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DISCONTINUOUS GALERKIN METHOD FOR FRACTIONAL CONVECTION-DIFFUSION EQUATIONS*

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Abstract. We propose a discontinuous Galerkin method for fractional convection-diffusion equations with a superdiffusion operator of order $\alpha(1 < \alpha < 2)$ defined through the fractional Laplacian. The fractional operator of order α is expressed as a composite of first order derivatives and a fractional integral of order $2 - \alpha$. The fractional convection-diffusion problem is expressed as a system of low order differential/integral equations, and a local discontinuous Galerkin method scheme is proposed for the equations. We prove stability and optimal order of convergence $\mathcal{O}(h^{k+1})$ for the fractional diffusion problem, and an order of convergence of $\mathcal{O}(h^{k+\frac{1}{2}})$ is established for the general fractional convection-diffusion problem. The analysis is confirmed by numerical examples.

Key words. fractional convection-diffusion equation, fractional Laplacian, fractional Burgers equation, discontinuous Galerkin method, stability, optimal convergence

AMS subject classifications. 26A33, 35R11, 65M60, 65M12

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1. Introduction.

We consider the fractional convection-diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon \left(-(-\Delta)^{\alpha/2}\right) u(x,t), & x \in \mathbb{R}, t \in (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where f is assumed to be Lipschitz continuous and the fractional diffusion is defined through the fractional Laplacian $-(-\Delta)^{\alpha/2}(\alpha \in (0,2])$, which can be defined using Fourier analysis as [24, 35, 42, 26]

$$-\widehat{(-\Delta)^{\alpha/2}} u(\xi) = (2\pi)^\alpha |\xi|^\alpha \hat{u}(\xi). \tag{1.2}$$

Equation (1.1) can be viewed as a generalization of the classical convection-diffusion equation. During the last decade, it has arisen as a suitable model in many application areas, such as geomorphology [27, 28, 2], overdriven detonations in gases [15, 1], signal processing [6], and anomalous diffusion in semiconductor growth [41]. With the special choice of $f(u) = u^2/2$, it is recognized as a fractional version of the viscous Burgers' equation. Fractional conservation laws, especially Burgers' equation, have been studied by many authors from a theoretical perspective, mainly addressing questions of well-posedness and regularity of the solutions [8, 29, 3, 33, 32, 22]. In the case of $\alpha < 1$, the solution is in general not smooth and shocks may appear even for smooth initial datum. Similar to the classical scalar conservation laws, an appropriate

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entropy formulation is needed to guarantee uniqueness and well-posedness [3, 4]. For the case of $\alpha > 1$, existence and uniqueness of a regular solution has been established in [3, 32]. In this case, the nonlocal term serves as a super-diffusion term, smoothing out discontinuities in the initial datum.

Numerical studies of partial differential equations with a nonlocal operator have attracted a lot of interest in recent years. Several authors [19, 45, 31, 21] have worked on numerical methods for fractional diffusion problems with fractional Laplacian operators or Riesz fractional derivatives. Others [10, 18, 34] have considered numerical methods for financial models with fractional Laplacian operators. However, for conservation laws with fractional Laplacian operators, the development of accurate and robust numerical methods remains limited. Droniou [23] appears to be the first to analyze a general class of difference methods for fractional conservation laws. Azerad and Bouharguane [7] proposed a class of finite difference schemes for solving a fractional antidiffusive equation, while Bouharguane [9] proposed Carles splitting methods for the nonlocal Fowler equation. For the case $\alpha < 1$, Cifani and others [13, 14] applied the discontinuous Galerkin method to the fractional conservation law and degenerate convection-diffusion equations and developed some error estimation. Unfortunately, their numerical results failed to confirm their analysis.

The discontinuous Galerkin method is a well-established method for classical conservation laws [16, 5, 30, 44]. For application of the method to fractional problems, Mustapha and McLean [36, 37, 38] have developed and analyzed discontinuous Galerkin methods for time fractional diffusion and wave equations, while Cifani and others [13, 14] applied such methods to the fractional conservation law. However, for equations containing higher order spatial derivatives, discontinuous Galerkin methods cannot be directly applied [17, 44]. A careless application of the discontinuous Galerkin method to a problem with high order derivatives may yield an inconsistent or unstable method [30]. The idea of local discontinuous Galerkin (LDG) methods for time-dependent partial differential equations with higher derivatives is to rewrite the equation into a first order system and then apply the discontinuous Galerkin method to the system [17]. A key ingredient for the success of this method is the correct design of interface numerical fluxes. These fluxes must be designed to guarantee stability and local solvability of the auxiliary variables, introduced to approximate the derivatives of the solution.

In this paper, we consider fractional convection-diffusion equations with a fractional Laplacian operator of order α $(1 < \alpha < 2)$. For $1 < \alpha < 2$, it is conceptually similar to a fractional derivative with an order between 1 and 2. To obtain a consistent and high-accuracy method for this problem, we rewrite the fractional operator as a composite of first order derivatives and a fractional integral and convert the fractional convection-diffusion equation into a system of low order equations. This allows us to apply the LDG method.

This paper is organized as follows. In section 2, we introduce some basic definitions and recall a few central results. In section 3, we derive the discontinuous Galerkin formulation for the fractional convection-diffusion problem, and in section 4, we present a stability and convergence analysis for fractional diffusion and convection-diffusion equations. Section 5 presents some numerical examples to support the analysis and illustrate the efficiency and flexibility of the scheme. A few concluding remarks are offered in section 6.

2. Definitions and background. Apart from the definitions of the fractional Laplacian based on the Fourier and the integral form, it can also be defined using ideas of fractional calculus [24, 35, 45], as

$$-(-\Delta)^{\alpha/2}u(x) = \frac{\partial^\alpha}{\partial|x|^\alpha}u(x) = -\frac{{}_{-\infty}D_x^\alpha u(x) + {}_xD_\infty^\alpha u(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)}, \tag{2.1}$$

where ${}_{-\infty}D_x^\alpha$ and ${}_xD_\infty^\alpha$ refer to the left and right Riemann–Liouville fractional derivatives, respectively, of αth order. This definition is also known as a Riesz derivative. In this paper, we will base our developments and analysis on this definition. To prepare we introduce a few definitions and recall some properties of fractional integrals and derivatives.

The left and right fractional integrals of order α are defined as

$$_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}u(t)dt, \tag{2.2}$$

$$_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}u(t)dt. \tag{2.3}$$

This allows for the definition of the left and right Riemann–Liouville fractional derivatives of order α $(n-1<\alpha<n)$ as

$$_{-\infty}D_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_{-\infty}^x (x-t)^{n-1-\alpha}u(t)dt = D^n({}_{-\infty}I_x^{n-\alpha}u(x)), \tag{2.4}$$

$$_xD_\infty^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^\infty (t-x)^{n-1-\alpha}u(t)dt = (-D)^n({}_xI_\infty^{n-\alpha}u(x)). \tag{2.5}$$

Fractional integrals and derivatives satisfy the following properties.

LEMMA 2.1 (linearity [39]).

$$_{-\infty}I_x^\alpha(\lambda f(x)+\mu g(x)) = \lambda\, {}_{-\infty}I_x^\alpha f(x) + \mu\, {}_{-\infty}I_x^\alpha g(x), \tag{2.6}$$

$$_{-\infty}D_x^\alpha(\lambda f(x)+\mu g(x)) = \lambda\, {}_{-\infty}D_x^\alpha f(x) + \mu\, {}_{-\infty}D_x^\alpha g(x). \tag{2.7}$$

LEMMA 2.2 (semigroup property [39]). *For any α, $\beta > 0$, we have the following semigroup property for the fractional integral operator:*

$$_{-\infty}I_x^{\alpha+\beta}f(x) = {}_{-\infty}I_x^\alpha\left({}_{-\infty}I_x^\beta f(x)\right) = {}_{-\infty}I_x^\beta\left({}_{-\infty}I_x^\alpha f(x)\right). \tag{2.8}$$

LEMMA 2.3 (see [39]). *Suppose $u^{(j)}(x) = 0$ for $x \to \pm\infty$. Then $\forall\, 0 \leqslant j \leqslant n$ $(n-1<\alpha<n)$, it holds that*

$$_{-\infty}D_x^\alpha u(x) = D^n\left({}_{-\infty}I_x^{n-\alpha}u(x)\right) = {}_{-\infty}I_x^{n-\alpha}\left(D^n u(x)\right), \tag{2.9}$$

$$_xD_\infty^\alpha u(x) = (-D)^n\left({}_xI_\infty^{n-\alpha}u(x)\right) = {}_xI_\infty^{n-\alpha}\left((-D)^n u(x)\right). \tag{2.10}$$

