Finite Dimensional Methods for Differential Flatness

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A mes parents

"I'll eat my head."

Mr. Grimwig

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Résumé

Dans la théorie de *l'automatique*, les équations différentielles ordinaires à temps continu et de dimension finie forment une discipline importante. Lorsqu'on aborde un problème de commande, l'étape initiale consiste à déterminer à quelle catégorie appartient un tel système d'équations. En effet, ces systèmes peuvent souvent être transformés d'une forme dans une autre, et prendre alors des apparences très différentes. Dès lors, un système de commande n'est plus associé à un unique système d'équations, mais à un ensemble de systèmes d'équations, "équivalents" les uns aux autres par transformation. Dans ce contexte, une propriété vérifiée par une représentation particulière d'un système peut être attribuée aussi bien au système lui-même, qu'à toutes les équations qui le représentent de façon équivalente. Vérifier une telle propriété peut être un problème ardu.

Dans cette thèse, et comme d'autres avant nous, nous étudions la question de savoir si un système de contrôle multi-entrées donné est *plat*, c'est-à-dire s'il peut être transformé en un système linéaire contrôlable par l'intermédiaire d'une extension dynamique dite endogène, suivie d'un changement de coordonnées. Ce problème est difficile et il a déjà été abondamment étudié. Ainsi, vouloir proposer une approche nouvelle ou une solution complètement originale serait, dans une certaine mesure, illusoire. Dès lors, une part substantielle du texte est consacrée à la présentation de concepts et de résultats existants, avec parfois une approche alternative ou un point de vue original. Une autre partie de la thèse est dévolue à des questions liées à la platitude, une autre encore à une version très simplifiée du problème.

Dans un premier temps, les systèmes de commande sont modélisés en tant que plongement d'une variété fibrée dans l'espace des jets de l'état. Le système extérieur, ou système Pfaffien, correspondant aux équations différentielles se présente alors naturellement. La prolongation des entrées est ensuite introduite en tant que relèvement du plongement mentionné ci-dessus. Diverses filtrations et leur application à la linéarisation par retour d'état statique sont ensuite traitées.

Un chapitre entier est consacré à l'approche géométrique de dimension infinie et à l'utilisation d'opérateurs différentiels matriciels. Un théorème désormais classique est présenté. Il lie la platitude à l'intégrabilité de la base d'un certain module après application d'un opérateur différentiel adéquat. La condition ainsi obtenue peut être décomposée en un problème fermé par différentiation des équations. Le problème qui en résulte est connu,

mais nous en présentons une alternative où une condition assimilable à une courbure est trivialisée.

Un sous-problème de la platitude qui a attiré l'attention des chercheurs est le suivant. On suppose donné un système de commande auquel on impose de satisfaire à certaines contraintes d'état; ce système est-t-il plat? Dans ce contexte, un concept utile est celui de couverture d'un système par un autre, ainsi que les résultats qui l'accompagnent. En effet, il est connu qu'un système plat n'est susceptible que de couvrir un système plat. Ainsi, pour un système linéaire sujet à des contraintes non-linéaires, si la version non-contrainte du système couvre la version contrainte, la platitude du système sous contraintes s'ensuit. Nous donnons une condition suffisante pour qu'un système quelconque donné couvre une version sous contrainte de ce même système. A cet effet, nous définissons le concept d'invariance commandée dynamiquement qui généralise la notion classique d'invariance commandée. Un autre ingrédient nécessaire est l'algorithme d'extension dynamique, dont nous présentons une version "infinitésimale".

La modélisation de systèmes mécaniques par un ensemble de masses ponctuelles soumises à certaines forces et soumises à des contraintes quadratiques, est une méthode souvent efficace. La version non-contrainte du système résultant est représentée par des équations de type linéaire en l'état et les entrées, ainsi que bilinéaire en l'état et les multiplicateurs de Lagrange. Nous proposons un système de drapeau relatif qui possède certaines propriétés d'intégrabilité génériques habituellement réservées aux systèmes linéaires. Ce drapeau peut être calculé à l'aide d'un algorithme dédié particulièrement simple et efficace, et fournit une condition suffisante pour la platitude des systèmes bilinéaires mentionnés. Cette condition, combinée à celle de couverture, est utilisée pour prouver la platitude d'un ensemble de systèmes. On montre encore que les systèmes bien connus que sont la voiture non-holonôme et le pendule font partie de cet ensemble.

Mots-clés: Commande non-linéaire, systèmes non-linéaires, platitude, linéarisation par bouclage dynamique, systèmes sous contraintes.

Abstract

In *Control System Theory*, the study of continuous-time, finite dimensional, underdetermined systems of ordinary differential equations is an important topic. Classification of systems in different categories is a natural initial step to the analysis of a given control problem. Systems of equations can often be "transformed" into other "equivalent" ones. Then, a control system is associated to a set of equivalent equations. In this setting, a property of a control system can be defined as a property that has to be satisfied by some arbitrary system of equations in the set representing the control system. Assessing such a property can be a difficult task.

In this thesis, we review and study a number of ways to determine whether a multi-input nonlinear system is *flat*, i.e. whether it is equivalent to a linear system after some dynamical extension and change of coordinates. This is a difficult as well as a well studied problem. Therefore, coming up with some altogether new approach or solution is to a certain extent illusory. A substantial part of the text is devoted to describing existing approaches and sometimes to propose either an original alternative or an original point of view. Another part of the thesis is dedicated to the study of a drastically reduced version of the problem, where more can be said in an algorithmic way.

Nonlinear control systems are first modeled as the embedding of some fibered bundle to the first jet of the time-and-state-variables manifold. The exterior system, or Pfaffian system, corresponding to the ODE then arises naturally. Input prolongations are then introduced as lifts of the previously mentioned embedding. Various filtration techniques and their applications to static feedback linearization are discussed next.

A full chapter is devoted to the infinite dimensional approach involving matrix differential operators. A now classical theorem, linking integrability of the basis of a differential module after application of one such an operator, and the flatness property is discussed. The relations obtained can be decomposed in an equivalent set of differentially closed equations. We state a version of the resulting theorem where the "curvature equations" are trivial.

A subproblem that has attracted the attention of researchers is the question whether a given system — subject to some state constraints — is flat. In this setting, useful concepts are that of a covering of a system by another one and the accompanying result stating

that a flat system can cover only a flat system. Hence, if a "large" linear system is given together with a set of nonlinear constraints, flatness of the constrained system is assessed if the unconstrained system can be shown to cover the constrained one. Starting with the classical notion of controlled invariance and a generalized notion coined dynamic controlled invariance, sufficient conditions are discussed which also involve the notion of right invertibility and the dynamic extension algorithm.

Modeling of mechanical systems by free moving point masses subject to some control forces and quadratic constraints is often effective. The resulting unconstrained equations are linear and bilinear in the state and control/Lagrangian multiplier variables. We propose a relative derived flag that leads to a filtration with guaranteed integrability at each stage. This leads to a very effective sufficient condition for the flatness of the unconstrained model. The algorithm, together with the test described in the previous section, is used to show flatness of some generalized pendulum-like equations. They are also shown to specialize to the non-holonomic car equations and to the VTOL/pendulum equations when some specific parameters are suitably chosen.

Keywords: Nonlinear control, nonlinear systems, flatness, dynamic feedback linearization, constrained systems.

Contents

Introduction

As an enthusiastic tyre manufacturer claims "power is nothing without control" a somewhat more cautious control engineer might reply with a worried frown, "and even then. . . ". Indeed, a higher performance car often requires a driver with better skills. In automation, a similar pattern occurs. The more advanced a device is, the more control possibilities and performance potential it offers, the more difficult the control task becomes.

Control and system theory has grown to become a vast and interdisciplinary field of research. In one of its aspects, a model of a plant or device to be controlled is given as a set of ordinary differential equations comprising a number of control variables. Manipulating these variables, one wishes to steer the behavior of the system. This amounts to impose something on the solutions of the equations by means of the application of appropriate control profiles. One might for instance require stabilizing the system, or making its solutions follow a predefined trajectory.

A common ground to all branches of control engineering is the theory of finite- (small-) dimensional, deterministic, linear systems. From that perspective, a lot of problems have a clear-cut solution and one is easily tempted to believe that mastering a dynamical system amounts to just a few computations. However, seldom does such an idealization faithfully model the behavior of a real plant. Tackling larger systems, adding various kinds of uncertainties, or considering nonlinear equations, quickly and tremendously complicate matters.

Renouncing linearity is quickly followed by a kind of nostalgia and by the desire of retrieving it without compromising the qualities of the model at hand. This concrete engineering concern has first tickled mathematicians a long time ago so that the relevant mathematical branches are manifold, ranging from algebra to geometry.

Retrieving linearity in a system without altering its behavior in an essential way calls for some notion of equivalence between systems. A nonlinear system is then said to be linearizable[1] if it can be put into equivalence with a linear one in some way. Varying types of systems and varying types of equivalence result in varying levels of difficulty in establishing such a property. When the equivalence consists of the existence of an invertible change of coordinates and a state feedback parameterized by new inputs, the corresponding linearizability property is referred to as static. In the single-input case, the problem is solved by the Goursat normal form, see e.g. [13]. For systems with more than one input, solutions were given in [72, 69] and as an extended version of the Goursat normal form

[1] This is not to be confused with a linear approximation of a nonlinear system.

in [58]. When static feedback linearization fails, one may be able to couple the system with another one, coined a dynamic extension or a dynamic feedback so as to obtain a larger statically linearizable system. The corresponding equivalence notion is then of a different more general type. A closely related property is the one of differential flatness, [44, 47, 45]. Solutions of a flat system are parameterized by m arbitrary functions of time and their derivatives up to a certain order, where m is the number of inputs. Checking linearizability by dynamic feedback or flatness of a given control system in the general case remains an open issue in a number of aspects. Necessary conditions are known, [129, 117], as well as sufficient ones, e.g. [17, 104, 21]. Conditions equivalent to the definition of flatness have also been found, see for instance [5, 21, 24, 8, 88]. Procedural approaches have been proposed; besides the already cited references see e.g. [121, 2, 3].
The objective of this thesis is to contribute to the study of flatness and dynamic feedback linearization.

Main Contributions

- A known characterization of differential flatness, [5, 21, 8, 88], involves an "exterior" differential equation whose unknown is a matrix differential operator. The differentially closed counterpart is a system of equations comprising a condition akin to curvature. We propose, in the $\mathcal{C}^\infty$ setting, an equivalent characterization that is free of the latter condition.

- We generalize the classical condition of controlled invariance to what we coined dynamic controlled invariance. We show that given a distribution satisfying the condition, there corresponds a subsystem for which the original system is a dynamic feedback. Singularity (regularity) of this feedback is then characterized using an adapted version of the dynamic extension algorithm.

- Given a system and a subsystem, we use the previous characterization to check whether the system covers ([21]) its subsystem. We give a sufficient condition to decide when an unconstrained system covers the constrained system resulting from the addition of state constraints. This also leads to a sufficient condition for flatness of the constrained system, once flatness of the unconstrained system is assumed.

- A class of bilinear control system is presented along with an algorithm that efficiently computes a specific relative derived flag. This flag possesses some guaranteed integrability properties. This leads to the fact that the system is flat once the algorithm saturates to an empty set.

- A subclass of the studied bilinear systems is proposed. When imposed a quadratic state constraint, these systems transform to a set of "generalized pendulums". Setting parameters to some adequate values produces the well-known non-holonomic car and the VTOL/pendulum equations.

- Known and well-known topics are presented. Along the way, we prove a number of technical and intermediate results. We think that some of them are of interest on their own. We also hope to have approached certain known concepts with some degree of originality.

Organization Chapter 1 defines the description of a control system as a fiber bundle embedded in a jet bundle. Maps between two such bundles are discussed, followed by input prolongations (finite and infinite). The second section treats static feedback linearization by considering the conditions allowing successive input eliminations. The chapter closes with an informal definition of flatness.

Chapter 2 deals with the characterization of flatness by the use of differential operators. These operators are defined along with their action on the differential 1-forms of the infinitely prolonged bundle. They are then endowed with a graded structure, a product rule and an exterior derivative. The algorithm for the construction of a Brunovsky-basis of the differential module of forms associated to the system is presented. Finally, the mentioned characterization of flatness is described. The contribution of this chapter consists in a variant of that characterization. The obtained variant is used as a guide in the construction of flat outputs on some simple examples.

Chapter 3 first discusses various types of dynamic feedback and reviews some related results. The classical notion of controlled invariance is introduced. We show that if the system is statically feedback linearizable, to any controlled invariant distribution corresponds a statically feedback linearizable subsystem. This motivates the introduction of dynamic controlled invariance. It is then shown that the dynamic feedback associated to a dynamically controlled distribution potentially lacks an important property, it is not necessarily non-singular (or regular). This regularity is assessed using the dynamic extension algorithm. A condition for deciding if a surjective map defined on the state space manifold induces a covering is obtained. We then deal with the covering of constrained systems. Does the unconstrained counterpart of such a system cover the constrained one? A sufficient condition is obtained, also yielding a sufficient condition for the flatness of the constrained system. These considerations show many parallelisms with the notion of relative flatness, [108]. In the last section, we give an example of a system linearizable by singular static feedback that is not flat.

Chapter 4 describes a class of bilinear systems and an associated integrable filtration. A sufficient flatness condition is immediately deduced. The algorithm is then used on a particular set of equations. When an additional state constraint is considered, the systems remain flat, and for some specific parameter values, well known physical flat systems are obtained. Some simulation results are presented.

Chapter 1

Preliminaries

The geometric study of differential equations has a long history and has lead to a very rich theory. Its manifold impact on engineering in general and in the study of nonlinear control problems in particular has been tremendous, see e.g. [70, 6, 52] among many more. In this preliminary chapter, we shall content ourselves with some concepts useful for representing and analyzing a given control system. The framework will be that of jet bundles and their prolongations [102, 119, 59, 133]. More concretely, the equations describing a control system in explicit form are a system of ordinary differential equations with some input (or control) variables that may be assigned freely. The "admissible" changes of coordinates on the state and input variables lead to the definition of a bundle structure. Solutions of the systems then relate to certain sections on the bundle. In turn, these sections and their jets allow one to define the codistribution of contact forms, their dual vector fields and the system of exterior equations related to the control problem. From these structures, many properties of the problem at hand can be analyzed, such as feedback linearizability [72, 69, 58, 57] or generalizations of the Goursat normal form of the associated Pfaffian system [58, 143, 148]. When jets of infinite order are considered, the corresponding geometry becomes infinite dimensional. This calls for the language of diffieties, [153, 1, 49, 23]. In Section 1.1, with the equations of a control system, we associate a bundle structure that reflects the different natures of time-, state- and input-variables. Solutions are represented by sections satisfying the initial ODE. The Cartan distribution (respectively codistribution) is defined. Input prolongations and infinite dimensional representations are discussed. Section 1.2 deals with the problem of static feedback linearizability within the presented framework. Lastly, Section 1.3 contains an informal and preliminary definition of the notion of differential flatness.

1.1 Representations of Control Systems

In this section we describe what is meant by a *control system* and some of the ways one may represent it mathematically. Roughly, one can distinguish two kinds of mathematical representations, finite-dimensional, underdetermined on one side and infinite dimensional

on the other. Our starting point will usually be an underdetermined system of ordinary differential equations in the (state) variables $x = (x^1, \ldots, x^n)$ and independent (time) variable t. These may assume two different forms, the implicit form

$$F^k(t, x, \dot{x}) = 0 \qquad k = 1, \ldots, n - m \tag{1.1}$$

or the explicit form

$$\dot{x}^i = f^i(t, x, u) \qquad i = 1, \ldots, n \tag{1.2}$$

where $u = (u^1, \ldots, u^m)$ is the vector of control variables. In the above we assume that rank $\frac{\partial F}{\partial \dot{x}} = n - m$, rank $\frac{\partial f}{\partial u} = m$ and that $F^k(t, x, f(t, x, u)) = 0$ for all u in some open set. A solution will be a set of n functions of time $x^i(t)$ satisfying the equations (1.1) or such that there exists m additional time functions $u^l(t)$ and (1.2) is satisfied.

1.1.1 Finite Dimensions

From now on, we consider the system's "variables" as coordinates on manifolds. Since all our considerations are local, the manifolds in question can always be taken as open subsets of multiple Cartesian products of $\mathbb{R}$. The "time manifold" $\mathcal{B}$ shall have the coordinate t. The "time-and-state-space manifold" $\mathcal{M} = \mathcal{B} \times \mathcal{X}$ shall have the coordinates $t, x^1, \ldots, x^n$. We also have a natural projection map $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$ given simply by $\pi_{\mathcal{MB}} : (t, x) \mapsto (t)$. In this setting, a function of time $x(t)$ is a section on the fiber bundle $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$, i.e. a map $\sigma : \mathcal{B} \to \mathcal{M}$, reading $\sigma : (t) \mapsto (t, x(t))$ and $\sigma \in \Gamma\mathcal{M}$.
Next, we consider the space of first jets of sections in $\Gamma\mathcal{M}$. This is nothing but the manifold $\mathcal{M}$ augmented with the data corresponding to the derivatives of curves $x(t)$ with respect to t. Hence, $J^1\mathcal{M} \equiv \mathcal{M} \times \mathbb{R}^n$ shall be equipped with coordinates $t, x^1, \ldots, x^n, p^1, \ldots, p^n$. The lift of a section $\sigma \in \Gamma\mathcal{M}$ to $j(\sigma) \in \Gamma J^1\mathcal{M}$ is then given by a map $j(\sigma) : \mathcal{B} \to J^1\mathcal{M}$ and reads $j(\sigma) : (t) \mapsto (t, x(t), p = \frac{\partial x(t)}{\partial t})$.
The discussion above implies that the coordinates t, x and p are related in some way. These relations can be encoded by a codistribution on $T(J^1\mathcal{M})^*$ called the Cartan codistribution and spanned by so called *contact forms*. A 1-form $\theta \in T(J^1\mathcal{M})^*$ is a contact form if the pull-back $j(\sigma)^*\theta$ is the zero form on $T\mathcal{B}^*$ for any section $\sigma \in \Gamma\mathcal{M}$. The Cartan codistribution on $T(J^1\mathcal{M})^*$ is spanned by the forms

$$\theta^i = dx^i - p^i dt \qquad i = 1, \ldots, n.$$

Indeed

$$j(\sigma)^*(dx^i - p^i dt) = d(x^i(t)) - \frac{\partial x^i(t)}{\partial t} dt = 0.$$

By duality, one can define the Cartan distribution on $T(J^1\mathcal{M})$ spanned by all vector fields (sections of $T(J^1\mathcal{M})$) annihilating the forms θ^i. The Cartan distribution is hence spanned by

$$\frac{\partial}{\partial t} + p^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \frac{\partial}{\partial p^1}, \ldots, \frac{\partial}{\partial p^n}.$$

Remark 1.1. The previous distribution does not qualify as a Cartan distribution as defined in the context of diffieties, see [153, 1, 49, 23], as it is not involutive.

In our geometric setting, a solution to the equations (1.1), is given by a section $\sigma \in \Gamma\mathcal{M}$ satisfying

$$F^k \circ j(\sigma)(t) = 0 \qquad k = 1, \ldots, n-m \quad t \in \mathcal{B}. \tag{1.3}$$

In other words, the image of t through $j(\sigma)$ must lie on the submanifold of $J^1\mathcal{M}$ defined by the equations

$$F^k(t, x, p) = 0.$$

We shall denote this submanifold by $\mathcal{E}$:

$$\mathcal{E} = \{(t, x, p) \in J^1\mathcal{M} \mid F^k(t, x, p) = 0,\ k = 1, \ldots, n-m\}.$$

Now, the explicit system equations (1.2) provides us with a parameterization of $\mathcal{E}$ in $J^1\mathcal{M}$ in the form of an embedding from some manifold $\mathcal{U}$ to $J^1\mathcal{M}$

$$f : \mathcal{U} \to J^1\mathcal{M} \qquad f : (t, x, u) \mapsto (t, x, p^i = f^i(t, x, u)) \tag{1.4}$$

with $f(\mathcal{U}) = \mathcal{E} \subset J^1\mathcal{M}$. The manifold $\mathcal{U}$ can also be considered as a fiber bundle, sharing the same base manifold $\mathcal{M}$ as the jet manifold $J^1\mathcal{M}$. The projection map $\pi_{\mathcal{U}\mathcal{M}} : \mathcal{U} \to \mathcal{M}$ is given by $\pi_{\mathcal{U}\mathcal{M}} = \pi_{10} \circ f$, leading to the commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "f"] \arrow[dr, "\pi_{\mathcal{U}\mathcal{M}}"'] & J^1\mathcal{M} \arrow[d, "\pi_{10}"] \\ & \mathcal{M} \arrow[d, "\pi_{\mathcal{M}\mathcal{B}}"] \\ & \mathcal{B} \end{tikzcd} \tag{1.5}$$

and explicitely

$$\pi_{\mathcal{U}\mathcal{M}} : (t, x, u) \mapsto (t, x).$$

A section $\sigma \in \Gamma\mathcal{M}$, $\sigma : (t) \mapsto (t, x(t))$ can be lifted to a section $\hat{\sigma}$ on $\pi_{\mathcal{M}\mathcal{B}} \circ \pi_{\mathcal{U}\mathcal{M}} : \mathcal{U} \to \mathcal{B}$ by additionally specifying time functions for the variables u^l,

$$\hat{\sigma} : (t) \mapsto (t, x(t), u(t)).$$

Such a section is then a solution of the explicit system of equations (1.2) if

$$f \circ \hat{\sigma}(t) = j(\sigma)(t) \qquad t \in \mathcal{B}. \tag{1.6}$$

Clearly, (1.6) implies (1.3). The infinitesimal relations imposed on solutions can again be encoded in a Cartan codistribution defined on $T\mathcal{U}^*$. A 1-form $\omega \in T\mathcal{U}^*$ is a contact form if for any section $\hat{\sigma}$ lifted from $\sigma \in \Gamma\mathcal{M}$ and satisfying (1.6), the pullback $\hat{\sigma}^*\omega$ is the zero form on $T\mathcal{B}^*$.

Note that since $\mathcal{U}$ is homeomorphic to its image $\mathcal{E}$ in $J^1\mathcal{M}$ through f, any form on $T\mathcal{U}^*$ is the pullback of some form on $T(J^1\mathcal{M})^*$. Now, assume $\sigma \in \Gamma\mathcal{M}$ satisfies (1.3) and assume $\hat{\sigma} \in \Gamma\mathcal{U}$ is a lift of σ satisfying (1.6). Let ϑ be any 1-form on $T(J^1\mathcal{M})^*$, then

$$\hat{\sigma}^*(f^*\vartheta) = (f \circ \hat{\sigma})^*\vartheta \overset{(1.6)}{=} j(\sigma)^*\vartheta.$$

Therefore, a 1-form on $T\mathcal{U}^*$ is a contact form if and only if it is the pullback of some contact form on $J^1\mathcal{M}$ through f. Hence, the n-dimensional Cartan codistribution on $T\mathcal{U}^*$ is spanned by the forms

$$\omega^i = f^*\theta^i = dx^i - f^i(t,x,u)dt \qquad i = 1,\dots,n \tag{1.7}$$

and the $(1+m)$-dimensional Cartan distribution by the vectors

$$\frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i} \quad \text{and} \quad \frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^m}.$$

Note that Remark 1.1 also applies here.
We close this section by a last remark on the properties of the Cartan distribution on $T\mathcal{U}$. The tangent space $T\mathcal{B}$ is spanned by $\frac{\partial}{\partial t}$. Let again σ and $\hat{\sigma}$ be a (solution) section in $\Gamma\mathcal{M}$ and its lift to $\Gamma\mathcal{U}$ such that (1.6) is satisfied. Then the pushforward of $\frac{\partial}{\partial t}$ by $\hat{\sigma}$ reads

$$\begin{aligned}\hat{\sigma}_*\frac{\partial}{\partial t} &= \frac{\partial}{\partial t} + \frac{\partial x^i(t)}{\partial t}\frac{\partial}{\partial x^i} + \frac{\partial u^l(t)}{\partial t}\frac{\partial}{\partial u^l} \\ &\overset{(1.6)}{=} \frac{\partial}{\partial t} + f^i(t,x(t),u(t))\frac{\partial}{\partial x^i} + \frac{\partial u^l(t)}{\partial t}\frac{\partial}{\partial u^l}\end{aligned}$$

and is a vector of the Cartan distribution at the point $\hat{\sigma}(t) = (t,x(t),u(t)) \in \mathcal{U}$. Therefore, with any solution $\hat{\sigma}$, there is an associated vector field in the Cartan distribution on $T\mathcal{U}$ that is $\hat{\sigma}$-related to $\frac{\partial}{\partial t} \in T\mathcal{B}$. This vector field precisely represents time differentiation of functions on $\mathcal{U}$ along the solution trajectory of (1.2) given by $\hat{\sigma}$.

1.1.1.1 Bundle Maps and Static Feedback Transformations

We now describe the effect of changing the coordinates on a control system in a way that preserves time. To this end, consider two fiber bundles $\pi_{\mathcal{NB}} : \mathcal{N} \to \mathcal{B}$ and $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$ over the same base $\mathcal{B}$, with local coordinates (t,z) and (t,x) respectively. Assume that $\phi : \mathcal{N} \to \mathcal{M}$ is an invertible smooth bundle map of the form $\phi : (t,z) \mapsto (t, x = \phi(t,z))$. Then, a section $s \in \Gamma\mathcal{N}$ defines a section $\sigma \in \Gamma\mathcal{M}$ by $\sigma = \phi \circ s$. This in turn induces an (invertible) map between the two corresponding jet bundles $J^1\phi : J^1\mathcal{N} \to J^1\mathcal{M}$ which is required to satisfy

$$J^1\phi\,(j(s)) = j(\phi \circ s)$$

where

$$\begin{aligned} j(s) : (t) &\mapsto (t,\ z(t),\ q = \frac{\partial z(t)}{\partial t}) \\ j(\phi\circ s) : (t) &\mapsto (t,\ \phi(t,z(t)),\ p = \frac{\partial \phi(t,z(t))}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z(t)}{\partial t}) \end{aligned}$$

for any section $s \in \Gamma\mathcal{N}$ and therefore reads

$$J^1\phi : (t,z,q) \mapsto (t,\ x = \phi(t,z),\ p = \frac{\partial \phi(t,z)}{\partial t} + \frac{\partial \phi(t,z)}{\partial z}q).$$

Recall that the implicit equations (1.1) define a sub-manifold $\mathcal{E} \subset J^1\mathcal{M}$. Through $J^1\phi$, the real valued functions $F^k(t,x,p)$ pull back to real valued functions $G^k(t,z,q) = F^k \circ J^1\phi(t,z,q)$ and define a sub-manifold $\mathcal{H} \subset J^1\mathcal{N}$ by setting $G^k(t,z,q) = 0$. The relations $G^k(t,z,\dot{z}) = 0$ correspond to (1.1), rewritten in the new state variable z.
One may now consider any bundle $\pi_{\mathcal{VN}} : \mathcal{V} \to \mathcal{N}$ over $\mathcal{N}$ together with an (invertible) map $\varphi : \mathcal{V} \to \mathcal{U}$ over ϕ of the form $\varphi : (t,z,v) \mapsto (t, x = \phi(t,z), u = \varphi(t,z,v))$. Because φ and $J^1\phi$ are invertible, the embedding $f : \mathcal{U} \to \mathcal{E} \subset J^1\mathcal{M}$ uniquely induces an embedding $g : \mathcal{V} \to \mathcal{H} \subset J^1\mathcal{N}$ by requiring

$$J^1\phi \circ g = f \circ \varphi.$$

The map $g = (J^1\phi)^{-1} \circ f \circ \varphi$ corresponds to the explicit equations (1.2) transformed in the new state and input variables z and v. The situation is summarized by the following commutative diagram (the sections and their lifts have been omitted):

$$\begin{tikzcd}
J^1\mathcal{N} \arrow[rrr, bend left, "J^1\phi"] \arrow[dr, "\pi_{10}"'] & \mathcal{V} \arrow[l, "g"'] \arrow[r, "\varphi"] \arrow[d, "\pi_{\mathcal{VN}}"] & \mathcal{U} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{UM}}"'] & J^1\mathcal{M} \arrow[dl, "\pi_{10}"] \\
& \mathcal{N} \arrow[r, "\phi"] \arrow[dr, "\pi_{\mathcal{NB}}"'] & \mathcal{M} \arrow[d, "\pi_{\mathcal{MB}}"] & \\
& & \mathcal{B} &
\end{tikzcd}$$

The transformation on f, induced by φ and leading to the map g encodes the classic notion of a *static (state and input-) feedback transformation*.
Finally, it should be clear that the contact forms on $T(\mathcal{V})^*$ can be pulled back from the contact forms ω^i on $T(\mathcal{U})^*$ and are hence given by $\varphi^*\omega^i, i = 1, \ldots, n$.

1.1.1.2 Pfaffian System Associated to a Control System

A section σ and its lift $\hat{\sigma}$ satisfying (1.6)

$$\hat{\sigma} : (t) \mapsto (t, x(t), u(t))$$

can also be seen as describing a 1-dimensional embedding of $\mathcal{B}$ in $\mathcal{U}$. Since the contact forms ω^i of (1.7) are precisely such that $\hat{\sigma}^*\omega^i = 0$, the solution submanifold is an integral manifold of the Pfaffian system $\Omega = \{\omega^1, \ldots, \omega^n\}$. Moreover, since t is a local coordinate on such a submanifold, solutions are 1-dimensional integral manifolds of (Ω, dt), the Pfaffian system Ω with independence condition dt. Indeed, $\hat{\sigma}^*dt = dt \neq 0$.
Considering the ideal in $\Lambda T(\mathcal{U})^*$ generated by Ω (or its dual) is the starting point of a number of algorithms used to analyze properties of control system such as static feedback linearizability in the Gardner-Shadwick algorithm [58] or equivalence to partial prolongations of contact distributions through the (generalized or extended) Goursat normal form [143, 148], to cite only a few.

