

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE HELMHOLTZ EQUATIONS WITH SIGN CHANGING COEFFICIENTS

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ABSTRACT. This paper is devoted to the study of the behavior of the unique solution $u_\delta \in H_0^1(\Omega)$, as $\delta \rightarrow 0$, to the equation

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta = s_0 f \text{ in } \Omega,$$

where Ω is a smooth connected bounded open subset of \mathbb{R}^d with $d = 2$ or 3 , $f \in L^2(\Omega)$, k is a non-negative constant, A is a uniformly elliptic matrix-valued function, Σ is a real function bounded above and below by positive constants, and s_δ is a complex function whose **real part takes the values 1 and -1** and whose imaginary part is positive and converges to 0 as δ goes to 0. This is motivated from a result of Nicorovici, McPhedran, and Milton; another motivation is the concept of complementary media. After introducing the reflecting complementary media, complementary media generated by reflections, we characterize f for which $\|u_\delta\|_{H^1(\Omega)}$ remains bounded as δ goes to 0. For such an f , we also show that u_δ converges weakly in $H^1(\Omega)$ and provide a formula to compute the limit.

1. INTRODUCTION

Negative index materials (NIMs) were first investigated theoretically by Veselago in [22] and were innovated by Pendry in [15]. The existence of such materials was confirmed by Shelby, Smith, and Schultz in [20] (see also [21]). Cloaking space and illusion optics using NIMs were discussed in [7, 8] (see also [14]) based on the concept of complementary and in [2, 3, 5, 6, 11–14] based on the anomalous localized resonance. Perfect lens using NIMs was studied in [13, 14, 16, 19].

The first motivation for this work comes from the following two-dimensional result of Nicorovici et al. in [14]. Let $0 < r_1 < r_2 < R$, and $f \in L^2(B_R)$. Here and in what follows, for $r > 0$, B_r denotes the ball centered at 0 of radius r . Set $r_3 = r_2^2/r_1$. Assume that $R > r_3$ and $\operatorname{supp} f \cap \{x \mid |x| < r_3\} = \emptyset$. Let $u_\delta \in H_0^1(B_R)$ be the unique solution to the equation

$$(1.1) \quad \operatorname{div}(\varepsilon_\delta \nabla u_\delta) = f \text{ in } B_R.$$

Here

$$\varepsilon_\delta = \varepsilon + i\mathbf{1}_{r_1 < |x| < r_2},$$

where

$$\varepsilon(x) = \begin{cases} -1 & \text{if } r_1 < |x| < r_2, \\ 1 & \text{otherwise.} \end{cases}$$

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Physically, the imaginary part of ε_δ is the loss of the medium. It is shown in [14] by separation of variables that

$$(1.2) \quad u_\delta \rightarrow \mathcal{U} \text{ for } |x| > r_3,$$

where $\mathcal{U} \in H_0^1(B_R)$ is the unique solution to the equation

$$\Delta \mathcal{U} = f \text{ in } B_R.$$

The surprising fact in this result is that (1.2) holds for **any** f with $\text{supp } f \cap B_{r_3} = \emptyset$. From (1.2), one might say that the region $\{r_2 < |x| < r_3\}$ is canceled by the one in $\{r_1 < |x| < r_2\}$ and the total system is effectively equal to the free space; invisibility is achieved.

The following questions naturally arise:

- What happens if $\text{supp } f \cap B_{r_3} \neq \emptyset$?
- Is the radial symmetry necessary? If not, what are conditions on ε ?
- What happens in the finite frequency regime?
- Do similar phenomena hold in three dimensions? If yes, under which conditions?

Another motivation for this work is the concept of complementary media which was suggested in [7, 18, 19] (see also [14, 15]). This concept has played an important role in the study of NIMs and its applications such as cloaking, perfect lens, and illusion optics (see [7, 8, 12, 15, 19]). Although many examples have been suggested, this concept has not been defined in a precise manner. A common point in examples studied is

$$(1.3) \quad F_* a = -b \text{ in } D_2,$$

for some diffeomorphism $F : D_1 \rightarrow D_2$ if a matrix a defined in a region D_1 is complementary to a matrix b defined in a region D_2 . Here

$$F_* a(y) = \frac{DF(x)a(x)DF^T(x)}{J(x)}, \text{ where } x = F^{-1}(y) \text{ and } J(x) = |\det DF(x)|.$$

It is easy to verify in two dimensions that if $F : \{r_1 < |x| < r_2\} \rightarrow \{r_2 < |x| < r_3\}$ with $r_3 = r_2^2/r_1$ defined by $F(x) = r_2^2 x/|x|^2$, then F is a diffeomorphism and

$$F * (-I) = -I \text{ in } \{r_2 < |x| < r_3\}.$$

In other words, the medium $-I$ in $\{r_1 < |x| < r_2\}$ is complementary to the medium I in $\{r_2 < |x| < r_3\}$: complementary media appears in the setting of Nacorovici et al.

In this paper, we address the above questions. To this end, we first introduce the notion of reflecting complementary media. Similar phenomena as in (1.2) take place for media that inherits this property in the quasistatic and the finite frequency regimes. Two media (two matrices in two regions in the quasistatic regime) are called reflecting complementary if they are complementary and the complementary is generated by a reflection which satisfies some mild conditions. The motivation for the definition of this notion comes from the reflection technique used in a heuristic argument for (1.2) in Section 2.1. We then establish results in the spirit of (1.2) for media of this property. The analysis again has roots in the heuristic argument. The key to the analysis is the derivation of two Cauchy problems for elliptic equations by the reflection technique using the reflecting complementary property. Concerning the analysis, we characterize f (based on the compatibility condition in Definition 2)

for which $\|u_\delta\|_{H^1}$ remains bounded as $\delta \rightarrow 0_+$; moreover, we show that for such a function f , the limit of u_δ exists as $\delta \rightarrow 0_+$ (Theorem 1). We also provide a formula to compute the limit which involves only the solutions of standard elliptic equations (no sign changing coefficients), and show that the limit has properties in the spirit of (1.2) (Theorem 2). To our knowledge, the results presented in this paper are new even in the 2d-quasistatic regime.