To rewrite the fractional Laplacian in system form, which will be useful later, we apply Lemma 2.3 to obtain the following result:

$$\begin{aligned} -(-\Delta)^{\alpha/2}u(x) &= -\frac{1}{2\cos(\alpha\pi/2)}\frac{d^2}{dx^2}\left({}_{-\infty}I_x^{2-\alpha}u(x) + {}_xI_\infty^{2-\alpha}u(x)\right) \\ &= -\frac{1}{2\cos(\alpha\pi/2)}\frac{d}{dx}\left({}_{-\infty}I_x^{2-\alpha}\frac{du(x)}{dx} + {}_xI_\infty^{2-\alpha}\frac{du(x)}{dx}\right) \\ &= -\frac{1}{2\cos(\alpha\pi/2)}\left({}_{-\infty}I_x^{2-\alpha}\frac{d^2u(x)}{dx^2} + {}_xI_\infty^{2-\alpha}\frac{d^2u(x)}{dx^2}\right) \end{aligned} \tag{2.11}$$

for any continuous function u with $\lim_{x\to\pm\infty} u^{(j)}(x) = 0$ $(0 \leqslant j \leqslant 2)$ and $1 < \alpha < 2$. In (2.11), if $\alpha < 0$, the fractional Laplacian becomes the fractional integral operator. In this case, for any $0 < s < 1$, we define

$$\Delta_{-s/2}u(x) = -\frac{{}_{-\infty}D_x^{-s}u(x) + {}_xD_\infty^{-s}u(x)}{2\cos\left(\frac{(2-s)\pi}{2}\right)} = \frac{{}_{-\infty}I_x^{s}u(x) + {}_xI_\infty^{s}u(x)}{2\cos\left(\frac{s\pi}{2}\right)}. \tag{2.12}$$

When $1 < \alpha < 2$, using (2.11) and (2.12), we can rewrite the fractional Laplacian in the following form:

$$-(-\Delta)^{\alpha/2}u(x) = \frac{d^2}{dx^2}\left(\Delta_{(\alpha-2)/2}u\right) = \frac{d}{dx}\left(\Delta_{(\alpha-2)/2}\frac{du}{dx}\right). \tag{2.13}$$

To carry out the analysis, we introduce the appropriate fractional spaces.

DEFINITION 2.4 (left fractional space [25]). *We define the seminorm*

$$|u|_{J_L^\alpha(\mathbb{R})} = \|\,{}_{-\infty}D_x^\alpha u\|_{L^2(\mathbb{R})} \tag{2.14}$$

and the norm

$$\|u\|_{J_L^\alpha(\mathbb{R})} = \left(|u|^2_{J_L^\alpha(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \tag{2.15}$$

and let $J_L^\alpha(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_L^\alpha(\mathbb{R})}$.

DEFINITION 2.5 (right fractional space [25]). *We define the seminorm*

$$|u|_{J_R^\alpha(\mathbb{R})} = \|\,{}_xD_\infty^\alpha u\|_{L^2(\mathbb{R})} \tag{2.16}$$

and the norm

$$\|u\|_{J_R^\alpha(\mathbb{R})} = \left(|u|^2_{J_R^\alpha(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \tag{2.17}$$

and let $J_R^\alpha(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_R^\alpha(\mathbb{R})}$.

DEFINITION 2.6 (symmetric fractional space [25]). *We define the seminorm*

$$|u|_{J_S^\alpha(\mathbb{R})} = \left|\left({}_{-\infty}D_x^\alpha u, {}_xD_\infty^\alpha u\right)_{L^2(\mathbb{R})}\right|^{\frac{1}{2}} \tag{2.18}$$

and the norm

$$\|u\|_{J_S^\alpha(\mathbb{R})} = \left(|u|^2_{J_S^\alpha(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \tag{2.19}$$

and let $J_S^\alpha(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_S^\alpha(\mathbb{R})}$.

Using these definitions, we obtain the following result.

LEMMA 2.7 (adjoint property [40, 25]).

$$({}_{-\infty}I_x^\alpha u, u)_{\mathbb{R}} = (u, {}_xI_\infty^\alpha u)_{\mathbb{R}}\,. \tag{2.20}$$

LEMMA 2.8 (see [25]).

$$({}_{-\infty}I_x^\alpha u, {}_xI_\infty^\alpha u)_{\mathbb{R}} = \cos(\alpha\pi)|u|^2_{J_L^{-\alpha}(\mathbb{R})} = \cos(\alpha\pi)|u|^2_{J_R^{-\alpha}(\mathbb{R})}. \tag{2.21}$$

From Lemmas 2.7 and 2.8, we obtain the following lemma.

LEMMA 2.9. *For any $0 < s < 1$, the fractional integral satisfies the following property:*

$$(\Delta_{-s}u, u)_{\mathbb{R}} = |u|^2_{J_L^{-s}(\mathbb{R})} = |u|^2_{J_R^{-s}(\mathbb{R})}. \tag{2.22}$$

Generally, we consider the problem in a bounded domain instead of $\mathbb{R}$. Hence, we restrict the definition to the domain $\Omega = [a, b]$.

DEFINITION 2.10. *Define the spaces $J^\alpha_{L,0}(\Omega), J^\alpha_{R,0}(\Omega), J^\alpha_{S,0}(\Omega)$ as the closures of $C_0^\infty(\Omega)$ under their respective norms.*

For these fractional spaces, we have the following theorem [20].

THEOREM 2.11. *If $-\alpha_2 < -\alpha_1 < 0$, then $J^{-\alpha_1}_{L,0}(\Omega)(or\, J^{-\alpha_1}_{R,0}(\Omega)\, or\, J^{-\alpha_1}_{S,0}(\Omega))$ is embedded into $J^{-\alpha_2}_{L,0}(\Omega)(or\, J^{-\alpha_2}_{R,0}(\Omega)\, or\, J^{-\alpha_2}_{S,0}(\Omega))$, and $L^2(\Omega)$ is embedded into both of them.*

LEMMA 2.12 (fractional Poincaré–Friedrichs, [25]). *For $u \in J^\mu_{L,0}(\Omega)$ and $\mu \in \mathbb{R}$, we have*

$$\|u\|_{L^2(\Omega)} \leqslant C|u|_{J^\mu_{L,0}(\Omega)},$$

and for $u \in J^\mu_{R,0}(\Omega)$, we have

$$\|u\|_{L^2(\Omega)} \leqslant C|u|_{J^\mu_{R,0}(\Omega)}.$$

From the definition of the left and right fractional integrals, we obtain the following lemma.

LEMMA 2.13. *Suppose the fractional integral is defined on $[0, b]$ and let $g(y) = f(b - y)$. Then*

$${}_xI^\alpha_b f(x) \overset{y=b-x}{=} {}_0I^\alpha_y g(y). \tag{2.23}$$

LEMMA 2.14. *Suppose $u(x)$ is a smooth function defined on $\Omega \subset \mathbb{R}$. Ω_h is a discretization of the domain with interval width h, $u_h(x)$ is an approximation of u in P^k_h. For all i, $u_h(x) \in I_i$ is a polynomial of degree up to order k, and $(u, v)_{I_i} = (u_h, v)_{I_i}\, \forall v \in P^k$. k is the degree of the polynomial. Then for $-1 < \alpha \leqslant 0$, we have*

$$\|\Delta_{\alpha/2}u(x) - \Delta_{\alpha/2}u_h(x)\|_{L_2(\Omega)} \leqslant Ch^{k+1},$$

where C is a constant independent of h.

Proof. We consider the approximation error for a fractional integral $\|\, {}_aI^{-\alpha}_x u(x) - {}_aI^{-\alpha}_x u_h(x)\|_{L_2(\Omega)}$.

Suppose $x \in \Omega_{h,i}$, and recall that $\|u(x) - u_h(x)\|_{L_2} = \mathcal{O}(h^{k+1})$ from classical approximation theory. We have the following estimate for $r^i = {}_aI^{-\alpha}_x(\mathcal{O}(h^{k+1}))$:

$$\begin{aligned} r^i &= \frac{1}{\Gamma(-\alpha)} \int_a^x (x-s)^{-\alpha-1} \mathcal{O}(h^{k+1}) ds \\ &\leqslant \frac{(b-a)^{-\alpha}\mathcal{O}(h^{k+1})}{\Gamma(1-\alpha)}. \end{aligned}$$

Recalling Lemma 2.13, the case $-1 < \alpha \leqslant 0$ is proved. □

LEMMA 2.15 (inverse and trace properties [12]). *Suppose V_h is a finite element space spanned by polynomials up to degree k. For any $u_h \in V_h$, there exists a positive constant C independent of u_h and h such that*

$$\|\partial_x u_h\|_{L_2(\Omega)} \leqslant Ch^{-1}\|u_h\|_{L_2(\Omega)}, \quad \|u_h\|_{\Gamma_h} \leqslant Ch^{-\frac{1}{2}}\|u_h\|_{L_2(\Omega)},$$

where Γ_h represents the trace.