1.1.2 Infinite Dimensions

1.1.2.1 First Input Prolongation

In this section, we describe the first prolongation of a control system. Prolongation consists in augmenting the set of equations describing the system by differentiating the relations at hand with respect to time. The augmented set of equations involves the initial set of variables and new variables representing the time derivatives of the initial ones. From the new variables, $n-m$ of these can be eliminated, and m cannot. When the set of variables that is not discarded consist in the time derivatives $u^{(1)}$ of the (initial) input variables u, one speaks of the first *input prolongation*.
We now describe how one may construct the *first prolonged bundle* $\mathcal{U}^1$, describing the system with the inputs u added to the state and the inputs derivative $u^{(1)}$ taken as new input variables.
The implicit equations $F^k(t,x,p_1)=0$ on $J^1\mathcal{M}$ are differentiated to obtain equations on $J^2\mathcal{M}$

$$F^k(t,x,p_1)=0 \tag{1.8a}$$

$$\frac{\partial F^k}{\partial t}+\frac{\partial F^k}{\partial x^i}p_1^i+\frac{\partial F^k}{\partial p_1^j}p_2^j=0 \qquad k=1,\dots,n-m \tag{1.8b}$$

where (t,x,p_1,p_2) are coordinates on $J^2\mathcal{M}$. Denote by $\mathcal{E}^1\subset J^2\mathcal{M}$ the submanifold of $J^2\mathcal{M}$ consisting of the points satisfying (1.8). Remember that the embedding f is a parameterization of the solution set $\mathcal{E}$ of $F(t,x,p_1)=0$ in $J^1\mathcal{M}$. Note that $\dim J^2\mathcal{M}=\dim J^1\mathcal{M}+n$ and there are $n-m$ new equations in (1.8) involving the n new variables p_2. An initial assumption is that $\operatorname{rank}\frac{\partial F(t,x,p_1)}{\partial p_1}=\operatorname{rank}\frac{\partial F(t,x,\dot{x})}{\partial \dot{x}}=n-m$. Therefore, any solution of (1.8a), parameterized by f, can be lifted to a solution of (1.8) by solving for a subset of p_2 and keeping m of them free. Hence, $\mathcal{E}^1$ is an $(1+n+2m)$-dimensional submanifold of $J^2\mathcal{M}$ and projects through the canonical projection $\pi_{21}:J^2\mathcal{M}\to J^1\mathcal{M}$ to the $(1+n+m)$-dimensional submanifold $\mathcal{E}\subset J^1\mathcal{M}$.
In the same manner as in Section 1.1.1, we proceed in describing an embedding f^1 from some manifold $\mathcal{U}^1$ to $\mathcal{E}^1$, i.e. $f^1:\mathcal{U}^1\to\mathcal{E}^1\subset J^2\mathcal{M}$.
Consider a section $\sigma\in\Gamma\mathcal{M}$ and its two first prolongations $j^1(\sigma)=j(\sigma)\in\Gamma J^1\mathcal{M}$ and $j^2(\sigma)\in\Gamma J^2\mathcal{M}$

$$\sigma:(t)\mapsto(t,x(t)) \qquad j^1(\sigma):(t)\mapsto(t,x(t),p_1=\frac{\partial x(t)}{\partial t})$$
$$j^2(\sigma):(t)\mapsto(t,x(t),p_1=\frac{\partial x(t)}{\partial t},p_2=\frac{\partial^2 x(t)}{\partial t^2})$$

Next, consider $J^1\mathcal{U}$. $J^1\mathcal{U}$ stands for the first jet of $\mathcal{U}$ considered as a bundle over the basis $\mathcal{B}$, i.e $J^1\mathcal{U}=J^1(\pi_{\mathcal{M}\mathcal{B}}\circ\pi_{\mathcal{U}\mathcal{M}}:\mathcal{U}\to\mathcal{B})$ with coordinates $(t,x,u,p_1,u^{(1)})$. Remember that $\hat{\sigma}$ is a lift of σ to a section in $\Gamma\mathcal{U}$. In turn, $\hat{\sigma}$ can be prolonged (lifted) to a section $j^1(\hat{\sigma})\in\Gamma J^1\mathcal{U}$:

$$\hat{\sigma}:(t)\mapsto(t,x(t),u(t)) \qquad j^1(\hat{\sigma}):(t)\mapsto(t,x(t),p_1=\frac{\partial x(t)}{\partial t},u^{(1)}=\frac{\partial u(t)}{\partial t}).$$

We now define the prolonged bundle $\mathcal{U}^1$ over $\mathcal{U}$. This definition depends on the initial choice of input coordinates u on $\mathcal{U}$. As a manifold, $\mathcal{U}^1$ shall have coordinates $t, x, u, u^{(1)}$ and we have the following projection maps

$$\begin{aligned} \pi_{1,\mathcal{U}^1} &: J^1\mathcal{U} \to \mathcal{U}^1 & \pi_{1,\mathcal{U}^1} &: (t, x, u, p_1, u^{(1)}) \mapsto (t, x, u, u^{(1)}) \\ \pi_{\mathcal{U},10} &: \mathcal{U}^1 \to \mathcal{U} & \pi_{\mathcal{U},10} &: (t, x, u, u^{(1)}) \mapsto (t, x, u) \end{aligned}$$

and

$$\pi_{\mathcal{U},10} \circ \pi_{1,\mathcal{U}^1} = \pi_{10} : J^1\mathcal{U} \to \mathcal{U}.$$

The dimension of $\mathcal{U}^1$ is $1+n+2m$, the same as $\mathcal{E}^1 \in J^2\mathcal{M}$. Let the section σ satisfy (1.6) so that it represents a system solution. We can define the embedding $f^1 : \mathcal{U}^1 \to \mathcal{E}^1 \subset J^2\mathcal{M}$ as the unique map satisfying

$$\pi_{21} \circ f^1 \circ \pi_{1,\mathcal{U}^1} \circ j^1(\hat{\sigma})(t) = f \circ \hat{\sigma}(t) \qquad t \in \mathcal{B}.$$

The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
J^1\mathcal{U} \arrow[r, "\pi_{1,\mathcal{U}^1}"] \arrow[dr, "\pi_{10}"'] & \mathcal{U}^1 \arrow[r, "f^1"] \arrow[d, "\pi_{\mathcal{U},10}"] & J^2\mathcal{M} \arrow[d, "\pi_{21}"] \\
 & \mathcal{U} \arrow[r, "f"] \arrow[dr, "\pi_{\mathcal{U}\mathcal{M}}"'] & J^1\mathcal{M} \arrow[d, "\pi_{10}"] \\
 & & \mathcal{M} \arrow[d, "\pi_{\mathcal{M}\mathcal{B}}"] \\
 & & \mathcal{B} \arrow[uuu, dotted, bend left, "j^1(\hat{\sigma})"] \arrow[uu, dotted, bend left, "\hat{\sigma}"] \arrow[u, dotted, bend left, "\sigma"]
\end{tikzcd}$$

The canonical projection $\pi_{10} : J^1\mathcal{U} \to \mathcal{U}$ reads $\pi_{10} : (t, x, u, p_1, u^{(1)}) \mapsto (t, x, u)$.
It is easy to verify that f^1 takes the form

$$\begin{aligned} f^1(t, x, u, u^{(1)}) \mapsto (t, x, u & \\ p_1^i &= f^i(t, x, u) \\ p_2^j &= \frac{\partial f^j(t, x, u)}{\partial t} + \frac{\partial f^j(t, x, u)}{\partial x^i} f^i(t, x, u) + \frac{\partial f^j(t, x, u)}{\partial u^l} u^{l(1)}). \end{aligned} \tag{1.9}$$

To define the Cartan codistribution on $\mathcal{U}^1$, one first writes

$$\hat{\sigma}^1 \in \Gamma\mathcal{U}^1 \qquad \hat{\sigma}^1 := \pi_{1,\mathcal{U}^1} \circ j^1(\hat{\sigma})$$

and then, exactly as for the Cartan codistribution on $\mathcal{U}$, $\omega \in T(\mathcal{U}^1)^*$ shall be a contact form if

$$\hat{\sigma}^{1*}\omega = 0, \quad \forall \sigma \text{ satisfying (1.6)}.$$

Computation shows that the $(n+m)$-dimensional Cartan codistribution on $\mathcal{U}^1$ is spanned by

$$dx^i - f^i(t, x, u)dt \qquad \text{and} \qquad du^l - u^{l(1)}dt \qquad i = 1, \ldots, n \;\; l = 1, \ldots, m$$

and the $(1+m)$-dimensional Cartan distribution by the vector fields

$$\frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i} + u^{l(1)}\frac{\partial}{\partial u^l} \qquad \text{and} \qquad \frac{\partial}{\partial u^{l(1)}} \qquad l = 1,\ldots,m.$$

Note also that the contact forms are pulled back from contact forms on $T(J^2\mathcal{M})^*$ through f^{1*}. Indeed, from (1.9):

$$\begin{aligned}
f^{1*}(dx^i - p_1^i) &= dx^i - f^i(t,x,u)dt = \omega^i \\
f^{1*}(dp_1^j - p_2^j dt) &= d(f^j(t,x,u)) \\
&\quad - \left(\frac{\partial f^j(t,x,u)}{\partial t} + \frac{\partial f^j(t,x,u)}{\partial x^i} f^i(t,x,u) + \frac{\partial f^j(t,x,u)}{\partial u^l} u^{l(1)}\right) dt \\
&= \frac{\partial f^j}{\partial x^i}\omega^i + \frac{\partial f^j}{\partial u^l}(du^l - u^{l(1)}dt)
\end{aligned}$$

and an initial assumption is that $\frac{\partial f}{\partial u}$ is full rank. Finally note that

$$\begin{aligned}
\hat{\sigma}^1_* \frac{\partial}{\partial t} &= \frac{\partial}{\partial t} + \frac{\partial x^i(t)}{\partial t}\frac{\partial}{\partial x^i} + \frac{\partial u^l(t)}{\partial t}\frac{\partial}{\partial u^l} + \frac{\partial u^{l(1)}(t)}{\partial t}\frac{\partial}{\partial u^{l(1)}} \\
&= \frac{\partial}{\partial t} + f^i(t,x(t),u(t))\frac{\partial}{\partial x^i} + \frac{\partial u^l(t)}{\partial t}\frac{\partial}{\partial u^l} + \frac{\partial^2 u^l(t)}{\partial t^2}\frac{\partial}{\partial u^{l(1)}}
\end{aligned}$$

and as in the end of Section 1.1.1, $\frac{\partial}{\partial t} \in T\mathcal{B}$ is both pushed forward to a vector annihilating the contact forms and is such that it represents time differentiation of functions on $\mathcal{U}^1$ along the trajectory $\hat{\sigma}^1$ of the system (1.2) lifted to $\mathcal{U}^1$.

1.1.2.2 Higher Order and Infinite Prolongations

It should be clear that the process of the previous section can be repeated any number of times, leading to the following "prolonged" diagram

$$\begin{tikzcd}
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] \\
J^2\mathcal{U} \arrow[r, "\pi_{2,\mathcal{U}^2}"] \arrow[d, "\pi_{21}"'] & \mathcal{U}^2 \arrow[r, "f^2"] \arrow[d, "\pi_{\mathcal{U},21}"] & J^3\mathcal{M} \arrow[d, "\pi_{32}"] \\
J^1\mathcal{U} \arrow[r, "\pi_{1,\mathcal{U}^1}"] \arrow[dr, "\pi_{10}"'] & \mathcal{U}^1 \arrow[r, "f^1"] \arrow[d, "\pi_{\mathcal{U},10}"] & J^2\mathcal{M} \arrow[d, "\pi_{21}"] \\
 & \mathcal{U} \arrow[r, "f"] \arrow[dr, "\pi_{\mathcal{U}\mathcal{M}}"'] & J^1\mathcal{M} \arrow[d, "\pi_{10}"] \\
 & & \mathcal{M} \arrow[d, "\pi_{\mathcal{M}\mathcal{B}}"] \\
 & & \mathcal{B}
\end{tikzcd}$$

At the k-th step, $\mathcal{U}^k$ is a $(1+n+(k+1)m)$-dimensional manifold. The contact forms on $\mathcal{U}^k$ are spanned by

$$dx^i - f^i(t,x,u)dt,\ du^l - u^{l(1)}dt,\ \ldots,\ du^{l(k-1)} - u^{l(k)}dt \qquad i=1,\ldots,n \ \ l=1,\ldots,m$$

and the annihilating vector fields by

$$\frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i} + u^{l(1)}\frac{\partial}{\partial u^l} + \ldots + u^{l(k)}\frac{\partial}{\partial u^{l(k-1)}}\ ,$$
$$\frac{\partial}{\partial u^{1(k)}} \qquad l=1,\ldots,m.$$

The projective (inverse) limit

$$\mathcal{U}^\infty = \varprojlim_k \mathcal{U}^k$$

is an infinite dimensional manifold. A countable basis of contact forms for the Cartan codistribution is given by

$$dx^i - f^i(t,x,u)dt,\ du^{l(r)} - u^{l(r+1)}dt \qquad i=1,\ldots,n \ \ l=1,\ldots,m, \ \ r=0,\ldots$$

The 1-dimensional Cartan distribution is spanned by the vector field

$$D = \frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i} + u^{l(r+1)}\frac{\partial}{\partial u^{l(r)}} \tag{1.10}$$

where the sum on r ranges from 0 to infinity. If D is additionally chosen so as to satisfy $D \lrcorner\, dt = D(t) = 1$, then it is uniquely given by (1.10) . Compared to the case $\mathcal{U}^k$ with k finite, the limit case $\mathcal{U}^\infty$ has an important additional property. Indeed, if ω is a contact form, $D\omega$ is also a contact form. Moreover, since the Cartan distribution is spanned by one vector field, it is involutive. In this situation, the pair $(\mathcal{U}^\infty, D)$ is a diffiety and D generates the Cartan distribution on $\mathcal{U}^\infty$, [153, 1, 49, 23].

1.2 Static Feedback Linearization

In this section, we deal with the following question. Given a control system of the form (1.2), is it possible to transform it into a linear controllable system using an invertible state and input transformation of the type described in Section 1.1.1.1?

1.2.1 Eliminating Inputs

In Section 1.1.2.1, we have seen that one can always augment a system by adding its input variables to the state and by devising new input variables corresponding to the old ones' time derivatives. Can the opposite process also be carried on? That is, can one discard the input variables and choose new inputs among the state variables so as to obtain a smaller system, whose prolongation produces the original one?

1.2.1.1 Linear Case

To illustrate the idea, consider the linear time-invariant system

$$\dot{x} = Ax + Bu \qquad A \in \mathbb{R}^{n\times n} \quad B \in \mathbb{R}^{n\times m} \quad \operatorname{rank} B = m. \tag{1.11}$$

Choosing any rank $n-m$ matrix $N \in \mathbb{R}^{(n-m)\times n}$ such that $NB = 0$, leads to the implicit equations

$$N\dot{x} = NAx. \tag{Step 1}$$

Next, picking any $\bar{N} \in \mathbb{R}^{m\times n}$ with rows independent of the rows of N and further setting $M \in \mathbb{R}^{n\times(n-m)}$ and $\bar{M} \in \mathbb{R}^{n\times m}$ such that $\begin{pmatrix} M & \bar{M} \end{pmatrix} \begin{pmatrix} N \\ \bar{N} \end{pmatrix} = I_n$, we may split the state variables into two sets

$$y = Nx \quad v = \bar{N}x \tag{Step 2}$$

that is $x = My + \bar{M}v$ and obtain the dynamics

$$\dot{y} = NAMy + NA\bar{M}v \qquad \operatorname{card} y = n-m \quad \operatorname{card} v = m. \tag{1.12}$$

Hence the reduced system has state dimension $n-m$. However, since $\operatorname{rank} \bar{M} = m$, we have that $\operatorname{rank} NA\bar{M} = m-\rho,\ \rho \geq 0$; from the new inputs v, ρ of them can be eliminated. To do so, choose a $G \in \mathbb{R}^{m\times m-\rho}$ such that $\operatorname{rank} NA\bar{M} = \operatorname{rank} NA\bar{M}G$ and obtain

$$\dot{y} = NAMy + NA\bar{M}Gw \qquad \operatorname{card} y = n-m \quad \operatorname{card} w = m-\rho. \tag{1.13}$$

Note that the input prolongation of (1.12) leads to a system equivalent to (1.11) under a linear instance of the bundle mapping of Section 1.1.1.1. The input prolongation of (1.13) leads to a "subsystem" of (1.11), with ρ less inputs. We will not define what we mean by a subsystem precisely.

1.2.1.2 Elimination from the Contact Codistribution

In the reduction process illustrated in the previous section, Step 1 corresponds to transforming the explicit system equations (1.2) into the implicit equations (1.1). This step extends to the nonlinear case without condition. However, Step 2 is not always possible in the nonlinear setting, there are integrability conditions. Before discussing reduction for nonlinear systems, we will re-describe the linear case in a way that is consistent with the description of control systems we have given so far. On the bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ with coordinates expressions $\pi_{\mathcal{UM}} : (t, x, u) \mapsto (t, x)$, the contact forms encoding the infinitesimal relations between coordinates are (in a vectorial notation that should be clear)

$$\Omega = \{dx - (Ax + Bu)dt\}.$$

These contact forms (because the coefficient functions depend on u and although given by combinations of the differentials dt, dx) are forms in $T\mathcal{U}^*$ and not in $T\mathcal{M}^*$. Elimination Step 1 can then be reformulated as looking for all infinitesimal relations that can be expressed using variables on $\mathcal{M}$ only. These are given by the codistribution $\hat{\Omega} \subset \Omega$

$$\hat{\Omega} = \{Ndx - (NAx)dt\}. \tag{1.14}$$

Step 2, leading to the reduced state y is obtained using the relation

$$Ndx = d(Nx). \tag{1.15}$$

Note that this relation is actually an *integration step* and is always possible and trivial in this linear time-invariant setting. After (linear) algebraic transformations identical to those of the previous section, we obtain the codistribution

$$\hat{\Omega} = \{dy - (NAMy + NA\bar{M}Gw)dt\}$$

where (t, y, w) are all coordinates on $\mathcal{M}$. The forms of $\hat{\Omega}$ are contact forms on a bundle $\pi_{\mathcal{M}\mathcal{Y}} : \mathcal{M} \to \mathcal{Y}$ with $\pi_{\mathcal{M}\mathcal{Y}} : (t, y, w) \mapsto (t, y)$ representing the system (1.13).

1.2.1.3 Nonlinear Case

We now discuss the construction of $\hat{\Omega}$ (similarly to (1.14)) in the nonlinear case. Recall the contact forms on $\mathcal{U}$ are given by

$$\Omega \;=\; \{\omega^i = dx^i - f^i(t, x, u)dt\} \;\subset\; T\mathcal{U}^* \qquad i = 1, \ldots, n. \tag{1.16}$$

Redefine $\hat{\Omega}$ as the largest codistribution on $\mathcal{M}$ satisfying

$$\hat{\Omega} \subset T\mathcal{M}^* \qquad \text{and} \qquad \pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega} \subset \Omega$$

where $\pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega}$ is the codistribution in $T\mathcal{U}^*$ generated by the elements of $\hat{\Omega}$ pulled back though $\pi^*_{\mathcal{U}\mathcal{M}}$.

Lemma 1.2. *The codistribution $\pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega} \subset \Omega$ satisfies*

$$\pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega} \subset \Omega^{(1)} := \{\omega \in \Omega \mid \frac{\partial}{\partial u^l} \lrcorner\, d\omega \in \Omega,\ l = 1, \ldots, m\}.$$

moreover, assume $\Omega^{(1)} + \{dt\}$ is integrable, then $\pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega} = \Omega^{(1)}$.

Proof. By Lemma A.12 and from $\pi_{\mathcal{U}\mathcal{M}*} = \{\frac{\partial}{\partial u^l}\}$, $\pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega}$ is obtained by computing the sequence of nested codistributions

$$\Omega^{(0)} = \Omega \qquad \Omega^{(r+1)} = \{\omega \in \Omega^{(r)} \mid \frac{\partial}{\partial u^l} \lrcorner\, d\omega \in \Omega^{(r)}, l = 1, \ldots, m\}. \tag{1.17}$$

which for some $r = r^*$, saturates to $\Omega^{(r^*)} = \pi^*_{\mathcal{U}\mathcal{M}}\hat{\Omega}$. We must show that if $\Omega^{(1)} + \{dt\}$ is integrable, then $r^* = 1$.
Assume $\Omega^{(1)} + \{dt\}$ is integrable, then there are functions y^j and g^j on $\mathcal{U}$ such that

$$\Omega^{(1)} = \{\mu^j = dy^j - g^j dt\}.$$

And by (1.17),

$$\frac{\partial}{\partial u^l} \lrcorner\, d\mu^j = \frac{\partial}{\partial u^l} \lrcorner\, dg^j \wedge dt = \frac{\partial g^j}{\partial u^l} dt \;\in\; \Omega, \quad l = 1, \ldots, m.$$

But $dt \notin \Omega$ implies $\frac{\partial g^j}{\partial u^l} = 0$, $l = 1, \ldots, m$ so that $\frac{\partial}{\partial u^l} \lrcorner\, d\mu^j = 0$. Any $\mu \in \Omega^{(1)}$ is of the form $\mu = \alpha_j \mu^j$ for some functions α_j on $\mathcal{U}$. Therefore

$$\frac{\partial}{\partial u^l} \lrcorner\, d\mu = \frac{\partial \alpha_j}{\partial u^l} \mu^j \in \Omega^{(1)}.$$

Hence $\Omega^{(2)} = \Omega^{(1)}$, i.e. $r^* = 1$. □

A successful example for the previous lemma is given by any linear system. Before going on, let us give a negative example.

Example 1.3. *Consider the control system given by*

$$\dot{x}^1 = x^2 + (u)^2 \qquad \dot{x}^2 = u.$$

The contact forms on $\mathcal{U}$ are spanned by

$$\Omega = \{\ \omega^1 = dx^1 - (x^2 + (u)^2)dt, \quad \omega^2 = dx^2 - udt\ \}$$

and $\ker \pi_{\mathcal{U}\mathcal{M}*} = \{\frac{\partial}{\partial u}\}$. *The codistribution $\Omega^{(1)}$ is spanned by*

$$\bar{\omega} = \omega^1 - 2u\omega^2 = (dx^1 - 2udx^2) - (x^2 - (u)^2)dt.$$

Indeed

$$\frac{\partial}{\partial u} \lrcorner\, d\bar{\omega} = 2dx^2 - 2udt = 2\omega^2.$$

However, ω^2 is independent of $\bar{\omega}$, hence $\frac{\partial}{\partial u} \lrcorner\, d\bar{\omega} \notin \Omega^{(1)}$ so that $\Omega^{(2)} = \{0\} \neq \Omega^{(1)}$. There is no non-zero 1-form in $T\mathcal{M}^*$ that pulls back to a form in Ω through $\ker \pi_{\mathcal{U}\mathcal{M}}{}^*$. *But we also see that*

$$\Omega^{(1)} + \{dt\} = \{dx^1 - 2u\ dx^2, dt\}$$

is not an integrable codistribution. Note that this example still admits a reduction/linearization of some kind. See Example 4 in [143].

Lemma 1.4. *The codistribution $\Omega^{(1)}$ as computed in Lemma 1.2 has dimension $n - m$.*

Proof. With the contact forms $\omega^1, \ldots, \omega^n$ spanning Ω given by (1.16), we have $\frac{\partial}{\partial u^l} \lrcorner\, d\omega^i = -\frac{\partial f^i}{\partial u^l} dt$. It follows that $\Omega^{(1)}$ is spanned by

$$\hat{\Omega} = \{\eta_i^k \omega^i\} = \{\eta_i^k dx^i - \eta_i^k f^i dt\} \qquad k = 1, \ldots, n-m$$
$$\text{s.t.} \qquad \eta_i^k(t,x,u) \frac{\partial f^i(t,x,u)}{\partial u^l} = 0 \qquad \operatorname{rank} \eta_i^k = n - m$$

since rank $\frac{\partial f}{\partial u}$ is assumed to be m. □

Lemma 1.5. *Let the codistribution $\Omega^{(1)}$, as computed in Lemma 1.2, be such that $\Omega^{(1)} + \{dt\}$ is integrable. Then $\Omega^{(1)} = \pi^*_{\mathcal{U}\mathcal{M}} \hat{\Omega}$ for some $\hat{\Omega} \subset T\mathcal{M}^*$ and $\hat{\Omega} + \{dt\} \subset T\mathcal{M}^*$ is integrable.*

Proof. The equality $\Omega^{(1)} = \pi^*_{\mathcal{UM}}\hat{\Omega}$ follows from Lemma 1.2. Clearly $dt = \pi^*_{\mathcal{UM}}dt$. Hence $\Omega^{(1)} + \{dt\} = \pi^*_{\mathcal{UM}}(\hat{\Omega} + \{dt\})$ and the result follows from Corollary A.9. □

Let $\Omega^{(1)} \subset \Omega$ be such that $\Omega^{(1)} + \{dt\}$ is integrable, so that by Lemma 1.2, there is a codistribution $\hat{\Omega} \subset T\mathcal{M}$ satisfying $\Omega^{(1)} = \pi^*_{\mathcal{UM}}\hat{\Omega}$. Then by Lemma 1.4 and since $\ker \pi^*_{\mathcal{UM}} = 0$, $\dim \hat{\Omega} = n - m$. Next, from Lemma 1.5, $\hat{\Omega} + \{dt\} \subset T\mathcal{M}^*$ is integrable so that $\hat{\Omega} = \{dy^j - g^j dt\}$ where y^j, g^j are functions on $\mathcal{M}$.

Recall from Section 1.1.1 that a section $\sigma \in \Gamma\mathcal{M}$, $\sigma : \mathcal{B} \to \mathcal{M}$ and its lift $\hat{\sigma} \in \Gamma\mathcal{U}$, $\hat{\sigma} : \mathcal{B} \to \mathcal{U}$ that satisfy the original system equations (1.2) are such that

$$\hat{\sigma}^*\Omega = \{0\}$$

and since $\sigma = \pi_{\mathcal{UM}} \circ \hat{\sigma}$ and $\pi^*_{\mathcal{UM}}\hat{\Omega} \subset \Omega$, we have

$$\sigma^*\hat{\Omega} = \hat{\sigma}^*(\pi^*_{\mathcal{UM}}\hat{\Omega}) = \{0\}.$$

Therefore, on any solution of the system (1.2), the functions y^k and g^k on $\mathcal{M}$ satisfy the equations

$$\dot{y}^k = g^k(t, x), \qquad k = 1, \ldots, n - m.$$

As in the linear case, (and without loss of generality) we may use the relations $y^k = y^k(t, x)$ to eliminate $x^1, \ldots, x^{n-m}$ so as to obtain the equations

$$\dot{y}^k = g^k(t, y, x^{n-m+1}, \ldots, x^n), \qquad k = 1, \ldots, n - m$$

analogous to (1.12). Clearly, $\operatorname{rank} \frac{\partial g}{\partial x} = m - \rho$, $\rho \geq 0$, so that we may further discard $x^{n-m+1}, \ldots, x^{n-m+\rho}$ to obtain the equations analogue to (1.13)

$$\dot{y}^k = g^k(t, y, w), \qquad k = 1, \ldots, n - m \quad w = (x^{n-m+\rho+1}, \ldots, x^n).$$

Finally, consider the set of equations

$$\begin{aligned} \dot{y}^k &= g^k(t, y, w) & k &= 1, \ldots, n - m && (1.18a)\\ \dot{w}^s &= \bar{w}^s & s &= 1, \ldots, m - \rho && (1.18b)\\ \dot{x}^{n-m+\sigma} &= \bar{w}^{m-\rho+\sigma} & \sigma &= 1, \ldots, \rho. && (1.18c) \end{aligned}$$

It is easy to check that the system described by (1.18) is statically feedback equivalent to the system described by (1.2), i.e. equivalent under a bundle map as presented in Section 1.1.1.1. Denote this map by φ, we also have

$$\{dy^k - g^k dt,\ dw^s - \bar{w}^s dt,\ dx^{n-m+\sigma} - \bar{w}^{m-\rho+\sigma} dt\} = \{\varphi^*(dx^i - f^i dt)\} = \varphi^*\Omega.$$

Importantly, the equations (1.18b)-(1.18c) are linear. Hence, if one is able to recursively repeat the elimination procedure on (1.18a) until the nonlinear part is "empty", then one has established a complete equivalence between the nonlinear system (1.2) and a linear system. Actually, it is possible to verify a priori whether the recursive procedure leads to the desired result, i.e. before performing any integration, by applying a recursive test. This is the object of *static feedback linearization.*

1.2.2 Static Feedback Linearization

In this section, we state the conditions under which a control system (1.2) may be transformed into a linear one using a transformation of the type described in Section 1.1.1.1. These transformations are also called *static feedback transformations*.
Recall once more that the Cartan codistribution of contact forms on $T\mathcal{U}^*$ is spanned by

$$\Omega = \{ \ \omega^i = dx^i - f^i(t,x,u)dt \ \} \qquad i = 1, \dots, n$$

and its annihilating distribution on $T\mathcal{U}$, $V = \perp_{T\mathcal{U}^*} \Omega$ by

$$V = \{D, \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}\} \qquad D = \frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i}. \tag{1.19}$$

It is useful to split V into two components. Indeed V is not involutive and contains an m dimensional involutive subdistribution $\ker \pi_{\mathcal{U}\mathcal{M}_*} \subset V$ spanned by

$$\ker \pi_{\mathcal{U}\mathcal{M}_*} = \{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}\}.$$

Hence,

$$V = \ker \pi_{\mathcal{U}\mathcal{M}_*} + \{D\}.$$

Note that the vector D is not unique. We may chose any representative of $V/\ker \pi_{\mathcal{U}\mathcal{M}_*}$ and we scale it so as to have

$$D(t) = D \lrcorner \, dt = 1.$$

1.2.2.1 Other Ways to Compute $\Omega^{(1)}$

Lemma 1.6. *With D chosen as in* (1.19)*, the computation of $\Omega^{(1)}$ from Lemma 1.17 takes any of the following three equivalent forms*

$$\Omega^{(1)} = \{\omega \in \Omega \mid \frac{\partial}{\partial u^l} \lrcorner \, d\omega \in \Omega, \ l = 1, \dots, m\} \tag{1.20}$$

$$= \{\omega \in \Omega \mid D \lrcorner \, d\omega \in \Omega\} \tag{1.21}$$

$$= \{\omega \in \Omega \mid d\omega \in \Omega\} \tag{1.22}$$

where in the last equality, Ω also represents the ideal it generates in $\Lambda T\mathcal{U}^*$.

Proof. With $\omega^i = dx^i - f^i dt$ the generators of Ω, expand $d\omega^i$ as

$$\begin{aligned} d\omega^i &= -df^i \wedge dt \\ &= -\left(\frac{\partial f^i}{\partial x^k}dx^k + \frac{\partial f^i}{\partial u^l}du^l + \frac{\partial f^i}{\partial t}dt\right) \wedge dt \\ &= -\left(\frac{\partial f^i}{\partial x^k}(\omega^k + f^k dt) + \frac{\partial f^i}{\partial u^l}du^l\right) \wedge dt \\ &= -\frac{\partial f^i}{\partial x^k}\omega^k \wedge dt - \frac{\partial f^i}{\partial u^l}du^l \wedge dt \end{aligned} \tag{1.23}$$

$$D \lrcorner\, d\omega^i = \frac{\partial f^i}{\partial x^k}\omega^k + \frac{\partial f^i}{\partial u^l}du^l \tag{1.24}$$

$$\frac{\partial}{\partial u^l} \lrcorner\, d\omega^i = -\frac{\partial f^i}{\partial u^l}dt. \tag{1.25}$$

Since neither du^l nor dt are in Ω, the result follows easily from (1.23), (1.24) and (1.25). □

Remark 1.7. Lemma 1.6 still holds verbatim if D is replaced by any other representative of $V/\ker \pi_{\mathcal{UM}_*}$.

