On the Constants in hp-Finite Element Inverse Inequalities

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Abstract

We derive trace inverse inequalities for hp-finite elements. Utilizing orthogonal polynomials, we show how to derive explicit expressions for the constants when considering triangular and tetrahedral elements. We also discuss how to generalize this technique to the general d-simplex.

Key words: hp-Finite Element, Trace Inverse Inequality

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1 Introduction

In the late seventies, Ciarlet [1] summarized a unified framework for the construction of inverse-inequalities on finite element spaces utilizing arguments of norm equivalence, i.e. the equivalence of any two finite dimensional norms. The well known result takes the form

Theorem 1 *For a planar triangle domain* D *the following result holds* $\forall u \in \mathsf{P}_p^2(\mathsf{D})$*:*

$$\|u\|_{\partial\mathsf{D}} \leq C \frac{p}{\sqrt{h}} \|u\|_{\mathsf{D}} \quad ,$$

where $h = diam(\mathsf{D})$*.*

Here $\mathsf{P}_p^2(\mathsf{D})$ is the space of two-dimensional polynomials of order p defined on D.

Unfortunately, this result does not explicitly illuminate the bounding coefficients relating the norms and, thus, leaves C unknown. Later work by Harari et al. [3] explicitly calculated the constants for some inverse inequalities. However, the techniques used limits the order of the finite element spaces considered. More recently, Schwab [7] discusses these inverse inequalities for general order and geometries, although also leaving the question of the bounding constants open.

These inverse inequalities have proven pivotal in the analysis of modern numerical schemes including continuous and discontinuous versions of the Galerkin method, see e.g. [9], as well as deriving approximate penalty parameters involved in stabilizing these schemes and others.

In this paper we shall loosely follow the work of Verfurth [11] and present explicit, sharp bounds for the finite element trace inverse inequality on d-simplices. The results are sharp with respect to the geometry of the elements (the so called h-dependence), with respect to the polynomial order, p, of the finite element space, as well as the physical dimension, d, of the element.

What remains of the paper is formatted as follows. In the following section, Sec. 2, we recall the notation used subsequently. Section 3 discusses in detail the analysis of the triangular element, while Sec. 4 discusses the extension to the tetrahedron and, subsequently, the d-simplex. In Sec. 5 we discuss circumstances under which the inequalities become equalities while Sec. 6 provides a brief conclusion.

2 Notation

We shall consider the development of inequalities of the type

$$\|u\|_{\partial \mathsf{D}} \leq C(h,p)\,\|u\|_{\mathsf{D}} \quad ,$$

where D is the element. In particular, we shall pay attention to the case where D is the d-dimensional simplex with planar faces, i.e., it can be affinely mapped to the canonical d-dimensional simplex, T^d, defined by

$$\mathsf{T}^d = \left((x_1, x_2, \ldots, x_d) \in \mathsf{R}^d | |x_i| \leq 1 \ , \ \sum_{i=1}^{d} x_i \leq 2 - d \right) \ .$$

We assume that $u(\boldsymbol{x}) \in \mathsf{P}_p^d(\mathsf{D})$, where $\mathsf{P}_p^d(\mathsf{D})$ is the space of d-dimensional p'th order polynomials defined in D. In the special case where D is a d-simplex, we have

$$\mathsf{P}_p^d(\mathsf{D}) = \text{span} \left\{ x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d} \mid i_k \geq 0 \ , \ \sum_k i_k \leq p \right\} \ .$$

The dimensional of the space, $\mathsf{P}_p^d(\mathsf{D})$, is in this case

$$N = \dim \mathsf{P}_p^d(\mathsf{D}) = \begin{pmatrix} p + d \\ \\ p \end{pmatrix} \ .$$

We generally assume that

$$u(\boldsymbol{x}) = \sum_{n=0}^{N-1} \hat{u}_n \psi_n(\boldsymbol{x}) \ ,$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^T$ and that an $L^2(\mathsf{D})$-orthonormal basis, $\psi_n(\boldsymbol{x})$, is given, i.e.,

$$(\psi_i, \psi_j)_{\mathsf{D}} = \int_{\mathsf{D}} \psi_i(\boldsymbol{x}) \psi_j(\boldsymbol{x}) \, d\boldsymbol{x} = \delta_{ij} \ ,$$

such that the expansion coefficients are given as

$$\hat{u}_n = \int_{\mathsf{D}} u(\boldsymbol{x})\psi_n(\boldsymbol{x})\, d\boldsymbol{x} \quad .$$

Orthogonal basis sets for the canonical T^d-simplex are derived in [6,4,2] and take the form of products of Jacobi polynomials [10]. For the simplest case of $\mathsf{D} = \mathsf{T}^1$, i.e., a line, the polynomial basis consists of the classical Legendre polynomials, i.e.,

$$\psi_n(x) = \frac{P_n(x)}{\sqrt{\frac{2}{2n+1}}} \quad .$$

3 Inequality for the Planar Triangle

To highlight the procedure to obtain the constants, let us first consider the inverse inequality which bounds the value of a p'th order polynomial at the end point of an interval by its integral over the same interval.