3. LDG scheme for the fractional convection-diffusion equation. Let us consider the fractional convection-diffusion equation with $1 < \alpha < 2$. To obtain a high order discontinuous Galerkin scheme for the fractional derivative, we rewrite the fractional derivative as a composite of first order derivatives and a fractional integral to recover the equation to a low order system.

Following (2.13), we introduce two variables p, q and set

$$\begin{aligned} q &= \Delta_{(\alpha-2)/2} p, \\ p &= \sqrt{\varepsilon}\frac{\partial}{\partial x} u. \end{aligned}$$

Then, the fractional convection-diffusion problem can be rewritten as

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) &= \sqrt{\varepsilon}\frac{\partial}{\partial x} q, \\ q &= \Delta_{(\alpha-2)/2} p, \\ p &= \sqrt{\varepsilon}\frac{\partial}{\partial x} u, \end{aligned}$$

where $\Delta_{(\alpha-2)/2}$ is a fractional integral operator, as defined in (2.13).

Consider $\Omega = [a, b]$ with a partition $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{K+\frac{1}{2}} = b$; we denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$.

We consider the solution in a polynomial space V_h, which is embedded into the fractional space according to Theorem 2.11. The piecewise polynomial space V_h is defined as

$$V_h = \{v : v \in P^k(I_j), x \in I_j\}.$$

We seek an approximation $(u_h, p_h, q_h) \in V_h$ to (u, p, q) such that for any $v, w, z \in V_h$ and any element I_i, we have

$$\begin{aligned} \left(\frac{\partial u_h(x,t)}{\partial t}, v(x)\right)_{I_i} + \left(\frac{\partial}{\partial x} f(u_h), v(x)\right)_{I_i} &= \sqrt{\varepsilon}\left(\frac{\partial q_h}{\partial x}, v(x)\right)_{I_i}, \\ (q_h, w(x))_{I_i} &= \left(\Delta_{(\alpha-2)/2} p_h, w(x)\right)_{I_i}, \\ (p_h, z(x))_{I_i} &= \sqrt{\varepsilon}\left(\frac{\partial u_h}{\partial x}, z(x)\right)_{I_i}, \\ (u_h(x,0), v(x))_{I_i} &= (u_0(x), v(x))_{I_i}. \end{aligned} \tag{3.1}$$

To complete the LDG scheme, we introduce some notation and the numerical flux. Define

$$u^{\pm}(x_j) = \lim_{x \to x_j^{\pm}} u(x), \quad \{\{u\}\} = \frac{u^+ + u^-}{2}, \quad [\![u]\!] = u^+ - u^-$$

and the numerical flux as

$$\hat{u} = h_u(u^-, u^+), \quad \hat{q} = h_q(q^-, q^+), \quad \hat{f}_h = \hat{f}(u_h^-, u_h^+).$$

For the high order derivative part, a good choice is the alternating direction flux [17, 44], defined as

$$\hat{u}_{i+\frac{1}{2}} = u^-_{i+\frac{1}{2}}, \quad \hat{q}_{i+\frac{1}{2}} = q^+_{i+\frac{1}{2}}, \ 0 \leqslant i \leqslant K-1,$$

or the alternative choice

$$\hat{u}_{i+\frac{1}{2}} = u^+_{i+\frac{1}{2}}, \quad \hat{q}_{i+\frac{1}{2}} = q^-_{i+\frac{1}{2}},\ 1 \leqslant i \leqslant K.$$

For the nonlinear part, $\hat{f}$, any monotone flux can be used [30].

Applying integration by parts to (3.1), and replacing the fluxes at the interfaces by the corresponding numerical fluxes, we obtain

$$((u_h)_t, v)_{I_i} + \left(\hat{f}_h v - \sqrt{\varepsilon}\hat{q_h} v\right)\Big|_{x^+_{i-\frac{1}{2}}}^{x^-_{i+\frac{1}{2}}} - \left(f(u_h) - \sqrt{\varepsilon} q_h, v_x\right)_{I_i} = 0, \tag{3.2}$$

$$(q_h, w(x))_{I_i} - \left(\Delta_{(\alpha-2)/2} p_h, w(x)\right)_{I_i} = 0, \tag{3.3}$$

$$(p_h, z(x))_{I_i} - \sqrt{\varepsilon}\hat{u_h} z\Big|_{x^+_{i-\frac{1}{2}}}^{x^-_{i+\frac{1}{2}}} + \sqrt{\varepsilon}\,(u_h, z_x)_{I_i} = 0, \tag{3.4}$$

$$(u_h(x,0), v(x))_{I_i} - (u_0(x), v(x))_{I_i} = 0. \tag{3.5}$$

Remark 3.1. Originally, the problem is defined in $\mathbb{R}$. However, for numerical purposes, we assume there exists a domain $\Omega = [a,b] \subset \mathbb{R}$ in which u has compact support and restrict the problem to this domain Ω. As a consequence, we impose homogeneous Dirichlet boundary conditions for $u \in \mathbb{R}\backslash\Omega$ to obtain

$$-(-\Delta)^{\alpha/2} u(x) = -\frac{{}_{-\infty}D^\alpha_x u(x) + {}_x D^\alpha_\infty u(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)} = -\frac{{}_a D^\alpha_x u(x) + {}_x D^\alpha_b u(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)}. \tag{3.6}$$

For the flux at the boundary, we use the flux introduced in [11], defined as

$$\hat{u}_{K+\frac{1}{2}} = u(b,t), \quad \hat{q}_{K+\frac{1}{2}} = q^-_{K+\frac{1}{2}} + \frac{\beta}{h}[\![u_{K+\frac{1}{2}}]\!]$$

for the right boundary or

$$\hat{u}_{\frac{1}{2}} = u(a,t), \quad \hat{q}_{\frac{1}{2}} = q^-_{\frac{1}{2}} + \frac{\beta}{h}[\![u_{\frac{1}{2}}]\!]$$

for the left boundary, where β is a positive constant.

4. Stability and error estimates. In the following we discuss stability and accuracy of the proposed scheme, both for the fractional diffusion problem and the more general convection-diffusion problem.

4.1. Stability. In order to carry out the analysis of the LDG scheme, we define

(4.1)

$$\begin{aligned}\mathscr{B}(u,p,q;v,w,z) = &\int_0^T \sum_{i=1}^K (u_t, v)_{I_i}\, dt + \int_0^T \sum_{i=1}^K \left(\hat{f} v - \sqrt{\varepsilon}\hat{q} v\right)\Big|_{x^+_{i-\frac{1}{2}}}^{x^-_{i+\frac{1}{2}}} dt \\ &- \int_0^T \sum_{i=1}^K \left(f(u) - \sqrt{\varepsilon} q, v_x\right)_{I_i} dt + \int_0^T \sum_{i=1}^K (q, w(x))_{I_i}\, dt \\ &- \int_0^T \sum_{i=1}^K \left(\Delta_{(\alpha-2)/2} p, w(x)\right)_{I_i} dt - \int_0^T \sum_{i=1}^K \sqrt{\varepsilon}\hat{u} z\Big|_{x^+_{i-\frac{1}{2}}}^{x^-_{i+\frac{1}{2}}} dt \\ &+ \int_0^T \sum_{i=1}^K (p, z(x))_{I_i}\, dt + \int_0^T \sum_{i=1}^K \sqrt{\varepsilon}\,(u, z_x)_{I_i}\, dt - \int_0^T \mathscr{L}(v,w,z)dt,\end{aligned}$$

where $\mathscr{L}$ contains the boundary term, defined as

$$\mathscr{L}(v,w,z) = \sqrt{\varepsilon}u(a,t)z^+_{\frac{1}{2}} - \frac{\sqrt{\varepsilon}\beta}{h}u(b,t)v^-_{K+\frac{1}{2}}dt + \sqrt{\varepsilon}u(b,t)z^-_{K+\frac{1}{2}} = 0. \tag{4.2}$$