1.2.2.2 Condition for Static Feedback Linearizability

Proposition 1.8. *Consider the two filtrations defined on the codistributions of contact forms $\Omega \subset T\mathcal{U}^*$*

$$\Omega_a^{(0)} := \Omega \qquad \Omega_a^{(r+1)} = \{\omega \in \Omega_a^{(r)} \mid D \lrcorner\, d\omega \in \Omega_a^{(r)}\} \tag{1.26}$$

$$\Omega_b^{(0)} := \Omega \qquad \Omega_b^{(r+1)} = \{\omega \in \Omega_b^{(r)} \mid \quad d\omega \in \Omega_b^{(r)} \quad \} \tag{1.27}$$

Assume either $\Omega_a^{(r)} + \{dt\}$ is integrable for all $r \geq 0$ or $\Omega_b^{(r)} + \{dt\}$ is integrable for all $r \geq 0$. Then $\Omega_a^{(r)} = \Omega_b^{(r)} =: \Omega^{(r)}$ for all $r \geq 0$ and there is a r^* such that $\Omega^{(r^*+k)} = \Omega^{(r^*)}$ for all $k \geq 0$. Further assume that $\Omega^{(r^*)} = \{0\}$. Then, the control system described by (1.2) *is locally equivalent under a static feedback transformation to a linear controllable system.*
Conversely, if the system (1.2) *is locally static feedback equivalent to a controllable linear system, then the filtrations* (1.26) *and* (1.27) *agree and $\Omega^{(r)} + \{dt\}$ is integrable for all r, and $\Omega^{(r^*)} = \{0\}$ for some r^*.*

Remark 1.9. If the integrability condition of Proposition 1.8 is not satisfied, then in general, the filtrations (1.26) and (1.27) do not agree for all r.

Sketch of proof. Use Lemma 1.6 to show that each step allows recursively to build a reduction of the form (1.18). Use $\Omega^{(r^*)} = 0$ to show that in the last reduction step, (1.18a) is empty, leaving only linear equations. To show the converse, take a linear controllable system, (1.15) implies that integrability modulo dt is alway satisfied. Use the rank of the controllability matrix to show that $\Omega^{(r^*)} = 0$. Finally, verify that the codistributions $\Omega^{(r)}$ are invariant under invertible bundle maps of the type of section 1.1.1.1. □

The result of Proposition 1.8 is found with equivalent or similar formulations in [72, 69, 58, 57] to cite a few.

1.2.2.3 Remark on Regularity

In the preceeding paragraphs, we ignored some important aspects regarding regularity. Indeed, in all computations, it was implicitly assumed that all distributions (respectively codistributions) were well defined, at least locally. This might fail and lead to difficulties even if the system equations satisfy the initial regularity assumptions. For instance, consider the simple equations

$$\dot{x}^1 = f^1 = x^1 x^2 \qquad \dot{x}^2 = f^2 = u.$$

The condition rank $\frac{\partial f}{\partial u} = 1 = m$ is satisfied everywhere. The system is equivalent to a double integrator except when $x^1 = 0$, where x^1 becomes uncontrolled. In the next chapter, we shall avoid this kind of situations by restricting the study of the system to the neighborhoods of so called Brunovsky-regular points.

1.3 Flatness

The relevance of the theory of differential algebra [116, 79] in the mathematical aspects of automatic control was recognized in [40]. Shortly afterwards, in [42], the same author reformulated many structural properties of linear systems using differential fields and module theory. The notion of *differential flatness* was then introduced in this differential algebraic context in [44]. In [47, 48], the module theoretic aspects of flatness in nonlinear systems reappeared by means of Kähler differentials and the introduction of the tangent linear system.
Shortly after the differential algebra point of view, the differential geometric description of flatness was introduced, [45, 46]. This approach describes control systems on diffieties and equivalence between these by Lie-Bäcklund transformations.
The notion of *differential flatness* is central throughout our text, but the concepts of differential algebra are essentially absent. In this regard, we cannot give a definition of flatness in strict accordance with the chronology of its development. However, the main idea can be grasped quite easily and an informal definition requires no particularly sophisticated apparatus. This is the object of the present section. We will then rely on the result from [5], given by Proposition 2.17, as a more rigorous definition.

1.3.1 Informal Definition

Consider a control system in explicit form (1.2) and its infinite input prolongation as described in Section 1.1.2. The system inputs variables $u^1, \ldots, u^m$ are free, i.e. they are not required to satisfy any differential equations. This means that within their domain of definition and on some time interval, one may assign any (smooth) time functions $u^1(t), \ldots, u^m(t)$. All the variables $u^{l(r)}$ for $r > 0$ are then obtained by time differentiation

and there therefore exists time functions (solutions) $x^i(t)$ such that the system equations are satisfied. However, the variables x^i are not completely specified by the choice of inputs. For instance, the initial conditions are free.
At the same time, the basic assumption that rank $\frac{\partial f}{\partial u} = m$ implies that given any trajectory $\sigma : (t) \mapsto (t, x^i(t))$ satisfying the implicit equations (1.1), the inputs u^l may be solved for and expressed as algebraic functions of t and $x^1, \ldots, x^n$. By time differentiation, all the variables $u^{l(r)}$ for $r > 0$ are specified too. Hence, all system variables $t, x, u, \dot{u}, \ldots$ can be computed from the variables t and x. However, these variables are not free, since they are required to satisfy the system equations (1.2).
A control system is *flat* if there exists variables with both properties, i.e. free of any differential relations *and* allowing the reconstruction of all the other system variables from their values and their time derivatives.
More precisely, consider the control system (1.2) and its infinite input prolongation on $\mathcal{U}^\infty$. The system is *flat* if there exist m functions $z^1, \ldots, z^m$ on $\mathcal{U}^\infty$

$$z^s = z^s(t, x, u, \dot{u}, \ldots, u^{(L-1)}) \qquad s = 1, \ldots, m$$

for some finite L, called the *flat outputs*, together with a map $\Phi : \mathbb{R}^{1+mR} \to \mathbb{R}^n$ such that

$$x^i = \Phi^i(t, z, \dot{z}, \ldots, z^{(R-1)}) \qquad i = 1, \ldots, n$$

for some finite R.

Remark 1.10. If the basic assumption rank $\frac{\partial f}{\partial u} = m$ is dropped, one additionally requires the existence of a map $\Psi : \mathbb{R}^{1+mR_u} \to \mathbb{R}^m$, allowing the reconstruction of u, i.e. such that $u^l = \Psi^l(t, z, \dot{z}, \ldots, z^{(R_u-1)})$ for $l = 1, \ldots, m$. In the following, we shall always assume that rank $\frac{\partial f}{\partial u} = m$.

Using notions that will be approached in Chapter 3, flatness can be defined precisely as follows.

Definition 1.11. *A control system with m inputs is said* differentially flat *or* flat *if it is Lie-Bäcklund equivalent (equivalent under some Lie-Bäcklund isomorphism) to the system described by the equations $\dot{y}^1 = u^1, \ldots, \dot{y}^m = u^m$.*

Any statically feedback linearizable system is flat, but the converse does not hold. Deciding if a given control system is flat is a difficult problem that has received a lot of attention over the years. We shall be concerned with related questions in the following chapters.

Chapter 2

Matrix Differential Operators

This chapter mainly reviews concepts borrowed from the literature on differential equations and differentially flat systems. Nevertheless, the last section presents an equivalent form of a known characterization of flat systems that is original to the best of our knowledge.
The point of view adopted here is the infinite dimensional geometric description of control systems. This approach has been introduced in the study of flatness in [45, 46]. Another important ingredient is the concept of the tangent linear system, the properties of which are reflected by those of a differential module of 1-forms associated to the control system under investigation, [47, 48]. Indeed, the problem of finding the flat outputs of a flat system is difficult, but its infinitesimal counterpart is algorithmic and involves the construction of a basis of the mentioned module [5]. An integrability problem then remains to be solved. The obtained basis may be transformed in another one by a matrix differential operator, the action of which does not preserve intergrability in general. Differential operators are of interest in the more general study of partial differential equations, see e.g. [150, 19, 20] and regarding their relevance in the study of flat systems, see [5, 21, 8, 88]. Once a basis of the module is obtained, it is possible to set up a system of differential equations, the unknown of which are differential operators, and for which the existence of a solution coincides with the flatness of the studied control system.

The differential module of 1-forms encoding the properties of the tangent linear system together with the differential operators acting on it are defined in Section 2.1. These operators are given a graded structure and an exterior derivative is defined. Section 2.2 describes the algorithmic procedure that produces a basis of the differential module. The decomposition of the module and some properties it shares with the original control system are also discussed. In Section 2.3, matrices of operators together with their action on module bases and their link to flatness is developed. The characterization of flatness mentioned earlier is also presented. Lastly, Section 2.4 gives an equivalent version of the flatness characterization result where curvature equations are absent. A worked out example closes the chapter.

2.1 Differential Operators

Consider the bundle $\mathcal{U}^\infty$ given by the projective (inverse) limit of the composite bundle

$$\cdots \xrightarrow{\pi_{\mathcal{U},32}} \mathcal{U}^2 \xrightarrow{\pi_{\mathcal{U},21}} \mathcal{U}^1 \xrightarrow{\pi_{\mathcal{U},10}} \mathcal{U} \xrightarrow{\pi_{\pi_{\mathcal{U}\mathcal{M}}}} \mathcal{M} \xrightarrow{\pi_{\pi_{\mathcal{M}\mathcal{B}}}} \mathcal{B}$$

and remember that coordinates on $\mathcal{U}^\infty$ are given by

$$(t, x, u, u^{(1)}, u^{(2)}, \ldots)$$

Let $\mathcal{R}$ denote the ring (also an $\mathbb{R}$-algebra) of smooth real-valued functions on $\mathcal{U}^\infty$. Functions in $\mathcal{R}$ are required to depend on a finite number of variables, more precisely

$$r \in \mathcal{R} \qquad \Rightarrow \qquad \exists k \text{ finite and } \exists \tilde{r} \in \mathcal{C}^\infty(\mathcal{U}^k, \mathbb{R}) \text{ s.t. } r = \pi^*_{\mathcal{U},\infty k}\, \tilde{r}.$$

In other words, $r = r(t, x, u, u^{(1)}, \ldots, u^{(k)})$ and is $\mathcal{C}^\infty$ in all its arguments.
In the following, D represents the infinitely prolonged Cartan vector field on $T\mathcal{U}^\infty$

$$D = \frac{\partial}{\partial t} + f^i(t, x, u)\frac{\partial}{\partial x^i} + u^{l(k+1)}\frac{\partial}{\partial u^{l(k)}}$$

where we assume that $f^i \in \mathcal{R}$. For any $r \in \mathcal{R}$, the Lie derivative Dr is in $\mathcal{R}$. Note that although D is given by an infinite sum, the computation of Dr can be performed with finitely many partial derivative operations. One can now define the ring of polynomials in D with coefficients in $\mathcal{R}$ denoted by $\mathcal{R}[D]$. An element $a \in \mathcal{R}[D]$ is a finite sum

$$a = a_0 + a_1 D + \ldots + a_A D^A \qquad a_i \in \mathcal{R}.$$

The *noncommutative* products $r \cdot D$ and $D \cdot r$ of D and $r \in \mathcal{R} \subset \mathcal{R}[D]$ read

$$\begin{aligned} r \cdot D &= rD \\ D \cdot r &= Dr + rD \end{aligned}$$

The ring $\mathcal{R}[D]$ is a ring of operators, i.e. $\forall a \in \mathcal{R}[D]$, $a : \mathcal{R} \to \mathcal{R}$. For $a, b \in \mathcal{R}[D]$ and $r \in \mathcal{R}$ one verifies that

$$(a \cdot b)(r) = a(b(r)).$$

From now on, let $\Lambda^p T\mathcal{U}^{k*}$ denote the $\mathcal{R}$-module of p-forms on $\mathcal{U}^k$. Differential p-forms on $\mathcal{U}^\infty$ are required to be finite in the following sense.

$$\omega \in \Lambda^p T\mathcal{U}^{\infty *} \qquad \Rightarrow \qquad \exists k, \ \exists \tilde{\omega} \in \Lambda^p T\mathcal{U}^{k*} \quad \text{s.t.} \quad \omega = \pi^*_{\mathcal{U},\infty k}\, \tilde{\omega}.$$

Explicitly, the p-form above takes the form

$$\omega = \sum_I^{\text{finite}} \alpha^I dy^{I_1} \wedge \ldots \wedge dy^{I_p} \qquad \alpha^I \in \mathcal{R}, \quad y^{I_i} \in \{x, u, \ldots, u^{(k)}\}.$$

The wedge product is as usual

$$\wedge : \ \Lambda^p T\mathcal{U}^{\infty *} \times \Lambda^q T\mathcal{U}^{\infty *} \to \Lambda^{p+q} T\mathcal{U}^{\infty *}$$

and the algebra of forms of all degree on $\mathcal{U}^\infty$ will be denoted by $\Lambda T\mathcal{U}^{\infty *}$. The Lie derivative along D gives a map $\Lambda^p T\mathcal{U}^{\infty *} \to \Lambda^p T\mathcal{U}^{\infty *}$. Therefore, the action of elements in $\mathcal{R}[D]$ can be extended naturally to $\Lambda^p T\mathcal{U}^{\infty *}$ and $\Lambda T\mathcal{U}^{\infty *}$:

$$a \in \mathcal{R}[D] \qquad a : \Lambda T\mathcal{U}^{\infty *} \to \Lambda T\mathcal{U}^{\infty *}$$

such that if $a = a_0 + a_1 D + \ldots + a_A D^A$ and $\omega \in \Lambda T\mathcal{U}^{\infty *}$ then

$$a(\omega) = a_0\omega + a_1 D\omega + \ldots + a_A D^A \omega$$

where $D^l\omega$ is the l-fold Lie derivative along D. For $a, b \in \mathcal{R}[D]$ and $\omega \in \Lambda T\mathcal{U}^{\infty *}$ we have

$$(a \cdot b)(\omega) = a(b(\omega)).$$

We now consider $\Lambda^p T\mathcal{U}^{\infty *}$ over the ring of operators $\mathcal{R}[D]$. Doing so, we obtain a new module. We shall denote this module by $\mathcal{A}^p$, the $\mathcal{R}[D]$-module of differential p-forms on $\mathcal{U}^\infty$.

Remark 2.1. For $p = 1$ and as a $\mathcal{R}$-module, $\Lambda^1 T\mathcal{U}^{\infty *}$ is generated by an infinite number of independent elements. But since $Ddu^{l(k)} = du^{l(k+1)}$, the $\mathcal{R}[D]$-module $\mathcal{A}^1$ is generated by

$$dt, dx^1, \ldots, dx^n, du^1, \ldots, du^m.$$

However, this is not a basis in the sense that $\mathcal{A}^1$ may possibly be generated by fewer elements. See Section 2.2.

2.1.1 Graded Differential Operators

The ring of differential operators $\mathcal{R}[D]$ can be extended to a graded structure $\Lambda\mathcal{R}[D]$. (See also [8]). We shall write $\Lambda^p\mathcal{R}[D]$ for the set of polynomials in D with coefficients in $\Lambda^p T\mathcal{U}^{\infty *}$. As the set of 0-forms on $\mathcal{U}^\infty$, $\mathcal{R}$ can be identified with $\Lambda^0 T\mathcal{U}^{\infty *}$. Therefore we also identify $\mathcal{R}[D]$ with $\Lambda^0\mathcal{R}[D]$. An operator $\alpha \in \Lambda^p\mathcal{R}[D]$ is a finite sum

$$\alpha = \alpha_0 \wedge 1 + \alpha_1 \wedge D + \ldots + \alpha_A \wedge D^A \qquad \alpha_i \in \Lambda^p T\mathcal{U}^{\infty *}$$

and is said of *order* A and *degree* p. For α as above and $\omega \in \Lambda^q T\mathcal{U}^{\infty *}$, we have

$$\alpha : \Lambda^q T\mathcal{U}^{\infty *} \to \Lambda^{p+q} T\mathcal{U}^{\infty *}$$
$$\alpha(\omega) = \alpha_0 \wedge \omega + \alpha_1 \wedge D\omega + \ldots + \alpha_A \wedge D^A\omega.$$

Note that α immediately extends to a map of modules,

$$\alpha : \mathcal{A}^p \to \mathcal{A}^{p+q}.$$

The wedge product on $\Lambda T\mathcal{U}^{\infty *}$ can in turn be extended to

$$\wedge : \ \Lambda^p\mathcal{R}[D] \ \times \ \Lambda^q\mathcal{R}[D] \ \to \ \Lambda^{p+q}\mathcal{R}[D]. \tag{2.1}$$

using the rule

$$(\alpha \wedge \beta)(\omega) = \alpha(\beta(\omega)).$$

Remark 2.2. The wedge product $\wedge$ is not anti-commutative on $\Lambda^1\mathcal{R}[D]$ as it is on $\Lambda^1 T\mathcal{U}^{\infty *}$.

Example 2.3. *Take* $\alpha, \beta \in \Lambda^1\mathcal{R}[D]$ *and* $\omega \in \Lambda^1 T\mathcal{U}^{\infty *}$ *such that*

$$\alpha = dx^1 D \qquad \beta = dx^2 \qquad \omega = dx^3.$$

Then

$$\begin{aligned}(\alpha \wedge \beta)(\omega) = \alpha(\beta(\omega)) &= dx^1 D(dx^2 \wedge dx^3) \\ &= dx^1 \wedge dDx^2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge dDx^3\end{aligned}$$

whereas

$$\begin{aligned}(\beta \wedge \alpha)(\omega) &= \beta(\alpha(\omega)) \\ &= -dx^1 \wedge dx^2 \wedge dDx^3.\end{aligned}$$

Let us finally write

$$\Lambda\mathcal{R}[D] = \bigoplus_p \Lambda^p\mathcal{R}[D].$$

The wedge product on $\Lambda\mathcal{R}[D]$ is associative and distributive over the addition. Hence $\Lambda\mathcal{R}[D]$ is a ring of differential operators and $\mathcal{R}[D] = \Lambda^0\mathcal{R}[D]$ is a subring of $\Lambda\mathcal{R}[D]$.

2.1.2 Exterior Derivative on Graded Differential Operators

In this section, and as in [8], we extend the exterior derivative d to $\Lambda\mathcal{R}[D]$ as a map

$$d : \Lambda^p\mathcal{R}[D] \to \Lambda^{p+1}\mathcal{R}[D]$$

In [88], this operator is directly defined on matrices with differential operators as entries and is denoted by $\mathfrak{d}$. In [21], the same behavior is obtained using a notation involving commutators and anticommutators of operators.
Given $\alpha \in \Lambda^p\mathcal{R}[D]$ and $\omega \in \Lambda^q T\mathcal{U}^{\infty *}$, the operator $d\alpha \in \Lambda^{p+1}\mathcal{R}[D]$ is such that

$$d(\alpha(\omega)) = (d\alpha)(\omega) + (-1)^p \alpha(d\omega).$$

Explicitly, the α above is a finite sum of the form

$$\alpha = \alpha_i \wedge D^i \qquad \alpha_i \in \Lambda^p T\mathcal{U}^{\infty *}$$

and thus

$$\begin{aligned}d(\alpha(\omega)) &= d(\alpha_i \wedge D^i\omega) \\ &= d\alpha_i \wedge D^i\omega + (-1)^p \alpha_i \wedge dD^i\omega \\ &= d\alpha_i \wedge D^i\omega + (-1)^p \alpha_i \wedge D^i d\omega\end{aligned}$$

so that $d\alpha \in \Lambda^{p+1}\mathcal{R}[D]$ is the operator given by

$$d\alpha = d\alpha_i D^i.$$

Hence, to compute the exterior derivative of an operator in $\Lambda^p\mathcal{R}[D]$, one simply applies the exterior derivative to the form coefficients. Clearly, $dd\alpha = 0$ for any $\alpha \in \Lambda^p\mathcal{R}[D]$.

2.2 Basis for the Module $\mathcal{A}^1$

As will be shown in forthcomming sections, important properties of a control system are reflected by properties of its associated module of differential 1-forms $\mathcal{A}^1$. A first task is to obtain a basis of $\mathcal{A}^1$. By a basis, we mean a minimal set of elements in $\mathcal{A}^1$, generating $\mathcal{A}^1$ as a $\mathcal{R}[D]$-module. Because a basic assumption is that rank $\frac{\partial f}{\partial u} = m$ in the explicit system equations $\dot{x} = f(t, x, u)$, the m variables u can be solved for as algebraic functions of t, x and $\dot{x} = Dx$. This implies, as already noted, that a basis of $\mathcal{A}^1$ is contained (as a $\mathcal{R}$-submodule) in

$$H^{(0)} = \{dt, dx^1, \ldots, dx^n\}.$$

Next, define the following filtration of the $\mathcal{R}$-module $H^{(0)}$ for $k \geq 0$

$$H^{(0)} = \{dt, dx\} \qquad H^{(k+1)} = \{\omega \in H^{(k)} \mid D\omega \in H^{(k)}\} \tag{2.2}$$

Remark 2.4. The above derived flag is defined using the infinitely prolonged vector field D, nevertheless, it can be computed using finite objects only. Denoting $\mathcal{C}(T\mathcal{U}^k)$ the Cartan distribution on $\mathcal{U}^k$ as defined in Chapter 1, the same filtration as (2.2) is obtained as

$$H^{(0)} = \{dt, dx\}$$
$$H^{(k+1)} = \{\omega \in H^{(k)} \mid D\omega \in H^{(k)}, D \in \mathcal{C}(T\mathcal{U}^k) \text{ s.t. } D \lrcorner\, dt = 1\}.$$

(Recall that the Cartan distribution $\mathcal{C}(T\mathcal{U}^k)$ on $\mathcal{U}^k$ is of dimension $1+m$ and that contrarily to the case of $\mathcal{U}^\infty$, the choice of $D \in \mathcal{C}(T\mathcal{U}^k)$ s.t. $D \lrcorner\, dt = 1$ is not unique. However, the obtained filtration is independent of this choice.) Therefore, despite the fact that $H^{(k)}$ is contained in $\{dt, dx\}$ as a $\mathcal{R}$-submodule, elements of $H^{(k)}$ are not necessarily elements of $T\mathcal{M}^*$ but are at least in $T\mathcal{U}^{k-1*}$. Indeed, the coefficients in the 1-forms may depend on $u, \ldots, u^{(k-1)}$.

Let E be a $\mathcal{R}$-submodule of $\Lambda^1 T\mathcal{U}^{\infty *}$. The dimension of E at a point $p \in \mathcal{U}^\infty$ is the dimension on the $\mathbb{R}$-vector space $E|_p$. The following definition is equivalent to the one given in [21].

Definition 2.5. *A point $p \in \mathcal{U}^\infty$ is said* Brunovsky-regular*, if the dimensions of the modules $H^{(k)}$ and $H^{(k)} + DH^{(k)}$ are constant in some neighborhood of p for all $k \geq 0$.*

In the above definition, one should be precise about what is meant by a neighborhood of $p \in \mathcal{U}^\infty$. A neighborhood of $p \in \mathcal{U}^\infty$ is taken as the pre-image through $\pi_{\mathcal{U},\infty q}$ of an open neighborhood of $\tilde{p} = \pi_{\mathcal{U},\infty q}(p)$ in $\mathcal{U}^q$ for some $q \geq 0$.

Remark 2.6. From Remark 2.4, $H^{(k)}$ corresponds to a $\mathcal{C}^\infty(\mathcal{U}^{k-1})$-submodule of $\Lambda^1 T\mathcal{U}^{k-1*}$. Therefore, around a Brunovsky-regular point, we may identify $H^{(k)}$ and $H^{(k)} + DH^{(k)}$ with codistributions on $\mathcal{U}^k$ (and codistributions on $\mathcal{U}^\infty$).

Example 2.7. *Consider the system described by*

$$\dot{x}^1 = x^1 x^2 \qquad \dot{x}^2 = u.$$

The system satisfies the basic regularity assumption $\frac{\partial f}{\partial u} = m = 1$ *at every point. The filtration defined above is easily computed as*

$$\begin{aligned} H^{(0)} &= \{dt, dx^1, dx^2\} \\ H^{(1)} &= \{dt, dx^1\} \\ H^{(2)} &= \begin{cases} \{dt\} & \text{if } x^1 \neq 0 \\ \{dt, dx^1\} & \text{if } x^1 = 0 \end{cases} . \end{aligned}$$

And $H^{(2+k)} = H^{(2)}$ *for all* $k \geq 0$*. One also obtains*

$$\begin{aligned} H^{(0)} + DH^{(0)} &= \{dt, dx^1, dx^2, du\} \\ H^{(1)} + DH^{(1)} &= \begin{cases} \{dt, dx^1, dx^2\} & \text{if } x^1 \neq 0 \\ \{dt, dx^1\} & \text{if } x^1 = 0 \end{cases} \\ H^{(2)} + DH^{(2)} &= \begin{cases} \{dt\} & \text{if } x^1 \neq 0 \\ \{dt, dx^1\} & \text{if } x^1 = 0 \end{cases} . \end{aligned}$$

Hence, the Brunovsky-regular points are all points $(t, x^1, x^2, u, u^{(1)}, u^{(2)}, \ldots) \in \mathcal{U}^\infty$ *such that* $x^1 \neq 0$*.*

Let us show

Lemma 2.8. *With the filtration defined by* (2.2) *and around any Brunovsky-regular point,*

i) There is a $k^* \geq 0$ *s.t.* $\forall k < k^* : \dim H^{(k+1)} < \dim H^{(k)}$ *and* $\forall k > 0 : H^{(k^*+k)} = H^{(k^*)}$*. Also* $D(H^{(k^*)}) \subset H^{(k^*)}$*.*

ii) For all $k \geq 0$

$$\dim\left((H^{(k)} + DH^{(k)})/H^{(k)}\right) = \dim\left(H^{(k)}/H^{(k+1)}\right)$$

iii) For all $k \geq 0$

$$\begin{aligned} \dim\left((H^{(k)} + DH^{(k)})/H^{(k)}\right) = &\dim\left(H^{(k)}/(H^{(k+1)} + DH^{(k+1)})\right) \\ &+ \dim\left(H^{(k+1)}/(H^{(k+2)} + DH^{(k+2)})\right) \\ &+ \ldots \\ &+ \dim\left(H^{(k^*-2)}/(H^{(k^*-1)} + DH^{(k^*-1)})\right) \\ &+ \dim\left(H^{(k^*-1)}/(H^{(k^*)})\right) \end{aligned}$$

iv) For all $k \geq 0$*, let* $\{\omega^{i,k}\} \subset H^{(0)}$ *be an independent set of representatives of*

$$H^{(k)}/(H^{(k+1)} + DH^{(k+1)})$$

in $H^{(0)}$ *for all* $k \geq 0$*. Then the* 1*-forms* $\omega^{i,k}$ *are independent and generate an* m*-dimensional* $\mathcal{R}$*-submodule of* $H^{(0)}$*.*

Proof. The condition about Brunovsky-regularity implies that the codistributions defined in *i),ii),iii)* and *iv)* are all locally well defined and spanned by some local sections of the respective cotangent spaces.
i) Follows directly from the construction of $H^{(k)}$ and the fact that the dimension on $H^{(0)}$ is finite.
ii) Since $H^{(k+1)} \subset H^{(k)}$, one can build adapted bases $H^{(k+1)} = \{\mu^i\}$ and $H^{(k)} = \{\mu^i, \lambda^j\}$. By construction, $D\mu^i \in H^{(k)}$. Next assume that the $D\lambda^j$ are not independent modulo $H^{(k)}$, then there are functions α_j such that $\alpha_j D\lambda^j \in H^{(k)}$. But this implies that the element $\alpha_j\lambda^j$ is in $H^{(k+1)}$, indeed $D(\alpha_j\lambda^j) = D(\alpha_j)\lambda^j + \alpha_j D(\lambda^j)$ is in $H^{(k)}$ by assumption. But this contradicts the fact that $\{\mu^i\}$ and $\{\mu^i, \lambda^j\}$ are the claimed adapted bases. Hence $D\lambda^j$ are independent modulo $H^{(k)}$.
From these adapted bases it is clear that $\left(H^{(k)}/H^{(k+1)}\right)$ is represented by the independent set $\{\lambda^j\}$. And since $D\lambda^j$ are independent modulo $H^{(k)}$ whereas $D\mu^i \in H^{(k)}$, $\left((H^{(k)} + DH^{(k)})/H^{(k)}\right)$ also has the independent set $\{\lambda^j\}$ as representatives.
iii) For $k \geq k^*$, $H^{(k+1)} = H^{(k)}$ and $DH^{(k)} \subset H^{(k)}$ so that the relation is trivially satisfied. For $0 \leq k < k^*$, and on each side of the equation, we subtract the case $k+1$ to the case k obtaining

$$\begin{aligned} \dim\left((H^{(k)} + DH^{(k)})/H^{(k)}\right) - \dim\left((H^{(k+1)} + DH^{(k+1)})/H^{(k+1)}\right) \\ = \dim\left(H^{(k)}/(H^{(k+1)} + DH^{(k+1)})\right) \end{aligned} \tag{2.3}$$

If one can verify (2.3), then the result follows by induction on k. Note that by construction

$$H^{(k+1)} \subset H^{(k+1)} + DH^{(k+1)} \subset H^{(k)} \subset H^{(k)} + DH^{(k)}. \tag{2.4}$$

Given three nested codistributions $C \subset B \subset A$, note the identity $\dim(A/B) + \dim(B/C) = \dim(A/C)$. Hence, (2.3) reduces to

$$\dim\left((H^{(k)} + DH^{(k)})/H^{(k)}\right) = \dim\left(H^{(k)}/H^{(k+1)}\right)$$

which is *ii)*.
iv) Representatives of the spaces $H^{(k)}/(H^{(k+1)} + DH^{(k+1)})$ are independent because of (2.4) and because for two nested codistributions $B \subset A$, representatives of A/B in A are independent of B. Hence the forms $\omega^{i,k}$ are all independent. It follows that $\dim\{\omega^{i,k}\}$ is given by setting $k = 0$ in *iii)*. This gives $\dim\{\omega^{i,k}\} = \dim\left((H^{(0)} + DH^{(0)})/H^{(0)}\right) = \dim(\{dt, dx, du\}/\{dt, dx\}) = m$. □

Besides the algebraic properties of the filtration (2.2) given by the previous lemma, the module $H^{(k^*)}$, to which the constructions saturates, also enjoys an important "differential" property. See [91, 92, 140, 5] for original results and more details. We content ourselves with the following lemma.