The use of reflections to study NIMs has been considered previously in [12]. However, there is a big difference between the use of reflections in [12] and in this paper. In [12], the authors used reflections as a change of variables to obtain a new simple setting from an old more complicated one and hence the analysis of the old problem becomes simpler. In this paper, we use reflections to derive the two Cauchy problems. This derivation essentially makes use of the complementary property of media. The global or non-global existence of the solutions to these Cauchy problems will determine the boundedness or unboundedness of the H^1 -norm of the solutions.

The goals and the setting of this paper are different from Ammari et al.'s in [2] and Kohn et al.'s in [6]. In this paper, we introduce the concept of reflecting complementary media and we study the boundedness of $\|u_\delta\|_{H^1}$ and the limit of u_δ in the whole domain as $\delta \rightarrow 0$ in the quasistatic and the finite frequency regimes. In [2, 6], the authors investigate the unboundedness of $\delta^{1/2}\|u_\delta\|_{H^1}$ for piecewise constant media (up to a diffeomorphism in [6]) in the quasistatic case. It is clear that the boundedness of $\|u_\delta\|_{H^1}$ implies the boundedness of $\delta^{1/2}\|u_\delta\|_{H^1}$ and the unboundedness of $\delta^{1/2}\|u_\delta\|_{H^1}$ implies the unboundedness of $\|u_\delta\|_{H^1}$. The media considered in [6] are of the reflecting complementary property; however the ones in [2] are, in general, not (the radial setting considered in [2, Section 5] is an exclusion). In [2], the authors also deal with the boundedness of u_δ in some region. To make use of their results mentioned above, one needs detailed information on the spectral properties of certain boundary integral operators. This information is difficult to come by in general. The method in this paper is different from and more elementary than the spectral one in [2] and the variational one in [6].

The method in this paper is used in [9] and developed in [10]. In [9] the authors study the complete resonance and localized resonance in plasmonic structures whereas in [10] the author investigates the cloaking via complementary media.

Our paper is organized as follows. Section 2 contains two subsections devoted to the concept of reflecting complementary media. In the first subsection, we present the heuristic argument to obtain (1.2) and to motivate the definition of this concept. The second subsection is devoted to the definition. In Section 3, we state and prove properties on the reflecting complementary media. More precisely, we state and prove Theorems 1 and 2 and present their two corollaries.

2. REFLECTING COMPLEMENTARY MEDIA

In this section, we introduce the notion of reflecting complementary media. To motivate the definition, we first present a heuristic argument, in Section 2.1, to obtain (1.2) based on the reflecting technique. The definition of reflecting complementary media is introduced in Section 2.

2.1. Motivation - a heuristic argument for (1.2). In this section, we present a heuristic (elementary) argument to obtain (1.2). This argument motivates not only the notion of the reflecting complementary media but also the analysis in Section 3.

In this section, we *assume* that $u_\delta \rightarrow u \in H^1(B_R)$. It follows that $u \in H_0^1(B_R)$ is a solution to the equation

$$\operatorname{div}(\varepsilon \nabla u) = f \text{ in } B_R.$$

Let F defined in $\{|x| < r_2\}$ be the Kelvin transform w.r.t. ∂B_{r_2} , i.e.,

$$F(x) = \frac{r_2^2}{|x|^2} x \text{ for } |x| < r_2.$$

Let u_1 defined in $\{|x| > r_2\}$ be the Kelvin transform F of u , i.e.,

$$(2.1) \quad u_1(x) = u \circ F^{-1}(x) \quad \text{for } |x| > r_2.$$

Then, by the transmission condition on ∂B_{r_2} , we have

$$(2.2) \quad u_1 = u \quad \text{and} \quad \partial_r u_1 \Big|_{r \rightarrow r_{2+}} = \partial_r u \Big|_{r \rightarrow r_{2+}} \quad \text{for } |x| = r_2.$$

Since F is a Kelvin transform and $\operatorname{supp} f \cap \{|x| < r_3\} = \emptyset$, it follows that

$$(2.3) \quad \operatorname{div}(\hat{\varepsilon} \nabla u_1) = 0 \text{ for } |x| > r_2,$$

where

$$(2.4) \quad \hat{\varepsilon}(x) = \begin{cases} 1 & \text{if } r_2 < |x| < r_3, \\ -1 & \text{if } |x| > r_3. \end{cases}$$

(Note that F transforms ∂B_{r_1} into ∂B_{r_3} .) By the unique continuation principle, we have

$$(2.5) \quad u_1 = u \text{ in } \{r_2 < |x| < r_3\}.$$

Let G defined in $\{|x| > r_3\}$ be the Kelvin transform w.r.t. ∂B_{r_3} , i.e.,

$$G(x) = \frac{r_3^2}{|x|^2} x \text{ for } |x| > r_3.$$

Define u_2 in $\{|x| < r_3\}$, the Kelvin transform G of u_1 , as

$$(2.6) \quad u_2(x) = u_1 \circ G^{-1}(x) \quad \text{for } |x| < r_3.$$

Similar to (2.2), we have

$$(2.7) \quad u_2 = u_1 \quad \text{and} \quad \partial_r u_2 \Big|_{r \rightarrow r_{3-}} = \partial_r u_1 \Big|_{r \rightarrow r_{3-}} \quad \text{for } |x| = r_3.$$

It follows from (2.5) that

$$(2.8) \quad u_2 = u \quad \text{and} \quad \partial_r u_2 \Big|_{r \rightarrow r_{3-}} = \partial_r u \quad \text{for } |x| = r_3.$$

We also have

$$(2.9) \quad \Delta u_2 = 0 \text{ in } |x| < r_3,$$

by the property of the Kelvin transforms. Define \mathcal{U} by

$$(2.10) \quad \mathcal{U}(x) = \begin{cases} u(x) & \text{if } |x| > r_3, \\ u_2(x) & \text{if } |x| < r_3. \end{cases}$$

Since $\Delta u = f$ for $|x| > r_3$, it follows from (2.8) and (2.9) that

$$\Delta \mathcal{U} = f \text{ in } B_R.$$

Therefore, we obtain (1.2).