If (u,p,q) is a solution, then $\mathscr{B}(u,p,q;v,w,z) = 0$ for any (v,w,z). Using the fluxes $\hat{u}_{i+\frac{1}{2}} = u^-_{i+\frac{1}{2}}, \hat{q}_{i+\frac{1}{2}} = q^+_{i+\frac{1}{2}}$ and the flux at the boundaries we obtain

$$\begin{aligned}
&\mathscr{B}(u,p,q;v,w,z) \\
&\quad= \int_0^T \sum_{i=1}^K (u_t,v)_{I_i}\,dt - \int_0^T \sum_{i=1}^K (f(u),v_x)_{I_i}\,dt + \int_0^T \sum_{i=1}^K \left(\sqrt{\varepsilon}q,v_x\right)_{I_i}dt \\
&\quad+ \int_0^T \sum_{i=1}^K \sqrt{\varepsilon}\,(u,z_x)_{I_i}\,dt + \int_0^T \sum_{i=1}^K (q,w(x))_{I_i}\,dt \\
&\quad- \int_0^T \sum_{i=1}^K \left(\Delta_{(\alpha-2)/2}p,w(x)\right)_{I_i}dt + \int_0^T \sum_{i=1}^K (p,z(x))_{I_i}\,dt \\
&\quad- \int_0^T \sum_{i=1}^{K-1} \hat{f}_{i+\frac{1}{2}}[\![v]\!]_{i+\frac{1}{2}}dt + \int_0^T \sum_{i=1}^{K-1} \sqrt{\varepsilon}q^+_{i+\frac{1}{2}}[\![v]\!]_{i+\frac{1}{2}}dt \\
&\quad+ \int_0^T \sum_{i=1}^{K-1} \sqrt{\varepsilon}u^-_{i+\frac{1}{2}}[\![z]\!]_{i+\frac{1}{2}}dt - \int_0^T (\hat{f}_{\frac{1}{2}}v^+_{\frac{1}{2}} - \hat{f}_{K+\frac{1}{2}}v^-_{K+\frac{1}{2}})dt \\
&\quad+ \int_0^T \sqrt{\varepsilon}(q^+_{\frac{1}{2}}v^+_{\frac{1}{2}} - q^-_{K+\frac{1}{2}}v^-_{K+\frac{1}{2}})dt + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h}u^-_{K+\frac{1}{2}}v^-_{K+\frac{1}{2}}dt.
\end{aligned} \tag{4.3}$$

LEMMA 4.1. *Set $(v,w,z) = (u,-p,q)$ in (4.3), and define $\Phi(u) = \int^u f(u)du$. Then the following result holds:*

$$\begin{aligned}
&\mathscr{B}(u,p,q;u,-p,q) \\
&\quad= \|u(x,T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)} + \int_0^T (\Delta_{(\alpha-2)/2}p,p)dt + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h}(u^-_{K+\frac{1}{2}})^2dt \\
&\quad+ \int_0^T \left(\Phi(u)_{\frac{1}{2}} - \Phi(u)_{K+\frac{1}{2}} - (\hat{f}u)_{\frac{1}{2}} + (\hat{f}u)_{K+\frac{1}{2}}\right)dt \\
&\quad+ \int_0^T \sum_{j=1}^{K-1} \left([\![\Phi(u)]\!]_{j+\frac{1}{2}} - \hat{f}[\![u]\!]_{j+\frac{1}{2}}\right)dt.
\end{aligned}$$

Proof. Set $(v,w,z) = (u,-p,q)$ in (4.3), and consider the integration by parts formula $(q,u_x)_{I_i} + (u,q_x)_{I_i} = (uq)|^{x^-_{i+\frac{1}{2}}}_{x^+_{i-\frac{1}{2}}}$, to obtain the interface condition

$$\begin{aligned}
&\sum_{i=1}^K \left(\sqrt{\varepsilon}q,v_x\right)_{I_i} + \sum_{i=1}^K \left(\sqrt{\varepsilon}u,z_x\right)_{I_i} + \sum_{i=1}^{K-1} \sqrt{\varepsilon}q^+_{i+\frac{1}{2}}[\![v]\!]_{i+\frac{1}{2}} + \sum_{i=1}^{K-1} \sqrt{\varepsilon}u^-_{i+\frac{1}{2}}[\![z]\!]_{i+\frac{1}{2}} \\
&\quad= \sum_{i=1}^K \sqrt{\varepsilon}\,(uq)\,|^{x^-_{i+\frac{1}{2}}}_{x^+_{i-\frac{1}{2}}} + \sum_{i=1}^{K-1} \sqrt{\varepsilon}q^+_{i+\frac{1}{2}}[\![u]\!]_{i+\frac{1}{2}} + \sum_{i=1}^{K-1} \sqrt{\varepsilon}u^-_{i+\frac{1}{2}}[\![q]\!]_{i+\frac{1}{2}} \\
&\quad= -\sqrt{\varepsilon}q^+_{\frac{1}{2}}u^+_{\frac{1}{2}} + \sqrt{\varepsilon}q^-_{K+\frac{1}{2}}u^-_{K+\frac{1}{2}}.
\end{aligned}$$

Substituting this equation into (4.3), we have

$$\begin{aligned}\mathscr{B}(u,p,q;u,-p,q) &= \int_0^T \sum_{i=1}^K (u_t,u)_{I_i}\,dt - \int_0^T \sum_{i=1}^K (f(u),u_x)_{I_i}\,dt \\ &+ \int_0^T \sum_{i=1}^K \left(\Delta_{(\alpha-2)/2}p,p\right)_{I_i}dt - \int_0^T \sum_{i=1}^{K-1} \hat{f}_{i+\frac12}[\![u]\!]_{i+\frac12}dt \\ &- \int_0^T (\hat{f}_{\frac12}u^+_{\frac12} - \hat{f}_{K+\frac12}u^-_{K+\frac12})dt + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h}\left(u^-_{K+\frac12}\right)^2 dt.\end{aligned} \tag{4.4}$$

Define $\Phi(u) = \int^u f(u)du$; then

$$\sum_{i=1}^K (f(u),u_x)_{I_i} = \sum_{i=1}^K \Phi(x)\Big|_{x^+_{i-\frac12}}^{x^-_{i+\frac12}} = -\sum_{i=1}^{K-1}[\![\Phi(u)]\!]_{i+\frac12} - \Phi(u)_{\frac12} + \Phi(u)_{K+\frac12}. \tag{4.5}$$

Combining (4.4) and (4.5) proves the lemma. ∎

THEOREM 4.2. *The semidiscrete scheme* (3.2)–(3.5) *is stable, and* $\|u_h(x,T)\|_{L^2(\Omega)} \leqslant \|u_0(x)\|_{L^2(\Omega)}$ *for any* $T > 0$.

Proof. From the properties of the monotone flux, we know that $\hat{f}(u^-,u^+)$ is a nondecreasing function of its first argument and a nonincreasing function of its second argument. Hence, we have $[\![\Phi(u_h)]\!]_{j+\frac12} - \hat{f}_h[\![u_h]\!]_{j+\frac12} > 0,\ 1 \leqslant j \leqslant K-1$. By Galerkin orthogonality, $\mathscr{B}(u_h,p_h,q_h;u_h,-p_h,q_h) = 0$. Lemma 4.1 yields

$$\begin{aligned}&\|u(x,T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)} + \int_0^T (\Delta_{(\alpha-2)/2}p,p)dt + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h}(u^-_{K+\frac12})^2dt \\ &+ \int_0^T \Phi(u)_{\frac12} - \Phi(u)_{K+\frac12} - (\hat{f}u)_{\frac12} + (\hat{f}u)_{K+\frac12}dt \leqslant 0.\end{aligned}$$

Considering the boundary condition and Lemma 2.9, we obtain $\|u_h(x,T)\| \leqslant \|u_0(x)\|$, hence completing the proof. ∎