Lemma 2.9. *Around a Brunovsky-regular point, the codistribution $H^{(k^*)}$ of Lemma 2.8 is involutive and $H^{(k^*)} \subset T\mathcal{M}^*$. Moreover $dt \in H^{(k^*)}$.*

Proof. This proof is adapted from the proof of Proposition 3.3 point 2 in [5].
By Remark 2.4 and Lemma 2.8, we may work in finite dimensions by considering only $\mathcal{U}^{k^*}$

and use the vector

$$D = \frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i} + u^{l(1)}\frac{\partial}{\partial u^{l(0)}} + \ldots + u^{l(k^*)}\frac{\partial}{\partial u^{l(k^*-1)}} \tag{2.5}$$

Define $V \subset T\mathcal{U}^{k^*}$ as the annihilator of $H^{(k^*)}$, i.e. $V = \perp_{T\mathcal{U}^{k^*}} H^{(k^*)}$. Next, write $\hat{V} \subset V$ the space of Cauchy characteristic vector fields of $H^{(k^*)}$ which, by Lemma A.3, is involutive and given by

$$\begin{aligned} \hat{V} &= \{X \in V \mid [X,V] \subset V\} \\ &= \{X \in V \mid X \lrcorner\, d\omega \in H^{(k^*)},\ \forall \omega \in H^{(k^*)}\}. \end{aligned} \tag{2.6}$$

Define also $\hat{H} = \perp_{T\mathcal{U}^{k^*}} \hat{V}$, the retracting space of $H^{(k^*)}$. Clearly, $H^{(k^*)} \subset \hat{H}$ and $\hat{H}$ is integrable.
We now show that $D(\hat{H}) \subset \hat{H}$. Take any $X \in \hat{V}$ and any $\omega \in H^{(k^*)}$. From the Cartan formula we have

$$[D,X] \lrcorner\, \omega = D(X \lrcorner\, \omega) - X \lrcorner\, D\omega = 0 \tag{2.7}$$

since $X \lrcorner\, \omega = 0$ and $D\omega \in H^{(k^*)}$. Again from the Cartan formula

$$[D,X] \lrcorner\, d\omega = D(X \lrcorner\, d\omega) - X \lrcorner\, Dd\omega.$$

The first term in the r.h.s of the last relation is in $H^{(k^*)}$ since $X \lrcorner\, d\omega \in H^{(k^*)}$ and for the second term, we have that $X \lrcorner\, Dd\omega = X \lrcorner\, dD\omega \in H^{(k^*)}$ by (2.6). Hence

$$[D,X] \lrcorner\, d\omega \in H^{(k^*)}. \tag{2.8}$$

But (2.7) and (2.8) imply that $[D,X] \in \hat{V}$, so that $[D,\hat{V}] \subset \hat{V}$. By Lemma A.17, this last relation implies that $D(\hat{H}) \subset \hat{H}$.
Next, we verify that $H^{(k^*)}$ is not only the largest D-invariant sub-codistribution of $H^{(0)} = \{dt, dx\}$, but also the largest D-invariant sub-codistribution of

$$\{dt, dx, du, \ldots, du^{(k^*-1)}\}.$$

To this end, we extend the filtration of H in the other direction as

$$\begin{aligned} H^{(0)} &= \{dt, dx\} \\ H^{(-1)} &= \{dt, dx, du\} \\ &\vdots \\ H^{(-k)} &= \{dt, dx, du, \ldots, du^{(k-1)}\} \\ &\vdots \\ H^{(-k^*)} &= \{dt, dx, du, \ldots, du^{(k^*-1)}\} \end{aligned}$$

We see that with this choice, the relations $H^{(k+1)} = \{\omega \in H^{(k)} \mid D\omega \in H^{(k)}\}$ are verified for all $k \geq -k^*$. Therefore, $H^{(k^*)}$ is the largest D-invariant codistribution inside $H^{(-k^*)}$.

Now, we show $\hat{H} \subset H^{(-k^*)}$. From the expressions of D, see (2.5), we deduce that for $k > 0$, $H^{(k)}$ possess a basis of the form

$$H^{(k)} = \{\alpha(t, x, u, \ldots, u^{(k-1)})dt,\ \beta_i^r(t, x, u, \ldots, u^{(k-1)})dx^i\} \quad r = 1, \ldots, \dim H^{(k)}.$$

Therefore, the retracting space of $H^{(k)}$ for $k > 0$ is contained in

$$\{dt, dx, du, \ldots, du^{(k-1)}\} = H^{(-k)}$$

and in particular, $\hat{H} \subset H^{(-k^*)}$.
We hence have verified that $\hat{H}$ is D-invariant and that $\hat{H} \subset H^{(-k^*)}$; but we also showed that $H^{(k^*)}$ is the largest D-invariant in $H^{(-k^*)}$. It follows that $\hat{H} \subset H^{(k^*)}$, so that $\hat{H} = H^{(k^*)}$ and $H^{(k^*)}$ is integrable. The codistribution generated by dt is D-invariant, hence $dt \in H^{(k^*)}$. As an integrable codistribution contained in $\{dt, dx\}$, $H^{(k^*)}$ may also be identified with a codistribution in $T\mathcal{M}^*$. □

Lemmas 2.8 and 2.9 provide a basis of the module $\mathcal{A}^1$ through the following proposition. The result is found in [5] for the autonomous (time invariant) case. The non-autonomous $\mathcal{C}^\infty$ case is presented in [21] where a set of generators of $\mathcal{A}^1/\{dt\}$ is constructed. In [88], a basis of $\mathcal{A}^1/\{dt\}$ for the case $H^{(k^*)} = \{dt\}$ is obtained by other means (more on this later).

Proposition 2.10. *Around a Brunovsky-regular point, let $\rho = \dim H^{(k^*)}$ and $\chi^1, \ldots, \chi^{\rho-1}$ functions on $\mathcal{M}$ such that $H^{(k^*)} = \{dt, d\chi^1, \ldots, d\chi^{\rho-1}\}$. Let also $\{\omega^1, \ldots, \omega^m\} = \{\omega^{i,k}\}$ with $\omega^{i,k}$ as in Lemma 2.8. Then*

$$dt,\ d\chi^1, \ldots, d\chi^{\rho-1},\ \omega^1, \ldots, \omega^m$$

generate the $\mathcal{R}[D]$-module $\mathcal{A}^1$ and

$$\omega^1, \ldots, \omega^m$$

are representatives in $\mathcal{A}^1$ of a basis of the $\mathcal{R}[D]$-module $\mathcal{A}^1/H^{(k^*)} = \mathcal{A}^1/\{dt,\ d\chi^1, \ldots, d\chi^{\rho-1}\}$.

Proof. By Lemma 2.8, the $\omega^j \in \{\omega^{i,k}\}$ are representatives of bases of the spaces

$$H^{(k)}/(H^{(k+1)} + DH^{(k+1)})$$

for $k \geq 0$. Therefore, for each $k \geq 0$

$$H^{(k)} \equiv \{\omega^{i,k}\} + H^{(k+1)} + DH^{(k+1)}.$$

Hence, because $H^{(k^*+l)} = H^{(k^*)}$, $\forall l \geq 0$ and by induction, the elements of $H^{(0)}$ are generated by the elements (together with their successive Lie-derivatives along D) of

$$\{\omega^1, \ldots, \omega^m\} + H^{(k^*)}.$$

But $H^{(0)}$ generates $\mathcal{A}^1$. The set $\{\omega^1, \ldots, \omega^m\}$ is minimal because in $\mathcal{A}^1/H^{(k^*)}$, it generates the $\mathcal{R}[D]$-submodule $\{du\}/H^{(k^*)}$ for which $\{du^1, \ldots, du^m\}$ is clearly an independent basis; and a basis of a module cannot be smaller than a basis of one of its submodules. □

Remark 2.11. We do not speak of a basis of $\mathcal{A}^1$ when referring to $\{dt,\ d\chi^1, \ldots, d\chi^{\rho-1}, \omega^1, \ldots, \omega^m\}$ because $\{dt,\ d\chi^1, \ldots, d\chi^{\rho-1}\}$ generate $H^{(k^*)}$ as a $\mathcal{R}$-module, but as a $\mathcal{R}[D]$-module, $H^{(k^*)}$ may be generated by fewer elements contained in $\{dt,\ d\chi^1, \ldots, d\chi^{\rho-1}\}$.

2.2.1 Decomposition of $\mathcal{A}^1$

In the following, we observe that around a Brunovsky-regular point, the module $\mathcal{A}^1$ decomposes as $\mathcal{A}^1 = \mathcal{T} \oplus \mathcal{F}$ where $\mathcal{T}$ is torsion and $\mathcal{F}$ is free. Moreover $\mathcal{T}$ is unique and $\mathcal{F} \equiv \mathcal{A}^1/\mathcal{T}$. For similar facts in the case where the ring $\mathcal{R}$ is a field, see [42, 88, 28].
In our setting, an element τ in $\mathcal{A}^1$ is torsion (is an element of the torsion submodule of $\mathcal{A}^1$) if and only if there exists a differential operator $r \in \mathcal{R}[D]$ such that $r(\tau) = 0$. On the other hand, the set $\mu^j \in \mathcal{A}^1, j = 1, \ldots, s$ generates a free submodule $S \subset \mathcal{A}^1$ if and only if there exist no set of (not all zero) operators $r_j \in \mathcal{R}[D], j = 1, \ldots, s$ such that $\sum_{j=1}^{s} r_j(\mu^j) = 0$.

Example 2.12. *Consider the system with one independent input u and three states x^1, x^2, x^3*

$$\dot{x}^1 = x^2 \qquad \dot{x}^2 = x^1 \qquad \dot{x}^3 = u.$$

The system is linear, hence, all points of $\mathcal{U}^\infty$ are Brunovsky-regular. Next, dx^1 and dx^2 are torsion, indeed, with $r = D^2 - 1$, $r(dx^1) = r(dx^2) = 0$. A free module $\mathcal{F}$ such that $\mathcal{A}^1 = \{dx^1, dx^2\} \oplus \mathcal{F}$ is generated by dx^3. Another choice could be $dx^3 + dx^1$. The element $du = Ddx^3$ also generates a free submodule, but not such as to decompose $\mathcal{A}^1$ as desired. Note also that the fact that the proposed generators of $\mathcal{F}$ are integrable is specific to the example.

These observations, together with Lemmas 2.8 and 2.9 and Proposition 2.10 lead to

Corollary 2.13. *Around a Brunovsky-regular point, the $\mathcal{R}[D]$-module $\mathcal{A}^1$ decomposes as*

$$\mathcal{A}^1 = \mathcal{T} \oplus \mathcal{F}$$

where $\mathcal{T}$ is the torsion submodule generated by $H^{(k^*)}$, i.e. by

$$dt, d\chi^1, \ldots, d\chi^{\rho-1}$$

and $\mathcal{F} \equiv \mathcal{A}^1/\mathcal{T}$ has basis elements represented by

$$\omega^1, \ldots, \omega^m.$$

Proof. We just need to show that $H^{(k^*)}$ contains only torsion elements. Take any $h \in H^{(k^*)}$. Since $\dim H^{(k^*)} = \rho$, the elements $h, Dh, \ldots, D^\rho h$ in $H^{(k^*)}$ must be $\mathcal{R}$-linearly dependent. Suppose $\alpha_0 h + \alpha_1 Dh + \ldots + \alpha_\rho D^\rho h = 0$, then $r \in \mathcal{R}[D]$ given by $r = \alpha_0 + \alpha_1 D + \ldots + \alpha_\rho D^\rho$ is such that $r(h) = 0$. □

The next lemma states that the ideal generated by torsion elements is stable under the action of D.

Lemma 2.14. *Around a Brunovsky-regular point, the ideal $\mathcal{I}_\mathcal{T}$ in $\Lambda T\mathcal{U}^{\infty *}$ generated by $H^{(k^*)}$ is invariant under the action of any differential operator in $\Lambda\mathcal{R}[D]$.*

Proof. It is enough to consider the action of D. A homogeneous element $\tau \in \mathcal{I}_\mathcal{T} \bigcap \Lambda^q T\mathcal{U}^{\infty *}$ has the form $\tau = \psi^k \wedge \alpha_k,\ k = 1, \ldots, \rho$ with $\{\psi_k\}$ a basis of $H^{(k^*)}$ and $\alpha_k \in \Lambda^{q-1} T\mathcal{U}^{\infty *}$. Then $D(\tau) = D(\psi^k) \wedge \alpha_k + \psi^k \wedge D(\alpha_k)$, but $D(\psi^k) \in H^{(k^*)}$. □

We close this section by insisting on the fact that the torsion submodule $\mathcal{T}$ of $\mathcal{A}^1$ corresponds to the state-space of the "largest uncontrolled subsystem" of (1.2), [42, 5]. Since $D(H^{(k^*)}) \subset H^{(k^*)}$, there are (local) functions $g^1, \ldots, g^{\rho-1} \in \mathcal{C}^\infty(\mathcal{O}, \mathbb{R})$ for some open set $\mathcal{O} \subset \mathcal{M}$ such that

$$\begin{aligned} \dot{\chi}^1 &= g^1(t, \chi^1, \ldots, \chi^{\rho-1}) \\ &\vdots \\ \dot{\chi}^{\rho-1} &= g^{\rho-1}(t, \chi^1, \ldots, \chi^{\rho-1}). \end{aligned}$$

and of course $\dot{t} = 1$.
In Example 2.12, we may take $\chi^1 = x^1$ and $\chi^2 = x^2$.

2.3 Matrix Differential Operators

The modules we are considering are all generated by finite sets of elements. For this reason, it is convenient to gather these together in a vectorial notation. We will call these objects *vectors of forms* and write for instance

$$\omega = \begin{pmatrix} \omega^1 & \cdots & \omega^m \end{pmatrix}^T.$$

See also e.g. [5]. A differential operator transforming a s-length vector of p-forms into a v-length vector of $(p+l)$-forms may be represented by a v-by-s matrix of operators in $\Lambda^l\mathcal{R}[D]$. Let P be such an operator whose matrix entries are $p_i^j \in \Lambda^l\mathcal{R}[D]$ and μ a s-length vector of p-forms $\mu = \begin{pmatrix} \mu^1 & \cdots & \mu^s \end{pmatrix}^T$. Then the v-length vector of $(p+l)$-forms $P(\mu)$, simply noted $P\mu$, is

$$P\mu = \begin{pmatrix} \sum_i p_i^1(\mu^i) \\ \vdots \\ \sum_i p_i^v(\mu^i) \end{pmatrix}.$$

We will denote by $\mathcal{M}_{v,s}^l[D]$ the set of such matrix operators. I.e.

$$\mathcal{M}_{v,s}^l[D] : \ \left(\Lambda^p T\mathcal{U}^{\infty *}\right)^s \ \rightarrow \ \left(\Lambda^{p+l} T\mathcal{U}^{\infty *}\right)^v.$$

Two operators of compatible shapes can be composed together. Let $P \in \mathcal{M}_{v,s}^l[D]$ and $Q \in \mathcal{M}_{s,k}^e[D]$. Then the composition of P and Q, noted $P \wedge Q$ or simply PQ is the operator in $\mathcal{M}_{v,k}^{l+e}[D]$ with entries

$$p_i^j \wedge q_r^i.$$

Summation is on $i = 1, \ldots, s$ and the binary operation $\wedge$ is the one of (2.1). The wedge operator (respectively the matrix product) is hence a map

$$\wedge : \ \mathcal{M}_{v,s}^l[D] \times \mathcal{M}_{s,k}^e[D] \ \rightarrow \ \mathcal{M}_{v,k}^{l+e}[D].$$

Therefore, s-by-s operators of all degree in $\mathcal{M}_{s,s}[D]$ form a ring. Operators in $\mathcal{M}_{s,s}^0[D]$ form a subring of $\mathcal{M}_{s,s}[D]$.
Thanks to these two ring structures, vector of forms can be considered as elements of modules. We will write

- $\mathcal{A}^{p,s}$ for the $\mathcal{M}^0_{s,s}[D]$-module of s-length vectors of p-forms
- $\mathcal{A}^{*,s}$ for the $\mathcal{M}_{s,s}[D]$-module of s-length vectors of forms of any degree.

2.3.1 Invertible Operators

In the set of square differential operators, those locally invertible by another operator are of special interest. Note that such operators can only be of degree zero. Also, invertible operators are two-sided:

Lemma 2.15. [21] *Let $\bar{U}, U \in \mathcal{M}^0_{s,s}[D]$, then $\bar{U}U = I_s$ implies $U\bar{U} = I_s$ where I_s is the s-by-s identity matrix.*

Proof. The proof in not trivial, see [21]. In the setting of [88], where $\mathcal{R}$ is a field, the decomposition theorem (Smith normal form) shows that an invertible matrix is the product of elementary matrices. It is easily verified that those elementary matrices satisfy the lemma and the result for all invertible operators follows. □

Note also that the (local) inverse to an invertible operator is (locally) unique. Following [88], define the set of (locally) *invertible* or *unimodular operators* or *unimodular matrices* $\mathcal{U}_s[D] \subset \mathcal{M}^0_{s,s}[D]$ as

$$\mathcal{U}_s[D] := \{U \in \mathcal{M}^0_{s,s}[D] \mid \exists \bar{U} \in \mathcal{M}^0_{s,s}[D],\ \bar{U}U = I_s\}.$$

2.3.2 Flatness and Integrable Module Bases

The following result from [5] provides an equivalent condition for a control system to be flat. The condition is not essentially easier to verify than the definition of flatness itself, but is appealing for the insight it gives on the problem. Firstly, the condition is formulated in the differential algebraic framework described earlier in this chapter. Secondly, it shows that assessing flatness of a system is also an integrability problem, as in static feedback linearization but of a more difficult kind. Indeed in the latter case, one can build a certain basis in a first step (which is an algorithmic task) and then perform an integrability test in a second step (which is also algorithmic). If the integrability test fails, the system cannot be static-feedback linearized.
The test for flatness also involves the construction of a basis, a basis of its associated module $\mathcal{A}^1$. A preliminary necessary condition is that the torsion submodule of $\mathcal{A}^1$ must be generated by dt alone. Next, one computes a basis $\omega \in \left(\Lambda^1 T\mathcal{U}^{\infty *}\right)^m$ of the free module $\mathcal{A}^1/dt$. If the forms $\omega^1, \ldots, \omega^m, dt$ constitute an integrable codistribution, then the system is flat. However, if this is not the case, the system is not necessarily non-flat. The reason is that given a basis ω for $\mathcal{A}^1/dt$, and any invertible (unimodular) matrix operator $P \in \mathcal{U}_m[D]$, the length-m vector of forms $\mu \in \left(\Lambda^1 T\mathcal{U}^{\infty *}\right)^m$ given by $\mu = P\omega$ is also a basis of $\mathcal{A}^1/dt$. Unfortunately, the action of invertible operators does not preserve integrability (as opposed to the case of non differential, i.e. order zero operators):

Example 2.16. *Assume that the four variables $x, y, z, \dot{x} = Dx$ are algebraically independent, then*

$$\begin{pmatrix} 1 & 0 \\ zD & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dx \\ zd\dot{x} + dy \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ zD & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zD & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But $\{dx, dy\}$ is integrable whereas $\{dx, zd\dot{x} + dy\}$ is not.

Therefore, if the integrability test on ω fails, there might still exist an invertible P such that $P\omega$ is integrable. We now state the mentioned result.

Proposition 2.17. [5] *Consider the control system described by* (1.2) *or* (1.1) *and the associated differential module $\mathcal{A}^1$. Compute $H^{(k^*)}$ and compute ω a basis representative for $\mathcal{A}^1/H^{(k^*)}$ as in Lemma 2.8. The control system is flat around a Brunovsky-regular point, if and only if the two following equivalent conditions are satisfied*

- *The codistribution $H^{(k^*)} = \{dt\}$ and there exists an operator $P \in \mathcal{U}_m[D]$ such that*

$$d(P\omega) = 0 \mod dt$$

- *The codistribution $H^{(k^*)} = \{dt\}$ and there exists an operator $\hat{P} \in \mathcal{U}_{m+1}[D]$ such that*

$$d\left(\hat{P}\begin{pmatrix} \omega \\ dt \end{pmatrix}\right) = 0$$

Where for a 2-form π, $\pi = 0 \mod dt$ if $\pi = \alpha \wedge dt$ for some 1-form α.

Sketch of proof. Since dt is torsion and $Ddt = 0$, it is easy to check that one may always choose

$$\hat{P} = \begin{pmatrix} P & h \\ 0 & 1 \end{pmatrix}$$

where h is a column of m functions in $\mathcal{R}$. Therefore, $\hat{P}$ is invertible if and only if P is. It also follows that $\{\mu = P\omega, dt\}$ is a set of $m+1$ integrable 1-forms. Remember that the entries of ω are forms on $\Lambda^1 T\mathcal{U}^{k^**}$. Suppose the order of P is r, then $\mu \in \Lambda^1 T\mathcal{U}^{(k^*+r)*}$. Hence, there exists m independent functions $z^1, \ldots, z^m$ (flat outputs) on $\mathcal{U}^{k^*+r}$, all independent of t such that $\{dz, dt\} = \{\mu, dt\}$. By the invertibility of P, dz represents a basis of $\mathcal{A}^1/dt$. Therefore, all state-variables x can be computed as functions $x^i = \phi^i(t, z, \ldots, z^{(A)})$ for some finite A, and the system is flat. Conversely, if the system is flat, there exists such functions z implying that dz is a basis of $\mathcal{A}^1/dt$. Hence there must be an invertible operator between dz, dt and ω, dt. □

Remark 2.18. The order of the required operator P is not known a priori. For a fixed chosen order, one may in principle set up a system of PDE and study existence of solutions. However, if there is no solution, there might still be one for the same problem with the order fixed higher. One may also fix the maximum number of input derivatives on which the flat output $z(t, x, u, u^{(1)} \ldots)$ may depend. See e.g. [50, 104, 3].

2.3.3 A Closed System of Equations

The equations of Proposition 2.17 are concise but are not differentially closed. Expanding the exterior derivative of the relations leads to new relations. Indeed

$$\begin{aligned} & d(P\omega) = 0 \mod dt \\ \Rightarrow \quad & dP\ \omega + P\ d\omega = 0 \mod dt. \end{aligned}$$

Assuming $H^{(k^*)} = \{dt\}$ and P invertible, by Lemma 2.14, we may multiply the expression above by P^{-1} and obtain

$$\begin{aligned} d\omega &= -P^{-1}\ dP\ \omega \mod dt \\ &= \Pi\ \omega \mod dt \end{aligned}$$

with $\Pi \in \mathcal{M}^1_{m,m}[D]$, $\Pi = -P^{-1}dP$, which may also be rewritten as $dP = -P\Pi$. Further differentiation leads to

$$d\Pi = -d(P^{-1})\ dP$$

and $0 = d(P^{-1}\ P) = d(P^{-1})\ P + P\ d(P^{-1})$ so that $d(P^{-1}) = -P^{-1}\ dP\ P^{-1}$. Hence

$$\begin{aligned} d\Pi &= P^{-1}\ dP\ P^{-1}\ dP \\ &= \Pi\ \Pi. \end{aligned}$$

We can therefore state the following proposition. The result is found in [21, 8, 88] with varying notations.

Proposition 2.19. *Consider the control system described by* (1.2) *or* (1.1) *and the associated differential module* $\mathcal{A}^1$*. Compute* $H^{(k^*)}$ *and compute* ω *a basis representative for* $\mathcal{A}^1/H^{(k^*)}$ *as in Lemma 2.8. The control system is flat, if and only if the codistribution* $H^{(k^*)} = \{dt\}$ *and there exist two operators* $P \in \mathcal{U}_m[D]$ *and* $\Pi \in \mathcal{M}^1_{m,m}[D]$ *satisfying*

$$d\omega = \Pi\ \omega \mod dt \tag{2.9a}$$

$$d\Pi = \Pi\ \Pi \tag{2.9b}$$

$$dP = -P\ \Pi. \tag{2.9c}$$

Remark 2.20. In contrast with the equations of Proposition 2.17, the system (2.9) is closed under the exterior derivative d. Indeed, applying d to the left-hand side of (2.9a)-(2.9c) gives zero since $dd\alpha = 0$ for all α. The derivatives of the right-hand sides read

$$\begin{aligned} d(\Pi\ \omega) &= d\Pi\ \omega - \Pi\ d\omega = \Pi\ \Pi\ \omega - \Pi\ \Pi\ \omega = 0 \mod dt \\ d(\Pi\ \Pi) &= d\Pi\ \Pi - \Pi\ d\Pi = \Pi\ \Pi\ \Pi - \Pi\ \Pi\ \Pi = 0 \\ d(-P\ \Pi) &= -dP\ \Pi - P\ d\Pi = P\ \Pi\ \Pi - P\ \Pi\ \Pi = 0. \end{aligned}$$

This means that differentiating the relations (2.9) does not lead to new algebraically independent relations. See also Corollary 2 in [88].

2.4 A System without Curvature Equations

The goal of this section is to state an equivalent of Proposition 2.19 "without curvature equations", i.e. where (2.9b) is trivial. To this end we first go along the line of [88] by introducing the notion of hyper-regularity and by defining an operator P_F obtained from the implicit system equations (1.1).

We will say that an operator $H \in \mathcal{M}^0_{p,q}[D]$ is (locally) *hyper-regular* if

- There exists a $U \in \mathcal{U}_p[D]$ s.t. $UH = \begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix}$ in the case $p > q$
- There exists a $U \in \mathcal{U}_q[D]$ s.t. $HU = \begin{pmatrix} I_p & 0_{p,q-p} \end{pmatrix}$ in the case $p < q$

and the set of p-by-q hyper-regular matrices will be denoted by $\mathcal{H}_{p,q}[D]$.

In [88], where $\mathcal{R}$ is a field, the set $\mathcal{H}_{p,q}[D]$ is defined as those operators whose diagonal Smith reduction leads to $\begin{pmatrix} I_p \\ 0_{p-q,p} \end{pmatrix}$ respectively $\begin{pmatrix} I_p & 0_{p,q-p} \end{pmatrix}$. We now state some simple properties of unimodular and hyper-regular operators.

Lemma 2.21. *The only elements of $\mathcal{R}[D]$ possessing a (local) inverse in $\mathcal{R}[D]$ are (locally) the non-zero elements of $\mathcal{R}$.*

Proof. Suppose $r = r_0 + \ldots + r_p \left(\frac{d}{dt}\right)^p$ has an inverse $s = s_0 + \ldots + s_q \left(\frac{d}{dt}\right)^q$ with $r_p \neq 0$ and $s_q \neq 0$. The product rs has a highest degree term in $\left(\frac{d}{dt}\right)^{p+q}$ and coefficient $r_p s_q \neq 0$. But since $rs = 1 = 1\left(\frac{d}{dt}\right)^0 + 0\left(\frac{d}{dt}\right)^1 + \ldots + 0\left(\frac{d}{dt}\right)^{p+q}$, it follows that $p + q = 0$. Thus, $p = 0$ and $q = 0$. We conclude that $r = r_0 \neq 0$ and $s = s_0 = 1/r \neq 0$. □

The next lemma states that block triangular matrices are invertible if and only if the diagonal blocks are invertible.

Lemma 2.22. *Let $M \in \mathcal{M}^0_{p+q,p+q}[D]$, $A \in \mathcal{M}^0_{p,p}[D]$, $B \in \mathcal{M}^0_{q,q}[D]$, $R \in \mathcal{M}^0_{p,q}[D]$ and $S \in \mathcal{M}^0_{q,p}[D]$ such that*

$$M = \begin{pmatrix} A & R \\ 0 & B \end{pmatrix} \qquad \texttt{or} \qquad M = \begin{pmatrix} A & 0 \\ S & B \end{pmatrix}$$

then $A \in \mathcal{U}_p[D]$ and $B \in \mathcal{U}_q[D]$ if and only if $M \in \mathcal{U}_{p+q}[D]$.

Proof. We show the first case. Suppose M is unimodular, then there is an inverse $M^{-1} \in \mathcal{U}_{p+q}[D]$

$$M^{-1} = \begin{pmatrix} \bar{A} & P \\ Q & \bar{B} \end{pmatrix}$$

and

$$MM^{-1} = \begin{pmatrix} A\bar{A} + RQ & AP + R\bar{B} \\ BQ & B\bar{B} \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

But then, using associativity and Lemma 2.15 (i.e. the two-sidedness of inverses)

$$\begin{aligned} B\bar{B} = I_q \quad &\Rightarrow \quad B \in \mathcal{U}_q[D] \\ BQ = 0 \quad &\Rightarrow \quad \bar{B}BQ = Q = 0 \\ A\bar{A} = I_p \quad &\Rightarrow \quad A \in \mathcal{U}_p[D]. \end{aligned}$$

The proof of the other case is similar. □

Lemma 2.23. *A triangular matrix $R \in \mathcal{M}^0_{n,n}[D]$ is (locally) unimodular if and only if the diagonal elements are (locally) non-zero elements of $\mathcal{R}$.*

Proof. Suppose R is upper triangular and write R as

$$R = \begin{pmatrix} \delta^1 & r^{1,1} & \dots & r^{1,n-1} \\ & \delta^2 & & \vdots \\ & & \ddots & r^{n-1,n-1} \\ & & & \delta^n \end{pmatrix} = \begin{pmatrix} \hat{R} & r \\ 0 & \delta^n \end{pmatrix}$$

where $\hat{R} \in \mathcal{M}^0_{n-1,n-1}[D]$ is again upper-triangular. By Lemma 2.22, R is unimodular if and only if $\hat{R}$ and δ^n are. By Lemma 2.21, δ^n is a 1×1 unimoduar matrix if and only if it is a non-zero element of $\mathcal{R}$. Repeating $n-1$ induction steps on $\hat{R}$ finishes the proof. The proof of the lower-triangular case is identical.

□

Lemma 2.24. *Let $M \in \mathcal{M}^0_{n,n}[D]$, $M = \sum_{i=0}^{r} M_i D^i$ such that $M_r \neq 0$ and $r > 0$. Then, if M_r is invertible, M is not unimodular.*

Proof. For any matrix $\bar{M} \in \mathcal{M}^0_{n,n}[D]$, $\bar{M} = \sum_{j=0}^{\bar{r}} \bar{M}_j D^j$ such that $\bar{M}_{\bar{r}} \neq 0$, we have that $\bar{M}M = \sum_{k=0}^{\bar{r}+r} N_k D^k$ with $N_{\bar{r}+r} = \bar{M}_{\bar{r}} M_r$. (The terms of lower degree are more complicated). If $\bar{M}$ is an inverse to M, then $N_1 = \ldots = N_{\bar{r}+r} = 0$. The two statements follow easily. □

Now we state an equivalent condition for a rectangular matrix to be hyper-regular.

