δ sufficiently small. It is natural to ask whether (1.1) holds for every $0 < \delta < 2$. In this paper, we show that (1.1) holds if and only if $0 < \delta < \sqrt{2 + 2/(N+1)}$; moreover, the constant C in this assertion can be chosen independently of δ.

More precisely, set

$$\ell_N = \sqrt{2 + \frac{2}{N+1}}. \tag{1.2}$$

Our main result is the following

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 101 (2007)
DOI 10.1007/s11854-007-0014-0

Theorem 1. *There exists a positive constant $C = C(N)$, depending only on N, such that for every $g \in C(\mathbb{S}^N, \mathbb{S}^N)$ and every $\delta \in (0, \ell_N)$,*

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy. \tag{1.3}$$

Furthermore, there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset C(\mathbb{S}^N, \mathbb{S}^N)$ such that

$$\deg g_k = 1, \quad \forall\, k \ge 1, \tag{1.4}$$

and

$$\lim_{k\to\infty} \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g_k(x)-g_k(y)|>\ell_N} \frac{1}{|x-y|^{2N}}\,dx\,dy = 0. \tag{1.5}$$

The following corollary is a consequence of (1.3) (in fact, it is equivalent to (1.3)).

Corollary 1. *There exists a positive constant $C = C(N)$, depending only on N, such that for every $g \in C(\mathbb{S}^N, \mathbb{S}^N)$,*

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|\ge\ell_N} \frac{1}{|x-y|^{2N}}\,dx\,dy.$$

In fact, from (1.3), we have

$$|\deg g| \le C_N \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, 0<\delta<\ell_N$$

for some positive constant $C = C(N)$ depending only on N. Thus, applying Lebesgue's dominated convergence theorem, one gets

$$|\deg g| \le C \liminf_{\delta\nearrow\ell_N} \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy = C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|\ge\ell_N} \frac{1}{|x-y|^{2N}}\,dx\,dy.$$

More generally, we show that if $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ satisfies

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy < +\infty \tag{1.6}$$

for some $0<\delta<\ell_N$, then we can define the degree of g as in [5] and, moreover, assertion (1.3) holds, i.e.,

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy \tag{1.7}$$

for some positive constant $C = C(N)$.

The proof of assertion (1.3) develops the idea used in [3]. To prove it, we first follow the strategy used in [3]. Then we establish a “generalized" version of [3, Lemma 2.1] (Lemma 6). We note that [3, Lemma 2.1] plays an important role in the proof of (1.1) (see [3]) and that it only holds for $\delta < \sqrt{2}$. Lemma 6 can be seen as a “continuous" version of Corollary 4 which is a consequence of Lemma 5. We use Corollary 3 in the proof of Lemma 5. Corollary 3 is a direct consequence of Lemma 3, dealing with a nice geometric property of $\mathbb{S}^N$, and Carathéodory’s theorem (see [6]).

The idea of the construction of the sequence $(g_k)_{k\in\mathbb{N}}$ satisfying (1.4) and (1.5) is to construct continuous maps g_k ($k \in \mathbb{N}$), homotopic to the identity map, whose images are more and more concentrated on the set $\{A_i : 1 \le i \le N+2\}$, where $A_1, \dots, A_{N+2}$, which lie on $\mathbb{S}^N$, are vertices of a $(N+2)$-vertex regular polyhedron.

The plan of the paper is as follows.

Section 2 discusses a weak version of assertion (1.3) of Theorem 1. In this section, we show that there exists a positive constant $C = C(N,\delta)$, depending only on N and δ, such that

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N,\mathbb{S}^N). \tag{1.8}$$

In Section 3, we construct a sequence $\{g_k\}_{k\in\mathbb{N}} \subset C(\mathbb{S}^N,\mathbb{S}^N)$ satisfying (1.4) and (1.5).

In Section 4, we prove that the constant C $(= C(N,\delta))$ in (1.8) can be chosen independently of δ. This completes the proof of Theorem 1.

Finally, in Section 5, we show that if $g \in L^\infty(\mathbb{S}^N,\mathbb{S}^N)$ satisfies (1.6) for some $0 < \delta < \ell_N$, then $\deg g$ is well-defined in a manner similar to the one as in [5]. Inequality (1.7) is also proved in this section.

2 Proof of assertion (1.8)

2.1 A useful lemma. We begin this section with the following lemma, whose consequence (Corollary 2) is useful in the proof of (1.8).

Lemma 1. *Let $g \in L^\infty(\mathbb{S}^N,\mathbb{S}^N)$, $\mathcal{D}$ be a measurable subset of $\mathbb{S}^N$ with $|\mathcal{D}| > 0$, and $\lambda : \mathcal{D} \to [0,+\infty)$ be a measurable function. Assume that there exists a constant $\beta \ge 1$ such that $1/\beta \le \lambda(s) \le \beta$ for every $s \in \mathcal{D}$ and*

$$\left| \fint_{\mathcal{D}} \lambda(s) g(s)\,ds \right| \le \frac{1}{8\beta(N+2)} \left[\ell_N^2 - \left(\frac{\ell_N + \delta}{2} \right)^2 \right].$$

Then

$$\operatorname{meas}\left(\{(\xi,\eta) \in \mathcal{D}\times\mathcal{D} :\, |g(\xi)-g(\eta)| > \delta\}\right) \geq C|\mathcal{D}|^2 \tag{2.1}$$

for some $C = C(\delta, \beta, N)$.

Henceforth, $|\mathcal{D}|$ denotes the Lebesgue measure of $\mathcal{D}$ on $\mathbb{S}^N$ for any $\mathcal{D}$ measurable subset of $\mathbb{S}^N$.

Proof. Set

$$\begin{cases} \alpha = \frac{1}{8\beta(N+2)}\left[\ell_N^2 - \left(\frac{\ell_N+\delta}{2}\right)^2\right], \\ \varepsilon_1 = \frac{1}{16\beta^2(N+2)}\left[\ell_N^2 - \left(\frac{\ell_N+\delta}{2}\right)^2\right], \\ \varepsilon = \frac{1}{16(N+2)}\left[\ell_N^2 - \left(\frac{\ell_N+\delta}{2}\right)^2\right]. \end{cases}$$

Then

$$\left| \fint_{\mathcal{D}} g(s)\, ds \right| \leq \alpha \tag{2.2}$$

and

$$\varepsilon = \beta^2 \varepsilon_1 = \beta\alpha/2. \tag{2.3}$$

Let $\Omega_1, \dots, \Omega_{k_1}$, be measurable subsets of $\mathbb{S}^N$ such that $\mathbb{S}^N = \bigcup_{i=1}^{k_1} \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$ for every $1 \leq i \neq j \leq k_1$, and $\operatorname{diam}(\Omega_i) \leq \varepsilon_1/2$ for every $1 \leq i \leq k_1$. Set

$$\mathcal{D}_i = g^{-1}(\Omega_i) \cap \mathcal{D}, \quad \forall\, 1 \leq i \leq k_1. \tag{2.4}$$

Define

$$J = \left\{ 1 \leq i \leq k_1 :\, |\mathcal{D}_i| \geq \frac{1}{m_1}|\mathcal{D}| \right\}, \tag{2.5}$$

where

$$m_1 = 32(N+2)\beta^2 k_1 \Bigg/ \left[\ell_N^2 - \left(\frac{\ell_N+\delta}{2}\right)^2\right]. \tag{2.6}$$

Without loss of generality, we assume that $J = \{1, \dots, k\}$ for some $k \leq k_1$. Since $\{\mathcal{D}_i\}_{i=1}^{k_1}$ is a partition of $\mathcal{D}$,

$$\fint_{\mathcal{D}} \lambda(s) g(s)\, ds = \frac{1}{|\mathcal{D}|}\sum_{i=1}^{k} \int_{\mathcal{D}_i} \lambda(s) g(s)\, ds + \frac{1}{|\mathcal{D}|}\sum_{i=k+1}^{k_1} \int_{\mathcal{D}_i} \lambda(s) g(s)\, ds.$$

This implies

$$\left|\frac{1}{|\mathcal{D}|}\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)g(s)\,ds\right| \leq \left|⨍_{\mathcal{D}}\lambda(s)g(s)\,ds\right| + \left|\frac{1}{|\mathcal{D}|}\sum_{i=k+1}^{k_1}\int_{\mathcal{D}_i}\lambda(s)g(s)\,ds\right|.$$

Thus, since $i \notin J$ for all $k < i \leq k_1$ and $|\lambda(s)| \leq \beta$ for $s \in \mathcal{D}$, it follows from (2.2) and (2.5) that

$$\left|\frac{1}{|\mathcal{D}|}\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)g(s)\,ds\right| \leq \alpha + \beta k_1/m_1. \tag{2.7}$$