4.2. Error estimates. To estimate the error, we first consider fractional diffusion with the Laplacian operator, i.e., the case with $f = 0$. For fractional diffusion, (3.2)–(3.5) reduce to

$$((u_h)_t,v)_{I_i} - (\sqrt{\varepsilon}\hat{q_h}v)\Big|_{x^+_{i-\frac12}}^{x^-_{i+\frac12}} + \left(\sqrt{\varepsilon}q_h,v_x\right)_{I_i} = 0, \tag{4.6}$$

$$(q_h,w(x))_{I_i} - \left(\Delta_{(\alpha-2)/2}p_h,w(x)\right)_{I_i} = 0, \tag{4.7}$$

$$(p_h,z(x))_{I_i} - \sqrt{\varepsilon}\hat{u_h}z\Big|_{x^+_{i-\frac12}}^{x^-_{i+\frac12}} + \sqrt{\varepsilon}\,(u_h,z_x)_{I_i} = 0, \tag{4.8}$$

$$(u_h(x,0),v(x))_{I_i} - (u_0(x),v(x))_{I_i} = 0. \tag{4.9}$$

Correspondingly, we have the compact form of the scheme as

$$\begin{aligned}&\mathscr{B}(u,p,q;v,w,z) \\ &= \int_0^T \sum_{i=1}^K (u_t,v)_{I_i}\,dt + \int_0^T \sum_{i=1}^K \sqrt{\varepsilon}\,(q,v_x)_{I_i}\,dt + \int_0^T \sum_{i=1}^K \sqrt{\varepsilon}\,(u,z_x)_{I_i}\,dt\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&+\int_0^T \sum_{i=1}^{K} (q, w(x))_{I_i}\, dt - \int_0^T \sum_{i=1}^{K} \left(\Delta_{(\alpha-2)/2} p, w(x)\right)_{I_i} dt \\
&+\int_0^T \sum_{i=1}^{K} (p, z(x))_{I_i}\, dt + \int_0^T \sum_{i=1}^{K-1} \sqrt{\varepsilon} q^+_{i+\frac{1}{2}} [\![v]\!]_{i+\frac{1}{2}} dt \\
&+\int_0^T \sum_{i=1}^{K-1} \sqrt{\varepsilon} u^-_{i+\frac{1}{2}} [\![z]\!]_{i+\frac{1}{2}} dt + \int_0^T \sqrt{\varepsilon} q^+_{\frac{1}{2}} v^+_{\frac{1}{2}} dt \\
&+\int_0^T \frac{\sqrt{\varepsilon}\beta}{h} u^-_{K+\frac{1}{2}} v^-_{K+\frac{1}{2}} dt - \int_0^T \sqrt{\varepsilon} q^-_{K+\frac{1}{2}} v^-_{K+\frac{1}{2}} dt.
\end{aligned}
$$

In the case $f(u) = 0$, we define $\mathscr{R}(u, p, q; v, w, z) = \mathscr{B}(u, p, q; v, w, z)$. Clearly, $\mathscr{R}$ is a bilinear operator.

To prepare for the main result, we first obtain a few central lemmas. We define special projections, $\mathscr{P}^\pm$ and $\mathscr{Q}$ into V_h, which satisfy, for each j,

$$
\int_{I_j} (\mathscr{P}^\pm u(x) - u(x)) v(x) dx = 0 \quad \forall v \in P^{k-1} \quad \text{and} \quad \mathscr{P}^\pm u_{j+\frac{1}{2}} = u(x^\pm_{j+\frac{1}{2}}), \tag{4.11}
$$

$$
\int_{I_j} (\mathscr{Q} u(x) - u(x)) v(x) dx = 0 \quad \forall v \in P^k. \tag{4.12}
$$

Denote $e_u = u - u_h, e_p = p - p_h, e_q = q - q_h$; then

$$
\mathscr{P}^- e_u = \mathscr{P}^- u - u_h, \mathscr{P}^+ e_q = \mathscr{P}^+ q - q_h, \mathscr{Q} e_p = \mathscr{Q} p - p_h.
$$

For any $(v, w, z) \in H^1(\Omega, \mathcal{T}) \times L^2(\Omega, \mathcal{T}) \times L^2(\Omega, \mathcal{T})$,

$$
\mathscr{R}(u, p, q; v, w, z) = \mathscr{L}(v, w, z). \tag{4.13}
$$

Hence, $\mathscr{R}(e_u, e_p, e_q; v, w, z) = 0$ and we obtain

$$
\begin{aligned}
&\mathscr{R}(\mathscr{P}^- e_u, \mathscr{Q} e_p, \mathscr{P}^+ e_q; \mathscr{P}^- e_u, -\mathscr{Q} e_p, \mathscr{P}^+ e_q) \\
&\quad = \mathscr{R}(\mathscr{P}^- e_u - e_u, \mathscr{Q} e_p - e_p, \mathscr{P}^+ e_q - e_q; \mathscr{P}^- e_u, -\mathscr{Q} e_p, \mathscr{P}^+ e_q) \\
&\quad = \mathscr{R}(\mathscr{P}^- u - u, \mathscr{Q} p - p, \mathscr{P}^+ q - q; \mathscr{P}^- e_u, -\mathscr{Q} e_p, \mathscr{P}^+ e_q).
\end{aligned} \tag{4.14}
$$

Substitute $(\mathscr{P}^- u - u, \mathscr{Q} p - p, \mathscr{P}^+ q - q; \mathscr{P}^- e_u, -\mathscr{Q} e_p, \mathscr{P}^+ e_q)$ into (4.10) to obtain the following lemma.

LEMMA 4.3. *For the bilinear form* (4.10), *we have*

$$
\begin{aligned}
&\mathscr{R}(\mathscr{P}^- u - u, \mathscr{Q} p - p, \mathscr{P}^+ q - q; \mathscr{P}^- e_u, -\mathscr{Q} e_p, \mathscr{P}^+ e_q) \\
&\quad \leqslant \int_0^T \sum_{i=1}^{K} \left((\mathscr{P}^- u)_t - u_t, \mathscr{P}^- e_u\right)_{I_i} dt + C_{T,a,b} h^{2k+2} + \frac{1}{C_{T,a,b}} \int_0^T \sum_{i=1}^{K} \|\mathscr{Q} e_p\|^2_{L^2(I_i)} dt \\
&\qquad + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h} |(\mathscr{P}^- e_u)^-_{K+\frac{1}{2}}|^2 dt,
\end{aligned}
$$

where $C_{T,a,b}$ is independent of h but may depend on T and Ω.

Proof. From (4.10) we have

$$
\begin{aligned}
&\mathscr{R}(\mathscr{P}^-u-u,\mathscr{Q}p-p,\mathscr{P}^+q-q;\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q)\\
&\quad=\int_0^T\sum_{i=1}^K\big((\mathscr{P}^-u)_t-u_t,\mathscr{P}^-e_u\big)_{I_i}dt+\int_0^T\sum_{i=1}^K\sqrt{\varepsilon}\big(\mathscr{P}^+q-q,(\mathscr{P}^-e_u)_x\big)_{I_i}dt\\
&\qquad+\int_0^T\sum_{i=1}^K\sqrt{\varepsilon}\big(\mathscr{P}^-u-u,(\mathscr{P}^+e_q)_x\big)_{I_i}dt-\int_0^T\sum_{i=1}^K\big(\mathscr{P}^+q-q,\mathscr{Q}e_p\big)_{I_i}dt\\
&\qquad+\int_0^T\sum_{i=1}^K\big(\Delta_{(\alpha-2)/2}(\mathscr{Q}p-p),\mathscr{Q}e_p\big)_{I_i}dt+\int_0^T\sum_{i=1}^K\big(\mathscr{Q}p-p,\mathscr{P}^+e_q\big)_{I_i}dt\\
&\qquad+\int_0^T\sum_{i=1}^{K-1}\sqrt{\varepsilon}(\mathscr{P}^+q-q)^+_{i+\frac12}[\![\mathscr{P}^-e_u]\!]_{i+\frac12}\\
&\qquad+\int_0^T\sum_{i=1}^{K-1}\sqrt{\varepsilon}(\mathscr{P}^-u-u)^-_{i+\frac12}[\![\mathscr{P}^+e_q]\!]_{i+\frac12}\\
&\qquad+\int_0^T\sqrt{\varepsilon}(\mathscr{P}^+q-q)^+_{\frac12}[\![\mathscr{P}^-e_u^+]\!]_{\frac12}dt+\int_0^T\frac{\sqrt{\varepsilon}\beta}{h}(\mathscr{P}^-u-u)^-_{K+\frac12}[\![\mathscr{P}^-e_u^-]\!]_{K+\frac12}dt\\
&\qquad-\int_0^T\sqrt{\varepsilon}(\mathscr{P}^+q-q)^-_{K+\frac12}[\![\mathscr{P}^-e_u^-]\!]_{K+\frac12}dt.
\end{aligned}
$$

Since $(\mathscr{P}^-e_u)_x\in P^{k-1},(\mathscr{P}^+e_q)_x\in P^{k-1},(\mathscr{Q}e_p)_x\in P^{k-1},\mathscr{Q}e_p\in P^k$, by the properties of the projection $\mathscr{P}^{\pm},\mathscr{Q}$, we obtain

$$
\begin{aligned}
&\big(\mathscr{P}^+q-q,(\mathscr{P}^-e_u)_x\big)_{I_i}=0,\quad \big(\mathscr{P}^-u-u,(\mathscr{P}^+e_q)_x\big)_{I_i}=0,\\
&\big(\mathscr{Q}p-p,\mathscr{P}^+e_q\big)_{I_i}=0,\quad \big(\mathscr{Q}p-p,(\mathscr{P}^+e_q)_x\big)_{I_i}=0,\\
&(\mathscr{P}^-u-u)_{i+\frac12}=0,\quad (\mathscr{P}^+q-q)_{i+\frac12}=0.
\end{aligned}
$$