2.2. Reflecting complementary media. In this section, we introduce the notion of reflecting complementary media. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$ be smooth connected bounded open subsets of \mathbb{R}^d ($d = 2, 3$).

Let A be a measurable matrix function and let Σ be a measurable real function defined in Ω such that

$$(2.11) \quad \frac{1}{\Lambda}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d,$$

for a.e. $x \in \Omega$ and for some $0 < \Lambda < +\infty$, and

$$(2.12) \quad 0 < \operatorname{ess\,inf}_{\Omega} \Sigma \leq \operatorname{ess\,sup}_{\Omega} \Sigma < +\infty.$$

Set

$$(2.13) \quad s_{\delta}(x) = \begin{cases} -1 + i\delta & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ 1 & \text{otherwise.} \end{cases}$$

We are later interested in the behavior of the unique solution $u_{\delta} \in H_0^1(\Omega)$ to the equation

$$(2.14) \quad \operatorname{div}(s_{\delta}A\nabla u_{\delta}) + k^2 s_0 \Sigma u_{\delta} = s_0 f \text{ in } \Omega,$$

as $\delta \rightarrow 0$.

We are ready to give

Definition 1 (Reflecting complementary media). The media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are said to be reflecting complementary if there exists a diffeomorphism $F : \Omega_2 \setminus \bar{\Omega}_1 \rightarrow \Omega_3 \setminus \bar{\Omega}_2$ such that

$$(2.15) \quad F_*A(x) = A(x), \quad F_*\Sigma(x) = \Sigma(x) \text{ for } x \in \Omega_3 \setminus \bar{\Omega}_2,$$

$$(2.16) \quad F(x) = x \text{ on } \partial\Omega_2,$$

and the following two conditions hold:

- 1) There exists a diffeomorphism extension of F , which is still denoted by F , from $\Omega_2 \setminus \{x_1\} \rightarrow \Omega_4 \setminus \bar{\Omega}_2$ for some $x_1 \in \Omega_1$ and some smooth open subset Ω_4 of \mathbb{R}^d with $\Omega_3 \subset \Omega_4$.
- 2) There exists a diffeomorphism $G : \Omega_4 \setminus \bar{\Omega}_3 \rightarrow \Omega_3 \setminus x_2$ for some $x_2 \in \Omega_3$ such that¹

$$(2.17) \quad G(x) = x \text{ on } \partial\Omega_3$$

and

$$(2.18) \quad G \circ F : \Omega_1 \rightarrow \Omega_3 \text{ is a diffeomorphism if one sets } G \circ F(x_1) = x_2.$$

Here and in what follows, we use the standard notation:

$$(2.19) \quad \mathcal{F}_*A(y) = \frac{D\mathcal{F}(x)A(x)D\mathcal{F}^T(x)}{J(x)}, \quad \mathcal{F}_*\Sigma(y) = \frac{\Sigma(x)}{J(x)}, \quad \text{and} \quad \mathcal{F}_*f(y) = \frac{f(x)}{J(x)},$$

where $x = \mathcal{F}^{-1}(y)$ and $J(x) = |\det D\mathcal{F}(x)|$.

Some comments on the definition are:

- i) If $k = 0$, then the condition on Σ is irrelevant in Definition 1.

¹In (2.16) and (2.17), F and G denote some diffeomorphism extensions of F and G in a neighborhood of $\partial\Omega_2$ and of $\partial\Omega_3$.

- ii) Condition (2.15) implies that (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are complementary in the usual sense. The term “reflecting” in the definition comes from (2.16) and the assumption $\Omega_1 \subset \Omega_2 \subset \Omega_3$. Conditions (2.15) and (2.16) are the main assumptions in the definition.
- iii) Condition (2.15) makes sure that u (the “solution” for $\delta = 0$) and $u_1 := u \circ F$ satisfy the same equation in $\Omega_3 \setminus \Omega_2$; hence the reflecting technique in Section 2.1 can be used. Conditions (2.16) and (2.17) assure that $u = u_1$ on $\partial\Omega_2$ and $u_2 = u_1$ on $\partial\Omega_3$ where $u_2 = u_1 \circ G^{-1}$. Condition (2.18) is a technical one which is required by the proof.
- iv) Conditions 1) and 2) in the definition are mild assumptions. Introducing G in the definition makes the analysis more accessible as in Section 2.1 (see the paragraph below (3.14) for other comments on G).
- v) In general, it is difficult to verify whether (2.15) holds for some F . In practice, to obtain the reflecting complementary in $\Omega_2 \setminus \Omega_1$ for (A, Σ) in $\Omega_3 \setminus \Omega_2$, it suffices to choose a diffeomorphism $F : \Omega_2 \setminus \{x_1\} \rightarrow \Omega_4 \setminus \bar{\Omega}_2$ for some $x_1 \in \Omega_1$ and for some smooth bounded open subset Ω_4 containing Ω_3 , and define $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ by $(-F_*^{-1}A, -F_*^{-1}\Sigma)$.