For each i, fix $A_i \in \Omega_i$. Then

$$\left|\frac{1}{|\mathcal{D}|}\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)\,ds\,A_i\right| \leq \left|\frac{1}{|\mathcal{D}|}\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)\left(A_i - g(s)\right)\,ds\right| + \left|\frac{1}{|\mathcal{D}|}\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)g(s)\,ds\right|. \tag{2.8}$$

Since $\operatorname{diam}(\Omega_i) \leq \varepsilon_1/2$ and $A_i \in \Omega_i$, it follows from (2.4) that $|A_i - g(s)| \leq \varepsilon_1$ for all $s \in \mathcal{D}_i$. Thus, since $|\lambda(s)| \leq \beta$ for all $s \in \mathcal{D}$, we deduce from (2.7) and (2.8) that

$$\left|\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)\,ds\,A_i\right| \leq (\beta\varepsilon_1 + \alpha + \beta k_1/m_1)|\mathcal{D}|,$$

which shows that (since $\lambda(s) \geq 1/\beta$ for all $s \in \mathcal{D}$)

$$\left|\sum_{i=1}^{k}c_iA_i\right| \leq (\beta\varepsilon_1 + \alpha + \beta k_1/m_1)\frac{\beta|\mathcal{D}|}{\sum_{i=1}^{k}|\mathcal{D}_i|},$$

where

$$c_i = \left(\int_{\mathcal{D}_i}\lambda(s)\,ds\right) \Bigg/ \left(\sum_{i=1}^{k}\int_{\mathcal{D}_i}\lambda(s)\,ds\right).$$

Consequently,

$$\left|\sum_{i=1}^{k}c_iA_i\right| \leq \frac{\beta(\beta\varepsilon_1 + \alpha + \beta k_1/m_1)}{1 - k_1/m_1}.$$

On the other hand, since $\varepsilon \leq 1/4$ and $\beta \geq 1$, it follows from (2.3) and (2.6) that

$$\frac{\beta(\beta\varepsilon_1 + \alpha + \beta k_1/m_1)}{1 - k_1/m_1} = \frac{\varepsilon + 2\varepsilon + \varepsilon/2}{1 - \varepsilon/(2\beta^2)} \leq \frac{\varepsilon + 2\varepsilon + \varepsilon/2}{1 - 1/8} = 4\varepsilon,$$

which yields

$$\left|\sum_{i=1}^{k}c_iA_i\right| \leq 4\varepsilon.$$

Set

$$A = \sum_{i=1}^{k} c_i A_i.$$

Then

$$|A| \le 4\varepsilon = \frac{1}{4(N+2)} \left[\ell_N^2 - \left(\frac{\ell_N + \delta}{2} \right)^2 \right]. \tag{2.9}$$

By Carathéodory's Theorem (see [6]), there exists $I \subset J = \{1, \dots, k\}$ such that $\mathrm{card}(I) \le N + 2$ and A is a convex combination of $\{A_i : i \in I\}$. We claim that there exist $i, j \in I$ such that

$$|A_i - A_j| \ge (\delta + \ell_N)/2. \tag{2.10}$$

Without loss of generality, we assume that

$$I = \{1, \dots, N+2\}.$$

Then take $\{d_i\}_{i=1}^{N+2}$ such that $d_i \ge 0$ for all $1 \le i \le N+2$, $\sum_{i=1}^{N+2} d_i = 1$, and

$$A = \sum_{i=1}^{N+2} d_i\, A_i.$$

It is possible to achieve this, since A is a convex combination of $\{A_i : 1 \le i \le N+2\}$. For notational ease, assume as well that $d_1 = \max\{d_i :\ i \in I\}$ and $A_1 = (1, 0, \dots, 0)$. Then since $\sum_{i=1}^{N+2} d_i = 1$,

$$d_1 \ge \frac{1}{N+2} \quad \text{and} \quad \sum_{i=2}^{N+2} d_i \le \frac{N+1}{N+2}. \tag{2.11}$$

Set

$$\gamma = \min\{\pi_1(A_i) :\ 2 \le i \le N+2\}, \tag{2.12}$$

where $\pi_1(.)$ denotes the first component of a point in $\mathbb{R}^{N+1}$. Since $|A| \le 1/(N+2)$ (see (2.9)) and $d_1 \ge 1/(N+2)$ (see (2.11)), it follows from (2.12) that

$$\gamma \le 0. \tag{2.13}$$

Thus, since

$$|\pi_1(A)| \ge \sum_{i=1}^{N+2} d_i \pi_1(A_i) \ge d_1 + \gamma \sum_{i=2}^{N+2} d_i,$$

it follows from (2.9), (2.11) and (2.13) that

$$\frac{1}{4(N+2)}\left[\ell_N^2-\left(\frac{\ell_N+\delta}{2}\right)^2\right]\geq\frac{1}{N+2}+\frac{N+1}{N+2}\gamma.$$

A simple computation yields

$$\gamma\leq\frac{1}{4}\left[\ell_N^2-\left(\frac{\ell_N+\delta}{2}\right)^2\right]-\frac{1}{N+1}. \tag{2.14}$$

Take $i_0\in\{1,\dots,N+2\}$ such that $\pi_1(A_{i_0})=\gamma$. Then

$$|A_1-A_{i_0}|^2=(1-\gamma)^2+(1-\gamma^2)=2-2\gamma. \tag{2.15}$$

On the other hand, from (2.14) and the definition of ℓ_N (see (1.2)),

$$2-2\gamma\geq 2+\frac{2}{N+1}-\frac{1}{2}\left[\ell_N^2-\left(\frac{\ell_N+\delta}{2}\right)^2\right]>\ell_N^2-\left[\ell_N^2-\left(\frac{\ell_N+\delta}{2}\right)^2\right].$$

Thus

$$2-2\gamma>\left(\frac{\delta+\ell_N}{2}\right)^2. \tag{2.16}$$

Combining (2.15) and (2.16) yields

$$|A_1-A_{i_0}|>\frac{\delta+\ell_N}{2}.$$

Since $\beta\geq 1$, it follows from the definition of ε_1 that

$$\varepsilon_1\leq\frac{1}{16(N+2)}\left[\ell_N^2-\left(\frac{\ell_N+\delta}{2}\right)^2\right]=\frac{1}{16(N+2)}\frac{(\ell_N-\delta)(3\ell_N+\delta)}{4}.$$

Thus

$$\varepsilon_1<\frac{\ell_N-\delta}{2},$$

which shows that (since $A_i\in\Omega_i$, $\operatorname{diam}(\Omega_i)\leq\varepsilon_1/2$ and $|A_1-A_{i_0}|>\frac{\delta+\ell_N}{2}$)

$$\begin{aligned}|x-y|&\geq|A_1-A_{i_0}|-|x-A_1|-|y-A_{i_0}|\\&>\frac{\delta+\ell_N}{2}-\varepsilon_1\\&>\frac{\delta+\ell_N}{2}-\frac{\ell_N-\delta}{2}=\delta,\quad\forall\,(x,y)\in\Omega_1\times\Omega_{i_0}.\end{aligned}$$

Therefore, from the construction of $\mathcal{D}_1$ and $\mathcal{D}_{i_0}$,

$$\operatorname{meas}\left(\{(\xi,\eta)\in\mathcal{D}\times\mathcal{D}:\ |g(\xi)-g(\eta)|>\delta\}\right)\geq\operatorname{meas}(\mathcal{D}_1\times\mathcal{D}_{i_0})\geq C|\mathcal{D}|^2. \qquad\square$$

Remark 1. Estimate (2.10) can be seen as a perturbation of the result obtained in Lemma 2. A sharper result is proved in Lemma 3.

The following consequence of Lemma 1 is used in the proof of assertion (1.8).

Corollary 2. *Let $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$, $x \in \mathbb{S}^N$, and $r \in (0, 2)$. Assume that*

$$\left| \fint_{B(x,r)} g(s)\, ds \right| \le \frac{1}{8(N+2)} \left[\ell_N^2 - \left(\frac{\ell_N + \delta}{2} \right)^2 \right],$$

where $B(x, r) = \{y \in \mathbb{S}^N : |y - x| \le r\}$.
Then

$$\text{meas}\left(\left\{ (\xi, \eta) \in [B(x,r)]^2 : |g(\xi) - g(\eta)| > \delta, \, |\xi - \eta| \ge \tau r \right\} \right) \ge C r^{2N} \tag{2.17}$$

for some positive constants $C = C(\delta, N)$ and $\tau = \tau(\delta, N)$.

Proof. Applying Lemma 1 with $\mathcal{D} = B(x,r)$ and $\beta = 1$, one gets

$$\text{meas}\left(\left\{ (\xi, \eta) \in [B(x,r)]^2 : |g(\xi) - g(\eta)| > \delta \right\} \right) \ge C r^{2N}, \tag{2.18}$$

where $C = C(\delta, N) > 0$.