Lemma 2.25.

i) A matrix $M \in \mathcal{M}^0_{p,q}[D]$, $p > q$, can be completed with a matrix $N \in \mathcal{M}^0_{p,p-q}[D]$ to a unimodular matrix $\Gamma = (M \ N) \in \mathcal{U}_p[D]$ if and only if M is hyper-regular. Moreover, N is also hyper-regular.

ii) A matrix $M \in \mathcal{M}^0_{p,q}[D]$, $p < q$, can be completed with a matrix $N \in \mathcal{M}^0_{q-p,q}[D]$ to a unimodular matrix $\Gamma = \begin{pmatrix} M \\ N \end{pmatrix} \in \mathcal{U}_q[D]$ if and only if M is hyper-regular. Moreover, N is also hyper-regular.

Proof. We show *i)*. Assume $M \in \mathcal{H}_{p,q}[D]$, then there exists $U \in \mathcal{U}_p[D]$ s.t. $UM = \begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix}$. Then set $N = U^{-1} \begin{pmatrix} 0_{q,p-q} \\ I_{p-q} \end{pmatrix}$. □

2.4.1 The Operator P_F

Consider the implicit system equations (1.1) that reads

$$F^k(t,x,\dot{x}) = 0 \qquad k = 1,\dots,n-m.$$

Build the following order-1 operator $P_F \in \mathcal{M}^0_{n-m,n}[D]$ with entries

$$(P_F)^k_i := \left.\frac{\partial F^k}{\partial \dot{x}^i}\right|_{\dot{x}=f(t,x,u)} D \;+\; \left.\frac{\partial F^k}{\partial x^i}\right|_{\dot{x}=f(t,x,u)}. \tag{2.10}$$

The column index is i and k is the row index. Clearly,

$$P_F\; dx = dF(t,x,f(t,x,u)) \mod dt \quad = \; 0 \mod dt \tag{2.11}$$

with $dF = \begin{pmatrix} dF^1 & \cdots & dF^{n-m} \end{pmatrix}^T$. In [88], the following result is shown using the Smith diagonal reduction of P_F. Our lemma is identical, only dropping the assumption that $\mathcal{R}$ is a field.

Lemma 2.26. *Let p be a Brunovsky-regular point. Let $\omega = \begin{pmatrix} \omega^1,\dots,\omega^m \end{pmatrix}^T$ be a basis of $\mathcal{A}^1/H^{(k^*)}$ around p, as constructed in Corollary 2.13, and set the matrix $M \in \mathcal{R}^{m\times n}$ such that*

$$\omega = M dx \mod dt.$$

There is a neighborhood of p where P_F is hyper-regular and $\begin{pmatrix} M \\ P_F \end{pmatrix}$ is unimodular if and only if around p, the torsion submodule of $\mathcal{A}^1$ is generated solely by dt.

Before proving the lemma, a simple negative example is given.

Example 2.27. *Consider the fully determined system (with no inputs, i.e. $m = 0$), given by implicit equations*

$$G^r(\dot{\gamma},\gamma) = \dot{\gamma}^r - g^r(\gamma) = 0 \qquad r = 1,\dots,\rho.$$

Then, $dG = P_G d\gamma$ with

$$P_G = I_\rho D - \frac{\partial g}{\partial \gamma}$$

and by Lemma 2.24, P_G is not unimodular (hence not hyper-regular). Indeed, the torsion submodule is generated by all variables $dt, d\gamma$.

Proof of Lemma 2.26. The following computations are all performed around a Brunovsky-regular point. Remember that the implicit and explicit equations (1.1) and (1.2) are such that $F^k(t,x,f(t,x,u)) = 0$ are identically zero for all u in some open set. Therefore, computing the exterior derivative $dF^k(t,x,f(t,x,u))$ leads to

$$\begin{aligned} 0 = &\left.\left(\frac{\partial F^k}{\partial t} + \frac{\partial F^k}{\partial \dot{x}^i}\frac{\partial f^i}{\partial t}\right)\right|_{\dot{x}=f(t,x,u)} dt \\ &+ \left.\left(\frac{\partial F^k}{\partial x^j} + \frac{\partial F^k}{\partial \dot{x}^i}\frac{\partial f^i}{\partial x^j}\right)\right|_{\dot{x}=f(t,x,u)} dx^j \\ &+ \left.\left(\frac{\partial F^k}{\partial \dot{x}^i}\frac{\partial f^i}{\partial u^l}\right)\right|_{\dot{x}=f(t,x,u)} du^l. \end{aligned}$$

From now on, we drop the evaluations $|_{\dot{x}=f(t,x,u)}$ from the notation, but they are still implicitly considered. By the independence of dt, dx^j, du^l, the previous computation imply the relations

$$\frac{\partial F^k}{\partial \dot{x}^i}\frac{\partial f^i}{\partial u^l} = 0 \tag{2.12}$$

and since rank $\frac{\partial F^k}{\partial \dot{x}^i} = n - m$ and $\frac{\partial f^i}{\partial u^l} = m$, $\frac{\partial f^i}{\partial u^l}$ is the kernel of $\frac{\partial F^k}{\partial \dot{x}^i}$. Moreover,

$$\frac{\partial F^k}{\partial x^j} = -\frac{\partial F^k}{\partial \dot{x}^i}\frac{\partial f^i}{\partial x^j}$$

so that

$$P_F = \frac{\partial F}{\partial \dot{x}}\left(I_n D - \frac{\partial f}{\partial x}\right). \tag{2.13}$$

By (2.12), the forms of relative degree 1, i.e. elements of $H^{(1)}$, are given by

$$H^{(1)} = \{dt, \frac{\partial F}{\partial \dot{x}}dx\} = \{dt, \frac{\partial F^1}{\partial \dot{x}^i}dx^i, \ldots, \frac{\partial F^{n-m}}{\partial \dot{x}^i}dx^i\}. \tag{2.14}$$

Now consider $\omega^1, \omega^2 \in \{dt, dx\}$ such that $D\omega^1 = \omega^2$, i.e. $\omega^1 \in H^{(1)}$. There are two n-length row vectors $m^1, m^2 \in \mathcal{R}^{1\times n}$ such that $\omega^1 = m^1 dx \mod dt$ and $\omega^2 = m^2 dx \mod dt$. Then

$$\begin{aligned}
\omega^2 = D\omega^1 &= D(m^1 dx) \mod dt \\
&= \big(D(m^1) + m^1 D\big)\, dx \mod dt \\
&= \big(D(m^1) + m^1 \frac{\partial f}{\partial x}\big) dx \mod dt. \\
\Rightarrow m^2 &= D(m^1) + m^1 \frac{\partial f}{\partial x}
\end{aligned}$$

so that

$$\begin{aligned}
D\omega^1 - \omega^2 &= D(m^1 dx) - m^2 dx \mod dt \\
&= \big(D(m^1) + m^1 D\big) dx - \big(D(m^1) + m^1 \frac{\partial f}{\partial x}\big) dx \mod dt \\
&= m^1 \big(I_n D - \frac{\partial f}{\partial x}\big) dx \mod dt
\end{aligned} \tag{2.15}$$

with m^1 in the row span (with coefficients in $\mathcal{R}$) of $\frac{\partial F}{\partial \dot{x}}$. By Lemmas 2.8, 2.9 and Proposition 2.10 there are $n - \rho$ forms $\omega^{k,r}$ spanning $H^{(0)}/H^{(k^*)}$ and ρ forms γ^s spanning $H^{(k^*)}$ such that around a Brunovsky-regular point, there are $n - \rho - m$ relations

$$D(\omega^{k,r}) = \omega^{k,r+1} \tag{2.16}$$

and ρ relations

$$D\gamma^s = G^s_v \gamma^v \mod dt \tag{2.17}$$

where $G \in \mathcal{R}^{\rho\times\rho}$. Define the row vectors $m^{k,r} \in \mathcal{R}^{1\times n}$ such that $\omega^{k,r} = m^{k,r}dx \mod dt$ and $g^s \in \mathcal{R}^{1\times n}$ such that $\gamma^s = g^s dx \mod dt$. Then consider the following construction

$$\left(\begin{array}{c} \omega^{1,0} \\ \vdots \\ \omega^{m,0} \\ \hline D(\omega^{1,r}) - \omega^{1,r+1} \\ \vdots \\ D(\omega^{m,r}) - \omega^{m,r+1} \\ D(\gamma^1) - G^1_v\gamma^v \\ \vdots \\ D(\gamma^\rho) - G^\rho_v\gamma^v \end{array}\right) \overset{(2.15)}{=} \left(\begin{array}{c} m^{1,0} \\ \vdots \\ m^{m,0} \\ \hline m^{1,r}(I_nD - \frac{\partial f}{\partial x}) \\ \vdots \\ m^{m,r}(I_nD - \frac{\partial f}{\partial x}) \\ g^1(I_nD - \frac{\partial f}{\partial x}) \\ \vdots \\ g^\rho(I_nD - \frac{\partial f}{\partial x}) \end{array}\right) dx \overset{(2.13)}{=} \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & N \end{array}\right) \left(\begin{array}{c} m^{1,0} \\ \vdots \\ m^{m,0} \\ \hline P_F \end{array}\right) dx.$$

for some full-rank matrix $N \in \mathcal{R}^{n-m,n-m}$ and for $k = 1, \dots, m$ and the equality is modulo dt. But the same vector also decomposes as

$$\left(\begin{array}{c} \omega^{1,0} \\ \vdots \\ \omega^{m,0} \\ \hline D(\omega^{1,r}) - \omega^{1,r+1} \\ \vdots \\ D(\omega^{m,r}) - \omega^{m,r+1} \\ \hline D(\gamma^1) - G^1_v\gamma^v \\ \vdots \\ D(\gamma^\rho) - G^\rho_v\gamma^v \end{array}\right) \overset{\substack{(2.16)\\(2.17)}}{=} \underbrace{\left(\begin{array}{c|ccc|c} I_m & & 0_{m,n-\rho-m} & & 0 \\ \hline ID & -I & & & 0 \\ & \ddots & \ddots & & \\ 0 & & ID & -I & 0 \\ \hline 0 & & 0_{\rho,n-\rho-m} & & I_\rho D - G \end{array}\right)}_{W} \underbrace{\left(\begin{array}{c} \omega^{1,0} \\ \vdots \\ \omega^{m,0} \\ \hline \omega^{1,r} \\ \vdots \\ \omega^{m,r} \\ \hline \gamma^1 \\ \vdots \\ \gamma^\rho \end{array}\right)}_{\Omega}.$$

By Lemmas 2.22 and 2.24, W cannot be invertible if $\rho \neq 0$ since it is block triangular and the lower right-block is $I_\rho D - G$. It is also easy to see that in the case $\rho = 0$, the matrix W is unimodular. Hence, $W \in \mathcal{U}_n[D]$ if and only if $H^{(k^*)} = \{dt\}$. On the other hand, around Brunovsky-regular points, $\{dt, dx\} = \{dt, \Omega\}$, so that there exists an invertible matrix $Q \in \mathcal{R}^{n\times n}$ satisfying $\Omega = QMdx \mod dt$. Therefore

$$\left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & N \end{array}\right) \left(\begin{array}{c} m^{1,0} \\ \vdots \\ m^{m,0} \\ \hline P_F \end{array}\right) dx = W\Omega = WQdx \mod dt$$

and we conclude

$$\left(\begin{array}{c} m^{1,0} \\ \vdots \\ m^{m,0} \\ \hline P_F \end{array}\right) = \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & N^{-1} \end{array}\right) WQ \mod dt.$$

Hence the matrix on the l.h.s. is unimodular if and only if W is, and by Lemma 2.25, P_F is hyper-regular under the same condition. Any other basis of $\mathcal{A}^1$ is obtained from $\begin{pmatrix}\omega^{1,0} & \cdots & \omega^{m,0}\end{pmatrix}^T$ by left multiplication by an element of $\mathcal{U}_m[D]$. □

We can now state the claimed equivalent to Proposition 2.19.

Proposition 2.28. *Consider the control system described by* (1.2) *or* (1.1) *and the associated operator* $P_F \in \mathcal{M}^0_{n-m,n}[D]$ *defined by* (2.10)*. The control system is flat around a Brunovsky-regular point if and only if there exists an operator* $\bar{P} \in \mathcal{M}^0_{m,n}[D]$ *such that*

$$\begin{pmatrix} \bar{P} \\ P_F \end{pmatrix} \in \mathcal{U}_n[D] \tag{2.18}$$

and

$$d\bar{P}\ dx = 0 \mod dt. \tag{2.19}$$

Proof. By (2.18), both $\bar{P}$ and P_F are hyper-regular. Hence, by Lemma 2.26, the torsion submodule of $\mathcal{A}^1$ is solely generated by dt. Let $N \in \mathcal{U}_n[D]$ be the inverse of the matrix (2.18) and decompose it as $N = (\bar{N}\ \tilde{N})$ with $\bar{N} \in \mathcal{H}_{n,m}[D]$ and $\tilde{N} \in \mathcal{H}_{n,n-m}[D]$. Then

$$dx = \begin{pmatrix}\bar{N} & \tilde{N}\end{pmatrix} \begin{pmatrix} \bar{P} \\ P_F \end{pmatrix} dx = \bar{N}\bar{P}dx + \tilde{N}P_F dx \overset{(2.11)}{=} \bar{N}\bar{P}dx \mod dt.$$

This shows that the vector of 1-forms $\omega = (\omega^1\ \cdots\ \omega^m)^T$, given by $\omega = \bar{P}dx$, is a basis of the free summand $\mathcal{A}^1/\{dt\}$ of $\mathcal{A}^1$. Next, by (2.19)

$$d\omega = d(\bar{P}dx) = d\bar{P}\ dx = 0 \mod dt$$

so that ω is exact modulo dt and the system is flat by Proposition 2.17. □

Remark 2.29. As in Remark 2.20, we note that (2.19) is closed under the exterior derivative d. Indeed, expanding

$$d(d\bar{P}\ dx) = dd\bar{P}\ dx - d\bar{P}\ ddx = 0$$

yields no new algebraically independent relation. Hence, there is no equivalent of the equation (2.9b) of Proposition 2.19.

2.4.2 Example

We now illustrate the result of Proposition 2.28 on a simple flat, non static feedback linearizable system.

Example 2.30. *Consider the following explicit system equations from [50].*

$$\dot{x}^1 = u^1 \qquad \dot{x}^2 = u^2 \qquad \dot{x}^3 = u^1 u^2$$

The corresponding implicit system reads $F(t, x, \dot{x}) = \dot{x}^1 \dot{x}^2 - \dot{x}^3 = 0$. *Hence*

$$P_F = \begin{pmatrix} u^2 D & u^1 D & -D \end{pmatrix}.$$

The filtration (2.2) *leads to* $k^* = 2$ *and*

$$\begin{aligned} H^{(0)} &= \{dt, dx^1, dx^2, dx^3\} \\ H^{(1)} &= \{dt, u^2 dx^1 + u^1 dx^2 - dx^3\} \\ H^{(2)} &= \{dt\} \end{aligned}$$

and

$$\begin{aligned} H^{(0)} + DH^{(0)} &= \{dt, dx^1, dx^2, dx^3, du^1, du^2\} \\ H^{(1)} + DH^{(1)} &= \{dt, u^2 dx^1 + u^1 dx^2 - dx^3, u^{2(1)} dx^1 + u^{1(1)} dx^2\} \\ H^{(2)} + DH^{(2)} &= \{dt\}. \end{aligned}$$

The Brunovsky-regular points are those where the 6 *previous codistributions have constant rank. Hence, all points are Brunovsky-regular, except for those where* $u^{2(1)} = u^{1(1)} = 0$. *Now consider the operator* $\bar{P} \in \mathcal{M}^0_{2,3}[D]$ *given by*

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 \\ -x^2 D & -u^1 & 1 \end{pmatrix}$$

and apply the following transformation to the composite matrix $U = \begin{pmatrix} \bar{P} \\ P_F \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -x^2 D & -u^1 & 1 \\ u^2 D & u^1 D & -D \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & u^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -x^2 D & 1 & 0 \\ u^2 D & -D & -u^{1(1)} \end{pmatrix}.$$

The second matrix of the l.h.s. has determinant -1. *The matrix on the r.h.s. is a diagonal operator and is therefore seen to be unimodular whenever* $u^{1(1)} \neq 0$. *If* $u^{1(1)}$ *happens to be zero, by the symmetry of the equations in* u^1, u^2, *one may swap their role and obtain a similar* U. *Hence, such a* U *exists whenever* $u^{1(1)}$ *and* $u^{2(1)}$ *are not both zero. Therefore, there is a unimodular* U *at all Brunovsky-regular points. Finally, in the above case, i.e.* $u^{1(1)} \neq 0$*:*

$$d\bar{P}\ dx = \begin{pmatrix} 0 & 0 & 0 \\ -dx^2 D & -du^1 & 1 \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = \begin{pmatrix} 0 \\ -dx^2 \wedge D dx^1 - du^1 \wedge dx^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The forms $\omega = \bar{P}\ dx$ *integrate to* $(x^1,\ x^3 - x^2u^1)$. *Finally, by Proposition 2.28, we conclude that the system is*

- *flat with flat output* $(x^1,\ x^3 - x^2u^1)$ *whenever* $u^{1(1)} \neq 0$
- *flat with flat output* $(x^2,\ x^3 - x^1u^2)$ *whenever* $u^{2(1)} \neq 0$
- *not Brunovsky-regular whenever* $u^{1(1)} = u^{2(1)} = 0$.

2.4.3 Applications

2.4.3.1 A Closer Look at the Condition $d\bar{P}\ dx = 0$

In the proof of Proposition 2.28, it was shown that the column of 1-forms ω given by $\omega = \bar{P}\ dx$ represents a basis of the free module $\mathcal{A}^1/\{dt\}$ and that the condition (2.19) then implies that

$$\omega = \bar{P}\ dx = dh \mod dt$$

for some functions $h = (h^1 \ \cdots \ h^m)$, the flat outputs. Moreover, there is a finite k such that $h = h(t, x, u, \ldots, u^{(k)})$. Hence, for $l = 1, \ldots, m$

$$dh^l = \frac{\partial h^l}{\partial t}dt + \frac{\partial h^l}{\partial x^i}dx^i + \frac{\partial h^l}{\partial u^l}du^l + \ldots + \frac{\partial h^l}{\partial u^{l(k)}}du^{l(k)}.$$

Next, under the basic regularity assumption rank $\frac{\partial f}{\partial u} = m$, the system equations can always be brought in the form

$$\begin{aligned}
\dot{x}^1 &= u^1 \\
&\vdots \\
\dot{x}^m &= u^m \\
\dot{x}^{m+1} &= \tilde{f}^1(t, x, u) \\
&\vdots \\
\dot{x}^n &= \tilde{f}^{n-m}(t, x, u)
\end{aligned}$$

so that for $r \geq 0$ and $l = 1, \ldots, m$

$$u^{l(r)} = D^{r+1}x^l \qquad \text{and} \qquad du^{l(r)} = D^{r+1}dx^l.$$

Therefore we may be assumed $\bar{P}$ to take the form

$$\bar{P} = \left(\begin{array}{cccccc}
h^1_{x^1} + h^1_{u^1}D + \ldots + h^1_{u^{1(k)}}D^{k+1} & \cdots & h^1_{x^m} + h^1_{u^m}D + \ldots + h^1_{u^{m(k)}}D^{k+1} & h^1_{x^{m+1}} & \cdots & h^1_{x^n} \\
\vdots & & \vdots & \vdots & & \vdots \\
h^m_{x^1} + h^m_{u^1}D + \ldots + h^m_{u^{1(k)}}D^{k+1} & \cdots & h^m_{x^m} + h^m_{u^m}D + \ldots + h^m_{u^{m(k)}}D^{k+1} & h^m_{x^{m+1}} & \cdots & h^m_{x^n}
\end{array}\right) \tag{2.20}$$

where the lower index indicates partial derivative. In the following, we use these remarks as a guide for the construction of flat outputs on some simple examples.

2.4.3.2 Nonholonomic Car

Consider the explicit equations of the nonholonomic car

$$\dot{x} = u^1 \cos\theta \qquad \dot{y} = u^1 \sin\theta \qquad \dot{\theta} = u^2$$

and their implicit form

$$F = \dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

The operator P_F then reads

$$P_F = \begin{pmatrix} \sin\theta D & -\cos\theta D & u^1 \end{pmatrix}.$$

By Lemma 2.22, the following ansatz

$$\left(\begin{array}{c} \bar{P} \\ \hline P_F \end{array}\right) = \left(\begin{array}{ccc} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ \hline \sin\theta D & -\cos\theta D & u^1 \end{array}\right) \in \mathcal{U}_3[D]$$

is unimodular if and only if the upper 2-by-2 block $S \in \mathcal{U}_2[D]$ and $u^1 \neq 0$. The condition $d\bar{P} \wedge dq = 0$ is easily satisfied by choosing $S = I_2$, i.e.

$$\left(\begin{array}{c} \bar{P} \\ \hline P_F \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline \sin\theta D & -\cos\theta D & u^1 \end{array}\right)$$

Hence, the differential of the flat outputs are given by

$$\begin{pmatrix} dh^1 \\ dh^2 \end{pmatrix} = \bar{P}\, dq = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

which integrates to the flat outputs

$$h^1 = x \qquad h^2 = y.$$

Other flat outputs We now attempt to transform the matrix P_F by left-multiplication by a (unimodular) rotation matrix V

$$\begin{aligned} &\begin{pmatrix} \underbrace{\sin\theta D \quad -\cos\theta D \quad u^1}_{P_F} \end{pmatrix} \underbrace{\begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{V} \\ &= \begin{pmatrix} 1 & D & u^1 \end{pmatrix} \end{aligned}$$

which has the effect of lowering the order of the leftmost element of P_F. Hence, by Lemma 2.22, we may now choose between two different ansätze ensuring unimodularity:

$$\left(\frac{\bar{P}}{P_F}\right) V = \left(\begin{array}{ccc} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ \hline 1 & D & u^1 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccc} 0 & s_{11} & s_{12} \\ 0 & s_{21} & s_{22} \\ \hline 1 & D & u^1 \end{array}\right).$$

Assume 0-flatness. This implies $h^{1,2} = h^{1,2}(x, y, \theta)$, i.e. from (2.20)

$$\bar{P} = \begin{pmatrix} h^1_x & h^1_y & h^1_\theta \\ h^2_x & h^2_y & h^2_\theta \end{pmatrix}$$

The two columns of zeros in

$$\bar{P}V \stackrel{!}{=} \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \end{pmatrix} \quad \text{and} \quad \bar{P}V \stackrel{!}{=} \begin{pmatrix} 0 & s_{11} & s_{12} \\ 0 & s_{21} & s_{22} \end{pmatrix}$$

lead to two set of PDE which read

$$\begin{array}{c} h^1_\theta = 0 \\ h^2_\theta = 0 \end{array} \quad \text{and} \quad \begin{array}{c} -\sin\theta\ h^1_y - \cos\theta\ h^1_x = 0 \\ -\sin\theta\ h^2_y - \cos\theta\ h^2_x = 0. \end{array}$$

These have solutions (both satisfying $\det S = 1 \neq 0$)

$$\begin{array}{c} h^1 = x \\ h^2 = y \end{array} \quad \text{and} \quad \begin{array}{c} h^1 = \theta \\ h^2 = y\cos\theta - x\sin\theta. \end{array}$$

The first solution is identical to the one obtained above, whereas the second represent another, algebraically independent set of flat outputs for the nonholonomic car.

2.4.3.3 Planar Pendulum

The explicit second order equations for the planar pendulum read

$$\ddot{x} = u^1 \qquad \ddot{y} = u^2 \qquad a\ddot{\theta} = -u^1\cos\theta + (u^2+1)\sin\theta.$$

The second order implicit equation is obtained as

$$F = a\ddot{\theta} + \ddot{x}\cos\theta - (\ddot{y}+1)\sin\theta = 0.$$

The associated order 2 operator P_F is then given by

$$P_F = \begin{pmatrix} \cos\theta D^2 & -\sin\theta D^2 & aD^2 - b \end{pmatrix}$$

where $a \in \mathbb{R}$ and $b = u^1\sin\theta + (u^2+1)\cos\theta \in \mathcal{R}$. As in the non-holonomic car example, we first apply a rotation matrix V_1

$$
\begin{aligned}
&\Big(\ \underbrace{\cos\theta D^2 \quad -\sin\theta D^2 \quad aD^2 - b}_{P_F}\ \Big) \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{V_1} \\
&= \Big(\ D^2 - \dot\theta^2 \quad 2\dot\theta D + \ddot\theta \quad aD^2 - b\ \Big)
\end{aligned}
$$

which results in two elements of order two with constant coefficients on the left and the right. Hence, a second (constant) transformation matrix V_2 allows us to obtain an order zero element, i.e. an element in $\mathcal{R}$

$$
\begin{aligned}
&\Big(\ D^2 - \dot\theta^2 \quad 2\dot\theta D + \ddot\theta \quad aD^2 - b\ \Big) \underbrace{\begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{V_2} \\
&= \Big(\ D^2 - \dot\theta^2 \quad 2\dot\theta D + \ddot\theta \quad \underbrace{a\theta^2 - b}_{\in\mathcal{R}}\ \Big)
\end{aligned}
$$

$$
V = V_1 V_2 \qquad V \in \mathcal{R}^{3\times 3} \qquad \det V = 1.
$$

We may now again try the following form of ansatz

$$
\left(\frac{\bar P}{P_F}\right) V = \left(\begin{array}{ccc} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ \hline D^2 - \dot\theta^2 & 2\dot\theta D + \ddot\theta & a\theta^2 - b \end{array}\right)
$$

which, by Lemma 2.22, ensures that $\begin{pmatrix} \bar P \\ P_F \end{pmatrix}$ is unimodular whenever S is and $a\theta^2 - b \neq 0$. Next, assuming $h^{1,2} = h^{1,2}(x, y, \theta)$ implies

$$
\bar P = \begin{pmatrix} h^1_x & h^1_y & h^1_\theta \\ h^2_x & h^2_y & h^2_\theta \end{pmatrix}
$$

and leads to the equations

$$
\det S = h^1_x h^2_y - h^1_y h^2_x \neq 0 \tag{2.21}
$$

$$
\begin{aligned}
h^1_\theta - a\cos\theta\ h^1_x + a\sin\theta\ h^1_y = 0 \\
h^2_\theta - a\cos\theta\ h^2_x + a\sin\theta\ h^2_y = 0.
\end{aligned} \tag{2.22}
$$

The PDE (2.22) has solution

$$
h^1 = x + a\sin\theta \qquad h^2 = y + a\cos\theta \tag{2.23}
$$

for which the unimodularity condition (2.21)

$$
\det S = h^1_x h^2_y - h^1_y h^2_x = 1 \neq 0
$$

is satisfied. The solution (2.23) represents the well known flat outputs.

2.4.3.4 A 1-Flat Example

Consider again the system from Example 2.30 given by

$$\dot{x}^1 = u^1 \qquad \dot{x}^2 = u^2 \qquad \dot{x}^3 = u^1 u^2$$

We will try to reconstruct a (local) set of flat outputs. The order of the rightmost element of P_F may be lowered by the following transformation

$$\begin{aligned} &\begin{pmatrix} \underbrace{u^2 D \quad u^1 D \quad -D}_{P_F} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & u^1 \end{pmatrix}}_{V} \\ &= \begin{pmatrix} u^2 D & u^1 D & -\dot{u}^1 \end{pmatrix} \end{aligned}$$

Next we try the ansatz

$$\bar{P}V = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{pmatrix}.$$

We suspect the system is 1-flat, i.e.

$$h^{1,2} = h^{1,2}(x^1, x^2, x^3, u^1, u^2).$$

Hence, from (2.20), the form

$$\bar{P} = \begin{pmatrix} h^1_{x^1} + h^1_{u^1} D & h^1_{x^2} + h^1_{u^2} D & h^1_{x^3} \\ \\ h^2_{x^1} + h^2_{u^1} D & h^2_{x^2} + h^2_{u^2} D & h^2_{x^3} \end{pmatrix}.$$

From the chosen ansatz for $\bar{P}$ and the chosen V

$$\bar{P}V = \begin{pmatrix} \cdot & \cdot & h^1_{u^2} D + h^1_{x^2} + u^1 h^1_{x^3} \\ \cdot & \cdot & h^2_{u^2} D + h^2_{x^2} + u^1 h^2_{x^3} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{pmatrix}. \tag{2.24}$$

This leads to the linear homgeneous PDE in 5 variables $x^{1,2,3}, u^{1,2}$

$$\begin{aligned} h^1_{u^2} &= 0 \\ h^2_{u^2} &= 0 \\ h^1_{x^2} + u^1 h^1_{x^3} &= 0 \\ h^2_{x^2} + u^1 h^2_{x^3} &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} dh^1, dh^2 \in &\ \{dx^1, du^1, dx^3 - u^1 dx^2\} \\ = &\ \{dx^1, du^1, d(x^3 - u^1 x^2)\} \end{aligned}$$

which for $h^{1,2}$, implies the form

$$h^1 = H^1(x^1,\ u^1,\ x^3 - u^1 x^2) \qquad h^2 = H^2(x^1,\ u^1,\ x^3 - u^1 x^2).$$

The left 2-by-2 block in (2.24) must be invertible and reads

$$\begin{pmatrix} (H^1_2 - x^2 H^1_3) D + H^1_1 & -u^1 H^1_3 \\ (H^2_2 - x^2 H^2_3) D + H^2_1 & -u^1 H^2_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x^2 D & -u^1 \end{pmatrix}$$

$$\text{for} \quad \begin{matrix} H^1_2 = H^1_3 = 0 & H^1_1 = 1 \\ H^2_1 = H^2_2 = 0 & H^2_3 = 1 \end{matrix}$$

making the block invertible (unimodular) if $u^1 \neq 0$. This integrates to the flat outputs

$$h^1 = x^1 \qquad h^2 = x^3 - u^1 x^2.$$

Note that lowering the order of the leftmost element of P_F then leads to a PDE without solution.

2.4.3.5 Remark

The approach sketched in the three previous example is very conservative. Indeed, the transformation matrix V, chosen so that $P_F V$ contains an $(n-m) \times (n-m)$ invertible sub-matrix (which by Lemma 2.26 is always possible for a flat system) is not necessarily such that the corresponding sub-matrix of $\bar{P}V$ is zero for some suitable $\bar{P}$. Nevertheless, it is noteworthy that the pendulum equations can be dealt with so easily.