Then the bilinear form (4.10) reduces to

$$
\begin{aligned}
&\mathscr{R}(\mathscr{P}^-u-u,\mathscr{Q}p-p,\mathscr{P}^+q-q;\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q)\\
&\quad=\int_0^T\sum_{i=1}^K\big((\mathscr{P}^-u)_t-u_t,\mathscr{P}^-e_u\big)_{I_i}dt-\int_0^T\sqrt{\varepsilon}(\mathscr{P}^+q-q^-)_{K+\frac12}(\mathscr{P}^-e_u)^-_{K+\frac12}dt\\
&\qquad+\int_0^T\sum_{i=1}^K\big(\Delta_{(\alpha-2)/2}(\mathscr{Q}p-p)-(\mathscr{P}^+q-q),\mathscr{Q}e_p\big)_{I_i}dt.
\end{aligned}
$$

Recalling the projection property and Lemma 2.14, we obtain $\|\Delta_{(\alpha-2)/2}(\mathscr{Q}p-p)-(\mathscr{P}^+q-q)\|\leqslant Ch^{k+1}$. Combining this with Young's inequality and (2.15), we obtain

$$
\begin{aligned}
&\mathscr{R}(\mathscr{P}^-u-u,\mathscr{Q}p-p,\mathscr{P}^+q-q;\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q)\\
&\quad\leqslant\int_0^T\sum_{i=1}^K\big((\mathscr{P}^-u)_t-u_t,\mathscr{P}^-e_u\big)_{I_i}dt+C_{T,a,b}h^{2k+2}+\frac{1}{C_{T,a,b}}\int_0^T\sum_{i=1}^K\|\mathscr{Q}e_p\|^2_{L^2(I_i)}dt\\
&\qquad+\int_0^T\frac{\sqrt{\varepsilon}\beta}{h}|(\mathscr{P}^-e_u)_{K+\frac12}|^2dt.
\end{aligned}
$$

This proves the lemma. $\square$

LEMMA 4.4 (see [11]). *Suppose that for all $t > 0$ we have*

$$\chi^2(t) + R(t) \leqslant A(t) + 2\int_0^t B(s)\chi(s)ds,$$

where R, A, B are nonnegative functions. Then, for any $T > 0$,

$$\sqrt{\chi^2(T) + R(t)} \leqslant \sup_{0\leqslant t\leqslant T} A^{1/2}(t) + \int_0^T B(t)dt.$$

THEOREM 4.5. *Let u be a sufficiently smooth exact solution to (1.1) on $\Omega \subset \mathbb{R}$ with $f(u) = 0$. Let u_h be the numerical solution of the semidiscrete LDG scheme (3.2)–(3.5). Then for small enough h, we have the following error estimates:*

$$\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^{k+1},$$

where C is a constant independent of h.

Proof. From Lemma 4.1 and the initial error $\|\mathscr{P}^- e_u(0)\|_{L^2(\Omega)} = 0$, we have

$$\begin{aligned}
&\mathscr{R}(\mathscr{P}^- e_u, \mathscr{Q}e_p, \mathscr{P}^+ e_q; \mathscr{P}^- e_u, -\mathscr{Q}e_p, \mathscr{P}^+ e_q) \\
&\quad = \frac{1}{2}\|\mathscr{P}^- e_u(T)\|^2_{L^2(\Omega)} + \int_0^T \left(\Delta_{(\alpha-2)/2}\mathscr{Q}e_p, \mathscr{Q}e_p\right)_{I_i} dt + \int_0^T \frac{\sqrt{\varepsilon}\beta}{h}|\mathscr{P}^- e_u|^2_{K+\frac{1}{2}} dt.
\end{aligned}$$

Combining this with Lemma 4.3 and (4.14), the following inequality holds:

$$\begin{aligned}
&\frac{1}{2}\|\mathscr{P}^- e_u(T)\|^2_{L^2(\Omega)} + \int_0^T \left(\Delta_{(\alpha-2)/2}\mathscr{Q}e_p, \mathscr{Q}e_p\right) dt \\
&\quad \leqslant \int_0^T \left((\mathscr{P}^- u)_t - u_t, \mathscr{P}^- e_u\right) dt + C_{T,a,b}h^{2k+2} + \frac{1}{C_{T,a,b}}\int_0^T \|\mathscr{Q}e_p\|^2_{L^2(\Omega)} dt.
\end{aligned}$$

Recalling the fractional Poincaré–Friedrichs Lemma 2.12, we get

$$\frac{1}{2}\|\mathscr{P}^- e_u(T)\|^2_{L^2(\Omega)} \leqslant \int_0^T \left((\mathscr{P}^- u)_t - u_t, \mathscr{P}^- e_u\right) dt + C_{T,a,b}h^{2k+2}.$$

Using Lemma 4.4 and the error associated with the projection error proves the theorem. ∎

For the more general fractional convection-diffusion problem, we introduce a few results and then give the error estimate.

LEMMA 4.6 (see [46]). *For any piecewise smooth function $\omega \in L^2(\Omega)$, on each cell boundary point we define*

$$\kappa(\hat{f};\omega) \equiv \kappa(\hat{f};\omega^-,\omega^+) \triangleq \begin{cases} [\omega]^{-1}(f(\bar{\omega}) - \hat{f}(\omega)) & \text{if } [\omega] \neq 0, \\ \frac{1}{2}|f'(\bar{\omega})| & \text{if } [\omega] = 0, \end{cases} \tag{4.15}$$

where $\hat{f}(\omega) \equiv \hat{f}(\omega^-,\omega^+)$ is a monotone numerical flux consistent with the given flux f. Then $\kappa(\hat{f},\omega)$ is nonnegative and bounded for any $(\omega^-,\omega^+) \in \mathbb{R}$. Moreover, we have

$$\frac{1}{2}|f'(\bar{\omega})| \leqslant \kappa(\hat{f};\omega) + C_*|[\omega]|, \tag{4.16}$$

$$-\frac{1}{8}f''(\bar{\omega})[\omega] \leqslant \kappa(\hat{f};\omega) + C_*|[\omega]|^2. \tag{4.17}$$

To estimate the nonlinear part, we define

$$\sum_{j=1}^{K}\mathscr{H}_j(f;u,u_h,v)=\sum_{j=1}^{K}\int_{I_j}(f(u)-f(u_h))v_x dx+\sum_{j=1}^{K}((f(u)-f(\bar{u}_h))[v])_{j+\frac{1}{2}}$$
$$+\sum_{j=1}^{K}((f(\bar{u}_h)-\hat{f})[v])_{j+\frac{1}{2}},$$

where $\bar{u}_h$ is the average.

LEMMA 4.7 (see [43]). *For $\mathscr{H}(f;u,u_h,v)$ defined above, we have the following estimate:*

$$\sum_{j=1}^{K}\mathscr{H}_j(f;u,u_h,v)\leqslant -\frac{1}{4}\kappa(\hat{f};u_h)[v]^2+(C+C_*(\|v\|_\infty+h^{-1}\|e_u\|_\infty^2))\|v\|_{L^2(\Omega)}^2$$
$$+(C+C_*h^{-1}\|e_u\|_\infty^2)h^{2k+1}.$$

To deal with the nonlinearity of the flux $f(u)$, we make the following assumption for h small enough and $k\geq 1$, which can be verified [43]:

$$\|e_u\|=\|u-u_h\|\leq h. \tag{4.18}$$

THEOREM 4.8. *Let u be the exact solution of* (1.1)*, which is sufficiently smooth on $\Omega\subset\mathbb{R}$, and assume $f\in C^3$. Let u_h be the numerical solution of the semidiscrete LDG scheme* (4.6)–(4.9) *and denote the corresponding numerical error by $e_u=u-u_h$. V_h is the space of piecewise polynomials of degree $k\geqslant 1$. Then for small enough h, we have the following error estimate:*

$$\|u-u_h\|_{L^2(\Omega)}\leqslant Ch^{k+\frac{1}{2}}.$$

Proof. From (4.3) and the boundary condition, we have for any $(v,w,z)\in H^1(\Omega,\mathcal{T})\times L^2(\Omega,\mathcal{T})\times L^2(\Omega,\mathcal{T})$,

$$\mathscr{B}(u,p,q;v,w,z)=\mathscr{B}(u_h,p_h,q_h;v,w,z)=0.$$

Then the following equality can be derived:

$$\begin{aligned}&\mathscr{B}(u,p,q;v,w,z)-\mathscr{B}(u_h,p_h,q_h;v,w,z)\\&\quad=\mathscr{R}(u,p,q;v,w,z)-\mathscr{R}(u_h,p_h,q_h;v,w,z)-\sum_{j=1}^{K}\mathscr{H}_j(f,u,u_h,v)\\&\quad=\mathscr{R}(u-u_h,p-p_h,q-q_h;v,w,z)-\sum_{j=1}^{K}\mathscr{H}_j(f,u,u_h,v)=0.\end{aligned}$$