It is easy to see that

$$\text{meas}\left(\left\{ (\xi, \eta) \in [B(x,r)]^2 : |\xi - \eta| \le \tau r \right\} \right) \le C_N \tau^N r^{2N}, \quad \forall \tau > 0. \tag{2.19}$$

Choose τ such that $C_N \tau^N = C/2$. Then it follows from (2.18) and (2.19) that

$$\text{meas}\left(\left\{ (\xi, \eta) \in [B(x,r)]^2 : |g(\xi) - g(\eta)| > \delta, \, |\xi - \eta| \ge \tau r \right\} \right) \ge (C/2) r^{2N}. \qquad \square$$

2.2 Proof of assertion (1.8).

Step 1. Proof of (1.3) when $g \in \text{Lip}(\mathbb{S}^N, \mathbb{S}^N)$.

As in [3], consider the function $u : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B} = \left\{ X \in \mathbb{R}^{N+1} : |X| \le 1 \right\}$, defined by

$$u(X) = \fint_{B(x,r)} g(s)\, ds, \quad \text{when } X \neq 0, \tag{2.20}$$

where $x = \dfrac{X}{|X|}$, $r = 2(1 - |X|)$, and

$$u(0) = \fint_{\mathbb{S}^N} g(s)\, ds.$$

We recall that $B(x, r) = \left\{ y \in \mathbb{S}^N : |y - x| \le r \right\}$.

For each $x \in \mathbb{S}^N$, let $\rho(x)$ be the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|u| > \alpha$ (possibly $\rho(x) = 1$), where $\alpha = \frac{1}{8(N+2)}\left[\ell_N^2 - \left(\frac{\ell_N+\delta}{2}\right)^2\right]$.

In [3], we proved that

$$|\deg g| \le \frac{C}{\alpha^{N+1}|\mathcal{B}|} \int_{\substack{\mathbb{S}^N \\ \rho(x)<1}} \frac{dx}{\rho(x)^N}, \tag{2.21}$$

where $C = C(N)$ is a constant depending only on N.

Take $x \in \mathbb{S}^N$ such that $\rho(x) < 1$. Then

$$\left| \fint_{B(x,2\rho(x))} g(s)\, ds \right| = \alpha.$$

Thus, by Corollary 2, there exists $\tau = \tau(\delta, N) > 0$ such that

$$\iint_{\substack{[B(x,2\rho(x))]^2 \\ |g(\xi)-g(\eta)|>\delta \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta \gtrsim 1.$$

Here and below, the notation $a \lesssim b$ means that there exists a constant c depending only on N and δ such that $a \le cb$. The notation $a \gtrsim b$ means that $b \lesssim a$ and the notation $a \simeq b$ means that $a \lesssim b$ and $b \lesssim a$.

Hence

$$\int_{\substack{\mathbb{S}^N \\ \rho(x)<1}} \frac{1}{\rho(x)^N}\, dx \lesssim \int_{\mathbb{S}^N} \frac{1}{\rho(x)^N} \iint_{\substack{[B(x,2\rho(x))]^2 \\ |g(\xi)-g(\eta)|>\delta \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta\, dx. \tag{2.22}$$

On the other hand, using Fubini's theorem, one gets

$$\int_{\mathbb{S}^N} \frac{1}{\rho(x)^N} \iint_{\substack{[B(x,2\rho(x))]^2 \\ |g(\xi)-g(\eta)|>\delta \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta\, dx = \tag{2.23}$$

$$\int_{\mathbb{S}^N} \int_{\substack{\mathbb{S}^N \\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}} \int_{\substack{\mathbb{S}^N \\ |x-\xi|\le 2\rho(x) \\ |x-\eta|\le 2\rho(x) \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{\rho(x)^N}\, dx\, d\xi\, d\eta.$$

However, if $|x-\xi| \le 2\rho(x)$, $|x-\eta| \le 2\rho(x)$ and $|\xi-\eta| \ge \tau\rho(x)$, then $|x-\xi| \le 2|\xi-\eta|/\tau$ and $|\xi-\eta| \le |x-\xi| + |x-\eta| \le 4\rho(x)$.

Thus

$$\int_{\substack{\mathbb{S}^N \\ |x-\xi|\le 2\rho(x) \\ |x-\eta|\le 2\rho(x) \\ |\xi-\eta|\ge \tau\rho(x)}} \frac{1}{\rho(x)^N}\,dx \le \int_{\substack{\mathbb{S}^N \\ |x-\xi|\le 2|\xi-\eta|/\tau \\ |\xi-\eta|\le 4\rho(x)}} \frac{1}{\rho(x)^N}\,dx \lesssim 1. \tag{2.24}$$

Combining (2.22), (2.23) and (2.24) yields

$$\int_{\substack{\mathbb{S}^N \\ \rho(x)<1}} \frac{1}{\rho(x)^N}\,dx \lesssim \int_{\mathbb{S}^N}\int_{\substack{\mathbb{S}^N \\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}}\,d\xi\,d\eta. \tag{2.25}$$

Therefore, (1.8) follows from (2.21) and (2.25).

Step 2. Proof of (1.8) when g is only a continuous function from $\mathbb{S}^N$ to $\mathbb{S}^N$.

The proof is the same as that of Step 2 in [3, Theorem 1.1].

3 Construction of a sequence $\{g_k\}_{k\in\mathbb{N}}$ satisfying (1.4) and (1.5).

We first recall that there exists a regular polyhedron Q_{N+2} which has $N+2$ vertices lying on $\mathbb{S}^N$ and each of its edges has length equal to $\ell_N = \sqrt{2+2/(N+1)}$.

Let A_1, A_2, ..., A_{N+2} be the vertices of one of these polyhedrons. Set

$$F = \bigcup_{i=1}^{N+2} i^{\text{th}}\text{-face } A_1\cdots A_{i-1}A_{i+1}\cdots A_{N+2},$$

and

$$P : F \to \mathbb{S}^N$$
$$X \mapsto \frac{X}{|X|},$$

where the i^{th}-face $A_1\cdots A_{i-1}A_{i+1}\cdots A_{N+2}$ is, by definition, the set of all the convex combinations of the points $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{N+2}$. Then P is a one-to-one continuous mapping and its inverse mapping is continuous.

Set

$$D_i = -A_i, \quad \forall\, 1 \le i \le N+2,$$

and

$$\Omega_i = \left\{P(X) :\ X \in i^{\text{th}}\text{-face } A_1\cdots A_{i-1}A_{i+1}\cdots A_{N+2}\right\}.$$

Then

$$|D_i - D_j| = \ell_N, \quad \forall\, 1 \le i \ne j \le N+2,$$

and

$$|x - D_i| \le \sqrt{2}, \quad \forall\, x \in \Omega_i,\ \forall\, 1 \le i \le N+2. \tag{3.1}$$

Moreover,

$$\begin{cases} \mathbb{S}^N = \bigcup_{i=1}^{N+2} \Omega_i, \\ |\partial\Omega_i| = 0, \forall\, 1 \le i \le N+2. \end{cases}$$

Here $\partial\Omega_i$ denotes the boundary of Ω_i corresponding to the usual topology on $\mathbb{S}^N$.

For each $x \in \partial\Omega_i$, set $\gamma_x^i = P \circ \hat{\gamma}_X^i : [0,1] \to \Omega_i$, where $X = P^{-1}(x)$ and $\hat{\gamma}_X^i$ is the line connecting X and $P^{-1}(D_i)$, i.e., $\hat{\gamma}_X^i(t) = (1-t)X + tP^{-1}(D_i)$ for all $t \in [0,1]$.

Take

$$n = 2N + 1.$$

For each $k \ge 1$, set $\varepsilon_k = \dfrac{1}{k+1}$ and define g_k as follows:

$$g_k(\gamma_x^i(t)) = \begin{cases} x & \text{if } 0 \le t \le \varepsilon_k, \\ \gamma_x^i\left((t-\varepsilon_k)/\varepsilon_k^n\right) & \text{if } \varepsilon_k < t \le \varepsilon_k + \varepsilon_k^n, \\ D_i & \text{if } \varepsilon_k + \varepsilon_k^n < t \le 1, \end{cases} \tag{3.2}$$

for all $x \in \partial\Omega_i$, for all $1 \le i \le N+2$. Then $g_k : \mathbb{S}^N \to \mathbb{S}^N$ is a continuous mapping and is homotopic to the identity:

$$\mathcal{I} : \mathbb{S}^N \to \mathbb{S}^N.$$

Thus

$$\deg g_k = 1. \tag{3.3}$$

Set (for notational ease)

$$\begin{cases} \Omega_i^1 = \left\{\gamma_x^i(t) : x \in \partial\Omega_i,\ \varepsilon_k + \varepsilon_k^n < t \le 1\right\}, \\ \Omega_i^2 = \left\{\gamma_x^i(t) : x \in \partial\Omega_i,\ \varepsilon_k < t \le \varepsilon_k + \varepsilon_k^n\right\}, \\ \Omega_i^3 = \left\{\gamma_x^i(t) : x \in \partial\Omega_i,\ 0 \le t \le \varepsilon_k\right\}, \end{cases} \qquad \forall\, 1 \le i \le N+2.$$

Step 1. Estimate of

$$\iint\limits_{\substack{\Omega_i \times \Omega_i \\ |g_k(x) - g_k(y)| > \ell_N}} \frac{1}{|x-y|^{2N}}\, dx\, dy,$$

for $1 \le i \le N+2$.