Theorem 2 *Trace Inverse Inequality on a Finite Interval*

For an interval $\mathsf{D} = [a, b]$ *the following result holds* $\forall u \in \mathsf{P}^1_p(\mathsf{D})$*:*

$$|u(a)| \leq \frac{(p+1)}{\sqrt{|b-a|}} \|u\|_{\mathsf{D}} \tag{1}$$

PROOF. Consider the reference interval, $\mathsf{T}^1 = [-1, 1]$, and the associated L^2-orthogonal polynomial, the classical Legendre polynomial. The value at one endpoint, e.g., $x = -1$, is given as

$$u^2(-1) = \hat{\boldsymbol{u}}^T \mathrm{F} \hat{\boldsymbol{u}} \quad ,$$

where $\hat{\boldsymbol{u}} = (\hat{u}_0, \ldots, \hat{u}_p)^T$ is the vector of expansion coefficients and the entries of the matrix, F are simply

$$\mathrm{F}_{ij} = \psi_i(-1)\psi_j(-1) \quad .$$

We note immediately that F is a symmetric rank one matrix, i.e., the spectral radius of F is

$$\rho(\mathrm{F}) = \sum_{i=0}^{p} \psi_i(-1)^2 = \sum_{i=0}^{p} \frac{2i+1}{2} = \frac{(p+1)^2}{2} \quad .$$

This yields the bound

$$u^2(-1) \leq \rho(\mathrm{F}) \hat{\boldsymbol{u}}^T \hat{\boldsymbol{u}} = \frac{(p+1)^2}{2} \|u\|^2_{\mathsf{T}^1} \quad ,$$

where the latter follows from Parsevals identity.

Using a scaling argument gives

$$u(a)^2 \leq \frac{(p+1)^2}{2} \frac{2}{b-a} \|u\|^2_{\mathsf{D}} \quad ,$$

and, thus, the result.

In the following we shall utilize this same approach to elucidate the explicit form of the constant in Theorem 1.

Theorem 3 *Trace Inverse Inequality for the Planar Triangle*

For a planar triangle element, D*, the following result holds* $\forall u \in \mathsf{P}^2_p(\mathsf{D})$ *:*

$$\|u\|_{\partial \mathsf{D}} \leq \sqrt{\frac{(p+1)(p+2)}{2} \frac{\textit{Perimeter Length}(\mathsf{D})}{\textit{Area}(\mathsf{D})}} \|u\|_{\mathsf{D}} \quad .$$

PROOF. We shall begin by considering the reference, right-angled, triangle:

$$\mathsf{T}^2 = (r, s| -1 \leq r, s \leq 1\, ;\, r + s \leq 0)\,.$$

An orthonormal, polynomial basis for T^2 was introduced by Proriol [6] and revived by Koornwinder [4], Dubiner [2], and Owens [5]. The basis is indexed by integer pairs $(i \geq 0; j \geq 0; i + j \leq p)$ and takes the form

$$\psi_{(ij)}(r, s) = P_i^{(0,0)}\left(a := \frac{2(r+1)}{1-s} - 1\right) \sqrt{\frac{2i+1}{2}} \left(\frac{1-s}{2}\right)^i P_j^{(2i+1,0)}(b := s) \sqrt{\frac{2(i+j)+2}{2}} \quad ,$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial of order n [10].