2.5 Conclusion

This chapter gave a review of the use of matrix differential operators in the context of the flatness problem. These operators, among which certain are invertible, were shown to act on bases of the differential module associated with a given control system. A now classical result was presented stating that given a basis, the flatness of the underlying system is equivalent to the two following conditions being satisfied: *i)* The torsion part of the module must be generated by the differential of the time variable dt alone and *ii)* there must exist an invertible matrix operator transforming the basis in a new basis composed of exact differential 1-forms. The conditions of the theorem were then further decomposed so as to obtain a closed system of equations characterizing the existence of flat outputs. The contribution of this chapter consists in a reformulation of the obtained characterization. This reformulation does not "solve" the problem either, but the absence of "curvature equations" may be seen as an appealing feature, as illustrated in the construction of flat outputs of some simple examples.

Chapter 3

Dynamic Controlled Invariance

In the preceding chapter and considering the characterization of flatness, the emphasis was put on establishing the existence of flat outputs. Another approach consists in attempting to verify the equivalence of the given control system with some controllable linear one. In general, doing so involves the usage of a dynamic feedback [16, 17, 51, 24]. The equivalence between two systems also means the existence of a specific type of reversible mapping, namely a Lie-Bäcklund isomorphism [48, 49, 108, 1]. In turn, these mappings are related to a specific type of dynamic feedback called endogenous [44, 47]. In [24], the introduction of the notion of covering of a system by another one allows to somewhat relax the conditions on the sought-after map. A covering is a map between infinite dimensional manifolds. It is then of interest to devise a test that decides whether a surjective map from the state-space of a given system to another manifold induces a covering between corresponding infinite input prolongations. A first step in that direction is made by generalizing the condition of controlled invariance [100, 62, 15] to something that will be called dynamic controlled invariance. This condition is however too loose on its own account and an additional condition is provided by an infinitesimal version of the dynamic extension algorithm [127, 128, 99, 97, 37, 105]. The developed theory then provides a convenient setting for the following problem: Given a control system and a set of feasible constraints, does the unconstrained system cover the constrained one? This problem is closely related to the notion of relative flatness [108]. Finally, the first aspect of our condition for a covering, i.e. dynamic controlled invariance, actually corresponds to a degenerate type of dynamic feedback. A simple example shows that linearization via that type of feedback introduces many additional difficulties.

In Section 3.1, after a brief discussion of dynamic feedback and endogenous dynamic feedback, we describe the kind of mappings relevant to the theory of flatness, namely Lie-Bäcklund mappings. We go on by reviewing the notion of a covering and the related results regarding flatness. Section 3.2 first reviews the condition of control invariance and then generalizes it to the one of dynamic controlled invariance. Next, the dynamic extension algorithm for a set of 1-forms is described. Brought together, dynamic controlled invariance and the DEA provide the condition for a (finite dimensional) map to induce a covering. In Section 3.3, we present a sufficient condition for a constrained system to be

covered by its unconstrained counterpart. We next show how this may be used to infer flatness of the constrained system. Finally, Section 3.4 gives the example of a system linearizable by singular static feedback that is not flat and discusses some implications.

3.1 Dynamic Feedback

Recall the general expression for the equations of a control system in explicit form

$$\dot{x}^i = f^i(t, x, u) \qquad \operatorname{card} x = n \quad \operatorname{card} u = m. \tag{3.1}$$

In Section 1.1, we considered changes of coordinates on systems described by (3.1) involving a map $x = \phi(t, z)$ and a change of inputs φ over ϕ of the form $u = \varphi(t, z, v)$. The maps where required to represent an invertible, time preserving bundle map, therefore $\operatorname{rank} \frac{\partial \phi}{\partial x} = n$ and $\frac{\partial \varphi}{\partial v} = m$. If φ is over the identity, i.e. $\phi = \mathrm{id}$, then the transformation represents a *static feedback* for system (3.1)

$$\dot{x}^i = f^i(t, x, u) \quad u = \varphi(t, x, v) \qquad \operatorname{card} x = n \quad \operatorname{card} u, v = m \tag{3.2}$$

with new inputs v. Let us insist on the fact that this definition of static feedback implies that $\operatorname{card} v = \operatorname{card} u$ and that the transformation is reversible.
A *dynamic feedback* for system (3.1) consists in assigning to the input u of system (3.1) the output of some other system. As inputs, this "added" system receives the state x of (3.1) and some new variables v.

$$\dot{x}^i = f^i(t, x, u) \tag{3.3a}$$

$$\dot{\xi}^j = a^j(t, \xi, x, v) \quad u^s = b^s(t, \xi, x, v) \tag{3.3b}$$

$$\operatorname{card} \xi = n_\xi \quad \operatorname{card} v = m_v \tag{3.3c}$$

At this point we make no assumption on $\operatorname{card} v$ but we require that

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f \circ b}{\partial v} \\ \frac{\partial a}{\partial v} \end{pmatrix} = m_v. \tag{3.4}$$

In the static case of (3.2), the condition $\operatorname{rank} \frac{\partial \varphi}{\partial v} = m$ guarantees that for any solution $\hat{\sigma} : (t) \mapsto (t, x(t), u(t))$ of equations (3.1), there is a solution $\varphi^{-1} \circ \hat{\sigma} : (t) \mapsto (t, x(t), v(t))$ of system (3.2). The analog in the case of a dynamic feedback is a bit more subtle.

3.1.1 Non-Singular Dynamic Feedback

Assume that for any (smooth) section $\hat{\sigma} \in \Gamma\mathcal{U}$, $\hat{\sigma} : (t) \mapsto (t, x(t), u(t))$, solution of system (3.3a) there exists a solution $\tilde{\sigma} : (t) \mapsto (t, \xi(t), x(t), v(t))$ of (3.3) satisfying $\hat{\sigma}(t) = (t, \tilde{\sigma}_x(t), b \circ \tilde{\sigma}(t))$. In other words, assume that any (smooth) solution of (3.3a) can be extended to a solution of (3.3). In this case, and following [16], we will call (3.3b) a *non-singular dynamic feedback* for system (3.3a). In [17], such a dynamic feedback is said *regular*.

Remark 3.1. The (non-)singularity of a dynamic feedback clearly depends on the system to which it is applied. In the literature, by *dynamic feedback* it is frequently meant a *non-singular dynamic feedback with* $m_v = m$ as for instance in [16, 51, 24] or non-singular with $m_v \geq m$ in [17] when it comes to linearizing the system (3.3a).

Lemma 3.2. *Consider the* s *independent* 1*-forms* $\alpha^1, \ldots, \alpha^s \in \Lambda^1 T\mathcal{U}^{\infty *}$ *and the sequences of* $\mathcal{R}$*-modules*

$$A_0 = \{\alpha^1, \ldots, \alpha^s\} \qquad A_k = \mathop{+}_{i=0}^{k} D^i A_0 = A_0 + DA_0 + \ldots + D^k A_0.$$

Then for $Q \geq 0$

$$\dim A_{Q+1} - \dim A_Q \leq \sigma \quad \Rightarrow \quad \dim A_{Q+q+1} - \dim A_{Q+q} \leq \sigma \quad \forall q \geq 0.$$

Proof. There exists a multi-index $I = (I_1, \ldots, I_\sigma)$ such that

$$A_{Q+1} = A_Q + \{\alpha^{I_1(Q+1)}, \ldots, \alpha^{I_\sigma(Q+1)}\}.$$

Hence

$$\begin{aligned} A_{Q+q} &= A_Q + \{\alpha^{I_1(Q+q)}, \ldots, \alpha^{I_\sigma(Q+q)}\} \\ A_{Q+q+1} &= A_Q + \{\alpha^{I_1(Q+q+1)}, \ldots, \alpha^{I_\sigma(Q+q+1)}\} \end{aligned}$$

and the result follows. □

Condider the sequence of $\mathcal{R}$-modules generated by

$$E_k = \{dt, dx, d\xi, db, \ldots, db^{(k)}\} \qquad k \geq 0.$$

In the language of Chapter 2, the non-singularity of a dynamic feedback can be characterized as follows.

Lemma 3.3. *Let* $\mathcal{V}^\infty$ *be the infinitely prolonged bundle and* $\mathcal{A}^1$ *the differential module associated to the system described by* (3.3) *so that* $\mathcal{A}^1$ *is generated by the* 1*-forms* $dt, dx^i, d\xi^j$*. Then, around a point* $p \in \mathcal{V}^\infty$ *where* $\{dt, dx, d\xi\}$ *and* $E_{n_\xi - 1}$ *have constant dimension,* (3.3b) *is a non-singular dynamic feedback for the system* (3.3a) *if and only if the* 1*-forms* $db^1, \ldots, db^m$ *are a basis of a free submodule of* $\mathcal{A}^1$*.*

Proof. Write $db = (db^1 \ \cdots \ db^m)^T$ and assume that $db^1, \ldots, db^m$ is not the basis of a free submodule of $\mathcal{A}^1$. Then there exists a non-zero operator $H \in \mathcal{M}^0_{m,m}[D]$ such that $H\ db = 0$ and where D is the infinitely prolonged Cartan vector field of system (3.3), i.e. $D = \frac{\partial}{\partial t} + f^i|_{u=b} \frac{\partial}{\partial x^i} + a^j \frac{\partial}{\partial \xi^j} + v^{s(k+1)} \frac{\partial}{\partial v^{s(k)}}$. Let $h^p_{qr} D^r, r = 0, \ldots, R$ be the entries of H, then $h^p_{qr} D^r db^q = h^p_{qr} d(D^r b^q) = 0$. This implies the existence of m not all trivial relations $\eta^p(b, Db \ldots, D^R b) = 0$ and $p = 1, \ldots, m$. Along solutions of 3.3, the implicit equations $\eta^p(b, \dot{b}, \ldots, b^{(R)}) = 0$ are hence satisfied. Choose any smooth trajectory $t \mapsto \bar{b}(t)$ not satisfying all relations $\eta^p(\bar{b}, \dot{\bar{b}}, \ldots, \bar{b}^{(R)}) = 0$, there clearly exists a solution section $\hat{\sigma}$ for (3.3a) such that $\hat{\sigma} : (t) \mapsto (t, x(t), u(t) = \bar{b}(t))$; but there is no solution $\tilde{\sigma}$ to (3.3) such

that $\hat{\sigma}(t) = (t, \tilde{\sigma}_x(t), b \circ \tilde{\sigma}(t))$ holds.
The converse needs a bit more work. From an arbitrary solution of (3.3a), we need to construct a solution of (3.3). Assume that $db^1, \ldots, db^m$ is the basis of a free submodule around p. Define

$$\begin{aligned} X_0 &= \{dt, dx\} \qquad X_k = \mathop{+}_{i=0}^{k} D^i X_0 \qquad k \geq 0 \\ B_k &= \{db, \ldots, db^{(k)}\} \qquad k \geq 0 \\ C_k &= \{dt, dx, db, \ldots, db^{(k)}\} = X_0 + B_k \qquad k \geq 0. \end{aligned}$$

Clearly $\dim B_k = (k+1)m$. We now show that $\dim C_k = 1 + n + (k+1)m$. By the assumption rank $\frac{\partial f}{\partial u} = m$ and by the equation $u = b(t, \xi, x, v)$, we have $\{dt, dx, dx^{(1)}\} = \{dt, dx, db\}$. Hence, $X_1 = C_0$. Differentiating k times, we obtain that $X_{k+1} = C_k$ for $k \geq 0$. Since $C_k = X_0 + B_k$, X_0 as finite dimension and $\dim B_k = (k+1)m$, there must exist a finite $P \geq 0$ such that

$$\dim C_{P+p+1} - \dim C_{P+p} = m \qquad \forall p \geq 0.$$

On the other hand, assume there is a $Q \geq 0$ such that $\dim X_{Q+1} - \dim X_Q < m$, then by Lemma 3.2

$$\dim X_{Q+q+1} - \dim X_{Q+q} < m \qquad \forall q \geq 0.$$

But since $X_{k+1} = C_k$, we have a contradiction. Therefore, using $\dim X_0 = 1 + n$, we indeed have that

$$\dim C_k = \dim X_{k+1} = 1 + n + (k+1)m. \tag{3.5}$$

Next, consider

$$E_k = \{dt, dx, d\xi, db, \ldots, db^{(k)}\} = C_k + \{d\xi\} \qquad k \geq 0.$$

we verify that any subset $\bar{\xi} \subset \xi$ such that $E_{n_\xi - 1} = C_{n_\xi - 1} + \{d\bar{\xi}\}$ (which exists, since $E_{n_\xi - 1}$ has constant dimension) is such that

$$dt, dx, d\bar{\xi}, db, \ldots, db^{(k)} \quad \text{are independent } \forall k \geq 0. \tag{3.6}$$

Choose a multi-index $I = (I_1, \ldots, I_m)$, $I_i \in \{1, \ldots, n\}$ such that

$$\{dt, dx, dx^{(1)}\} = \{dt, dx, dx^{I(1)}\} = \{dt, dx, db\}. \tag{3.7}$$

Define $C_{i,k} = \{dt, dx, dx^{I_i(1)}, \ldots, dx^{I_i(k+1)}\}$ such that by (3.7), $C_k = +_{i=1}^m C_{i,k}$. From (3.5), note that

$$\dim C_{i,k} = 1 + n + (k+1). \tag{3.8}$$

Since x, ξ are state variables for the system (3.3), by (3.8), and $\dim d\xi = n_\xi$ we see that

$$\{dt, dx, d\xi\} \cap C_{i,n_\xi - 1} = \{dt, dx, d\xi\} \cap C_{i,n_\xi + l} \qquad \forall l \geq 0.$$

Intersection distributes over union, therefore

$$\begin{aligned}\{dt, dx, d\xi\} \cap C_{n_\xi - 1} &= \{dt, dx, d\xi\} \cap \mathop{+}_{i=1}^{m} C_{i, n_\xi - 1} = \mathop{+}_{i=1}^{m} \{dt, dx, d\xi\} \cap C_{i, n_\xi - 1} \\ &= \mathop{+}_{i=1}^{m} \{dt, dx, d\xi\} \cap C_{i, n_\xi + l} = \{dt, dx, d\xi\} \cap \mathop{+}_{i=1}^{m} C_{i, n_\xi + l} \\ &= \{dt, dx, d\xi\} \cap C_{n_\xi + l} \qquad \forall l \geq 0.\end{aligned}$$

Hence $\bar{\xi}$ is as claimed. Choose a minimal subset $\tilde{\xi} \subset \xi$, complement of $\bar{\xi}$ in ξ. Clearly, $\{d\tilde{\xi}\} \subset C_{n_\xi - 1} + \{d\bar{\xi}\}$ so that there exist some function χ such that

$$\tilde{\xi} = \chi(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi - 1)}).$$

From (3.7), we also see that $x, b, \ldots, b^{(n_\xi - 1)}$ are local state coordinates for the n_ξ-th prolongation of system (3.3a). From (3.3b), $d\bar{\xi}^{(1)} \subset \{dt, dx, d\xi, dv\}$. And by the assumption (3.4), we may choose a minimal subset $\bar{v} \subset v$ such that

$$E_{n_\xi - 1} + DE_{n_\xi - 1} = \{dt, dx, d\bar{\xi}, db, \ldots, db^{(n_\xi)}, dv\} = \{dt, dx, d\bar{\xi}, db, \ldots, db^{(n_\xi)}, d\bar{v}\}$$

and $\tilde{v}$ a complement of $\bar{v}$ in v. There exists a function ν satisfying

$$\tilde{v} = \nu(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi)}, \bar{v}).$$

Hence, $t, x, \bar{\xi}, b, \ldots, b^{(n_\xi - 1)}$ are local coordinates for a system with inputs $b^{(n_\xi)}, \bar{v}$. The equations of this system are

$$\dot{x} = f(t, x, b) \tag{3.9a}$$

$$\dot{b} = b^{(1)} \quad \ldots \quad \dot{b}^{(n_\xi - 1)} = b^{(n_\xi)} \tag{3.9b}$$

$$\dot{\bar{\xi}} = \bar{a}(t, x, \bar{\xi}, \chi(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi - 1)}), \bar{v}, \nu(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi)}, \bar{v})). \tag{3.9c}$$

Clearly, the map

$$\pi : (t, x, \bar{\xi}, b, \ldots, b^{(n_\xi)}, \bar{v}) \mapsto (t, x, u = b)$$

transforms solutions of (3.9) to solutions of (3.3a) and the map

$$\theta : (t, x, \bar{\xi}, b, \ldots, b^{(n_\xi)}, \bar{v}) \mapsto \begin{cases} t \\ x \\ u = b \\ \xi = (\bar{\xi}, \tilde{\xi} = \chi(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi - 1)})) \\ v = (\bar{v}, \tilde{v} = \nu(t, x, \bar{\xi}, b, \ldots, b^{(n_\xi)}, \bar{v})) \end{cases}$$

transforms solutions of (3.9) to solutions of (3.3). Now consider any (smooth) solution of system (3.3a) given by the section $\hat{\sigma} : (t) \mapsto (t, x(t), u(t))$. This section always lifts to a solution $\hat{\sigma}^{n_\xi}$ of system (3.9a)-(3.9b) as

$$\hat{\sigma}^{n_\xi} : (t) \mapsto (t, x(t), b = u(t), \ldots, b^{(n_\xi)} = \frac{\partial^{n_\xi} u(t)}{\partial t^{n_\xi}}).$$

One may now choose any suitable initial condition for $\bar{\xi}$ and input $\bar{v}(t)$ (around p) and together with $\hat{\sigma}^{n_\xi}(t)$, (3.9c) is an ODE whose solution completes $\hat{\sigma}^{n_\xi}(t)$ to a solution of the system (3.9). This solution is mapped to a solution of (3.3) by the map θ. See also [24, 109] □

Lemma 3.3 has the following obvious consequence.

Corollary 3.4.

i) A static feedback (3.2) *with* $\operatorname{rank}\frac{\partial\varphi}{\partial v} = m$ *is a non-singular (dynamic) feedback.*

ii) A dynamic feedback (3.3) *with* $m_v < m$ *is singular.*

3.1.2 Endogenous Dynamic Feedback

One may further restrict the class of dynamic feedbacks (3.3b) with the notion of *endogenous dynamic feedback*. This notion was introduced in the framework of differential algebra not discussed here, see [44, 47]. In our setting, an endogenous dynamic feedback is a non-singular dynamic feedback (3.3b) for (3.3a), such that $m_v = m$ and satisfying the additional property that the state variables $\xi^1, \ldots, \xi^{n_\xi}$ of the compensator can all be expressed as functions of the variables $t, x, u, \dot{u}, \ldots, u^{(k)}$ for some finite k, see e.g. [5]. Equivalently, a dynamic feedback (3.3b) for (3.3a) is endogenous if the systems described by (3.3a) and (3.3) are Lie-Bäcklund equivalent, see for instance [87], p. 128.

Example 3.5. *The two following instances of* (3.3) *are non-singular dynamic feedbacks for the integrator* $\dot{x} = u$*, both with* $m = m_v = 1$*.*

$$a)\quad \begin{array}{c} \dot{x} = u \\ \dot{\xi}^1 = \xi^2 \quad \dot{\xi}^2 = v \end{array} \quad u = \xi^1 \qquad\qquad b)\quad \begin{array}{c} \dot{x} = u \\ \dot{\xi}^1 = v \quad \dot{\xi}^2 = v \end{array} \quad u = \xi^1$$

In a), the feedback is endogenous, indeed, $\xi^1 = \dot{u}$ *and* $\xi^2 = \ddot{u}$*. In b), the variable* ξ^2 *cannot be obtained as a function of* $x, u, \dot{u}, \ldots$ *In particular, the initial condition* $\xi^2|_{t=0}$ *can be assigned independently from* $x|_{t=0}, u|_{t=0}, \dot{u}|_{t=0}, \ldots$

An endogenous feedback can also be characterized as follows.

Lemma 3.6. *Let* $\mathcal{A}^1$ *be the differential module associated to the system described by* (3.3) *so that* $\mathcal{A}^1$ *is generated by the* 1*-forms* $dt, dx^i, d\xi^j$ *and let* $\mathcal{A}^1_x$ *be the submodule of* $\mathcal{A}^1$ *generated by the* 1*-forms* dt, dx^i*. Then, at a point* $p \in \mathcal{V}^\infty$*,* (3.3b) *is an endogenous dynamic feedback for the system* (3.3a) *if and only if it is non-singular,* $m_v = m$ *and* $\mathcal{A}^1_x = \mathcal{A}^1$ *around* p*.*

3.1.3 Mappings

Consider two control systems sharing the same independent time variable t described by the equations

$$\dot{x}^i = f^i(t, x, u) \qquad \operatorname{card} x = n_x,\ \operatorname{card} u = m_u \tag{3.10}$$

and

$$\dot{y}^j = g^j(t, y, v) \qquad \operatorname{card} y = n_y,\ \operatorname{card} v = m_v \tag{3.11}$$

respectively. Following the discussion of Section 1.1, one can define bundles $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ and $\pi_{\mathcal{VN}} : \mathcal{V} \to \mathcal{N}$ where $\pi_{\mathcal{UM}} : (t, x, u) \mapsto (t, x)$ and $\pi_{\mathcal{VN}} : (t, y, v) \mapsto (t, y)$. On these bundles, the system equations induce corresponding Cartan distributions and codistributions. The infinite-input prolongations then yield two diffieties $(\mathcal{U}^\infty, D)$ and $(\mathcal{V}^\infty, E)$ where

$$D = \frac{\partial}{\partial t} + f^i \frac{\partial}{\partial x^i} + u^{l(k+1)} \frac{\partial}{\partial u^{l(k)}} \qquad \text{and} \qquad E = \frac{\partial}{\partial t} + g^j \frac{\partial}{\partial y^j} + v^{q(r+1)} \frac{\partial}{\partial v^{q(r)}}$$

with $k, q = 0, 1, \ldots$ span the respective one-dimensional Cartan distributions.
Between the infinite bundles $\mathcal{U}^\infty$ and $\mathcal{V}^\infty$, considered as bundles over the same base (time manifold) $\mathcal{B}$, a smooth bundle map is a map $\Phi : \mathcal{U}^\infty \mapsto \mathcal{V}^\infty$ such $\Phi^* t = t$ and such that for any function $h \in \mathcal{R}(\mathcal{V}^\infty)$, $\Phi^* h \in \mathcal{R}(\mathcal{U}^\infty)$. See Chapter 2, p. 40, for the definition of the smooth functions in $\mathcal{R}$. A mapping Φ additionally satisfying

$$\Phi_*(D|_p) = E|_{\Phi(p)} \qquad \forall p \in \mathcal{U}^\infty \tag{3.12}$$

is called a *Lie-Bäcklund mapping*. The condition (3.12) says that the two Cartan vector fields D and E representing (total) time differentiation along system solutions are Φ-related. If there are local coordinates on $\mathcal{U}^\infty$ and $\mathcal{V}^\infty$ such that $\Phi(t, x, \ldots, y, \ldots) = (t, x, \ldots)$ then Φ is a *Lie-Bäcklund submersion*. Lie-Bäcklund submersions are connected to subsystems in [108]. If there are local coordinates on $\mathcal{U}^\infty$ and $\mathcal{V}^\infty$ such that $\Phi(t, x, \ldots) = (t, x, \ldots, 0, \ldots)$ then Φ is a *Lie-Bäcklund immersion*. If Φ is one-to-one and has a smooth inverse, it is called a *Lie-Bäcklund isomorphism*. [48, 49, 108, 1].

3.1.3.1 Coverings

The next definition and the related results are borrowed from [24]. A Lie-Bäcklund mapping $\Phi : \mathcal{U}^\infty \mapsto \mathcal{V}^\infty$ is called a *covering* if the tangent map $\Phi_*|_p$ is a $\mathbb{R}$-vector space epimorphism (a surjective map) and $\dim \ker \Phi_*|_p$, if finite, is constant for all p in $\mathcal{U}^\infty$. The *dimension* of a covering is the dimension of $\ker \Phi_*|_p$. The dimension may be finite or infinite. If $(\mathcal{U}^\infty, D)$ and $(\mathcal{V}^\infty, E)$ are the diffieties associated to systems (3.10) and (3.11) respectively and if $\Phi : \mathcal{U}^\infty \mapsto \mathcal{V}^\infty$ is a covering, then the system (3.10) is said to *cover* the system (3.11).

Proposition 3.7 ([24]).

i) A non-singular dynamic feedback (3.3b) *for system* (3.3a) *with* $m_v = m$ *defines a finite-dimensional covering of* (3.3a) *by* (3.3). *The dimension of the covering is smaller or equal to* n_ξ.

ii) Under some regularity assumptions (see [24]), if a system is covered by a flat system, then it is flat. The dimension of the covering can be finite or infinite. [1]

[1] It has been suggested that the proof of Theorem 6 in [24] contains an arguable step. In a private communication, the author of [24] sent a correction for his proof.

Proof. i) see Theorem 4 and *ii)* Theorem 6, both in [24]. □

Note that if the covered system has less inputs than the system covering it, then the dimension of the covering is infinite.
An interesting consequence of the previous Proposition is that given a flat system system (3.3), the flatness of (3.3a) can be guaranteed by verifying the non-singularity of the dynamic feedback (3.3b), i.e., without checking that the feedback is endogenous.
Another consequence, which is actually the point of [24], is that a system linearizeable by a non-singular dynamic feedback is necessarily flat. In Section 3.4, we give an example of a system linearizable by a singular (static) feedback that is not flat. To the best of our knowledge, the conditions under which a system linearizable by singular dynamic feedback is flat are not known.

3.2 Finite Dimensional Tests

In this section, we go back to finite dimensional descriptions of control systems, as presented in Chapter 1. In particular, we are interested in the behavior of surjective bundle maps from the state space of some given system to another manifold, fibered over the same base $\mathcal{B}$, i.e sharing the same time variable t. We will see that there always exists a control system defined on the codomain of the surjective map such that all trajectories of the original system map to solutions of the codomain system. However, some cases are more interesting than others, in particular we will be interested in knowing whether the considered map induces a covering between the corresponding prolonged systems. Indeed, in this case, and in this case only, not only do trajectories from the first system map to trajectories of the second one, but any solution of the second system can be "lifted" to trajectories of the first one.
The desired information can be gathered from the properties of the kernel of the induced tangent map. If the distribution defined by the kernel satisfies the classical *control invariance* property [100, 62, 15], then Corollary 3.11 shows that the map induces a covering and the test requires no system prolongation. However, regarding flatness of the covered system, Proposition 3.12 shows that this simple situation is not "interesting" in some sense. If the covering system is static feedback linearizable, then the covered system is static feedback linearizable too. The more appealing situation of a statically feedback linearizable system covering a flat but not static feedback linearizable system cannot happen when the kernel of the tangent map is controlled invariant.
Besides controlled invariance, more general situations have been studied in many different ways. In [141], the system defined on the codomain of the studied surjective map is said to describe a quotient system. In [81, 114], the related question of decomposition of a system in so called cascades is investigated. Various other structures are proposed in [151]. In [75], vector fields of the tangent of the state-space manifold that may be lifted to symmetries of the control system are characterized; these symmetries and their properties are then used to decompose the system.
We propose a characterization of the surjective bundle map that we coin *dynamic controlled invariance* because of the similarity it bears with the criterion for controlled invariance, which it generalizes, and its natural link with (singular) dynamic feedback. The criterion

can be applied on the unprolonged system. However, verifying that the corresponding system decomposition induces a covering necessitates a finite number of input prolongations and the application of the *dynamic extension algorithm* [128].

We first state a technical result useful in proving both Propositions 3.9 on controlled invariance and 3.13 on dynamic controlled invariance.

Lemma 3.8. *Let M be an m-dimensional manifold, $y = \{y^1, \ldots, y^m\}$ a local system of coordinates on M and $D \in TM^*$ a vector field on M. Let also $\{v^1, \ldots, v^{m_v}\} \subset y$, $\{w^1, \ldots, w^{m_w}\} \subset y$ and $\{z^1, \ldots, z^{m_z}\} \subset y$ be three subsets of the coordinate set y. Assume that*

i) the vector fields $\left[\frac{\partial}{\partial w^1}, D\right], \ldots, \left[\frac{\partial}{\partial w^{m_w}}, D\right], \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^{m_z}}$ are all indenpendant

ii) there are functions α_i^j and β_i^k in $\mathcal{C}^\infty(M)$ such that

$$\left[\frac{\partial}{\partial v^i}, D\right] = \alpha_i^j \left[\frac{\partial}{\partial w^j}, D\right] + \beta_i^k \frac{\partial}{\partial z^k} \tag{3.13}$$

Then the m_v independent vector fields

$$X_i = \frac{\partial}{\partial v^i} - \alpha_i^j \frac{\partial}{\partial w^j}$$

commute, i.e. $[X_i, X_k] = 0$.