Since $g_k(x) = D_i$ for all $x \in \Omega_i^1$, it follows from (3.1) that

$$|g_k(x) - g_k(y)| < \sqrt{2}, \quad \forall\, (x,y) \in \Omega_i^1 \times \Omega_i.$$

This implies

$$\iint\limits_{\substack{\Omega_i^1 \times \Omega_i \\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\, dx\, dy = 0.$$

Thus, since $\{\Omega_i^l\}_{1 \le l \le 3}$ is a partition of Ω_i,

$$\iint\limits_{\substack{\Omega_i \times \Omega_i \\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\, dx\, dy = \iint\limits_{\substack{(\Omega_i^2 \cup \Omega_i^3) \times (\Omega_i^2 \cup \Omega_i^3) \\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\, dx\, dy. \tag{3.4}$$

Since $\gamma_\xi^i(t)$ is continuous with respect to ξ and t for all $(\xi, t) \in \partial\Omega_i \times [0,1]$ and diam $\left(\left\{\gamma_\xi^i(t) : t \in [0,1]\right\}\right) < \sqrt{2} < \ell_N$, there exists a constant $\delta(N)$ such that

$$\text{if } |\xi - \eta| < \delta(N), \text{ then } |\gamma_\xi^i(t) - \gamma_\eta^i(s)| < \ell_N, \quad \forall\, (\xi, \eta) \in \partial\Omega_i,\ \forall\, s,t \in [0,1]. \tag{3.5}$$

Define the mapping G_i as

$$\begin{aligned} G_i :\ \Omega_i \setminus \{D_i\} &\longrightarrow \partial\Omega_i \\ x &\longmapsto G_i(x), \end{aligned}$$

where $G_i(x) \in \partial\Omega_i$ is the point such that $P^{-1}(x)$ is a convex combination of $P^{-1}(G_i(x))$ and $P^{-1}(D_i)$. Then, since G_i is a continuous map on $\Omega_i \setminus \{D_i\}$, there exists a constant C such that

$$\text{if } |x-y| < C, \text{ then } |G_i(x) - G_i(y)| < \delta(N), \quad \forall\, (x,y) \in (\Omega_i^2 \cup \Omega_i^3) \times (\Omega_i^2 \cup \Omega_i^3). \tag{3.6}$$

Henceforth in this proof, C denotes a positive constant depending only on N.

On the other hand, for each $(x,y) \in (\Omega_i^2 \cup \Omega_i^3) \times (\Omega_i^2 \cup \Omega_i^3)$, it follows from the definition of g_k that there exist some $s, t \in [0,1]$ such that

$$g_k(x) = \gamma_{G_i(x)}(t) \quad \text{and} \quad g_k(y) = \gamma_{G_i(y)}(s). \tag{3.7}$$

Hence, combining (3.5)–(3.7) yields that $|g_k(x) - g_k(y)| \le \ell_N$ whenever $|x-y| \le C$ and x, $y \in \Omega_i^2 \cup \Omega_i^3$.

Thus, since $|\Omega_i^2 \cup \Omega_i^3| \lesssim \varepsilon_k$, one deduces that

$$\iint\limits_{\substack{(\Omega_i^2 \cup \Omega_i^3) \times (\Omega_i^2 \cup \Omega_i^3) \\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\, dx\, dy = \iint\limits_{\substack{(\Omega_i^2 \cup \Omega_i^3) \times (\Omega_i^2 \cup \Omega_i^3) \\ |g_k(x)-g_k(y)|>\ell_N \\ |x-y| \ge C}} \frac{1}{|x-y|^{2N}}\, dx\, dy \lesssim \varepsilon_k^2. \tag{3.8}$$

Combining (3.4) and (3.8) yields

$$\iint\limits_{\substack{\Omega_i\times\Omega_i\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \lesssim \varepsilon_k^2. \tag{3.9}$$

Step 2. Estimate of

$$\iint\limits_{\substack{\Omega_i\times\Omega_j\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy,$$

for $1 \le i \ne j \le N+2$.

Since $|D_i - D_j| = \ell_N$, $g_k(x) = D_i$ and $g_k(y) = D_j$ for all $(x,y) \in \Omega_i^1 \times \Omega_j^1$,

$$\iint\limits_{\substack{\Omega_i^1\times\Omega_j^1\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy = 0. \tag{3.10}$$

On the other hand, since $|\Omega_i^2| \lesssim \varepsilon_k^n$ and $|x-y| \gtrsim \varepsilon_k$ for all $(x,y) \in \Omega_i^2 \times \Omega_j$, we infer that

$$\iint\limits_{\substack{\Omega_i^2\times\Omega_j\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \lesssim \varepsilon_k^{n-2N} = \varepsilon_k. \tag{3.11}$$

Set $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Then

$$|z - D_i| < \sqrt{2}, \quad \forall\, z \in \Gamma_{ij}.$$

Hence $|z - D_i| < \ell_N$ for all $z \in \Omega_j$ such that $\mathrm{dist}(z, \Gamma_{ij}) \le C$.
This implies (since $|\Omega_i^3| \lesssim \varepsilon_k$)

$$\iint\limits_{\substack{\Omega_i^1\times\Omega_j^3\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \lesssim \varepsilon_k. \tag{3.12}$$

Combining (3.10)–(3.12) yields

$$\iint\limits_{\substack{\Omega_i\times\Omega_j\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \lesssim \varepsilon_k. \tag{3.13}$$

Thus it follows from (3.9) and (3.13) that

$$\iint\limits_{\substack{\mathbb{S}^N\times\mathbb{S}^N\\ |g_k(x)-g_k(y)|>\ell_N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \lesssim \varepsilon_k. \tag{3.14}$$

Therefore, from (3.3) and (3.14), the sequence $\{g_k\}_{k\in N}$ satisfies (1.4) and (1.5). $\square$

Remark 2. Given any $\varphi \in C(\mathbb{R}\setminus\{0\})$, the estimate

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}_{|g(x)-g(y)|>\ell_N} \varphi(|x-y|)\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N, \mathbb{S}^N), \tag{3.15}$$

where C is a positive constant independent of g, fails. To see this, one can use the same construction of a sequence $\{g_k\}_{k\in N}$ as above with an appropriate choice of n.

Remark 3. Theorem 1 shows that the condition $\delta < \sqrt{2}$ is optimal, if we want a condition independent of N (see [3]).

Remark 4. In [4] H. Brezis raised the following interesting question related to the behavior of the constant $C = C(\delta, N)$ as δ goes to 0, in the estimate (1.8). Does there exist a positive constant $C = C(N)$ such that

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^N}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N, \mathbb{S}^N),\ \forall\, 0<\delta<1?$$

4 Proof of assertion (1.3)

In this section, we prove that the constant $C = C(\delta, N)$ in (1.8) can be chosen independently of δ. This completes the proof of Theorem 1. We first prove some preliminary results.

4.1 Preliminaries.

We begin this section with the following lemma, which deals with a nice geometric property of $\mathbb{S}^N$. The idea of the proof has already appeared in the proof of Lemma 1. In this section, O denotes the point $(0,\dots,0) \in \mathbb{R}^{N+1}$ and conv(.) denotes the convex hull of a subset of $\mathbb{R}^{N+1}$.