Again assuming that

$$u(r, s) = \sum_{ij} \hat{u}_{ij} \psi_{(ij)}(r, s) \quad ,$$

we compute the edge integral for the face, $s = -1$, i.e.,

$$\int_{-1}^{1} u^2(r, -1)\, dr = \hat{\boldsymbol{u}}^T \mathrm{F} \hat{\boldsymbol{u}} \quad .$$

The edge mass matrix, F, has the entries

$$
\begin{aligned}
\mathrm{F}_{(ij),(kl)} &= \int_{-1}^{1} \psi_{(ij)}(r,-1)\psi_{(kl)}(r,-1)\,dr \\
&= \int_{-1}^{1} P_i^{(0,0)}(r)\sqrt{\frac{2i+1}{2}}P_k^{(0,0)}(r)\sqrt{\frac{2k+1}{2}}\,dr \\
&\quad \times P_j^{(2i+1,0)}(-1)\sqrt{i+j+1}P_l^{(2k+1,0)}(-1)\sqrt{k+l+1} \\
&= \delta_{ik}(-1)^{j+l}\sqrt{(i+j+1)(k+l+1)} \quad ,
\end{aligned}
$$

where δ_{ik} is the Dirac delta function, appearing due to the L^2-orthogonality of the Legendre polynomials, $P_n^{(0,0)}(x)$.

It is simple to show that F is block diagonal, with a block for each $0 \le i \le p$. To compute the spectral radius of F, we can consider each block separately. This is made possible by realizing that for a fixed i the corresponding block is a rank one matrix since

$$
\mathrm{F}_{(i)} = \boldsymbol{v}^{(i)}\left(\boldsymbol{v}^{(i)}\right)^T \quad ,
$$

where

$$
\boldsymbol{v}_j^{(i)} = (-1)^j\sqrt{i+j+1} \;\; , \;\; 0 \le j \le p-i \quad .
$$

Thus, we immediately deduce that for a fixed i,

$$
\rho(\mathrm{F}_{(i)}) = \sum_{j=0}^{p-i}(i+j+1) = \frac{1}{2}(p-i+1)(p+i+2) \quad .
$$

As this is monotonically decreasing in i, we recover that

$$\rho(\mathrm{F}) = \frac{1}{2}(p+1)(p+2) \quad ,$$

such that

$$\int_{-1}^{1} u^2(r,-1)\, dr \leq \frac{1}{2}(p+1)(p+2)\, \|u\|^2_{\mathsf{T}^2} \quad ,$$

by using Parsevals identity for orthonormal complete expansions.

Using a standard scaling argument we generalize this to the arbitrary planar triangle, D, as

$$\frac{2}{\text{length}} \int_{\text{edge}} u^2(r,-1)\, dr \leq \frac{1}{2}(p+1)(p+2) \frac{2}{\text{Area}(\mathsf{D})}\, \|u\|^2_{\mathsf{D}} \quad ,$$

yielding

$$\int_{\text{edge}} u^2(r,-1)\, dr \leq \frac{1}{2}(p+1)(p+2) \frac{\text{length}}{\text{Area}(\mathsf{D})}\, \|u\|^2_{\mathsf{D}} \quad .$$

Rotating the coordinate system, this generalizes to any of the edges, resulting in the combined result

$$\|u\|^2_{\partial\mathsf{D}} \leq \frac{1}{2}(p+1)(p+2) \frac{\text{Perimeter Length}(\mathsf{D})}{\text{Area}(\mathsf{D})}\, \|u\|^2_{\mathsf{D}} \quad ,$$

and, thus, the stated result.

We note in particular that as

$$\frac{\text{Perimeter Length}(\mathsf{D})}{\text{Area}(\mathsf{D})} \simeq h^{-1} \quad ,$$

the result of Theorem 3 reflect the same hp-dependence as in Theorem 1.

4 Generalization to the d-Simplex

Let is briefly discuss the extension of the approach used to above, first to the tetrahedron and then to the general d-simplex.

Theorem 4 *Trace Inverse Inequality for the Tetrahedron*

For a tetrahedral element, D*, the following result holds* $\forall u \in \mathsf{P}_p^3(\mathsf{D})$ *:*

$$\|u\|_{\partial \mathsf{D}} \leq \sqrt{\frac{(p+1)(p+3)}{3} \frac{\textit{Surface Area}(\mathsf{D})}{\textit{Volume}(\mathsf{D})}} \|u\|_{\mathsf{D}} \quad .$$

PROOF. We consider first the standard tetrahedron

$$\mathsf{T}^3 = (r, s, t| -1 \leq r, s, t \leq 1\, ; r + s + t \leq -1)\, .$$

To prepare ourselves for the general d-simplex, we introduce the mapping

$$\begin{aligned} r &= \frac{(1+a)}{2}\frac{(1-b)}{2}(1-c) - 1 \\ s &= \frac{(1+b)}{2}(1-c) - 1 \\ t &= c \end{aligned}$$

where $-1 \le a, b, c \le 1$, i.e., the mapping collapses the unit cube into T^3. These coordinates are often known as the Duffy coordinates.