It is clear that the medium ε given in (2.4) has the reflecting complementary property with $\Omega_i = B_{r_i}$ (for $i = 1, 2, 3$) and $\Omega = B_R$ where $r_4 := +\infty$, and F and G are the Kelvin transforms w.r.t. ∂B_{r_2} and ∂B_{r_3} , resp.

Remark 1. Concerning reflecting complementary media, the 2d-quasistatic case is quite special in the sense that two constant media (two constant matrices) can be complementary (see the example in the introduction). In fact, in the 2d-finite frequency case, it seems that there do not exist two constant media (two constant matrices and two constant functions) which are complementary and in the 3d case there do not exist two constant media (two constant matrices) which are complementary.

3. PROPERTIES RELATED TO REFLECTING COMPLEMENTARY MEDIA

3.1. Statement of the results. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$ be smooth connected bounded open subsets of \mathbb{R}^d ($d = 2, 3$). Let A be a measurable matrix function and let Σ be a measurable real function defined in Ω such that

$$(3.1) \quad \frac{1}{\Lambda} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d,$$

for a.e. $x \in \Omega$ and for some $0 < \Lambda < +\infty$, and

$$(3.2) \quad 0 < \operatorname{ess\,inf}_\Omega \Sigma \leq \operatorname{ess\,sup}_\Omega \Sigma < +\infty.$$

We will assume that A is piecewise differentiable in Ω in three dimensions.²

Set

$$(3.3) \quad s_\delta(x) = \begin{cases} -1 + i\delta & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ 1 & \text{otherwise.} \end{cases}$$

We are interested in the behavior of the unique solution $u_\delta \in H_0^1(\Omega)$ to the equation

$$(3.4) \quad \operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta = s_0 f \text{ in } \Omega,$$

as $\delta \rightarrow 0$.

²This condition is necessary to obtain the uniqueness for the Cauchy problems.

Throughout this section, we will assume that:

(3.5) systems (3.6) and (3.7) have only zero solutions in $H^1(\Omega \setminus \bar{\Omega}_2)$ and $H^1(\Omega)$, resp., where

$$(3.6) \quad \begin{cases} \operatorname{div}(A\nabla v) + k^2\Sigma v = 0 & \text{in } \Omega \setminus \bar{\Omega}_2, \\ v = 0 & \text{on } \partial\Omega \cup \partial\Omega_2, \end{cases}$$

and

$$(3.7) \quad \begin{cases} \operatorname{div}(\hat{A}\nabla U) + k^2\hat{\Sigma}U = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases}$$

Here \hat{A} and $\hat{\Sigma}$ are defined as

$$(3.8) \quad \hat{A} := \begin{cases} A & \text{if } x \in \Omega \setminus \Omega_3, \\ G_*F_*A & \text{if } x \in \Omega_3, \end{cases} \quad \text{and} \quad \hat{\Sigma} := \begin{cases} \Sigma & \text{if } x \in \Omega \setminus \Omega_3, \\ G_*F_*\Sigma & \text{if } x \in \Omega_3. \end{cases}$$

We also define

$$(3.9) \quad \hat{f} := \begin{cases} f & \text{if } x \in \Omega \setminus \Omega_3, \\ G_*F_*f & \text{if } x \in \Omega_3. \end{cases}$$

Remark 2. The well-posedness of (3.6) and (3.7) always holds for $k = 0$. In the case $k > 0$, if one is interested the corresponding problem on the whole space in which outgoing solutions are considered, the well-posedness assumption is not necessary.

In what follows we assume that $k > 0$. The case $k = 0$ is similar and even easier to obtain. The first main result in this section is

Theorem 1. *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$ and let $u_\delta \in H_0^1(\Omega)$ be the unique solution to equation (2.14):*

$$\operatorname{div}(s_\delta A\nabla u_\delta) + k^2s_0\Sigma u_\delta = s_0f \text{ in } \Omega.$$

Assume that the media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary. We have

a) *Case 1: f is compatible with the medium. Then (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $u_0 \in H_0^1(\Omega)$, the unique solution to the equation*

$$(3.10) \quad \operatorname{div}(s_0\nabla u_0) + k^2s_0\Sigma u_0 = s_0f \text{ in } \Omega,$$

as $\delta \rightarrow 0$.

b) *Case 2: f is not compatible with the medium. We have*

$$(3.11) \quad \lim_{\delta \rightarrow 0} \|u_\delta\|_{H^1(\Omega)} = +\infty.$$

In the statement of Theorem 1, we use the following

Definition 2 (Compatibility condition). *Assume that the media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary. Then $f \in L^2(\Omega)$ is said*

to be compatible with the system if and only if there exist $U \in H^1(\Omega_3 \setminus \Omega_2)$ and $V \in H^1(\Omega_3 \setminus \Omega_2)$ such that

$$(3.12) \quad \begin{cases} \operatorname{div}(A\nabla U) + k^2\Sigma U = F_*f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U = 0 & \text{on } \partial\Omega_2, \\ A\nabla U \cdot \eta = 0 & \text{on } \partial\Omega_2, \end{cases}$$

and

$$(3.13) \quad \begin{cases} \operatorname{div}(A\nabla V) + k^2\Sigma V = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ V = W|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla V \cdot \eta = A\nabla W \cdot \eta|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases}$$

Here $W \in H^1(\Omega \setminus \partial\Omega_3)$ is the unique solution to the system

$$(3.14) \quad \begin{cases} \operatorname{div}(\hat{A}\nabla W) + k^2\hat{\Sigma}W = \hat{f} & \text{in } \Omega \setminus \partial\Omega_3, \\ W = 0 & \text{on } \partial\Omega, \\ [W] = -U & \text{on } \partial\Omega_3, \\ [\hat{A}\nabla W \cdot \eta] = -A\nabla U \cdot \eta & \text{on } \partial\Omega_3. \end{cases}$$

The compatibility condition is an intrinsic one, i.e., it does not depend on the choice of F and G . In fact, it is equivalent to the boundedness of $\|u_\delta\|_{H^1(\Omega)}$ as $\delta \rightarrow 0$. Given F , there are infinitely many choices of G . A choice of G that would make the compatibility condition more accessible is preferred. Problems (3.12) and (3.13) are the two Cauchy problems obtained from the reflecting technique. The reflecting complementary condition is necessary so that $(\|u_\delta\|_{H^1(\Omega)})$ explodes for some f (see [6]).