Lemma 2. *Let $A_i \in \mathbb{S}^N$, $1 \le i \le N+2$, be such that O is a convex combination of $\{A_i\}_{i=1}^{N+2}$. Then there exist $1 \le i,j \le N+2$ such that*

$$|A_i - A_j| \ge \ell_N.$$

Proof. Since O is a convex combination of A_i, there exist $\{d_i\}_{i=1}^{N+2}$ such that $d_i \ge 0$, for all $1 \le i \le N+2$, $\sum_{i=1}^{N+2} d_i = 1$, and

$$O = \sum_{i=1}^{N+2} d_i\, A_i. \tag{4.1}$$

Without loss of generality, we assume that $d_1 = \max\{d_i : 1 \le i \le N+2\}$ and $A_1 = (1, 0, \ldots, 0)$. Then, since $\sum_{i=1}^{N+2} d_i = 1$,

$$d_1 \ge \frac{1}{N+2} \quad \text{and} \quad \sum_{i=2}^{N+2} d_i \le \frac{N+1}{N+2}. \tag{4.2}$$

Set

$$\gamma = \min\{\pi_1(A_i) : 2 \le i \le N+2\},$$

where as before $\pi_1(.)$ denotes the first component of a point in $\mathbb{R}^{N+1}$. Then it is easy to see that

$$\gamma \le 0. \tag{4.3}$$

Thus, since

$$0 = \sum_{i=1}^{N+2} d_i \pi_1(A_i) \ge d_1 + \gamma \sum_{i=2}^{N+2} d_i,$$

it follows from (4.1), (4.2) and (4.3) that

$$0 \ge \frac{1}{N+2} + \frac{N+1}{N+2}\gamma.$$

A simple computation yields

$$\gamma \le -\frac{1}{N+1}.$$

Take $i_0 \in \{1, \ldots, N+2\}$ such that $\pi_1(A_{i_0}) = \gamma$. Then

$$|A_1 - A_{i_0}|^2 = (1-\gamma)^2 + (1-\gamma^2) = 2 - 2\gamma \ge 2 + \frac{2}{N+1}.$$

Therefore, $|A_1 - A_{i_0}| \ge \ell_N$ □

The following result is an important improvement of Lemma 2, which is used in the proof of Lemma 5.

Lemma 3. *Let $A_i \in \mathbb{S}^N$, $1 \le i \le N+2$. Assume that there exists a convex combination A of $\{A_i\}_{i=1}^{N+2}$ such that $|A| \le \frac{1}{N+1}$. Then there exist $1 \le i, j \le N+2$, such that*

$$|A_i - A_j| \ge \ell_N.$$

Proof. Set

$$\mathbf{K} = \operatorname{conv}\left(\{A_i : 1 \le i \le N+2\}\right).$$

Then $\mathbf{K}$ is a convex, closed, and non-empty subset of $\mathbb{R}^{N+1}$.

If $O \in \mathbf{K}$, then applying Lemma 2, one has

$$|A_i - A_j| \geq \ell_N,$$

for some $1 \leq i, j \leq N+2$.

We now suppose that $O \notin \mathbf{K}$. Let H be the projection of O onto $\mathbf{K}$, i.e.,

$$\frac{1}{2}|H|^2 = \frac{1}{2}\min_{X \in \mathbf{K}} |X|^2.$$

Let $\mathbf{P}$ be the hyperplane containing H such that OH is orthogonal to $\mathbf{P}$. Set

$$\mathbb{S} = \mathbf{P} \cap \mathbb{S}^N.$$

Then H is the center of $\mathbb{S}$; moreover, H is also a convex combination of $\{A_i : 1 \leq i \leq N+2,\ A_i \in \mathbf{P}\}$. Applying Lemma 2, one gets

$$|A_i - A_j|^2 \geq (1 - |H|^2)(2 + 2/N).$$

for some $1 \leq i, j \leq N+2$. On the other hand,

$$|H| \leq |A| \leq 1/(N+1).$$

Thus

$$|A_i - A_j|^2 \geq (2 + 2/(N+1)),$$

which shows that

$$|A_i - A_j| \geq \ell_N,$$

for some $1 \leq i, j \leq N+2$. $\square$

Remark 5. It is easy to see that the condition $|A| \leq \frac{1}{N+1}$ in Lemma 3 is optimal.

The following result is a consequence of Lemma 3 and Carathéodory's Theorem (see [6]).

Corollary 3. *Let I be a nonempty set and $A_i \in \mathbb{S}^N$, for all $i \in I$. Assume that there exists $A \in \operatorname{conv}(\{A_i :\ i \in I\})$, such that $|A| \leq \frac{1}{N+1}$. Then*

$$|A_i - A_j| \geq \ell_N$$

for some $i, j \in I$.

Proof. Since $A \in \operatorname{conv}(\{A_i;\, i \in I\})$, by Carathéodory's Theorem, there exist $\{i_m\}_{m=1}^{N+2} \subset I$ such that A is a convex combination of $\{A_{i_m}\}_{m=1}^{N+2}$. Thus, applying Lemma 2, one has

$$|A_{i_l} - A_{i_m}| \geq \ell_N,$$

for some $1 \leq l, m \leq N+2$. □

Another geometric property of $\mathbb{S}^N$ is given in the following lemma

Lemma 4. *Let* $A_i \in \mathbb{S}^N$, $1 \leq i \leq N+2$. *Then there exist* $1 \leq i \neq j \leq N+2$ *such that*

$$|A_i - A_j| \leq \ell_N.$$

Proof. We prove this by contradiction. Suppose that

$$|A_i - A_j| > \ell_N, \quad \forall\, 1 \leq i \neq j \leq N+2.$$

Then, from (1.2),

$$|A_i|^2 - 2A_i.A_j + |A_j|^2 > 2 + \frac{2}{N+1}, \quad \forall\, 1 \leq i \neq j \leq N+2,$$

where $A_i.A_j$ denotes the scalar product of A_i and A_j. Consequently,

$$A_i.A_j < -\frac{1}{N+1}, \quad \forall\, 1 \leq i \neq j \leq N+2.$$

This implies, for all $1 \leq i \leq N+2$,

$$\sum_{j=1}^{N+2} A_i.A_j = \sum_{\substack{j=1 \\ j \neq i}}^{N+2} A_i.A_j + |A_i|^2 < -\frac{N+1}{N+1} + 1 = 0.$$

Thus

$$\left| \sum_{i=1}^{N+2} A_i \right|^2 = \sum_{i=1}^{N+2} \sum_{j=1}^{N+2} A_i.A_j < 0.$$

We have a contradiction. □

We now introduce some positive constants depending only on N which are used later.

Define

$$c_N = 1 - \sup_{\substack{t \in [\frac{1}{N+1}, 1] \\ s \in [-1, \frac{1}{2(N+1)}]}} \frac{\sqrt{1-s^2}}{\sqrt{1-s^2+(t-s)^2}} > 0. \tag{4.4}$$

Next fix $C(N)$ and $b_N \in \mathbb{N}$, two positive constants depending only on N such that

$$|B(x,r)| \geq C(N) r^N, \quad \forall\, x \in \mathbb{S}^N, \forall\, 0 < r < 1,$$

and

$$|\mathbb{S}^N| < C(N) b_N \left[\frac{c_N}{2(N+1)}\right]^N, \tag{4.5}$$

where c_N is defined by (4.4).

The next lemma has a consequence (Corollary 4) which is useful in the proof of Lemma 6.

Lemma 5. *Let $k \in \mathbb{N}_+$ and $A_i \in \mathbb{S}^N$ for all $1 \leq i \leq k$. Assume that $k \geq 16 b_N(N+1)$ and*

$$\left| \sum_{i=1}^{k} \frac{1}{k} A_i \right| \leq \frac{1}{16(N+1)}. \tag{4.6}$$

Then there exists $1 \leq i_0 \leq k$ such that

$$\operatorname{card}\left(\{1 \leq i \leq k :\, |A_i - A_{i_0}| \geq \ell_N\}\right) \geq \frac{k}{4 b_N (N+1)}.$$

Proof. Set $I = \{i \in \mathbb{N} :\, 1 \leq i \leq k\}$ and

$$U_n = \{i \in I :\, |A_i - A_n| \geq \ell_N\}, \quad \forall\, n \in I.$$

We argue by contradiction. Suppose that

$$\text{card}(U_i) < \frac{k}{4 b_N (N+1)}, \quad \forall\, i \in I. \tag{4.7}$$

Take $i_n \in I$ and $V_n \subset I$, $1 \leq n \leq b_N$, such that $i_1 = 1$, $V_1 = I \setminus U_{i_1}$, $i_{n+1} \in I \setminus V_n$ satisfies

$$\begin{aligned} &\text{dist}\left(O, \text{conv}\left(\{A_{i_m} :\, 1 \leq m \leq n+1\}\right)\right) = \\ &\qquad\qquad \min_{j \in I \setminus V_n} \text{dist}\left(O, \text{conv}\left(\{A_{i_m} :\, 1 \leq m \leq n\} \cup \{A_j\}\right)\right), \end{aligned}$$

and

$$V_{n+1} = V_n \cup U_{i_{n+1}},$$

for all $1 \leq n < b_N$ (since $k \geq 16 b_N(N+1)$, it follows from (4.7) that $I \setminus V_n \neq \emptyset$, for all $1 \leq n \leq b_N$).