As above we utilize the orthonormal basis for the tetrahedron introduced by Dubiner [2]. The elements of the basis are indexed by integer triples (i, j, k) and take the form

$$\psi_{(ijk)}(r,s,t) = \left(\frac{P_i^{(0,0)}(a)}{\sqrt{\frac{2}{2i+1}}} \right) \left(\frac{\left(\frac{1-b}{2}\right)^i P_j^{(2i+1,0)}(b)}{\sqrt{\frac{2}{2(i+j)+2}}} \right) \left(\frac{\left(\frac{1-c}{2}\right)^{i+j} P_k^{(2(i+j)+2,0)}(c)}{\sqrt{\frac{2}{2(i+j+k)+3}}} \right) \quad .$$

We recall that $P_n^{(\alpha,\beta)}(x)$ is the n'th order Jacobi polynomial defined on $[-1, 1]$ and that (a, b, c) are related to (r, s, t) through the inverse of the mapping given above.

Proceeding as previously, we assume that

$$u(r,s,t) = \sum_{ijk} \hat{u}_{(ijk)} \psi_{(ijk)}(r,s,t) \quad ,$$

and consider the integration over one face, $c = -1$, as

$$\int_{\text{face}} u^2(r,s,-1)\, dr\, ds = \int_{-1}^{1}\int_{-1}^{1} u^2(a,b,-1)\, da\, db = \hat{\boldsymbol{u}}^T \mathrm{F} \hat{\boldsymbol{u}} \quad ,$$

where the face matrix, F, has the entries

$$\begin{aligned} \mathrm{F}_{(ijk)(lmn)} &= \int_{-1}^{1}\int_{-1}^{1} \psi_{(ijk)}(a,b,-1)\psi_{(lmn)}(a,b,-1)\, da\, db \\ &= \delta_{il}\delta_{jm} \left(\frac{P_k^{(2(i+j)+2,0)}(-1)}{\sqrt{\frac{2}{2(i+j+k)+3}}} \frac{P_n^{2(l+m)+2,0}(-1)}{\sqrt{\frac{2}{2(l+m+n)+3}}} \right) \\ &= \delta_{il}\delta_{jm}(-1)^{k+n} \sqrt{\frac{2(i+j+k)+3}{2}} \sqrt{\frac{2(l+m+n)+3}{2}} \end{aligned}$$

This again takes on a block diagonal form and the blocks indexed by $i+j$ are all rank one matrices, with the maximum eigenvalue, and thus the spectral radius of F, being

$$\rho\,(\mathrm{F}) = \frac{(p+1)(p+3)}{2} \quad .$$

for $i+j=0$.

As for the triangle this yields the following bound for $u \in P_p^3(\mathsf{T}^3)$

$$\int\limits_{\text{face}} u^2(r,s,-1)\,dr\,ds \leq \frac{(p+1)(p+3)}{2}\,\|u\|^2_{\mathsf{T}^3} \quad .$$

Using the affine nature of the simplex yields

$$\int\limits_{\text{face}} u^2(r,s,-1)\,d\boldsymbol{x} \leq \frac{(p+1)(p+3)}{2}\frac{2}{3}\frac{\text{Area(Face)}}{\text{Volume}(\mathsf{D})}\,\|u\|^2_{\mathsf{T}^3} \quad .$$

Rotating the coordinates to cover the other faces immediately yields the result.

We again have the same asymptotic behavior as in Theorem 1 since

$$\frac{\text{Surface Area}(\mathsf{D})}{\text{Volume}(\mathsf{D})} \sim h^{-1} \quad .$$

The technique utilized here generalizes to the general simplex.

Theorem 5 *Trace Inverse Inequality for the d-Simplex*

For a d-simplex, D*, the following result holds* $\forall u \in \mathsf{P}_p^d(\mathsf{D})$ *:*

$$\|u\|_{\partial \mathsf{D}} \leq \sqrt{\frac{(p+1)(p+d)}{d} \frac{Volume(\partial \mathsf{D})}{Volume(\mathsf{D})}} \|u\|_{\mathsf{D}} \quad .$$