The uniqueness of U and V follow from the unique continuation principle (see, e.g., [1, 17]). The existence and uniqueness of W are standard and follow from Fredholm’s theory (see, e.g., [4]) since system (3.7) is well-posed.

The next result is in the spirit of (1.2).

Theorem 2. *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$ and let $u_\delta \in H_0^1(\Omega)$ be the unique solution to equation (2.14):*

$$\operatorname{div}(s_\delta A\nabla u_\delta) + k^2s_0\Sigma u_\delta = s_0f \text{ in } \Omega.$$

Assume that the media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary and f is compatible with the system. Then (u_δ) converges weakly to $NI(f)$ in $H^1(\Omega)$, where

$$(3.15) \quad NI(f) = \begin{cases} W & \text{if } x \in \Omega \setminus \Omega_3, \\ V & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ (U + V) \circ F & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{if } x \in \Omega_1. \end{cases}$$

Here U, V , and W are given in Definition 2.

If $f = 0$ in Ω_3 , then $U = 0$. In this case, the compatibility condition is equivalent to the existence of $V \in H^1(\Omega_3 \setminus \bar{\Omega}_2)$ to the Cauchy problem

$$(3.16) \quad \begin{cases} \operatorname{div}(A\nabla V) + k^2\Sigma V = 0 & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ V = W|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla V \cdot \eta = A\nabla W \cdot \eta|_{\text{ext}} & \text{on } \partial\Omega_3, \end{cases}$$

where $W \in H_0^1(\Omega)$ is the unique solution to

$$\operatorname{div}(\hat{A}\nabla W) + k^2\hat{\Sigma}W = f \text{ in } \Omega.$$

We have

Corollary 1. *Let $d = 2, 3$ and $f \in L^2(\Omega)$ with $\operatorname{supp} f \cap \Omega_3 = \emptyset$. Assume that the media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary and there exists a solution $V \in H^1(\Omega_3 \setminus \bar{\Omega}_2)$ to (3.16). Then f is compatible to the system.*

It follows from Corollary 1 that if $\operatorname{supp} f \cap \Omega_3 = \emptyset$ and if $G_*F_*A(x) = A(x)$ and $G_*F_*\Sigma(x) = \Sigma(x)$ for $x \in \Omega_3 \setminus \bar{\Omega}_2$, then $V = W$. We obtain

Corollary 2. *Let $d = 2, 3$ and $f \in L^2(\Omega)$ with $\operatorname{supp} f \cap \Omega_3 = \emptyset$. Assume that the media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary and*

$$(3.17) \quad G_*F_*A(x) = A(x) \quad \text{and} \quad G_*F_*\Sigma(x) = \Sigma(x) \quad \text{for } x \in \Omega_3 \setminus \bar{\Omega}_2.$$

Then f is compatible to the system.

Remark 3. It is easy to verify that the 2d setting considered in the introduction satisfies the assumptions of Corollary 2.

Remark 4. In this paper, we characterize the behavior of u_δ as $\delta \rightarrow 0$ for compatible f . The paper [10] develops the method introduced here to deal with this problem without the assumption on the compatibility in the cloaking setting.

3.2. Proofs of Theorems 1 and 2. This section containing two subsections is devoted to the proof of Theorems 1 and 2. In the first subsection, we establish basic properties of solutions to equation (2.14) such as existence, uniqueness, and stability, and establish a result on the change of variables concerning reflections. The proofs of Theorems 1 and 2 are given in the second subsection.

3.2.1. Preliminaries. This section contains two lemmas. The first one is on the well-posedness of (2.14).

Lemma 1. *Let $d = 2, 3, k > 0, 0 < \delta < 1, g \in H^{-1}(\Omega)$ (the duality of $H_0^1(\Omega)$) and let s_δ be defined in (2.13). Assume that A and Σ satisfy (2.11) and (2.12), and (3.5) holds. Then there exists a unique solution $v_\delta \in H_0^1(\Omega)$ to the equation*

$$(3.18) \quad \operatorname{div}(s_\delta A\nabla v_\delta) + k^2s_\delta\Sigma v_\delta = g \text{ in } \Omega.$$

Moreover,

$$(3.19) \quad \|v_\delta\|_{H^1(\Omega)} \leq C \left(\frac{1}{\delta} \|g\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Omega_1)} + \|g\|_{L^2(\Omega_2 \setminus \bar{\Omega}_1)} + \|g\|_{L^2(\Omega \setminus \bar{\Omega}_2)} \right),$$

for some positive constant C independent of g and δ , as δ is small.