Set

$$\delta_n = \text{dist}\left(O, \text{conv}\left(\{A_{i_m} :\, 1 \leq m \leq n\}\right)\right).$$

Then, since $|A_{i_m} - A_{i_n}| < \ell_N$ for all $1 \leq m, n \leq b_N$, it follows by Corollary 3 that

$$\delta_n \geq \frac{1}{N+1}, \quad \forall\, 1 \leq n \leq b_N. \tag{4.8}$$

We claim that $|A_{i_m} - A_{i_n}| \geq c_N/(N+1)$ for all $1 \leq m < n \leq b_N$, where c_N is defined by (4.4).

In fact, suppose that

$$|A_{i_m} - A_{i_n}| < c_N/(N+1) \quad \text{for some } \; 1 \leq m < n \leq b_N. \tag{4.9}$$

Let H_n be the projection of O onto $\operatorname{conv}(\{A_{i_m} : 1 \leq m \leq n-1\})$. Without loss of generality, we assume that $H_n = (h_n, 0, \dots, 0)$ and $h_n \geq 0$. Then

$$H_n.(A_{i_l} - H_n) \geq 0, \quad \forall\, 1 \leq l \leq n-1.$$

Thus, from (4.8) and (4.9), this implies

$$\delta_n > h_n - c_N/(N+1) \geq h_n - c_N h_n. \tag{4.10}$$

Therefore,

$$\pi_1(A_i) \geq \frac{1}{2(N+1)}, \quad \forall\, i \in I \setminus V_{n-1}.$$

To see this, suppose that $\pi_1(A_i) < \frac{1}{2(N+1)}$, for some $i \in I \setminus V_{n-1}$. Then

$$\operatorname{dist}(O, \operatorname{conv}(\{A_{i_m} : 1 \leq m \leq n-1\} \cup \{A_i\})) \leq \operatorname{dist}(O, H_n A_i). \tag{4.11}$$

Here $H_n A_i$ denotes the segment whose end points are H_n and A_i, i.e.,

$$H_n A_i = \{tH_n + (1-t)A_i : 0 \leq t \leq 1\}.$$

Thus, from (4.4) and (4.8),

$$\operatorname{dist}(O, H_n A_i) \leq h_n(1 - c_N). \tag{4.12}$$

Combining (4.10), (4.11), and (4.12) yields

$$\operatorname{dist}(O, \operatorname{conv}(\{A_{i_m} : 1 \leq m \leq n-1\} \cup A_i)) < \delta_n.$$

This contradicts the definition of δ_n. Thus

$$\pi_1(A_i) \geq \frac{1}{2(N+1)}, \quad \forall\, i \in I \setminus V_{n-1}. \tag{4.13}$$

Hence one deduces from (4.7) and (4.13) that

$$\left|\sum_{i=1}^{k} \frac{1}{k} A_i\right| \geq \left|\sum_{i \in I \setminus V_{n-1}} \frac{1}{k} A_i\right| - \left|\sum_{i \in V_{n-1}} \frac{1}{k} A_i\right| \geq \frac{k - \operatorname{card}(V_{n-1})}{2(N+1)} - \frac{b_N}{k} \frac{k}{4b_N(N+1)}. \tag{4.14}$$

On the other hand, since $k \ge 16 b_N(N+1)$ and $n \le b_N$, it follows from (4.7) that

$$k - \text{card}(V_{n-1}) \ge k - b_N \frac{k}{4b_N(N+1)} = k - \frac{k}{4(N+1)} \ge \frac{2k}{3}.$$

Thus, from (4.14), one has

$$\left| \sum_{i=1}^{k} \frac{1}{k} A_i \right| \ge \frac{1}{3(N+1)} - \frac{1}{4(N+1)} = \frac{1}{12(N+1)}. \tag{4.15}$$

This contradicts (4.6). Hence

$$|A_{i_m} - A_{i_n}| \ge c_N/(N+1), \quad \forall\, 1 \le m \ne n \le b_N.$$

Thus we obtain a family $\left\{ B\left(A_{i_m}, \frac{c_N}{2(N+1)}\right)\right\}_{i=1}^{b_N}$ such that

$$B\left(A_{i_m}, \frac{c_N}{2(N+1)}\right) \cap B\left(A_{i_n}, \frac{c_N}{2(N+1)}\right) = \emptyset, \quad \forall\, 1 \le m \ne n \le b_N.$$

Therefore,

$$|\mathbb{S}^N| \ge C(N) b_N \left[\frac{c_N}{2(N+1)}\right]^N.$$

This contradicts the choice of b_N. $\square$

Remark 6. Applying Corollary 3, we deduce from (4.6) that there exist $1 \le i, j \le k$ such that $|A_i - A_j| \ge \ell_N$. However, under the condition (4.6), Lemma 5 gives more information: There exists $1 \le i_0 \le k$ such that card$\left(\{1 \le i \le k : |A_i - A_{i_0}| \ge \ell_N\}\right) \ge C_N k$ whenever k is sufficiently big, where C_N is a positive constant depending only on N.

Corollary 4. *Let $k \in \mathbb{N}_+$ and $A_i \in \mathbb{S}^N$, for $1 \le i \le k$. Assume that $k \ge 100 b_N(N+1)$ and*

$$\left| \sum_{i=1}^{k} \frac{1}{k} A_i \right| \le \frac{1}{64(N+1)}. \tag{4.16}$$

Then

$$\text{card}\left(\{(i,j) :\, 1 \le i,j \le k,\, |A_i - A_j| \ge \ell_N\}\right) \ge \frac{k^2}{1000 b_N^2 (N+1)^2}.$$

Proof. Set

$$I = \{i \in \mathbb{N} :\, 1 \le i \le k\}$$

and

$$U_n = \{i \in I :\, |A_i - A_n| \ge \ell_N\}, \quad \forall\, n \in I.$$

By Lemma 5, there exists $i_1 \in I$ such that

$$\text{card}\,(U_{i_1}) \geq \frac{k}{4b_N(N+1)}.$$

Suppose that there exist i_m, $1 \leq m \leq n$ ($n < \frac{k}{50(N+1)}$), such that $i_m \neq i_l$ for $m \neq l$ and $\text{card}\,(U_{i_m}) \geq \frac{1}{8b_N(N+1)}$. Set

$$I_n = \{i_m : 1 \leq m \leq n\}.$$

One has

$$\sum_{i \in I \setminus I_n} \frac{1}{k-n} A_i = \frac{k}{k-n} \sum_{i \in I} \frac{1}{k} A_i - \sum_{i \in I_n} \frac{1}{k-n} A_i. \tag{4.17}$$

Since $n < \frac{k}{50(N+1)}$, it follows that

$$\left| \frac{k}{k-n} \sum_{i \in I} \frac{1}{k} A_i \right| \leq \frac{2}{64(N+1)} = \frac{1}{32(N+1)} \tag{4.18}$$

(by (4.16)) and

$$\left| \sum_{i \in I_n} \frac{1}{k-n} A_i \right| \leq \frac{n}{k-n} \leq \frac{1}{32(N+1)}. \tag{4.19}$$

Combining (4.17), (4.18), and (4.19) yields

$$\left| \sum_{i \in I \setminus I_n} \frac{1}{k-n} A_i \right| \leq \frac{1}{16(N+1)}.$$

Thus, since $n < \frac{k}{50(N+1)}$ and $k \geq 100 b_N(N+1)$, it follows that $k - n \geq k/2$ and $k - n \geq 16 b_N(N+1)$. Hence, by Lemma 5, there exists $i_{n+1} \in I \setminus I_n$ such that

$$\text{card}\left(U_{i_{n+1}}\right) \geq \frac{k}{8b_N(N+1)}.$$

Therefore, there exist i_m, $1 \leq m \leq \frac{k}{50(N+1)}$, such that $i_m \neq i_l$ for $l \neq m$ and

$$\text{card}\,(U_{i_m}) \geq \frac{k}{8b_N(N+1)}, \quad \forall\, 1 \leq m \leq \frac{k}{50(N+1)}.$$

Consequently,

$$\text{card}\,(\{(i,j) : 1 \leq i,j \leq k,\, |A_i - A_j| \geq \ell_N\}) \geq \frac{k^2}{1000 b_N^2 (N+1)^2}. \qquad \square$$

The following result is a continuous version of Corollary 4.

Lemma 6. *Let $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$, $\mathcal{D}$ be a measurable subset of $\mathbb{S}^N$ with $|\mathcal{D}| > 0$, and $\lambda : \mathcal{D} \to [0, +\infty)$ be a measurable function. Assume that there exists a constant $\beta \geq 1$ such that $1/\beta \leq \lambda(s) \leq \beta$ for every $s \in \mathcal{D}$ and*

$$\left| ⨍_{\mathcal{D}} \lambda(s) g(s)\, ds \right| \leq \frac{1}{128\beta(N+1)}.$$

Then

$$\operatorname{meas}\left(\{(\xi, \eta) \in \mathcal{D} \times \mathcal{D} :\, |g(\xi) - g(\eta)| > \delta\}\right) \geq C|\mathcal{D}|^2$$

for some $C = C(\beta, N)$.