Setting $(v,w,z)=(\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q)$ and using that $u-u_h=u-\mathscr{P}^-u+\mathscr{P}^-e_u$, we obtain

$$\begin{aligned}&\mathscr{R}(\mathscr{P}^-e_u,\mathscr{Q}e_p,\mathscr{P}^+e_q;\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q)-\sum_{j=1}^{K}\mathscr{H}_j(f,u,u_h,\mathscr{P}^-e_u,)\\&\quad=\mathscr{R}(\mathscr{P}^-u-u,\mathscr{Q}p-p,\mathscr{P}^+q-q;\mathscr{P}^-e_u,-\mathscr{Q}e_p,\mathscr{P}^+e_q).\end{aligned}$$

From Lemmas 4.1, 4.3, and 4.7, we derive the following inequality:

$$\frac{1}{2}\|\mathscr{P}^- e_u(T)\|^2_{L^2(\Omega)} + \int_0^T \left(\Delta_{(\alpha-2)/2}\mathscr{Q}e_p, \mathscr{Q}e_p\right) dt + \frac{1}{4}\kappa(\hat{f};u_h)[v]^2$$
$$\leqslant \int_0^T \left((\mathscr{P}^- u)_t - u_t, \mathscr{P}^- e_u\right) dt + C_{T,a,b}h^{2k+2} + \frac{1}{C_{T,a,b}} \int_0^T \|\mathscr{Q}e_p\|^2 dt$$
$$+ \int_0^T ((C + C_*(\|\mathscr{P}^- e_u\|_\infty + h^{-1}\|e_u\|^2_\infty))\|\mathscr{P}^- e_u\|^2_{L^2(\Omega)}$$
$$+ (C + C_* h^{-1}\|e_u\|^2_\infty)h^{2k+1})dt.$$

Recalling Lemmas 2.12 and 4.6 and assumption (4.18), we have

$$\frac{1}{2}\|\mathscr{P}^- e_u(T)\|^2_{L^2(\Omega)}$$
$$\leqslant \int_0^T \left((\mathscr{P}^- u)_t - u_t, \mathscr{P}^- e_u\right) dt + C\int_0^T \|\mathscr{P}^- e_u\|^2_{L^2(\Omega)} dt + Ch^{2k+1}.$$

Using Gronwall's inequality and the error associated with the projection error proves the theorem. ∎

Remark 4.9. Although the order of convergence $k + \frac{1}{2}$ is obtained it can be improved if an upwind flux is chosen for $\hat{f}$.

Remark 4.10. For problems with mixed boundary conditions, another scheme may be more suitable for implementation. Suppose the problem has a Dirichlet boundary on the left and a Neumann boundary on the right. Let $p = \sqrt{\varepsilon}\frac{\partial u}{\partial x}, q = \sqrt{\varepsilon}\frac{\partial p}{\partial x}$; then we have

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \Delta_{(\alpha-2)/2} q,$$
$$q = \sqrt{\varepsilon}\frac{\partial p}{\partial x},$$
$$p = \sqrt{\varepsilon}\frac{\partial u}{\partial x}$$

and recover the scheme

$$((u_h)_t, v)_{I_i} + \hat{f}_h v|_{I_i} - (f(u_h), v_x)_{I_i} - \left(\Delta_{(\alpha-2)/2} q_h, v(x)\right)_{I_i} = 0, \tag{4.19}$$
$$(q_h, w(x))_{I_i} - \sqrt{\varepsilon}\hat{p_h} w|_{I_i} + \sqrt{\varepsilon}\,(p_h, w_x)_{I_i} = 0, \tag{4.20}$$
$$(p_h, z(x))_{I_i} - \sqrt{\varepsilon}\hat{u_h} z|_{I_i} + \sqrt{\varepsilon}\,(u_h, z_x)_{I_i} = 0, \tag{4.21}$$
$$(u_h(x,0), v(x))_{I_i} - (u_0(x), v(x))_{I_i} = 0. \tag{4.22}$$

In this scheme, a mixed boundary condition is imposed naturally. However, the analysis is more complicated. While computational results indicate excellent behavior and optimal convergence, we shall not discuss the theoretical aspect of this scheme.

Remark 4.11. In the above sections, we have developed an LDG scheme for the partial differential equation with a fractional Laplacian operator with $1 < \alpha < 2$. For $0 < \alpha < 1$, the fractional Laplacian operator can be rewritten as a composite of a fractional integral and a first order derivative, and the LDG scheme can be applied similarly. However, special attention is required to address the possible development of shocks.

5. Numerical examples. In the following, we present a few results to numerically validate the analysis.

The discussion so far has focused on the treatment of the spatial dimension with a semidiscrete form as

$$\frac{d\mathbf{u}_h}{dt} = \mathcal{L}_h(\mathbf{u}_h, t), \tag{5.1}$$

where $\mathbf{u}_h$ is the vector of unknowns. For the time discretization of this system, we use a fourth order low storage explicit Runge–Kutta (LSERK) method [30] of the form

$$\begin{aligned}
&\mathbf{p}^0 = \mathbf{u}^n,\\
&i \in [1, \ldots, 5] : \begin{cases} \mathbf{k}^i = a_i\mathbf{k}^{i-1} + \Delta t \mathcal{L}_h(\mathbf{p}^{i-1}, t^n + c_i\Delta t),\\ \mathbf{p}^{(i)} = \mathbf{p}^{(i-1)} + b_i\mathbf{k}^{(i)}, \end{cases}\\
&\mathbf{u}_h^{n+1} = \mathbf{p}^{(5)},
\end{aligned}$$

where a_i, b_i, c_i are coefficients of the LSERK method given in [30]. For explicit methods, a proper time-step Δt is needed. In our examples, the condition $\Delta t \leqslant C\Delta x_{min}^{\alpha} (0 < C < 1)$ is used to ensure stability.

Example 5.1. As the first example, we consider the fractional diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -(-\Delta)^{\alpha/2}u(x,t) + g(x,t) \quad \text{in } [0,1] \times (0, 0.5],\\ u(x,0) = u_0(x) \quad \text{on } [0,1] \end{cases} \tag{5.2}$$

with the initial condition $u_0(x) = x^6(1-x)^6$. We choose the source term

$$g(x,t) = e^{-t}\left(-u_0(x) + (-\Delta)^{\frac{\alpha}{2}}u_0(x)\right)$$

to obtain an exact solution $u(x,t) = e^{-t}x^6(1-x)^6$.

We solve the equation for several different α and polynomial orders. The errors and order of convergence are listed in Table 5.1, confirming optimal $\mathcal{O}(h^{k+1})$ order of convergence across $1 < \alpha < 2$.

Example 5.2. We consider the fractional Burgers' equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right) = \varepsilon(-(-\Delta)^{\alpha/2})u(x,t) + g(x,t) \quad \text{in } [-2,2] \times (0, 0.5],\\ u(x,0) = u_0(x) \quad \text{on } [-2,2] \end{cases} \tag{5.3}$$

with the initial condition

$$u_0(x) = \begin{cases} (1-x^2)^4/10, & -1 \leqslant x \leqslant 1,\\ 0 & \text{otherwise.} \end{cases}$$

We set the parameter $\varepsilon = 1$ and the source term as

$$g(x,t) = e^{-t}\left(-u_0(x) + e^{-t}u_0(x)u_0'(x) + \varepsilon(-\Delta)^{\frac{\alpha}{2}}u_0(x)\right)$$

to obtain the exact solution

$$u(x,t) = \begin{cases} e^{-t}(1-x^2)^4/10, & -1 \leqslant x \leqslant 1,\\ 0 & \text{otherwise.} \end{cases}$$

To complete the scheme, we choose a Lax–Friedrichs flux for the nonlinear term and an alternating direction flux for the linear term. The problem is solved for several different values of α, polynomial orders (N), and numbers of elements (K), and the results are shown in Table 5.2. Although an order of convergence of $N + 1/2$ is predicted by Theorem 4.8 for any monotone flux for the nonlinear term, optimal $\mathcal{O}(h^{N+1})$ order of convergence is observed. The reason for this improved rate of convergence is not understood but may be associated with the global nature of the fractional operator which exhibits optimal convergence.