Proof. The existence of v_δ follows from the uniqueness of v_δ by Fredholm’s theorem (see, e.g., [4]). We now establish the uniqueness of v_δ by showing that $v_\delta = 0$ if $v_\delta \in H_0^1(\Omega)$ is a solution to the equation

$$\operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = 0 \text{ in } \Omega.$$

Multiplying the above equation by \bar{v}_δ (the conjugate of v_δ) and integrating the obtained expression on Ω , we have

$$\int_\Omega s_\delta \langle A \nabla v_\delta, \nabla v_\delta \rangle dx - \int_\Omega k^2 s_0 \Sigma |v_\delta|^2 dx = 0.$$

This implies, by considering the imaginary part,

$$\int_{\Omega_2 \setminus \Omega_1} \langle A \nabla v_\delta, \nabla v_\delta \rangle dx = 0.$$

It follows from (2.11) that v_δ is constant in $\Omega_2 \setminus \Omega_1$. Thus $v_\delta = 0$ in $\Omega_2 \setminus \Omega_1$ since $\operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = 0$ in $\Omega_2 \setminus \Omega_1$. This implies $v_\delta = 0$ in $\Omega \setminus \Omega_2$ and in Ω_1 by (3.5). Here to establish $v_\delta = 0$ in Ω_1 , we considered the function V_δ defined in Ω as $V_\delta = v_\delta \circ F^{-1} \circ G^{-1}$ in Ω_3 and $V_\delta = 0$ in $\Omega \setminus \Omega_3$ and used the well-posedness of (3.7). The proof of the uniqueness is complete.

We next establish (3.19) by a contradiction argument. Assume that (3.19) is not true. Then there exists $(g_\delta) \subset H^{-1}(\Omega)$ such that

$$(3.20) \quad \|v_\delta\|_{H^1(\Omega)} = 1 \text{ and } \frac{1}{\delta} \|g_\delta\|_{H^{-1}} + \|g_\delta\|_{L^2(\Omega_1)} + \|g_\delta\|_{L^2(\Omega_2 \setminus \bar{\Omega}_1)} + \|g\|_{L^2(\Omega \setminus \bar{\Omega}_2)} \rightarrow 0,$$

as $\delta \rightarrow 0$, where $v_\delta \in H_0^1(\Omega)$ is the unique solution to the equation

$$(3.21) \quad \operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = g_\delta \text{ in } \Omega.$$

Multiplying this equation by \bar{v}_δ and integrating the obtained expression on Ω , we have

$$\int_\Omega s_\delta \langle A \nabla v_\delta, \nabla v_\delta \rangle dx - \int_\Omega k^2 s_0 \Sigma |v_\delta|^2 dx = - \int_\Omega g_\delta \bar{v}_\delta dx.$$

Considering the imaginary part and using the fact that

$$\frac{1}{\delta} \left| \int_\Omega g_\delta \bar{v}_\delta \right| \leq \frac{1}{\delta} \|g_\delta\|_{H^{-1}} \|v_\delta\|_{H^1(\Omega)} \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ by (3.20),}$$

we obtain, by (2.11),

$$(3.22) \quad \|\nabla v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Since $\operatorname{div}(A \nabla v_\delta) + k^2 \Sigma v_\delta = g_\delta$ in $\Omega_2 \setminus \bar{\Omega}_1$, it follows from (3.20) and a standard compactness argument that

$$(3.23) \quad \|v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

A combination of (3.22) and (3.23) yields

$$(3.24) \quad \|v_\delta\|_{H^1(\Omega_2 \setminus \Omega_1)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

In particular,

$$\|v_\delta\|_{H^{1/2}(\partial\Omega_2)} + \|v_\delta\|_{H^{1/2}(\partial\Omega_1)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We derive from the well-posedness of (3.6) and (3.7) that

$$(3.25) \quad \|v_\delta\|_{H^1(\Omega \setminus \Omega_2)} \rightarrow 0$$

and

$$(3.26) \quad \|v_\delta\|_{H^1(\Omega \setminus \Omega_1)} \rightarrow 0.$$

A combination of (3.24), (3.25), and (3.26) yields

$$\|v_\delta\|_{H^1(\Omega)} \rightarrow 0.$$

We have a contradiction by (3.20). The proof of (3.19) is complete. \square

The second lemma is on the change of variables for reflections.

Lemma 2. *Let $k \geq 0$, let D_1 and D_2 be two smooth open subsets of \mathbb{R}^d , let T be a diffeomorphism from D_1 onto D_2 , let $a \in [L^\infty(D_1)]^{d \times d}$ be a matrix function, and let $\sigma \in L^\infty(D_1)$ be a complex function. Fix $u \in H^1(D_1)$ and set $v = u \circ T^{-1}$. Then*

$$\operatorname{div}(a \nabla u) + k^2 \sigma u = f \text{ in } D_1$$

iff

$$\operatorname{div}(T_* a \nabla v) + k^2 T_* \sigma v = T_* f \text{ in } D_2.$$

Assume that Γ_1 and Γ_2 are open subsets of ∂D_1 and ∂D_2 such that Γ_1 and Γ_2 are smooth, $\Gamma_2 = T(\Gamma_1)$, and $\mathbf{T} := T|_{\Gamma_1} : \Gamma_1 \rightarrow \Gamma_2$ is a diffeomorphism.³ We have

$$a \nabla u \cdot \eta_1 = g_1 \text{ on } \Gamma_1$$

iff

$$T_* a \nabla v \cdot \eta_2 = g_2 \text{ on } \Gamma_2,$$

where⁴

$$g_2(y) = g_1(x) / |\det \nabla \mathbf{T}(x)| \text{ with } x = \mathbf{T}^{-1}(y).$$

Here η_1 and η_2 are the normal unit vectors on Γ_1 and Γ_2 directed to the exterior of D_1 and D_2 . In particular, if $\Gamma_1 = \Gamma_2$, $\mathbf{T}(x) = x$ on Γ_1 , $D_2 \cap D_1 = \emptyset$, then we have

$$(3.27) \quad T_* a \nabla v \cdot \eta_1 = -a \nabla u \cdot \eta_1 \text{ on } \Gamma_1 = \Gamma_2.$$

Proof. Lemma 2 is a consequence of the change of variables. The first equivalence relation is known and can be proved by using the weak formula for u and v . The second equivalence follows similarly. The details are left to the reader. \square

3.2.2. Proof of the first statement of Theorem 1 and Theorem 2. In this section, f is compatible. The proof is derived from the following steps:

Step 1. Let $v \in H_0^1(\Omega)$ be a solution to the equation

$$(3.28) \quad \operatorname{div}(s_0 A \nabla v) + k^2 s_0 \Sigma v = s_0 f \text{ in } \Omega.$$

We prove that $v = NI(f)$.