Proof. Let $\Omega_1, \dots, \Omega_{k_1}$ be the measurable subsets of $\mathbb{S}^N$ such that $\mathbb{S}^N = \bigcup_{i=1}^{k_1} \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$, for each $1 \leq i \neq j \leq k_1$, and

$$\operatorname{diam}(\Omega_i) < \min\left\{ \frac{1}{128\beta^2(N+1)}, \frac{\ell_N - \delta}{3} \right\}$$

for every $1 \leq i \leq k_1$. Set

$$\mathcal{D}_i = g^{-1}(\Omega_i) \cap \mathcal{D}, \quad \forall\, 1 \leq i \leq k_1.$$

We assume as well that

$$|\mathcal{D}_i| > 0, \quad \forall\, 1 \leq i \leq k,$$

for some $k \leq k_1$. Take $A_i \in \mathcal{D}_i$. Then

$$\left| \sum_{i=1}^{k} \int_{\mathcal{D}_i} \lambda(s)\, ds A_i \right| \leq \left| \sum_{i=1}^{k} \int_{\mathcal{D}_i} \lambda(s) g(s)\, ds \right| + \left| \sum_{i=1}^{k} \int_{\mathcal{D}_i} \lambda(s)(g(s) - A_i)\, ds \right|,$$

which shows (since $1/\beta \leq \lambda(s) \leq \beta$ for all $s \in \mathcal{D}$) that

$$\left| \sum_{i=1}^{k} c_i A_i \right| \leq \beta \left| ⨍_{\mathcal{D}} \lambda(s) g(s)\, ds \right| + \beta^2 \max_i \operatorname{diam}(\Omega_i),$$

where

$$c_i = \left(\int_{\mathcal{D}_i} \lambda(s)\, ds \right) \Bigg/ \left(\sum_{i=1}^{k} \int_{\mathcal{D}_i} \lambda(s)\, ds \right).$$

Consequently,

$$\left| \sum_{i=1}^{k} c_i A_i \right| < \frac{1}{64(N+1)}.$$

Without loss of generality, suppose that c_i, $1 \leq i \leq k$, is a rational number (otherwise, we may approximate c_i, $1 \leq i \leq k$, by rational numbers). Suppose that $c_i = p_i/q$, $p_i \in \mathbb{N}_+$ and $q \in \mathbb{N}_+$ ($q \gg N$), for all $1 \leq i \leq k$. We as well assume that

$p_i = 1$ for all $1 \le i \le k$ (if not, take Ω_i^j, $1 \le j \le p_i$, such that $\bigcup_{j=1}^{p_i} \Omega_i^j = \Omega_i$ and $\int_{\mathcal{D}_i^j} \lambda(s)\,ds$ approximates $1/q$, where $\mathcal{D}_i^j = g^{-1}(\Omega_i^j)$, and $A_i^j = A_i$, for $1 \le j \le p_i$). Thus we now suppose that $c_i = 1/k$ for all $1 \le i \le k$, and $k \ge 100 b_N (N+1)$. Then

$$\Big| \sum_{i=1}^{k} \frac{1}{k} A_i \Big| < \frac{1}{64(N+1)}.$$

Applying Corollary 4, one has

$$\operatorname{card}\left(\{(i,j) :\, 1 \le i,j \le k,\, |A_i - A_j| \ge \ell_N\}\right) \ge C_N k^2. \tag{4.20}$$

On the other hand, since $c_i = 1/k$ and $1/\beta \le \lambda(s) \le \beta$ for $s \in \mathcal{D}$, one sees from the definition of c_i that

$$\frac{1}{k} = c_i \le \frac{\beta^2 |\Omega_i|}{|\mathcal{D}|},$$

which shows

$$|\Omega_i| \ge \frac{1}{\beta^2 k} |\mathcal{D}|. \tag{4.21}$$

Since $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$, combining (4.20) and (4.21) yields

$$\operatorname{meas}\left(\{(x,y) :\, x \in \Omega_i, y \in \Omega_j,\, |A_i - A_j| \ge \ell_N\}\right) \ge C_N k^2 \frac{1}{\beta^4 k^2} |\mathcal{D}|^2 = C_{N,\beta} |\mathcal{D}|^2.$$

Therefore, since $\operatorname{diam}(\Omega_i) \le (\ell_N - \delta)/3$, it follows that

$$\operatorname{meas}\left(\{(\xi,\eta) \in \mathcal{D} \times \mathcal{D} :\, |g(\xi) - g(\eta)| > \delta\}\right) \ge C_{N,\beta} |\mathcal{D}|^2. \qquad \square$$

4.2 Proof of assertion (1.3).

Step 1. Proof of assertion (1.3) when $g \in \mathrm{Lip}(\mathbb{S}^N, \mathbb{S}^N)$.

As in [3], consider the function $u : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B} = \left\{X \in \mathbb{R}^{N+1} \,:\, |X| \le 1\right\}$, defined by

$$u(X) = \fint_{B(x,r)} g(s)\,ds, \quad \text{when } X \ne 0, \tag{4.22}$$

where $x = X/|X|$, $r = 2(1 - |X|)$, and

$$u(0) = \fint_{\mathbb{S}^N} g(s)\,ds.$$

For each $x \in \mathbb{S}^N$, let $\rho(x)$ be the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|u| > \frac{1}{128(N+1)}$ (possibly $\rho(x) = 1$).

Thus, as in [3], one has

$$|\deg g| \le C \int_{\substack{\mathbb{S}^N \\ \rho(x)<1}} \frac{dx}{\rho(x)^N}. \tag{4.23}$$

Hereafter C denotes a constant depending only on N in this proof. Take $x \in \mathbb{S}^N$ such that $\rho(x) < 1$. Then

$$\left| \fint_{B(x,2\rho(x))} g(s)\, ds \right| = \frac{1}{128(N+1)}.$$

Applying Lemma 6, one has

$$\text{meas}\left\{(\xi,\eta) \in [B(x,2\rho(x))]^2 : |g(\xi)-g(\eta)| \ge \delta\right\} \ge C\rho(x)^{2N}.$$

Hence there exists a constant $\tau = \tau(N)$, depending only on N, such that

$$\text{meas}\left\{(\xi,\eta) \in [B(x,2\rho(x))]^2 : |g(\xi)-g(\eta)| \ge \delta, |\xi-\tau| \ge \tau\rho(x)\right\} \ge C\rho(x)^{2N}.$$

This implies

$$\iint_{\substack{[B(x,2\rho(x))]^2 \\ |g(\xi)-g(\eta)|>\delta \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta \ge C. \tag{4.24}$$

Combining (4.23) and (4.24) yields

$$|\deg g| \le C \int_{\mathbb{S}^N} \frac{1}{\rho(x)^N} \iint_{\substack{[B(x,2\rho(x))]^2 \\ |g(\xi)-g(\eta)|>\delta \\ |\xi-\eta|\ge\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta\, dx.$$

A computation gives

$$|\deg g| \le C \int_{\substack{\mathbb{S}^N \\ |g(\xi)-g(\eta)|>\delta}} \int_{\mathbb{S}^N} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta.$$

Step 2. Proof of assertion (1.3) when g is merely a continuous function from $\mathbb{S}^N$ to $\mathbb{S}^N$.

The proof is the same as the proof of the Step 2 of [3, Theorem 1.1]. □

5 General class of maps from $\mathbb{S}^N$ into $\mathbb{S}^N$

In this section, we study the class of $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ satisfying (1.6) for some $0 < \delta < \ell_N$.

5.1 Definition of $\deg g$ when $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ satisfies (1.6) for some $0 < \delta < \ell_N$. We first recall a classical result.

Lemma 7 (cf. [7, Chapter 4])**.** *Let $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ and $\delta \in (0, +\infty)$. Assume that*

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\, dx\, dy < +\infty.$$

Then

$$\lim_{r\to 0} \sup_{x\in\mathbb{S}^N} \iint_{\substack{[B(x,r)]^2\\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta = 0.$$

Define for each $r > 0$,

$$g_r(x) = \fint_{B(x,r)} g(s)\, ds, \quad \forall\, x \in \mathbb{S}^N. \tag{5.1}$$

The following result is a consequence of Lemmas 6 and 7.

Corollary 5. *Let $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ and $\delta \in (0, \ell_N)$. Assume that*

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\, dx\, dy < +\infty.$$

Then there exists a positive constant r_0 such that

$$|g_r(x)| \geq \frac{1}{128(N+1)}, \quad \forall x \in \mathbb{S}^N, \forall\, 0 < r < r_0,$$

where g_r is defined by (5.1).