Proof. First note that for any two functions a^1, a^2, part of a local coordinate system on M, the Jacobi identity implies that the double bracket

$$\begin{aligned}\left[\frac{\partial}{\partial a^1}, \left[\frac{\partial}{\partial a^2}, D\right]\right] &= -\left[D, \left[\frac{\partial}{\partial a^1}, \frac{\partial}{\partial a^2}\right]\right] - \left[\frac{\partial}{\partial a^2}, \left[D, \frac{\partial}{\partial a^1}\right]\right] \\ &= \left[\frac{\partial}{\partial a^2}, \left[\frac{\partial}{\partial a^1}, D\right]\right]\end{aligned} \tag{3.14}$$

is symmetric in a^1 and a^2. Next use $\equiv_Z$ for the equality modulo Z, i.e. $A_1 \equiv_Z A_2$ iff there

exists a $z \in Z$ such that $A_1 = A_2 + z$ and expand the following double bracket

$$\begin{aligned}
\left[\frac{\partial}{\partial v^i}, \left[\frac{\partial}{\partial v^k}, D\right]\right] &\overset{(3.13)}{=} \left[\frac{\partial}{\partial v^i}, \alpha_k^j \left[\frac{\partial}{\partial w^j}, D\right] + \beta_k^l \frac{\partial}{\partial z^l}\right] \\
&\equiv_Z \left[\frac{\partial}{\partial v^i}, \alpha_k^j \left[\frac{\partial}{\partial w^j}, D\right]\right] \\
&= \alpha_k^j \left[\frac{\partial}{\partial v^i}, \left[\frac{\partial}{\partial w^j}, D\right]\right] + \frac{\partial \alpha_k^j}{\partial v^i} \left[\frac{\partial}{\partial w^j}, D\right] \\
&\overset{(3.14)}{=} \alpha_k^j \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial v^i}, D\right]\right] + \frac{\partial \alpha_k^j}{\partial v^i} \left[\frac{\partial}{\partial w^j}, D\right] \\
&\overset{(3.13)}{=} \alpha_k^j \left[\frac{\partial}{\partial w^j}, \alpha_i^r \left[\frac{\partial}{\partial w^r}, D\right] + \beta_i^s \frac{\partial}{\partial z^s}\right] + \frac{\partial \alpha_k^j}{\partial v^i} \left[\frac{\partial}{\partial w^j}, D\right] \\
&\equiv_Z \alpha_k^j \left[\frac{\partial}{\partial w^j}, \alpha_i^r \left[\frac{\partial}{\partial w^r}, D\right]\right] + \frac{\partial \alpha_k^j}{\partial v^i} \left[\frac{\partial}{\partial w^j}, D\right] \\
&= \alpha_k^j \alpha_i^r \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial w^r}, D\right]\right] + \alpha_k^j \frac{\partial \alpha_i^r}{\partial w^j} \left[\frac{\partial}{\partial w^r}, D\right] + \frac{\partial \alpha_k^j}{\partial v^i} \left[\frac{\partial}{\partial w^j}, D\right] \\
&= \alpha_k^j \alpha_i^r \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial w^r}, D\right]\right] + \left(\alpha_k^j \frac{\partial \alpha_i^r}{\partial w^j} + \frac{\partial \alpha_k^r}{\partial v^i}\right) \left[\frac{\partial}{\partial w^r}, D\right] \qquad (3.15)
\end{aligned}$$

The first term in the last line of (3.15) is symmetric in i, k:

$$\begin{aligned}
\alpha_k^j \alpha_i^r \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial w^r}, D\right]\right] &\overset{(3.14)}{=} \alpha_k^j \alpha_i^r \left[\frac{\partial}{\partial w^r}, \left[\frac{\partial}{\partial w^j}, D\right]\right] \\
\overset{j \leftrightarrow r}{=} \alpha_k^r \alpha_i^j \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial w^r}, D\right]\right] &= \alpha_i^j \alpha_k^r \left[\frac{\partial}{\partial w^j}, \left[\frac{\partial}{\partial w^r}, D\right]\right]
\end{aligned}$$

Again by the Jacobi identity (3.14), the whole expression (3.15) is symmetric in the indices i, k. Therefore, one obtains the relation $(3.15)_{i,k} - (3.15)_{k,i} \equiv_Z 0$ which reads

$$\left(\alpha_k^j \frac{\partial \alpha_i^r}{\partial w^j} - \alpha_i^j \frac{\partial \alpha_k^r}{\partial w^j} + \frac{\partial \alpha_k^r}{\partial v^i} - \frac{\partial \alpha_i^r}{\partial v^k}\right) \left[\frac{\partial}{\partial w^r}, D\right] \equiv_Z 0.$$

and by assumption *i)*, this implies

$$\alpha_k^j \frac{\partial \alpha_i^r}{\partial w^j} - \alpha_i^j \frac{\partial \alpha_k^r}{\partial w^j} + \frac{\partial \alpha_k^r}{\partial v^i} - \frac{\partial \alpha_i^r}{\partial v^k} = 0. \qquad (3.16)$$

Finally, compute the bracket

$$\begin{aligned}
[X_k, X_i] &= \left[\frac{\partial}{\partial v^k} - \alpha_k^r \frac{\partial}{\partial w^r}, \frac{\partial}{\partial v^i} - \alpha_i^j \frac{\partial}{\partial w^j}\right] \\
&\quad - \left[\frac{\partial}{\partial v^k}, \alpha_i^j \frac{\partial}{\partial w^j}\right] + \left[\frac{\partial}{\partial v^i}, \alpha_k^r \frac{\partial}{\partial w^r}\right] + \left[\alpha_k^r \frac{\partial}{\partial w^r}, \alpha_i^j \frac{\partial}{\partial w^j}\right] \\
&= \left(\alpha_k^j \frac{\partial \alpha_i^r}{\partial w^j} - \alpha_i^j \frac{\partial \alpha_k^r}{\partial w^j} + \frac{\partial \alpha_k^r}{\partial v^i} - \frac{\partial \alpha_i^r}{\partial v^k}\right) \frac{\partial}{\partial w^r} \overset{(3.16)}{=} 0.
\end{aligned}$$

□

3.2.1 Controlled Invariance

Throughout this section, we consider the bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ with adapted coordinates $\pi_{\mathcal{UM}} : (t, x, u) \mapsto (t, x)$ and the system described by equation (3.1). The manifold $\mathcal{M}$ itself has a bundle structure $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$ with adapted coordinates (t, x) and $\mathcal{B}$ is the time manifold with coordinate t. To simplify notations, we denote by $U \subset T\mathcal{U}$ the involutive *input distribution* defined as $U = \ker \pi_{\mathcal{UM}*}$. In the specified coordinates, $U = \{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}\}$. We shall also consider the vector field $D \in T\mathcal{U}$

$$D = \frac{\partial}{\partial t} + f^i(t, x, u)\frac{\partial}{\partial x^i}. \tag{3.17}$$

With the above choices, the Cartan distribution $\mathfrak{C}$ on $\mathcal{U}$ is spanned by

$$\mathfrak{C} = \{D\} + U.$$

A *controlled invariant distribution* Z is a (locally) constant dimensional involutive distribution in $T\mathcal{M}$ satisfying $Z \lrcorner\, dt = 0$ and such that there exist a lift $\hat{Z} \subset T\mathcal{U}$ and a vector field $\bar{D} \in \mathfrak{C}$ with

$$\pi_{\mathcal{UM}*}\hat{Z} = Z \quad \dim \hat{Z} = \dim Z \qquad \bar{D} \lrcorner\, dt = 1 \tag{3.18a}$$

$$\left[\bar{D}, \hat{Z}\right] \subset \hat{Z}. \tag{3.18b}$$

The next proposition is adapted from and generalizes Theorems 7.5 and 13.7 in [100]. See also e.g. [62].

Proposition 3.9. *Let $Z \subset T\mathcal{M}$ be an involutive ρ-dimensional distribution satisfying $Z \lrcorner\, dt = 0$ and let $U \in T\mathcal{U}$ be the input distribution. Then (locally), the following conditions are equivalent*

i) The distribution Z is controlled invariant.

ii) There are functions $y^i, z^k \in \mathcal{C}^\infty(\mathcal{M})$ and $v^q, w^p \in \mathcal{C}^\infty(\mathcal{U})$ such that $Z = \{\frac{\partial}{\partial z^k}\} \subset T\mathcal{M}$, (t, y, z) are coordinates on $\mathcal{M}$ and (t, y, z, v, w) are coordinates on $\mathcal{U}$. The Cartan distribution on $T\mathcal{U}$ is spanned by

$$\bar{D} = \frac{\partial}{\partial t} + g^k(t, z, y, v, w)\frac{\partial}{\partial z^k} + h^i(t, y, w)\frac{\partial}{\partial y^i} \quad \text{and} \quad \frac{\partial}{\partial v^q}, \ \frac{\partial}{\partial w^p}.$$

iii) The control system described by (3.1) is static feedback equivalent to the one described by

$$\dot{z}^k = g^k(t, z, y, v, w) \tag{3.19a}$$

$$\dot{y}^i = h^i(t, y, w) \tag{3.19b}$$

iv) Let $Z_i = \zeta_i^j \frac{\partial}{\partial x^j}$ with $\zeta_i^j \in \mathcal{C}^\infty(\mathcal{M})$ be a basis of the involutive distribution $Z \subset T\mathcal{M}$. With the coordinates (t, x, u) on $\mathcal{U}$, consider $Z \subset T\mathcal{U}$ the lift of $Z \subset T\mathcal{M}$ spanned by the (same) vectors $Z_i = (\pi^*_{\mathcal{UM}}\zeta_i^j)\frac{\partial}{\partial x^j} = \zeta_i^j \frac{\partial}{\partial x^j}$. The distribution $Z \subset T\mathcal{U}$ satisfies

$$[Z_i, D] \in [U, D] + U + Z \qquad i = 1, \dots, \rho. \tag{3.20}$$

Remark 3.10. In *iv)*, the criterion can be checked equivalently, replacing the proposed lift of $Z \subset T\mathcal{M}$ by any lift $\tilde{Z} \subset T\mathcal{U}$ satisfying $\dim \tilde{Z} = \dim Z$ and $\pi_{\mathcal{U}\mathcal{M}*}\tilde{Z} = Z \subset T\mathcal{M}$. To see this, replace Z_i in (3.20) by $Z_i + \rho_i^l \frac{\partial}{\partial u^l}$ for any $\rho_i^l \in \mathcal{C}^\infty(\mathcal{U})$.

Proof. We first show *i)* $\Rightarrow$ *iv)*. The assumption that $Z \subset T\mathcal{M}$ is controlled invariant implies that there are coordinates (t, z, y) on $\mathcal{M}$ and (t, z, y, u) on $\mathcal{U}$ such that $Z = \{\frac{\partial}{\partial z}\}$ and

$$\bar{D} = D + \gamma^l \frac{\partial}{\partial u^l} \qquad \hat{Z} = \left\{ \frac{\partial}{\partial z^i} + \epsilon_i^l \frac{\partial}{\partial u^l} \right\} \qquad \gamma^l, \epsilon_i^l \in \mathcal{C}^\infty(\mathcal{U}).$$

The condition $[\hat{Z}, \bar{D}] \subset \hat{Z}$ then implies that there exist $c_i^s \in \mathcal{C}^\infty(\mathcal{U})$ such that

$$\left[\frac{\partial}{\partial z^i} + \epsilon_i^l \frac{\partial}{\partial u^l}, D + \gamma^l \frac{\partial}{\partial u^l} \right] = c_i^s \left(\frac{\partial}{\partial z^s} + \epsilon_s^l \frac{\partial}{\partial u^l} \right)$$

$$\left[\frac{\partial}{\partial z^i}, D \right] = -\epsilon_i^l \left[\frac{\partial}{\partial u^l}, D \right] + \left(c_i^s \epsilon_s^l - \frac{\partial \gamma^l}{\partial z^i} + D(\epsilon_i^l) - \epsilon_i^p \frac{\partial \gamma^l}{\partial u^p} + \gamma^p \frac{\partial \epsilon_i^l}{\partial u^p} \right) \frac{\partial}{\partial u^l} + c_i^s \frac{\partial}{\partial z^s}$$

which implies *iv)*.
Next we show *iv)* $\Rightarrow$ *iii)*. By assumption, there are independent functions $z^1, \ldots, z^\rho$, $y^1, \ldots, y^{n-\rho}$ on $\mathcal{M}$ such that (t, z, y) are coordinates on $\mathcal{M}$, (t, z, y, u) are coordinates on $\mathcal{U}$ and $Z = \{\frac{\partial}{\partial z^i}\} \subset T\mathcal{M}$. With this choice of coordinates, the vectors $\frac{\partial}{\partial z^i} \in T\mathcal{U}$ are vectors lifted from the vectors $\frac{\partial}{\partial z^i} \in T\mathcal{M}$, indeed, $\pi_{\mathcal{U}\mathcal{M}*}\frac{\partial}{\partial z^i} = \frac{\partial}{\partial z^i}$. We shall let $Z \subset T\mathcal{U}$ denote the involutive distribution spanned by those lifted vectors $\frac{\partial}{\partial z^i}$. Set $\bar{m} \leq m$ such that $\dim([U, D] + U + Z) = \bar{m} + m + \rho$. One can split the set of input variables (previously rearranging them if necessary) as $\bar{u}^1 = u^1, \ldots, \bar{u}^{\bar{m}} = u^{\bar{m}}$ and $\tilde{u}^1 = u^{\bar{m}+1}, \ldots, \tilde{u}^{m-\bar{m}} = u^m$ such that

$$\left[\frac{\partial}{\partial u}, D \right] + U + Z = \left[\frac{\partial}{\partial \bar{u}}, D \right] + U + Z \qquad \left[\frac{\partial}{\partial \bar{u}}, D \right] \cap (U + Z) = 0. \tag{3.21}$$

Now, conditions (3.20) and (3.21) imply that there are functions $\alpha_i^l, \beta_i^s \in \mathcal{C}^\infty(\mathcal{U})$, $i \in 1 \ldots \rho$, $l \in 1 \ldots \bar{m}$ such that

$$\left[\frac{\partial}{\partial z^i}, D \right] = \alpha_i^l \left[\frac{\partial}{\partial \bar{u}^l}, D \right] + \beta_i^p \frac{\partial}{\partial u^p} + \beta_i^{m+k} \frac{\partial}{\partial z^k} \tag{3.22}$$
$$i, k = 1 \ldots \rho \quad l = 1 \ldots \bar{m} \quad p = 1 \ldots m.$$

Set the three bases of Lemma 3.8 as $\{\frac{\partial}{\partial z^k}\}$, $\{\frac{\partial}{\partial \bar{u}^l}\}$ and $\{\frac{\partial}{\partial u^p}, \frac{\partial}{\partial z^k}\}$ respectively. Condition *i)* of the lemma is satisfied because of (3.21). Relation (3.22) satisfies the condition *ii)*. Hence we conclude that the vector fields

$$X_i := \frac{\partial}{\partial z^i} - \alpha_i^l \frac{\partial}{\partial \bar{u}^l} \tag{3.23}$$

commute, i.e.

$$[X_i, X_k] = 0. \tag{3.24}$$

Note that the vectors $X_i \in T\mathcal{U}$ are (other) lifts of $\frac{\partial}{\partial z^i} \in T\mathcal{M}$ to $T\mathcal{U}$, indeed $\pi_{\mathcal{U}\mathcal{M}*}X_i = \pi_{\mathcal{U}\mathcal{M}*}\frac{\partial}{\partial z^i} = \frac{\partial}{\partial z^i}$. Now, recall that $y^1, \dots y^{n-\rho}$ are coordinates on $\mathcal{U}$ such that $\{dt, dy, d\bar{u}, d\tilde{u}\} =\perp_{T\mathcal{U}} Z$. Compute

$$\left[\frac{\partial}{\partial z^i}, D\right] y^j = \frac{\partial(Dy^j)}{\partial z^i} - D\frac{\partial y^j}{\partial z^i} = \frac{\partial(Dy^j)}{\partial z^i} \tag{3.25}$$

$$\overset{(3.22)}{=} \alpha_i^l \left[\frac{\partial}{\partial \bar{u}^l}, D\right] y^j$$

$$= \alpha_i^l \left(\frac{\partial(Dy^j)}{\partial \bar{u}^l} - D\frac{\partial y^j}{\partial \bar{u}^l}\right)$$

$$= \alpha_i^l \frac{\partial(Dy^j)}{\partial \bar{u}^l} \tag{3.26}$$

so as to obtain from (3.25), (3.26) and (3.23) that

$$0 = \left(\frac{\partial}{\partial z^i} - \alpha_i^l \frac{\partial}{\partial \bar{u}^l}\right) Dy^j = X_i \lrcorner\, dDy^j \tag{3.27}$$

Set $X \subset T\mathcal{U}$, $X = \{X_i\}$ and note that $\dim X = \dim Z = \rho$. Define the $\bar{m}$ independent 1-forms $\bar{\nu}^q := \alpha_r^q dz^r + d\bar{u}^q$, $q = 1 \dots \bar{m}$ and let the codistribution $J \subset T\mathcal{U}^*$ be given by $J = \{dt, dy^j, \bar{\nu}^q, d\tilde{u}^p\}$. Clearly, $X_i \lrcorner\, dt = 0$, $X_i \lrcorner\, dy^j = 0$, $X_i \lrcorner\, \nu^q = 0$, and $X_i \lrcorner\, d\tilde{u}^p = 0$. Hence

$$\dim T\mathcal{U} = n + m + 1 \qquad \dim J = n + 1 - \rho + m \qquad \dim X = \rho.$$
$$\perp_{T\mathcal{U}} J = X \tag{3.28}$$

It now follows from (3.24) and (3.28) that the codistribution J is completely integrable, i.e. there are $\bar{v}^q \in \mathcal{C}^\infty(\mathcal{U})$ such that

$$J = \{dt, dy^j, d\bar{v}^q, d\tilde{u}^p\} \qquad \operatorname{rank} \frac{\partial \bar{v}}{\partial \bar{u}} = \bar{m}$$

with the rank property following from the form of $\bar{\nu}^q$. From (3.27) we also see that

$$dDy^k = d\dot{y}^k \in J.$$

Hence, we have obtained adapted coordinates $(t, z, y, \bar{v}, \tilde{u})$ on $\mathcal{U}$ with (t, z, y) coordinates on $\mathcal{M}$ in which $\dot{y}^j$ can be expressed as functions independent of z^i. It follows that we can rearrange the new input variables $\bar{v}, \tilde{u}$ in two sets v, w and that in these coordinates, the vector field $\bar{D}$ can be taken of the form

$$\frac{\partial}{\partial t} + g^k(t, z, y, v, w)\frac{\partial}{\partial z^k} + h^i(t, y, w)\frac{\partial}{\partial y^i}$$

That *iii)* $\Rightarrow$ *ii)* should be clear from the discussion of Chapter 1.
We now show *ii)* $\Rightarrow$ *i)*. Let ϕ be the invertible bundle map $\phi : \mathcal{U} \to \mathcal{U}$ such that

$$x^i = \phi_x^i(t, z, y) \qquad u^l = \phi_u^l(t, z, y, v, w).$$

Note that

$$\begin{aligned}\phi_*\bar{D} = \frac{\partial}{\partial t} &+ \frac{\partial \phi_x^i}{\partial t}\frac{\partial}{\partial x^i} + \frac{\partial \phi_u^l}{\partial t}\frac{\partial}{\partial u^l} \\ &+ g^k \frac{\partial \phi_x^i}{\partial z^k}\frac{\partial}{\partial x^i} + g^k \frac{\partial \phi_u^l}{\partial z^k}\frac{\partial}{\partial u^l} \\ &+ h^j \frac{\partial \phi_x^i}{\partial y^j}\frac{\partial}{\partial x^i} + h^j \frac{\partial \phi_u^l}{\partial y^j}\frac{\partial}{\partial u^l} \\ = D &+ \gamma^l \frac{\partial}{\partial u^l}\end{aligned}$$

with

$$\gamma^l = \frac{\partial \phi_u^l}{\partial t} + g^k \frac{\partial \phi_u^l}{\partial z^k} + h^j \frac{\partial \phi_u^l}{\partial y^j}.$$

The expression of $\bar{D}$ from *ii)* shows that

$$\left[\frac{\partial}{\partial z^i}, \bar{D}\right] = \frac{\partial g^k}{\partial z^i}\frac{\partial}{\partial z^k} \subset \left\{\frac{\partial}{\partial z}\right\}.$$

Hence

$$\begin{aligned}\phi_*\left[\frac{\partial}{\partial z^i}, \bar{D}\right] &= \left[\phi_*\frac{\partial}{\partial z^i}, \phi_*\bar{D}\right] \\ &= \left[\phi_*\frac{\partial}{\partial z^i}, D + \gamma^l\frac{\partial}{\partial u^l}\right] \subset \left\{\phi_*\frac{\partial}{\partial z}\right\}.\end{aligned}$$

But

$$\phi_*\frac{\partial}{\partial z^i} = \frac{\partial \phi_x^s}{\partial z^i}\frac{\partial}{\partial x^s} + \frac{\partial \phi_u^l}{\partial z^i}\frac{\partial}{\partial u^l}$$

so that we clearly have

$$\pi_{\mathcal{U}\mathcal{M}*}\left\{\phi_*\frac{\partial}{\partial z^i}\right\} = \left\{\frac{\partial \phi_x^s}{\partial z^i}\frac{\partial}{\partial x^s}\right\} = \left\{\phi_{x*}\frac{\partial}{\partial z^i}\right\} = Z \subset T\mathcal{M}.$$

Hence $\hat{Z} = \phi_*\{\frac{\partial}{\partial z}\} \subset T\mathcal{U}$ is the sought after lift of $Z \subset T\mathcal{M}$ and $\phi_*\bar{D}$ the sought after element of $\mathfrak{C}$. □

An easy consequence of Proposition 3.9 is that controlled invariance implies that there is a subsystem described by the equations (3.19b) that is covered by the complete system (3.19), and therefore also covered by the static feedback equivalent system (3.1):

Corollary 3.11. *Let everything be as in Proposition 3.9. Then the system* (3.19) *covers the system* (3.19b).

Proof. Consider the infinite prolongation $\hat{\bar{D}} = \bar{D} + v^{l(s+1)}\frac{\partial}{\partial u^{s(l)}} + w^{r(p+1)}\frac{\partial}{\partial w^{r(p)}}$ and the vector field $\hat{\bar{D}}_y = \frac{\partial}{\partial t} + h^i(t,y,w)\frac{\partial}{\partial y^i} + w^{r(p+1)}\frac{\partial}{\partial w^{r(p)}}$. The map $\phi : (t,z,y,v,w,\dot{v},\dot{w},\ldots) \mapsto (t,y,w,\dot{w},\ldots)$ is such that $\phi_*\hat{\bar{D}} = \hat{\bar{D}}_y$. □

Corollary 3.11 together with Proposition 3.7 say that if one wants to construct a flat system, one may simply consider a controllable linear system and an arbitrary controlled invariant distribution Z. Then, any surjective map π such that $\ker \pi_* = Z$ will "project" the linear system to a smaller flat system. However, as the next proposition shows, nothing very interesting can happen because the obtained flat system is necessarily static feedback linearizable.

Proposition 3.12. *Let the conditions of Proposition 3.9 be satisfied and additionally assume that the control system described by* (3.1) *is static feedback linearizable. Then the system described by* (3.19b) *is statically feedback linearizable as well.*

Proof. By Proposition 3.9, there are coordinates (t, z, y, v, w) on $\mathcal{U}$ such that the Cartan distribution $\mathfrak{C}$ is spanned by

$$\mathfrak{C} = \{\frac{\partial}{\partial t} + g^k(t, z, y, v, w)\frac{\partial}{\partial z^k} + h^i(t, y, w)\frac{\partial}{\partial y^i},\ \frac{\partial}{\partial v^q},\ \frac{\partial}{\partial w^p}\}$$

and the Cartan codistribution Ω by

$$\Omega = \{dz^k - g^k(t, z, y, v, w)dt,\ dy^i - h^i(t, y, w)dt\}.$$

From the same result, we may define a surjective map $\pi : \mathcal{U} \to \mathcal{V}$ given in coordinates by $\pi : (t, z, y, v, w) \mapsto (t, y, w)$ where (t, y, w) are coordinates on $\mathcal{V}$. The equations (3.19b) describe a system on $\mathcal{V} \to \mathcal{N}$ with Cartan distribution

$$\mathfrak{C}_w = \{\frac{\partial}{\partial t} + h^i(t, y, w)\frac{\partial}{\partial y^i},\ \frac{\partial}{\partial w^p}\}$$

and Cartan codistribution

$$\Omega_w = \{dy^i - h^i(t, y, w)dt\}.$$

We also see that $\ker \pi_* = \{\frac{\partial}{\partial z^k},\ \frac{\partial}{\partial v^q}\}$ and that $\pi_*\mathfrak{C} = \mathfrak{C}_w$ in the sense of Section A.2.2. Hence, by Lemma A.9*ii)*, the two sequences defined by

$$\mathfrak{C}^{(0)} = \mathfrak{C} \qquad \mathfrak{C}^{(r+1)} = \mathfrak{C}^{(r)} + [\mathfrak{C}^{(r)}, \mathfrak{C}^{(r)}]$$

$$\mathfrak{C}_w^{(0)} = \mathfrak{C}_w \qquad \mathfrak{C}_w^{(r+1)} = \mathfrak{C}_w^{(r)} + [\mathfrak{C}_w^{(r)}, \mathfrak{C}_w^{(r)}]$$

are such that $\pi_*\mathfrak{C}^{(r)} = \mathfrak{C}_w^{(r)}$ for all $r \geq 0$. And by Lemma A.13 the two sequences

$$\Omega^{(1)} = \Omega \qquad \Omega^{(r+1)} := \{\omega \in \Omega^{(r)} \mid d\omega \in \Omega^{(r)}\}$$

$$\Omega_w^{(1)} = \Omega_w \qquad \Omega_w^{(r+1)} := \{\omega \in \Omega_w^{(r)} \mid d\omega \in \Omega_w^{(r)}\}$$

are such that $\Omega^{(r)} = \perp_{T\mathcal{U}} \mathfrak{C}^{(r)}$ and $\Omega_w^{(r)} = \perp_{T\mathcal{V}} \mathfrak{C}_w^{(r)}$ for all $r \geq 0$. Define the two distributions $K \subset T\mathcal{U}$ and $K_w \subset T\mathcal{V}$ such that $K = \perp_{T\mathcal{U}^*} dt = \perp_{T\mathcal{U}^*} \pi^* dt$ and $K_w = \perp_{T\mathcal{V}^*} dt$. Clearly $\pi_* K = K_w$. Since the system on $\mathcal{U}$ is static feedback linearizable, $\Omega^{(r)} + \{dt\}$ is integrable for all $r \geq 0$ by Proposition 1.8 and it follows that $\mathfrak{C}^{(r)} \cap K = \perp_{T\mathcal{U}^*} (\Omega^{(r)} + \{dt\})$ is involutive.

One may next verify that $\ker \pi_* \subset K$ implies that $\pi_*(\mathfrak{C}^{(r)} \cap K) = (\pi_*\mathfrak{C}^{(r)}) \cap (\pi_* K)$. Therefore, again by Lemma A.9*ii)*, $\pi_*(\mathfrak{C}^{(r)} \cap K) = \mathfrak{C}_w^{(r)} \cap K_w$ is involutive. But $C_w^{(r)} \cap K_w = \perp_{T\mathcal{V}^*} (\Omega_w^{(r)} + \{dt\})$ so that $\Omega_w^{(r)} + \{dt\}$ is integrable for all $r \geq 0$. The result follows by invoking Proposition 1.8 once more. □

3.2.2 Dynamic Controlled Invariance

Assume an involutive distribution $Z \subset T\mathcal{M}$ of constant dimension ρ is given but that the conditions of Proposition 3.9 are not satisfied. That is, Z is not controlled invariant. Then, one may still consider the surjective maps π with domain $\mathcal{M}$ and $\ker \pi_* = Z$. It is also still possible to lift such a map to a map $\hat{\pi} : \mathcal{U} \to \mathcal{V}$ with $\pi_{\mathcal{VN}} : \mathcal{V} \to \mathcal{N}$ and $\mathcal{N} \equiv \pi(\mathcal{M})$. Moreover, there are adapted coordinates such that

$$\pi_{\mathcal{UM}} : (t, y, z, u) \mapsto (t, y, z) \qquad \pi_{\mathcal{VN}} : (t, y, \kappa) \mapsto (t, y)$$
$$\pi : (t, y, z) \mapsto (t, y) \qquad \hat{\pi} : (t, y, z, u) \mapsto (t, y, \kappa(t, y, z, u))$$

and $Z = \{\frac{\partial}{\partial z}\}$. The situation is summarized in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\hat{\pi}"] \arrow[d, "\pi_{\mathcal{UM}}"] & & \mathcal{V} \arrow[d, "\pi_{\mathcal{VN}}"'] \\
\mathcal{M} \arrow[rr, "\pi"] \arrow[dr, "\pi_{\mathcal{MB}}"'] & & \mathcal{N} \arrow[dl, "\pi_{\mathcal{NB}}"] \\
& \mathcal{B} &
\end{tikzcd}$$

On the bundle $\mathcal{V}$, one may also define a control system such that any solution $t \mapsto \hat{\sigma}(t)$ of the system on $\mathcal{U}$ leads to a solution $t \mapsto \hat{\pi}(\hat{\sigma}(t))$ of the system on $\mathcal{V}$. However, in general, the converse is lost, i.e. a given solution of the system on $\mathcal{V}$ is not necessarily the projection of a solution on $\mathcal{U}$. Indeed, in the map $\hat{\pi}$, the functions κ are only such that

$$\operatorname{rank} \frac{\partial \kappa}{\partial u} \leq \operatorname{card} \kappa.$$

In [141], the system obtained on $\mathcal{V}$ has been coined a *quotient system* of the original one. We use the term *dynamic controlled invariance* for the similarity the criterion of the next proposition bears with the test for controlled invariance and its natural link to (singular) dynamic feedback. One should also stress the fact that in the following, the obtained system is not necessarily covered by the original one. To decide this, additional tests need to be performed. Indeed, one has to verify that the corresponding dynamic feedback is non-singular.
Let us reconsider the set of coordinates such that

$$\pi_{\mathcal{UM}} : (t, x, u) \mapsto (t, x).$$

The following is a weaker form of the controlled invariance criterium of Proposition 3.9.

Proposition 3.13. *Let $Z = \{\zeta^i_j \frac{\partial}{\partial x^i}\} \subset T\mathcal{M}$ be an involutive ρ-dimensional distribution satisfying $Z \lrcorner\, dt \equiv 0$. Let also $Z \subset T\mathcal{U}$ denote the distribution spanned by $\{\pi^*_{\mathcal{UM}}(\zeta^i_j)\frac{\partial}{\partial x^i}\} = \{\zeta^i_j \frac{\partial}{\partial x^i}\}$. Consider $D \in T\mathcal{U}$, the vector field given by* (3.17) *and $U \in T\mathcal{U}$, the input distribution spanned by $\{\frac{\partial}{\partial u^l} \mid l = 1 \ldots m\}$. Assume the distributions*

$$[U, D] + U + [Z, D] + Z, \qquad U + Z \quad \text{and} \qquad Z$$

have constant dimensions around some point of $\mathcal{U}$*. Then, around that point, the following conditions are equivalent*

i) The integrable codistribution $Y = \perp_{T\mathcal{M}} Z$*, satisfies* $\dim(Y + DY) = \dim Y + s$

ii) There are $n - \rho$ *functions* y^i *such that* (t, y, z) *are coordinates on* $\mathcal{M}$ *and independent functions* $\kappa^1, \ldots, \kappa^s \in \mathcal{C}^\infty(\mathcal{U})$*; in these coordinates,* $Z = \{\frac{\partial}{\partial z^k}\}$ *and the vector field* $D \in T\mathcal{U}$ *can be rewritten as*

$$D = \frac{\partial}{\partial t} + g^k(t, z, y, u)\frac{\partial}{\partial z^k} + h^i(t, y, \kappa(t, y, z, u))\frac{\partial}{\partial y^i}$$

$$\operatorname{rank}\frac{\partial \kappa}{\partial u} = s_u \qquad \operatorname{rank}\frac{\partial \kappa}{\partial z} = s_z \qquad \operatorname{rank}\frac{\partial h}{\partial \kappa} = s = s_u + s_z.$$

iii) The distribution Z *satisfies*

$$\dim\big(\ ([U, D] + U + [Z, D] + Z) \mod (U + Z)\ \big) = s. \tag{3.29}$$

Moreover, the largest choice for the number s_u *is given by*

$$s_u = \dim\big(\ ([U, D] + U + Z) \mod (U + Z)\ \big).$$

Remark 3.14. There is no (static) feedback involved, only a change of coordinates $(t, x) \mapsto (t, z, y)$ on $\mathcal{M}$, i.e. the system inputs u are kept unchanged.