Step 2. Define $u_0 := NI(f)$. We prove that $u_0 \in H_0^1(\Omega)$ is a solution to the equation

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = s_0 f \text{ in } \Omega.$$

Step 3. We prove that $(u_\delta)_{0 < \delta < 1}$ is bounded in $H^1(\Omega)$.

³We assume here that there is an extension of T in a neighborhood of ∂D_1 (which is also called T) such that it is a diffeomorphism.

⁴In the identity below, $\nabla \mathbf{T}$ stands for the gradient of a transformation from a $(d-1)$ -manifold into a $(d-1)$ -manifold, and $\det \nabla \mathbf{T}$ denotes the determinant of a $(d-1) \times (d-1)$ -matrix.

Step 4. We prove that (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 as δ goes to 0.

It is clear that the proof of the first statement of Theorem 1 and Theorem 2 is complete after these four steps. We now process these steps.

Step 1. Assume that $v \in H_0^1(\Omega)$ is a solution to the equation

$$\operatorname{div}(s_0 A \nabla v) + k^2 s_0 \Sigma v = s_0 f \text{ in } \Omega.$$

Set

$$(3.29) \quad v_1 = v \circ F^{-1} \text{ in } \Omega_4 \setminus \bar{\Omega}_2$$

and

$$(3.30) \quad \hat{s}_0 = \begin{cases} 1 & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ -1 & \text{if } x \in \Omega_4 \setminus \Omega_3. \end{cases}$$

Then $v_1 \in H^1(\Omega_3 \setminus \bar{\Omega}_2) \cap H_{loc}^1(\Omega_4 \setminus \bar{\Omega}_2)$ and, by Lemma 2, v_1 satisfies the equation

$$(3.31) \quad \operatorname{div}(\hat{s}_0 F_* A \nabla v_1) + k^2 \hat{s}_0 F_* \Sigma v_1 = \hat{s}_0 F_* f \text{ in } \Omega_4 \setminus \bar{\Omega}_2,$$

and

$$v_1 = v \text{ on } \partial\Omega_2 \quad \text{and} \quad F_* A \nabla v_1 \cdot \eta = A \nabla v \cdot \eta \Big|_{\text{ext}} \text{ on } \partial\Omega_2.$$

In the last identity, we use the facts that $F_* A \nabla v_1 \cdot \eta = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by (3.27), and that $A \nabla v \cdot \eta \Big|_{\text{ext}} = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by the transmission condition on $\partial\Omega_2$. Define

$$(3.32) \quad \mathbf{U} = v_1 - v \text{ in } \Omega_3 \setminus \bar{\Omega}_2.$$

Since $F_* A = A$ and $F_* \Sigma = \Sigma$ in $\Omega_3 \setminus \bar{\Omega}_2$, it follows that

$$(3.33) \quad \begin{cases} \operatorname{div}(A \nabla \mathbf{U}) + k^2 \Sigma \mathbf{U} = F_* f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ \mathbf{U} = 0 & \text{on } \partial\Omega_2, \\ A \nabla \mathbf{U} \cdot \eta = 0 & \text{on } \partial\Omega_2. \end{cases}$$

Applying the unique continuation principle (see, e.g., [1, 17]), from (3.12), we have

$$(3.34) \quad \mathbf{U} = U \text{ in } \Omega_3 \setminus \bar{\Omega}_2.$$

Define v_2 in Ω as

$$(3.35) \quad v_2(x) = \begin{cases} v_1 \circ G^{-1}(x) & \text{if } x \in \Omega_3, \\ v(x) & \text{if } x \in \Omega \setminus \Omega_3. \end{cases}$$

Using (2.18) and applying Lemma 2, we have

$$(3.36) \quad \operatorname{div}(\hat{A} \nabla v_2) + k^2 \hat{\Sigma} v_2 = \hat{f} \text{ in } \Omega \setminus \partial\Omega_3,$$

and, on $\partial\Omega_3$,

$$\begin{aligned} \hat{A} \nabla v_2 \cdot \eta \Big|_{\text{ext}} - \hat{A} \nabla v_2 \cdot \eta \Big|_{\text{int}} &= A \nabla v \cdot \eta \Big|_{\text{ext}} + F_* A \nabla v_1 \cdot \eta \Big|_{\text{ext}} && \text{(by (3.27))} \\ &= A \nabla v \cdot \eta \Big|_{\text{ext}} - F_* A \nabla v_1 \cdot \eta \Big|_{\text{int}} && \text{(by (3.31))} \\ &= A \nabla v \cdot \eta \Big|_{\text{ext}} - F_* A \nabla(v + \mathbf{U}) \cdot \eta \Big|_{\text{int}} && \text{(by (3.32)).} \end{aligned}$$

Since $F_*A = A$ in $\Omega_3 \setminus \bar{\Omega}_2$, it follows from (3.34) that

$$(3.37) \quad \hat{A}\nabla v_2 \cdot \eta \Big|_{\text{ext}} - \hat{A}\nabla v_2 \cdot \eta \Big|_{\text{int}} = -A\nabla U \cdot \eta \Big|_{\text{int}} \text{ on } \partial\Omega_3.$$