PROOF. Consider the d-simplex

$$\mathsf{T}^d = \left((r_1, r_2, \ldots, r_d) | |r_i| \leq 1 \ , \ \sum_{i=1}^{d} r_i \leq 2 - d \right) \quad ,$$

and introduce the canonically collapsed coordinate transform

$$\begin{aligned}
r_1 &= \frac{(1+a_1)}{2} \frac{(1-a_2)}{2} \frac{(1-a_3)}{2} ...(1-a_d) - 1 \quad , \\
r_2 &= \frac{(1+a_2)}{2} \frac{(1-a_3)}{2} \frac{(1-a_4)}{2} ...(1-a_d) - 1 \quad , \\
r_3 &= \frac{(1+a_3)}{2} \frac{(1-a_4)}{2} \frac{(1-a_5)}{2} ...(1-a_d) - 1 \quad , \\
&\vdots \\
r_d &= a_d \quad ,
\end{aligned}$$

where $|a_i| \leq 1$, i.e., the d-simplex is mapped to a bi-unit d-cube. The orthonormal, polynomial basis for the d-simplex is indexed by integer d-tuples $\boldsymbol{i} = (i_1, i_2, i_3, .., i_d)$ and takes the form

$$\psi_{(\boldsymbol{i}}(\boldsymbol{r})) = \frac{P_{i_1}^{(i_1,0)}(a_1)}{\sqrt{\frac{2}{2i_1+1}}} \prod_{l=2}^{l=d} \left(\frac{\left(\frac{1-a_l}{2}\right)^{N_l(\boldsymbol{i})} P_{i_l}^{(2N_l(\boldsymbol{i})+l,0}(a_l)}{\sqrt{\frac{2}{2N_l(\boldsymbol{i})+d}}} \right)$$

where

$$N_l(\boldsymbol{i}) = \sum_{j=1}^{l} i_j \quad .$$

As previously, we now consider the inner product over one $(d-1)$-simplex, associated with $a_d = -1$, for which the mass matrix, F, has the entries

$$
\begin{aligned}
\mathrm{F}_{(\boldsymbol{i})(\boldsymbol{i})} &= \int\limits_{a_d=-1} \psi_{(\boldsymbol{i})}\psi_{(\boldsymbol{j})}(\boldsymbol{a})\, da_1 \ldots a_{d-1} \\
&= \prod_{l=1}^{d-1} \delta_{i_l i_l} (-1)^{i_d} \sqrt{\frac{2N_d(\boldsymbol{i})+d}{2}} (-1)^{j_d} \sqrt{\frac{2N_d(\boldsymbol{j})+d}{2}} \quad .
\end{aligned}
$$

One again realizes that this is a block diagonal matrix. Indeed, when indexed by $i_1, i_2, .., i_{d-1}$ each have one non-zero eigenvalue and the maximum of all of these is

$$
\rho\,(\mathrm{F}) = \frac{(p+1)(p+d)}{2} \quad . \tag{2}
$$

Proceeding as previously, this yields for all $u \in \mathsf{P}_p^d(\mathsf{T}^d)$

$$
\int\limits_{\partial T^d|_{a_d=-1}} u^2\, d\boldsymbol{a} \le \frac{(p+1)(p+d)}{2}\, \|u\|^2_{\mathsf{T}^d} \quad .
$$

Observing that the volume of T^d is

$$
\mathrm{volume}(\mathsf{T}^d) = \frac{2^d}{d!} \quad ,
$$

and using the affine nature of the d-simplex, and rotation of the coordinates as previously, we recover the result for the general element, D, as stated.

5 Extremal Polynomials

While we so far have provided sharp bounds on the scaling constants we have not discussed the candidate polynomials which lead to equality in the bounds. We shall restrict this discussion to the two-dimensional triangle case as the same result for the d-simplex follows by immediate extension.

First recall that one can construct a quadrature for the triangle in the (a, b) coordinates using a tensor-product of the Gauss-Lobatto-Legendre (GLL) and the Gauss-Radau-Jacobi (GRJ) quadrature [8]. For the p'th order polynomial space we use a set of $p+1$ GLL nodes in the a direction and a set of $p+1$ GRJ nodes in the b direction. The GRJ quadrature weights include the necessary $(1-b)/2$ factor introduced by the Duffy mapping.