Proof. The proof is by contradiction. Suppose that there exists a sequence of positive numbers $\{r_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n = 0$, and a sequence of points $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^N$ such that

$$|g_{r_n}(x_n)| < \frac{1}{128(N+1)}.$$

Then by Lemma 6,

$$\liminf_{n\to\infty} \iint_{\substack{[B(x_n,r_n)]^2\\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta > 0.$$

However, this contradicts the fact (see Lemma 7) that

$$\lim_{r\to 0} \sup_{x\in\mathbb{S}^N} \iint_{\substack{[B(x,r)]^2\\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta = 0.$$

□

We now introduce the following

Definition Let $g \in L^\infty(\mathbb{S}^N, \mathbb{S}^N)$ be such that

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}{}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{2N}}\, dx\, dy < +\infty.$$

Take $0 < r < r_0$, where r_0 is the constant in Corollary 5; then $|g_r(x)| > 0$ for all $x \in \mathbb{S}^N$. Set

$$\widetilde{g}_r(x) = \frac{g_r(x)}{|g_r(x)|}, \quad \forall\, x \in \mathbb{S}^N, \tag{5.2}$$

and define

$$\deg g = \deg \widetilde{g}_r, \tag{5.3}$$

for any $0 < r < r_0$.

Since $\widetilde{g}_r$ is continuous, $\deg \widetilde{g}_r$ is well-defined. Furthermore, using the deformation $\widetilde{g}_{tr+(1-t)r'}$, $0 \le t \le 1$, we have $\deg \widetilde{g}_r = \deg \widetilde{g}_{r'}$ for every $0 < r, r' < r_0$. Thus $\deg g$ is well-defined by (5.3). This definition in the case $g \in$ VMO$(\mathbb{S}^N, \mathbb{S}^N)$ was presented in [5]. A survey of degree theory for maps in $VMO(\mathbb{S}^N, \mathbb{S}^N)$ can be also found there.

5.2 Proof of Inequality (1.7). Take $r = r_0/2$ and $\alpha \le 1/4$ (α depends only on N) sufficiently small to be defined later. Here r_0 denotes the constant in Corollary 5. Consider $\widetilde{g}_{r,\alpha} \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ such that $\deg \widetilde{g}_{r,\alpha} = \deg \widetilde{g}_r$ and $||\widetilde{g}_{r,\alpha} - \widetilde{g}_r||_{L^\infty(\mathbb{S}^N)} \le \alpha$, where $\widetilde{g}_r$ is defined by (5.2). As in the definition of u and ρ with respect to the map g in Step 1 of the proof of assertion (1.8), we define u and ρ for the map $\widetilde{g}_{r,\alpha}$ (for notational ease).

Recall (see [3]) that

$$|\deg \widetilde{g}_{r,\alpha}| \le \frac{C}{\alpha^{N+1}|\mathcal{B}|} \int_{\substack{\mathbb{S}^N \\ \rho(x)<1}} \frac{dx}{\rho(x)^N} \tag{5.4}$$

for some $C = C(N)$.

Fix $x \in \mathbb{S}^N$ such that $\rho(x) < 1$. Then from the definition of ρ,

$$\left| \fint_{B(x,2\rho(x))} \widetilde{g}_{r,\alpha}(y)\, dy \right| = \alpha. \tag{5.5}$$

Since $\|\widetilde{g}_{r,\alpha} - \widetilde{g}_r\|_{L^\infty(\mathbb{S}^N)} \le \alpha$, it follows from (5.5) that

$$\left| \fint_{B(x,2\rho(x))} \widetilde{g}_r(y)\, dy \right| \le \alpha + \alpha = 2\alpha. \tag{5.6}$$

Since $\alpha \le 1/4$, this implies

$$\left| \fint_{B(x,2\rho(x))} \widetilde{g}_r(y)\, dy \right| \le \frac{1}{2}. \tag{5.7}$$

On the other hand, by Corollary 5,

$$|g_r(y)| = \left| \fint_{B(y,r)} g(s)\, ds \right| \ge \frac{1}{128(N+1)}, \quad \forall\, y \in \mathbb{S}^N. \tag{5.8}$$

Combining (5.2), (5.7), and (5.8) yields

$$r \lesssim \rho(x). \tag{5.9}$$

Since

$$\fint_{B(x,2\rho(x))} \widetilde{g}_r(\xi)\, d\xi = \frac{1}{|B(x,2\rho(x))|} \int_{B(x,2\rho(x))} \fint_{B(\xi,r)} \frac{g(y)}{|g_r(\xi)|}\, dy\, d\xi,$$

it follows by Fubini's theorem that

$$\fint_{B(x,2\rho(x))} \widetilde{g}_r(\xi)\, d\xi = \frac{1}{|B(x,2\rho(x))|} \int_{B(x,2\rho(x)+r)} \lambda(y) g(y)\, dy, \tag{5.10}$$

where

$$\lambda(y) = \frac{1}{|B(x,r)|} \int\limits_{\substack{|\xi-y|\le r\\ |\xi-x|\le 2\rho(x)}} \frac{1}{|g_r(\xi)|}\, d\xi. \tag{5.11}$$

Combining (5.8), (5.9), and (5.11) yields

$$\begin{cases} |\lambda(y)| \simeq (1-t)^N, & \text{if } |y-x| = 2\rho(x) + tr, \quad \forall\, 0 < t < 1, \\ |\lambda(y)| \simeq 1, & \text{if } y \in B(x, 2\rho(x)). \end{cases} \tag{5.12}$$

On the other hand, for $0 < t < 1$, we have from (5.10)

$$\left| \int_{B(x,2\rho(x)+tr)} \lambda(y) g(y)\, dy \right| \le \\ \left| \int_{B(x,2\rho(x))} \widetilde{g}_r(\xi)\, d\xi \right| + \left| \int\limits_{B(x,2\rho(x)+r)\setminus B(x,2\rho(x)+tr)} \lambda(y) g(y)\, dy \right|.$$

Thus it follows from (5.6), (5.9), (5.11), and (5.12) that

$$\left| \fint_{B(x,2\rho(x)+tr)} \lambda(y) g(y)\, dy \right| \lesssim \alpha + (1-t)^{N+1}, \quad \forall\, 0 < t < 1.$$

Applying Lemma 6 with $\beta = \sup_{\substack{|y-x|=2\rho(x)+sr\\ 0<s<t}} \frac{1}{\lambda(y)}$ ($\beta \simeq (1-t)^{-N}$) and $\mathcal{D} = B(x, 2\rho(x) + tr)$, after fixing t sufficiently near to 1 ($0 < t < 1$) and taking $\alpha = (1-t)^{N+1}$, one has

$$\operatorname{meas}\{(\xi,\eta) \in B(x, 2\rho(x)+tr) : |g(\xi) - g(\eta)| > \delta\} \gtrsim |B(x, 2\rho(x)+tr)|^2.$$

Thus, just as in the proof of Corollary 5, there exists a positive constant $\tau = \tau(N)$ such that

$$\iint_{\substack{[B(x,2\rho(x)+tr)]^2\\ |g(\xi)-g(\eta)|>\delta\\ |\xi-\eta|\geq\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta \gtrsim 1. \tag{5.13}$$

On the other hand, from (5.9), there exists a constant $k = k(N)$, depending only on N, such that $k\rho(x) \geq 2\rho(x) + r$. Thus, it follows from (5.13) that

$$\iint_{\substack{[B(x,k\rho(x))]^2\\ |g(\xi)-g(\eta)|>\delta\\ |\xi-\eta|\geq\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta \gtrsim 1. \tag{5.14}$$

Combining (5.4) and (5.14) yields

$$|\deg \widetilde{g}_{r,\alpha}| \lesssim \int_{\mathbb{S}^N} \frac{1}{\rho(x)^N} \iint_{\substack{[B(x,k\rho(x))]^2\\ |g(\xi)-g(\eta)|>\delta\\ |\xi-\eta|\geq\tau\rho(x)}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta\, dx.$$

A computation yields

$$|\deg \widetilde{g}_{r,\alpha}| \lesssim \int_{\mathbb{S}^N}\int_{\substack{\mathbb{S}^N\\ |g(\xi)-g(\eta)|>\delta}} \frac{1}{|\xi-\eta|^{2N}}\, d\xi\, d\eta.$$

Therefore, inequality (1.7) follows, since $\deg g = \deg \widetilde{g}_r = \deg \widetilde{g}_{r,\alpha}$. $\square$

Acknowledgments. The author is deeply grateful to Prof. H. Brezis for his encouragement. He thanks Prof. H. Brezis for having drawn his attention to this subject and for indicating useful directions. He thanks Prof. P. Mironescu for useful comments and discussions and D. Chiron, R. Ignat, V. Millot and A. Ponce for interesting discussions.

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(Received June 19, 2005 and in revised form July 13, 2006)