Since $G(x) = x$ on $\partial\Omega_3$, we obtain, on $\partial\Omega_3$,

$$(3.38) \quad v_2 \Big|_{\text{ext}} - v_2 \Big|_{\text{int}} = v \Big|_{\text{ext}} - v_1 \Big|_{\text{ext}} = v \Big|_{\text{ext}} - v_1 \Big|_{\text{int}} = v \Big|_{\text{ext}} - (\mathbf{U} + v) \Big|_{\text{int}} = -U,$$

by (3.34). Combining (3.36), (3.37), (3.38), and (3.14), and applying the unique continuation principle, we have

$$(3.39) \quad v_2 = W \text{ in } \Omega.$$

Since $\text{div}(A\nabla v) + k^2\Sigma v = f$ in $\Omega_3 \setminus \bar{\Omega}_2$, it follows from (3.35) and (3.39) that

$$\begin{cases} \text{div}(A\nabla v) + k^2\Sigma v = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ v = W \Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla v \cdot \eta \Big|_{\text{int}} = A\nabla W \cdot \eta \Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases}$$

By the unique continuation principle, it follows from (3.13) that

$$(3.40) \quad v = V \text{ in } \Omega_3 \setminus \bar{\Omega}_2.$$

We claim that

$$v = \begin{cases} W & \text{in } \Omega \setminus \Omega_3, \\ V & \text{in } \Omega_3 \setminus \Omega_2, \\ (V + U) \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{in } \Omega_1. \end{cases}$$

In fact, the statement $v = W$ in $\Omega \setminus \Omega_3$ is a consequence of (3.35) and (3.39); the statement $v = V$ in $\Omega_3 \setminus \Omega_2$ follows from (3.40); the statement $v = (V + U) \circ F$ in $\Omega_2 \setminus \Omega_1$ is a consequence of the fact $v_1 = v + U = V + U$ in $\Omega_3 \setminus \Omega_2$, and the statement $v = v_1 \circ F$ in $\Omega_2 \setminus \Omega_1$; $v = W \circ G \circ F$ in Ω_1 is a consequence of the definition of v_1 and v_2 , and $v_2 = W$ in Ω_3 . The claim is proved. Therefore,

$$v = NI(f) \text{ in } \Omega.$$

The proof of Step 1 is complete.

Step 2. We claim that

$$(3.41) \quad \text{div}(A\nabla u_0) + k^2\Sigma u_0 = f \text{ in } \Omega \setminus (\partial\Omega_3 \cup \partial\Omega_2 \cup \partial\Omega_1),$$

where $u_0 := NI(f)$. Indeed, it is just a consequence of the definition of U, V and W and the facts that $F_*A = A$ and $F_*\Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$.

It remains to verify

$$(3.42) \quad [A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_3, \quad [s_0 A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_2,$$

and

$$(3.43) \quad [A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_1.$$

From the definitions of V in (3.13) and $NI(f)$ in (3.15), we have

$$[A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_3.$$

Since $U = 0$ and $A\nabla U \cdot \eta = 0$ on $\partial\Omega_2$, it follows from (3.27) that

$$[s_0 A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_2.$$

From (3.14) and (3.27), we have

$$[s_0 A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_1.$$

The proof of Step 2 is complete.

Step 3. Set

$$v_\delta = u_\delta - u_0 \text{ in } \Omega.$$

We have, in Ω ,

$$\operatorname{div}(s_\delta A\nabla v_\delta) + k^2 s_0 \Sigma v_\delta = \operatorname{div}(s_\delta A\nabla u_\delta) - \operatorname{div}(s_\delta A\nabla u_0) + k^2 s_0 \Sigma u_\delta - k^2 s_0 \Sigma u_0.$$

This implies

$$\operatorname{div}(s_\delta A\nabla v_\delta) + k^2 s_0 \Sigma v_\delta = \operatorname{div}[(s_0 - s_\delta)A\nabla u_0] \text{ in } \Omega.$$

By Lemma 1, we have

$$\|\nabla v_\delta\|_{L^2(\Omega)} \leq C\|\nabla u_0\|_{L^2(\Omega)},$$

which yields, since $u_\delta = v_\delta + u_0$,

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C\|\nabla u_0\|_{L^2(\Omega)}.$$

Since $u_\delta \in H_0^1(\Omega)$, by Poincaré’s inequality, it follows that

$$\|u_\delta\|_{H^1(\Omega)} \leq C\|\nabla u_0\|_{L^2(\Omega)}.$$

Step 3 is complete.

Step 4. The conclusion of Step 4 follows from Step 3 and the facts that the limit of u_δ (up to a subsequence) satisfies (3.28) and (3.28) has a unique solution in $H_0^1(\Omega)$ by Steps 1 and 2. □

3.2.3. Proof of the second statement of Theorem 1. In this section f is not compatible. We prove the second statement of Theorem 1 by contradiction. Assume that (3.11) is not true. Without loss of generality, there exists a bounded sequence (u_δ) in $H_0^1(\Omega)$ such that u_δ is the unique solution to the equation

$$\operatorname{div}(s_\delta A\nabla u_\delta) + k^2 s_0 \Sigma u_\delta = s_0 f \text{ in } \Omega,$$

and u_δ converges weakly to (some) $u \in H^1(\Omega)$ as $\delta \rightarrow 0$. It follows that $u \in H_0^1(\Omega)$ is a solution to the equation

$$\operatorname{div}(s_0 A\nabla u) + k^2 s_0 u = s_0 f \text{ in } \Omega.$$

Define

$$U = u \circ F^{-1} - u \text{ in } \Omega_3 \setminus \bar{\Omega}_2 \quad \text{and} \quad V = u \text{ in } \Omega_3 \setminus \bar{\Omega}_2.$$

As in Step 1 of Section 3.2.2, U and V satisfy (3.12) and (3.13) respectively. We have a contradiction since f is not compatible with the system.

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