Example 5.3. Let us finally consider the fractional Burgers' equation with a discontinuous initial condition,

Table 5.1

Error and order of convergence for Example 1: solving the fractional Laplacian problem with K elements and polynomial order N.

	$\alpha = 1.1$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
	K= 10	K= 20	order	K= 30	order	K= 40	order
1	1.64e-06	3.98e-07	2.04	1.73e-07	2.05	9.60e-08	2.05
2	1.24e-07	1.69e-08	2.87	5.13e-09	2.94	2.18e-09	2.97
3	1.14e-08	7.23e-10	3.98	1.41e-10	4.03	4.44e-11	4.03

	$\alpha = 1.3$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
	K= 10	K= 20	order	K= 30	order	K= 40	order
1	1.45e-06	3.46e-07	2.07	1.50e-07	2.06	8.33e-08	2.05
2	1.19e-07	1.55e-08	2.94	4.64e-09	2.98	1.96e-09	2.99
3	1.04e-08	6.48e-10	4.00	1.26e-10	4.03	3.97e-11	4.02

	$\alpha = 1.5$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
1	1.35e-06	3.24e-07	2.06	1.42e-07	2.04	7.90e-08	2.03
2	1.22e-07	1.51e-08	3.01	4.51e-09	2.99	1.90e-09	2.99
3	1.04e-08	6.28e-10	4.05	1.22e-10	4.04	3.85e-11	4.02

	$\alpha = 1.8$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
1	1.28e-06	3.11e-07	2.04	1.38e-07	2.01	7.74e-08	2.01
2	1.40e-07	1.51e-08	3.21	4.48e-09	3.00	1.89e-09	3.00
3	1.44e-08	6.72e-10	4.42	1.23e-10	4.19	3.83e-11	4.05

Table 5.2

Error and order of convergence for Example 2: solving the fractional Burgers' equation with K elements and polynomial order N.

	$\alpha = 1.01$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
	K= 10	K= 20	order	K= 30	order	K= 40	order
1	1.10e-03	2.81e-04	1.97	1.24e-04	2.03	6.90e-05	2.03
2	6.53e-05	1.00e-05	2.70	3.09e-06	2.90	1.33e-06	2.94
3	5.94e-06	4.00e-07	3.89	8.05e-08	3.96	2.58e-08	3.95

	$\alpha = 1.5$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
1	8.89e-04	2.15e-04	2.05	9.45e-05	2.03	5.28e-05	2.02
2	6.71e-05	8.62e-06	2.96	2.57e-06	2.99	1.09e-06	2.99
3	4.91e-06	3.34e-07	3.88	6.80e-08	3.93	2.16e-08	3.99

	$\alpha = 1.8$						
K	10	20		30		40	
N	$\Vert e_u\Vert_2$	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order	$\Vert e_u\Vert_2$	Order
1	8.43e-04	2.09e-04	2.01	9.25e-05	2.01	5.20e-05	2.00
2	6.78e-05	8.59e-06	2.98	2.56e-06	2.99	1.08e-06	2.99
3	4.80e-06	3.32e-07	3.85	6.84e-08	3.90	2.20e-08	3.94

$$u_0(x) = \begin{cases} \frac{1}{2}, & -1 \leqslant x \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

We consider (5.3) with parameters $\varepsilon = 0.04$, $g(x,t) = 0$. We fix $T = 1$ and solve the equation for several different values of α. The numerical solution $u_h(x,t)$ for

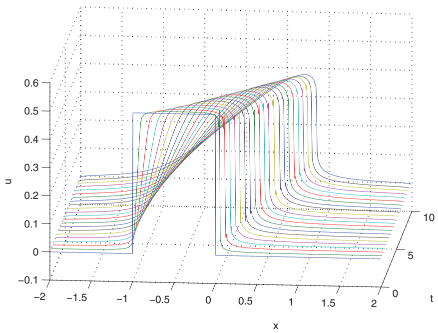

FIG. 5.1. *Solution of the fractional Burgers's equation with* $\epsilon = 0.04, \alpha = 1.005$.

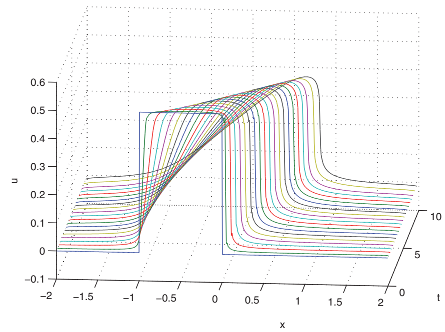

FIG. 5.2. *Solution of the fractional Burgers's equation with* $\epsilon = 0.04, \alpha = 1.1$.

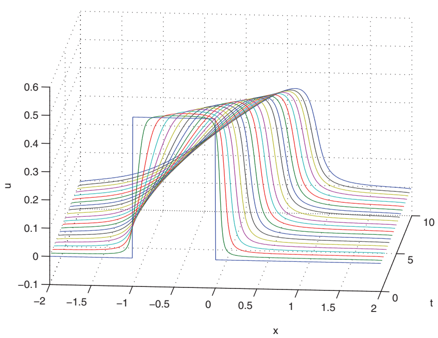

FIG. 5.3. *Solution of the fractional Burgers's equation with* $\epsilon = 0.04, \alpha = 1.5$.

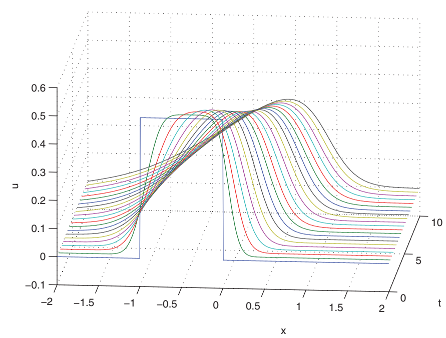

FIG. 5.4. *Solution of the fractional Burgers's equation with* $\epsilon = 0.04, \alpha = 2.0$.

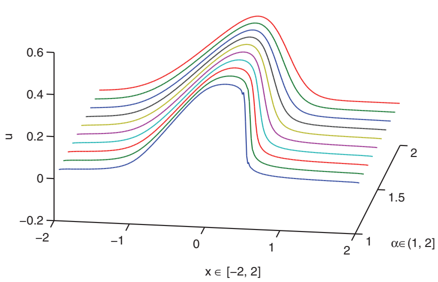

FIG. 5.5. *Solution of the fractional Burgers's equation with* $t = 2, \alpha \in (1, 2]$.

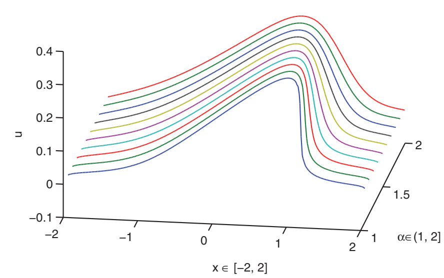

FIG. 5.6. *Solution of the fractional Burgers's equation with* $t = 5, \alpha \in (1, 2]$.

$\alpha = 1.005,\ 1.1,\ 1.5,\ 2.0$ is shown in Figures 5.1, 5.2, 5.3 and 5.4, respectively. Two results for showing the relation $u(x, \alpha)$ versus (x, α) for fixed t are given in Figures 5.5 and 5.6. From these figures it is clear that the dissipative effect increases with α and the classical case with $\alpha = 2$ is a limit of the fractional case.

6. Concluding remarks. We propose a discontinuous Galerkin method for fractional convection-diffusion problems in which fractional diffusion is expressed through a fractional Laplacian. To obtain high order accuracy, we rewrite the fractional Lapacian as a composite of first order derivatives and integrals and transform the problem to a low order system. We consider the equation in a domain Ω with homogeneous boundary conditions. An LDG method is proposed and stability and error estimations are presented. An optimal convergence order is proved for fractional diffusion. For the fractional convection-diffusion equation, an order of convergence of $k + 1/2$ is obtained for the LDG scheme with general monotone flux for the nonlinear term. Numerical experiments for fractional diffusion and fractional convection-diffusion confirm the analysis. As a last example, the fractional Burgers' equation with discontinuous initial conditions is solved for different values of α and results show that the dissipative effect increases as α increase, with the classical case $\alpha = 2$ as the limit.

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