Corollary 6 *Extremal polynomials For The Trace Inverse Inequality*

For a planar triangle element, D*, the only set of polynomials which turn the trace inverse inequality into an equality are scalar multiples of that polynomial which is constant in the* r *direction and is the Lagrange polynomial defined to be unity at the* $s=-1$ *GRJ node and zero at the other* p *GRJ nodes.*

PROOF. First consider the eigenpair (λ, u) where $\lambda = \frac{(p+1)(p+2)}{2}$ and $u \in \mathsf{P}^2_p(\mathsf{T}^2)$ such that the trace inverse inequality is an equality:

$$\int_{\text{edge}} u^2(a,-1)\, da = \lambda \, \|u\|^2_{\mathsf{T}^2} \quad .$$

By the definition of the eigenpair we know from the earlier analysis that the polynomial with the maximal eigenvalue is a constant in the a direction, i.e., $i = 0$, so the left hand side is trivially:

$$\int_{\text{edge}} u^2(a, -1)\, da = u(a = -1, b = -1)^2 \ .$$

Since u is included in $\mathsf{P}_p^2(\mathsf{T}^2)$ we are able to evaluate the right hand side norm using a minimal GLL-GRJ quadrature

$$\begin{aligned} \lambda \|u\|_{\mathsf{T}^2} &= \lambda \sum_{i=0}^{i=p} \sum_{j=0}^{j=p} w_i^{GLL} w_j^{GRJ} u\,(a_i, b_j)^2 \\ &= \lambda \sum_{j=0}^{j=p} w_j^{GRJ} u\,(a = -1, b_j)^2 \end{aligned}$$

where w_i^{GLL} and w_j^{GRJ} are the weights from the GLL and GRJ quadrature rules, respectively. The second step follows from u being independent of the a coordinate. Combining these results yields

$$u\,(a = -1, b = -1)^2 = \lambda w_0^{GRJ} u\,(a = -1, b = -1)^2 + \lambda \sum_{j=1}^{j=p} w_j^{GRJ} u\,(a = -1, b_j)^2$$

Now observe that

$$w_0^{GRJ} = \frac{2}{(p+1)(p+2)} = \frac{1}{\lambda} \ ,$$

from which it immediately follows that $u\,(a = -1, b_j) = 0$ for $j = 1, 2, .., p$. The stated result follows from $\lambda > 0$ and the uniqueness of the p'th order interpolant through $p + 1$ nodes.

6 Conclusion

We have derived explicit bounds for the finite element inverse trace inverse inequality. This was accomplished by using orthonormal polynomials on the d-simplex and realizing that a special ordering makes the associated face matrices block diagonal. Moreover, each of these blocks are rank one matrices, thus allowing for obtaining explicit expressions for their spectrum. The affine nature of the d-simplex allows the extension to the general simplex. The results are asymptotically sharp in the element geometry and the order of the polynomial. The analysis also reveals that structure of the polynomial for which the bound becomes an equality.

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References

[1] P. G. CIARLET, *The Finite-Element Method for Elliptic Problems*, North-Holland, The Netherlands, 1978.

[2] M. DUBINER, *Spectral Methods on Triangles and Other Domains*, J. Sci. Comput. **6**(1991), pp. 345-390.

[3] I. HARARI, AND T. J. R. HUGHES, *What are C and h?: Inequalities for the analysis and design of finite element methods* Comput. Methods Appl. Mech. Engin. **97**, pp. 157-192.

[4] T. KOORNWINDER, *Two-variable Analogues of the Classical Orthogonal Polynomials* in 'Theory and Application of Special Functions', R. A. Askey ed., Academic Press, 1975, pp. 435-495.

[5] R. G. OWENS , *Spectral approximations on the triangle* Proc. Roy. Soc. Lond. A, 454, pp. 857-872, 1998.

[6] J. PRORIOL, *Sur une Famille de Polynomes à deux Variables Orthogonaux dans un Triangle*, C. R. Acad. Sci. Paris **257**(1957), pp. 2459-2461.

[7] C. SCHWAB, *p- and hp-Finite Element Methods. Theory and Applications in Solid and Fluid Mechanics*, Numerical Mathematics and Scientific Computation. Clarendon Press, Oxford, 1998.

[8] S. J. SHERWIN, *Triangular and Tetrahedral Spectral/hp Element Methods for Fluid Dynamics*, Ph.D. thesis, Princeton University, 1995.

[9] E. Suli, C. Schwab, and P. Houston, *hp-DGFEM for Partial Differential Equations with Nonnegative Characteristic Form.* In "Discontinuous Galerkin Methods. Theory, Computation and Applications", B. Cockburn, G. E. Karniadakis, and C. W. Shu (Eds). Lecture Notes in Computational Science and Engineering **11**, Springer Verlag, Berlin, pp. 221-230.

[10] G. Szegö, *Orthogonal Polynomials.* Colloquium Publications **23**, American Mathematical Society, Providence, RI, 1939.

[11] R. Verfürth,*On the constants in some inverse inequalities for finite element functions*, preprint Ruhr-Universität Bochum.