Proof. The equivalence *i)* $\Leftrightarrow$ *ii)* should be clear. We now concentrate on *iii)* $\Rightarrow$ *ii)*. Define

$$s_u := \dim\big(\ ([U, D] + U + Z) \mod (U + Z)\ \big)$$
$$s_z := \dim\big(\ ([U, D] + U + [Z, D] + Z) \mod ([U, D] + U + Z)\ \big)$$

For any two $\mathbb{R}$-linear vector space A and B such that $B \subset A$, note that $\dim(A \mod B) = \dim A - \dim B$. Hence

$$s_u + s_z = s.$$

By assumption, there are functions $z^1, \ldots, z^\rho$ (part of a coordinate system on $\mathcal{M}$) and input variables $u^1, \ldots, u^m$ (part of a coordinate system on $\mathcal{U}$) such that $U = \{\frac{\partial}{\partial u^t}\}$ and $Z = \{\frac{\partial}{\partial z^i}\}$. The numbers s_u and s_z are such that (possibly rearranging the lists of variables), splitting the u and z variables each in two sets $(\bar{u}, \tilde{u})$ and $(\bar{z}, \tilde{z})$ as

$$\bar{u}^1 = u^1, \ldots, \bar{u}^{s_u} = u^{s_u} \qquad \tilde{u}^1 = u^{s_u+1}, \ldots, \tilde{u}^{m-s_u} = u^m$$
$$\bar{z}^1 = z^1, \ldots, \bar{z}^{s_z} = z^{s_z} \qquad \tilde{z}^1 = z^{s_z+1}, \ldots, \tilde{z}^{\rho-s_z} = z^\rho$$

leads to a basis of the distribution $[U, D] + [Z, D] + U + Z$ given by the independent elements

$$\Big\{ \left[\frac{\partial}{\partial \bar{u}^1}, D\right], \ldots, \left[\frac{\partial}{\partial \bar{u}^{s_u}}, D\right], \left[\frac{\partial}{\partial \bar{z}^1}, D\right], \ldots, \left[\frac{\partial}{\partial \bar{z}^{s_z}}, D\right], \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^m}, \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^\rho} \Big\} \tag{3.30}$$

For the brackets involving the remaining variables $\tilde{u}, \tilde{z}$, there must exists relations of the form

$$\left[\frac{\partial}{\partial \tilde{u}^w}, D\right] = a_w^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + b_w^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right] + c_w^p \frac{\partial}{\partial u^p} + e_w^i \frac{\partial}{\partial z^i} \tag{3.31}$$

$$\left[\frac{\partial}{\partial \tilde{z}^v}, D\right] = \alpha_v^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + \beta_v^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right] + \gamma_v^p \frac{\partial}{\partial u^p} + \epsilon_v^i \frac{\partial}{\partial z^i}. \tag{3.32}$$

Considering the three sets of variables $\{\tilde{u}^w, \tilde{z}^v\}$, $\{\bar{u}^k, \bar{z}^h\}$ and $\{u^p, z^i\}$, and using Lemma 3.8, we may conclude that the vector fields defined by

$$V_l := \frac{\partial}{\partial \tilde{u}^l} - a_l^k \frac{\partial}{\partial \bar{u}^k} - b_l^h \frac{\partial}{\partial \bar{z}^h}$$
$$X_i := \frac{\partial}{\partial \tilde{z}^i} - \alpha_i^k \frac{\partial}{\partial \bar{u}^k} - \beta_i^h \frac{\partial}{\partial \bar{z}^h}$$

commute, that is

$$[X_i, X_j] = 0 \qquad [X_i, V_l] = 0 \qquad [V_l, V_p] = 0. \tag{3.33}$$

Next, define the independent 1-forms

$$\mu^k := d\bar{u}^k + a_l^k d\tilde{u}^l + \alpha_i^k d\tilde{z}^i$$
$$\nu^h := d\bar{z}^h + b_l^h d\tilde{u}^l + \beta_i^h d\tilde{z}^i$$

Clearly,

$$V_l \lrcorner\, \mu^k = 0 \qquad V_l \lrcorner\, \nu^h = 0 \qquad X_i \lrcorner\, \mu^k = 0 \qquad X_i \lrcorner\, \nu^h = 0. \tag{3.34}$$

Now, choose $y^1, \dots y^{n-\rho}$ independent coordinates on $\mathcal{M}$ such that $\{dt, dy^j\} = \perp_{T\mathcal{M}} Z$. One has

$$V_l \lrcorner\, dt = 0 \qquad V_l \lrcorner\, dy^j = 0 \qquad X_i \lrcorner\, dt = 0 \qquad X_i \lrcorner\, dy^j = 0. \tag{3.35}$$

Define the codistribution $J := \{dt, dy^j, \mu^k, \nu^h\}$. Relations (3.34), (3.35) and a count of dimensions shows that

$$J = \perp_{T\mathcal{U}} \{V_l, X_i\}$$

and (3.33) imply that J is completely integrable. Therefore, there are independent functions $\upsilon^k, \zeta^h \in \mathcal{C}^\infty(\mathcal{U})$ such that

$$J = \{dt, dy^j, d\upsilon^k, d\zeta^h\} \qquad \operatorname{rank} \frac{\partial \upsilon}{\partial \bar{u}} = s_u \quad \operatorname{rank} \frac{\partial \zeta}{\partial \bar{z}} = s_z \tag{3.36}$$

where the rank properties follow from the form of μ^k, ν^h.
Now on one hand we have

$$\left[\frac{\partial}{\partial \tilde{u}^w}, D\right] y^j = \frac{\partial D y^j}{\partial \tilde{u}^w} = \frac{\partial}{\partial \tilde{u}^w} \lrcorner\, dDy^j \tag{3.37}$$

$$\left[\frac{\partial}{\partial \tilde{z}^v}, D\right] y^j = \frac{\partial D y^j}{\partial \tilde{z}^v} = \frac{\partial}{\partial \tilde{z}^v} \lrcorner\, dDy^j \tag{3.38}$$

and on the other hand

$$\begin{aligned}\left[\frac{\partial}{\partial \tilde{u}^w}, D\right] y^j &\overset{(3.31)}{=} \left(a_w^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + b_w^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right]\right) y^j \\ &= a_w^k \frac{\partial D y^j}{\partial \bar{u}^k} + b_w^h \frac{\partial D y^j}{\partial \bar{z}^h} \\ &= \left(a_w^k \frac{\partial}{\partial \bar{u}^k} + b_w^h \frac{\partial}{\partial \bar{z}^h}\right) \lrcorner\, dDy^j \end{aligned} \tag{3.39}$$

$$\begin{aligned}\left[\frac{\partial}{\partial \tilde{z}^v}, D\right] y^j &\overset{(3.32)}{=} \left(\alpha_v^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + \beta_v^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right]\right) y^j \\ &= \alpha_v^k \frac{\partial D y^j}{\partial \bar{u}^k} + \beta_v^h \frac{\partial D y^j}{\partial \bar{z}^h} \\ &= \left(\alpha_v^k \frac{\partial}{\partial \bar{u}^k} + \beta_v^h \frac{\partial}{\partial \bar{z}^h}\right) \lrcorner\, dDy^j. \end{aligned} \tag{3.40}$$

Computing the differences between the r.h.s. of (3.37) and (3.39) and between the r.h.s. of (3.38) and (3.40) shows that

$$V_l \lrcorner\, dDy^j = 0 \qquad X_i \lrcorner\, dDy^j = 0$$

which in turn implies

$$dDy^j = d\dot{y}^j \in J. \tag{3.41}$$

We now show that rank $\frac{\partial \dot{y}}{\partial \bar{u}} = s_u$ and rank $\frac{\partial \dot{y}}{\partial \bar{z}} = s_z$. Assume rank $\frac{\partial \dot{y}}{\partial \bar{u}} +$ rank $\frac{\partial \dot{y}}{\partial \bar{z}} < s_u + s_z = s$. This implies that there exist functions $\phi^k, \varphi^h \in \mathcal{C}^\infty(\mathcal{U})$, not all zero, such that

$$\phi^k \frac{\partial \dot{y}^j}{\partial \bar{u}^k} + \varphi^h \frac{\partial \dot{y}^j}{\partial \bar{z}^h} = 0.$$

This can be rewritten as

$$\begin{aligned} 0 &= \phi^k \frac{\partial D y^j}{\partial \bar{u}^k} + \varphi^h \frac{\partial D y^j}{\partial \bar{z}^h} \\ &= \phi^k \left(\frac{\partial D y^j}{\partial \bar{u}^k} - D \frac{\partial y^j}{\partial \bar{u}^k}\right) + \left(\varphi^h \frac{\partial D y^j}{\partial \bar{z}^h} - D \frac{\partial y^j}{\partial \bar{z}^h}\right) \\ &= \Bigg(\underbrace{\phi^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + \varphi^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right]}_{S}\Bigg) \lrcorner\, dy^j \end{aligned} \tag{3.42}$$

Clearly, we also have that $S \lrcorner\, dt = 0$. But $\{dt, dy^j\} = \perp_{T\mathcal{U}} \{\frac{\partial}{\partial u^l}, \frac{\partial}{\partial z^i}\}$, so that $S \in \{\frac{\partial}{\partial u^l}, \frac{\partial}{\partial z^i}\}$. Therefore, there are functions $\theta^l, \vartheta^i \in \mathcal{C}^\infty(\mathcal{U})$ satisfying the nontrivial relation

$$\phi^k \left[\frac{\partial}{\partial \bar{u}^k}, D\right] + \varphi^h \left[\frac{\partial}{\partial \bar{z}^h}, D\right] + \theta^l \frac{\partial}{\partial u^l} + \vartheta^i \frac{\partial}{\partial z^i} = 0$$

which contradicts the fact that (3.30) is a basis of independent elements. Hence indeed

$$\operatorname{rank}\frac{\partial \dot{y}}{\partial \bar{u}} + \operatorname{rank}\frac{\partial \dot{y}}{\partial \bar{z}} = s_u + s_z = s$$

and because card $\bar{u} = s_u$ and card $\bar{z} = s_z$

$$\operatorname{rank}\frac{\partial \dot{y}}{\partial \bar{u}} = s_u \qquad \operatorname{rank}\frac{\partial \dot{y}}{\partial \bar{z}} = s_z.$$

Lastly, from (3.36) and (3.41) and setting $\kappa^1 = \upsilon^1, \ldots, \kappa^{s_u} = \upsilon^{s_u}, \kappa^{s_u+1} = \zeta^1, \ldots, \kappa^s = \zeta^{s_z}$, we conclude that there are functions h^j such that

$$\dot{y}^j = h^j(t, y, \kappa) \qquad \kappa^p = \kappa^p(t, y, z, u)$$
$$\operatorname{rank}\frac{\partial h}{\partial \kappa} = s \qquad \operatorname{rank}\frac{\partial \kappa}{\partial u} = s_u \qquad \operatorname{rank}\frac{\partial \kappa}{\partial z} = s_z.$$

The converse statement *ii)* $\Rightarrow$ *iii)* is easily verified. □

3.2.3 Differentially Independent Codistributions

Proposition 3.13 states the conditions that need to be satisfied for the existence of coordinates on $\mathcal{M}$ such that the dynamics (3.1) take the form

$$\dot{z}^k = g^k(t, z, y, u) \tag{3.43a}$$

$$\dot{y}^i = h^i(t, y, \kappa(t, z, y, u)) \qquad \operatorname{rank}\frac{\partial h}{\partial \kappa} = s. \tag{3.43b}$$

However, in contrast to the case of controlled invariance, the conditions of Proposition 3.13 are not sufficient to conclude that the system described by (3.43) covers the system (3.43b) described by $\dot{y}^i = h^i(t, y, \kappa)$ with κ as input. Indeed, any solution $z(t), y(t), u(t)$ satisfying (3.43) leads to a solution of (3.43b). But given a solution $y(t), \tilde{\kappa}(t)$ of the equations

$$\dot{y}^i = h^i(t, y, \tilde{\kappa}) \tag{3.44}$$

there does not necessarily exist $z(t), u(t)$ such that $z(t), y(t), u(t)$ is a solution of (3.43) and $\tilde{\kappa} = \kappa(t, z, y, u)$. Notice that the equations (3.43) can be seen as a dynamic feedback on the system with equations $\dot{y}^i = h^i(t, y, \kappa)$. By Proposition 3.7*i)*, if this dynamic feedback is non-singular, (3.43) covers (3.43b). In turn, by Lemma 3.3, if $\mathcal{A}^1$ is the differential module associated to (3.43), the dynamic feedback is non-singular if and only if $d\kappa^1, \ldots, d\kappa^s$ (locally) generate a free sub-module of $\mathcal{A}^1$.
These additional conditions can be checked by verifying the right invertibility of system (3.43) with $\kappa(t, z, y, u)$ as output, also characterized by the differential independence (to be defined next) of the codistribution $\{d\kappa\}$.

Form Chapter 1, recall that for system (3.1) and for all $k \geq 0$, we may defined the manifold $\mathcal{U}^k$ with (local) coordinates $(t, x, u, \ldots, u^{(k)})$ on which the kth input prolongation of the system is defined. This set of manifolds has a composite bundle structure, for $q > k$ we have the projection map $\pi_{\mathcal{U},qk} : \mathcal{U}^k \to \mathcal{U}^q$ with coordinate expression

$\pi_{\mathcal{U},qk} : (t, x, u, \ldots, u^{(k)}, \ldots, u^{(q)}) \mapsto (t, x, u, \ldots, u^{(k)})$.

We will say a set of 1-forms $\mu = (\mu^1, \ldots, \mu^s)$, $\mu^i \in \mathcal{U}^k$, or the codistribution it spans, is *differentially independent* around a point $p \in \mathcal{U}^k$ if there is a point $\hat{p} \in \mathcal{U}^\infty$ such that $\pi_{\mathcal{U},\infty k}(\hat{p}) = p$ and $\{\pi^*_{\mathcal{U},\infty k}\mu^1, \ldots, \pi^*_{\mathcal{U},\infty k}\mu^s\}$ is a basis of a free module around $\hat{p}$.

Clearly, identifying $\pi^*_{\mathcal{U},\infty k}\mu^i$ with μ^i and working on $\mathcal{U}^\infty$, the set μ is differentially independent if and only if $\dim\{\mu|_{\hat{p}}, \dot{\mu}|_{\hat{p}}, \ldots, \mu^{(r)}|_{\hat{p}}\} = s(r+1)$ as a $\mathbb{R}$-linear space for all $r \geq 0$.

We begin with an easy but useful fact.

Lemma 3.15. *Let $\mu = (\mu^1, \ldots, \mu^p)$ be a set of independent 1-forms and $\hat{\mu} = (\hat{\mu}^1, \ldots, \hat{\mu}^p) = (\mu^{1(r_1)}, \ldots, \mu^{p(r_p)})$ for some numbers $r_l \geq 0$. Then μ is differentially independent if and only if $\hat{\mu}$ is.*

Proof. From the definition, it is obvious that if μ is differentially independent, so is $\hat{\mu}$. Next suppose $r_1 = 1, r_2 = 0, \ldots, r_p = 0$, i.e. $\hat{\mu} = (\dot{\mu}^1, \mu^2, \ldots, \mu^p)$ and assume $\hat{\mu}$ is differentially independent but μ is not. Therefore $\mu^1 = \gamma_{l,r}\mu^{l(r)}$ with $l = 1, \ldots, p$, $r = 0, \ldots, R$ and $\gamma_{1,0} = 0$. Differentiating the relation once gives $\mu^1 = \dot{\gamma}_{l,r}\mu^{l(r)} + \gamma_{l,r}\mu^{l(r+1)}$. In the last relation, $\dot{\gamma}_{1,0} = \gamma_{1,0} = 0$, it is hence a relation on $\hat{\mu}$, a contradiction. The result follows by induction. □

The next Lemma is an extension to (not necessarily integrable) codistributions of the notion of static state feedback.

Lemma 3.16. *Assume $\nu = (\nu^1, \ldots, \nu^m)$ is a set of differentially independent 1-forms and set $U = \{\nu\}$. Let X be a (finite dimensional) codistribution satisfying*

$$X \cap U = 0 \qquad X + \dot{X} \subset X + U.$$

Then, any set $\bar{\nu}^1, \ldots, \bar{\nu}^m$ of m representatives of a basis for

$$(U + X)/X$$

is differentially independent.

Proof. Differentiating k times shows that

$$X + \dot{X} \subset X + U \quad \Rightarrow \quad X + \ldots + X^{(k)} \subset X + U + \ldots + U^{(k-1)} \tag{3.45}$$

Let $\chi^1, \ldots, \chi^n$ form a basis of X. By assumption, there exist two matrices α, β such that

$$\bar{\nu}^l = \alpha^l_s \nu^s + \beta^l_i \chi^i \qquad \operatorname{rank} \alpha = m.$$

Hence

$$\bar{\nu}^{l(k)} = \alpha^l_s \nu^{s(k)} + \sum_{r=1}^{k} \binom{k}{r} \alpha^{l(r)}_s \nu^{s(k-r)} + \sum_{q=0}^{k} \binom{k}{q} \beta^{l(q)}_i \chi^{i(k-q)}$$

such that by (3.45) that

$$\bar{\nu}^{l(k)} = \alpha_s^l \nu^{s(k)} \mod X + U + \ldots + U^{(k-1)} \quad \forall k \geq 1.$$

Hence, by the full rank of α and after setting $\bar{U} = \{\bar{\nu}\}$ we have

$$V^k := X + \bar{U} + \ldots + \bar{U}^{(k)} = X + U + \ldots + U^{(k)} \quad \forall k \geq 0.$$

Since X is finite dimensional, there must be a $\hat{k} \geq 0$ such that for all $p \geq 0$ $\dim V^{\hat{k}+p+1} - \dim V^{\hat{k}+p} = m$. This in turn implies that $\bar{\nu}^{1(\hat{k})}, \ldots, \bar{\nu}^{m(\hat{k})}$ are differentially independent. Hence, $\bar{\nu}^1, \ldots, \bar{\nu}^m$ are differentially independent by Lemma 3.15. □

3.2.3.1 Dynamic Extension Algorithm

Verifying differential independence of a set of forms μ from the definition requires testing the (non differential) independence of an infinite set of 1-forms. If time differentiation is defined by the equations of a control system in classical form $\dot{x} = f(t, x, u)$, and if we restrict the definition of μ to $\mu^i \in T\mathcal{M}^*$, i.e. to $\mu^i \in \{dt, dx\}$, then we can devise a finite test for the differential independence of μ, requiring at most $n = \operatorname{card} x$ prolongations on the system inputs u.
The test consists in an "infinitesimal" version of the *dynamic extension algorithm (DEA)* [127, 128, 99, 97, 37, 105] which tests the right-invertibility of a control system with specified outputs [118, 68, 67]. If a system with (classical) outputs $y^1(t, x), \ldots, y^s(t, x)$ is right-invertible, the dynamic extension algorithm produces a non-singular dynamic extension with the property that the extended system has $y^{1(l_1)}, \ldots, y^{s(l_s)}$ as a subset of its inputs for some $l_j > 0$. Note that the transformation from the system to its extended form also happens to be a quasi-static feedback, see [34]. By "infinitesimal", we shall mean that instead of verifying the right-invertibility of a system with (classical) output $y^1(t, x), \ldots, y^s(t, x)$, we verify differential independence of the 1-forms $dy^1, \ldots, dy^s$, which is the same. To do so, we describe a version of the DEA that is applied to the 1-forms dy^j instead of the functions y^j. This modified algorithm works without regard to the integrability of the set of 1-forms to which it is applied.

We now define the recursive procedure of the Dynamic Extension Algorithm for a set of not necessarily integrable 1-forms

$$\mu = (\mu^1, \ldots, \mu^s) \subset T\mathcal{M}^*. \tag{3.46}$$

For every $k \geq 0$ define

$$X_k = \{dt, dx, \mu, \ldots, \mu^{(k)}\} \tag{3.47}$$

where $\{dt, dx\} = T\mathcal{M}^*$ and

$$U_0 = \{du^1, \ldots, du^m\}$$

such that $X_0 + U_0 = T\mathcal{U}^*$. Note that from (3.46) we have $X_0 = \{dt, dx, \mu\} = \{dt, dx\}$. Also, U_0 may be taken as spanned by any set of m 1-forms such that $X_0 + U_0 = T\mathcal{U}^*$.

The algorithm successively produces $U_k \subset T\mathcal{U} + \dot{U} + \ldots + U^{(k)}$, for $k = 1, \ldots$ such that the following three conditions are satisfied for all k:

$$U_k \text{ is differentially independent} \tag{3.48a}$$

$$X_k \cap (U_k + \ldots + U_k^{(p)}) = 0 \quad \forall p \geq 0 \tag{3.48b}$$

$$\dot{X}_k \subset X_k + U_k. \tag{3.48c}$$

Lemma 3.17. Dynamic Extension Algorithm
Assume X_k and U_k satisfy the conditions (3.48). *Suppose U_k is spanned by $\{\nu\} = \{\nu^1, \ldots, \nu^m\}$. Choose any subset $\bar{\mu} \subset \mu$ with* $\operatorname{card} \bar{\mu}$ *minimal such that*

$$X_{k+1} = \{dt, dx, \mu, \ldots, \mu^{(k)}, \mu^{(k+1)}\} = \{dt, dx, \mu, \ldots, \mu^{(k)}, \bar{\mu}^{(k+1)}\}.$$

Set $\bar{\nu} = \bar{\mu}^{(k+1)}$ and take a minimal subset $\tilde{\nu} \subset \nu$ such that $X_k + \{\bar{\nu}, \tilde{\nu}\} = X_k + U_k$. Build U_{k+1} as $U_{k+1} = \{\dot{\bar{\nu}}, \tilde{\nu}\}$. Then, X_{k+1} and U_{k+1} satisfy the conditions (3.48) *where k is replaced by $k+1$.*

Remark 3.18. If one additionally assumes that $\{\mu\}$ is integrable, then in each step of the Dynamic Extension Algorithm, one may choose $\bar{\mu}$ such that $\{\bar{\mu}\}$ is integrable too. This then produces integrable codistributions X_k and U_k for all k.

Proof. The choice of $\bar{\nu}$ is such that

$$(\bar{\mu}^{(k+1)})^r = \bar{\alpha}_l^r \nu^l \mod X_k \qquad r = 1, \ldots, \operatorname{card} \bar{\mu} \quad \operatorname{rank} \bar{\alpha} = \operatorname{card} \bar{\mu}$$

Define $\hat{U}_k = \{\bar{\nu}, \tilde{\nu}\}$. By Lemma 3.16, the pair $X_k, \hat{U}_k$ also satisfy condition (3.48a) and by Lemma 3.15, so does U_{k+1}. By construction, the pair X_{k+1}, U_{k+1} also satisfies (3.48b) and (3.48c). □

Proposition 3.19. *Consider a set (or codistribution) μ as in* (3.46) *and the sequence X_k as in* (3.47). *Define the sequence of codistributions*

$$Y_k = \{\mu, \dot{\mu}, \ldots, \mu^{(k)}\}.$$

Then the following conditions are equivalent

i) μ *is differentially independent*

ii) $\dim Y_{k+1} - \dim Y_k = s \quad \forall k = 1, \ldots, n-1$

iii) $\dim Y_n = s(n+1)$

iv) $\dim X_n - \dim X_{n-1} = s$

Proof. The main idea for this proof is adapted from [105]. Apply k steps of the algorithm of Lemma 3.17. Since

$$X_{k+1} = X_k + \{\bar{\nu}\}$$

we have that

$$X_{k+1} + \{\dot{\bar{\nu}}\} \subset X_{k+2}$$

and by the assumption (3.48b), $\dot{\bar{\nu}}$ is independent of X_{k+1}, hence $\dim(X_{k+1} + \{\dot{\bar{\nu}}\}) = \dim X_{k+1} + \operatorname{card} \bar{\mu}$. Therefore $\dim X_{k+2} - \dim X_{k+1} \geq \operatorname{card} \bar{\mu}$, so that

$$\dim X_{k+1} - \dim X_k \leq \dim X_{k+2} - \dim X_{k+1}. \tag{3.49}$$

We now turn our attention to the sequence Y_k. Assume $\dim Y_{k+1} - \dim Y_k = r$ for some $k \geq 0$. Then, there is a subset $\bar{\mu} \subset \mu$ with $\operatorname{card} \bar{\mu} = r$ satisfying

$$Y_{k+1} = Y_k + \{\bar{\mu}^{(k+1)}\}.$$

Set $\tilde{\mu} \subset \mu$ such that $\{\bar{\mu}, \tilde{\mu}\} = \{\mu\}$. We have

$$\tilde{\mu}^{(k+1)} = 0 \mod Y_k + \{\bar{\mu}^{(k+1)}\}$$

and this implies, differentiating, that

$$\tilde{\mu}^{(k+2)} = 0 \mod Y_{k+1} + \{\bar{\mu}^{(k+1)}, \bar{\mu}^{(k+2)}\}$$

which implies that $\dim Y_{k+2} - \dim Y_{k+1} \leq r$ (i.e. there are at least $s - r$ more relations between the elements $\mu, \ldots, \mu^{(k+2)}$ as relations between the elements $\mu, \ldots, \mu^{(k+1)}$) . Hence

$$\dim Y_{k+1} - \dim Y_k \geq \dim Y_{k+2} - \dim Y_{k+1}. \tag{3.50}$$

For $k \geq 0$, define the two sequences of numbers

$$p_k = \dim X_{k+1} - \dim X_k \qquad q_k = \dim Y_{k+1} - \dim Y_k$$

From $\operatorname{card} \mu = s$ and from (3.49) and (3.50) we see that

$$0 \leq p_k \leq p_{k+1} \leq s \qquad s \geq q_k \geq q_{k+1} \geq 0$$

hence both sequences must stabilize for $k \leq k^*$ for some finite k^*. Since $X_k = \{dt, dx\} + Y_k$

$$\dim Y_k - \dim X_k = -\dim\{dt, dx\} + \dim(\{dt, dx\} \cap Y_k)$$

leading to

$$\begin{aligned}(\dim Y_{k+1} - \dim Y_k) - (\dim X_{k+1} - \dim X_k) \\ &= q_k - p_k \\ &= \dim(\{dt, dx\} \cap Y_{k+1}) - \dim(\{dt, dx\} \cap Y_k) \geq 0\end{aligned} \tag{3.51}$$

so that $q_k \geq p_k$. For any $\omega \in \{dt, dx\}$, remember that either ω is torsion and then $\omega^{(k)} \in \{dt, dx\}$ $\forall k \geq 0$ and $\omega^{(n)} \in \{dt, \omega, \ldots, \omega^{(n-1)}\}$, or there is a $k_\omega \leq n$ such that $\omega^{(k)} \notin \{dt, dx\}$ $\forall k \geq k_\omega$. Therefore

$$\begin{aligned}\{dt, dx\} \cap \{\omega, \ldots, \omega^{(n-1)}\} = \{dt, dx\} \cap \{\omega, \ldots, \omega^{(n+r)}\} \\ \forall \omega \in \{dt, dx\} \quad \text{and} \quad \forall r \geq 0.\end{aligned} \tag{3.52}$$

Define $Y_{i,k} = \{\mu^i, \ldots, \mu^{i(k)}\}$ such that

$$Y_k = \mathop{+}_{i=1}^{s} Y_{i,k}.$$

Since $\mu^i \in \{dt, dx\}$ and by (3.52), we have

$$Y_{i,n-1} \cap \{dt, dx\} = Y_{i,n+r} \cap \{dt, dx\} \qquad i = 1, \ldots, s, \quad r \geq 0.$$

We now use the fact that intersection distributes over union, i.e. that $A \cap \bigcup_i B_i = \bigcup_i (A \cap B_i)$ to show that for $r \geq 0$

$$\begin{aligned}
\{dt, dx\} \cap Y_{n+r} &= \{dt, dx\} \cap \mathop{+}_{i=1}^{s} Y_{i,n+r} \\
&= \mathop{+}_{i=1}^{s} (\{dt, dx\} \cap Y_{i,n+r}) \\
&= \mathop{+}_{i=1}^{s} (\{dt, dx\} \cap Y_{i,n-1}) \\
&= \{dt, dx\} \cap \mathop{+}_{i=1}^{s} Y_{i,n-1} \\
&= \{dt, dx\} \cap Y_{n-1}.
\end{aligned}$$

Hence, from (3.51), we deduce that $q_{n-1} = p_{n-1}$. Finally, we see that $k^* \leq n-1$ and the result follows. □

We may now state an equivalent to Proposition 3.7*i)* the conditions of which are checkable using the DEA, i.e. in a finite number of steps.

Corollary 3.20. *The system described by* (3.43) *covers the system described by* (3.44) *if and only if the codistribution* $\{d\kappa\}$ *is differentially independent.*

Remark 3.21. The differential independence of $\{d\kappa\} \subset T\mathcal{U}^*$ can be checked by first prolonging the inputs once and then using Proposition 3.19.

Proof. Applying the Dynamic Extension Algorithm $n+m$ times (see above remark), one can construct successive prolongations of (3.43). At step k, the state space is made of $y, \tilde{z}, \kappa, \ldots, \kappa^{(k)}$, where $\tilde{z}$ is any minimal subset of the variables z satisfying $X_k = \{dt, dy, dz, d\kappa, \ldots, d\kappa^{(k)}\} = \{dt, dy, d\tilde{z}, d\kappa, \ldots, d\kappa^{(k)}\}$. By the differential independence of κ, after $n+m$ steps, U_{n+m} can be chosen so that $d\kappa^{(n+m+1)} \subset U_{n+m}$. Hence, by Corollary 3.11, the obtained system covers the system

$$\begin{aligned}
\kappa^{(n+m+1)} &= w \\
\dot{y}^i &= h^i(t, y, \kappa)
\end{aligned}$$

which in turn clearly covers the system (3.44). □

3.2.4 Bundle Maps and Coverings

We now turn our attention to the following situation. We are given a bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ and a control system is defined on $\mathcal{U}$ by the equations (3.1). A surjective bundle map ϕ from the bundle $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$ to a bundle $\pi_{\mathcal{NB}} : \mathcal{N} \to \mathcal{B}$ of the form $\phi : (t, x) \mapsto (t, y = \phi_x(t, x))$ is also specified. We want to verify whether or not, the map ϕ induces a covering by (3.1) of a control system on some bundle $\pi_{\mathcal{VN}} : \mathcal{V} \to \mathcal{N}$.
The following proposition is an adaptation of Propositions 3.13 and 3.19 and answers the question using only the data of ϕ in its domain.

Proposition 3.22. *Set $Z \subset T\mathcal{M}$ as $Z = \ker \phi_*$ and let $U \subset T\mathcal{U}$ be the input distribution $U = \{\frac{\partial}{\partial u^l}\}$. Pick $D \in T\mathcal{U}$ given by*

$$D = \frac{\partial}{\partial t} + f^i(t, x, u)\frac{\partial}{\partial x^i}.$$

On $T\mathcal{U}^{n+m}$ define

$$\hat{Z} = Z + \{\frac{\partial}{\partial u^{l(0)}}, \ldots, \frac{\partial}{\partial u^{l(n+m+1)}}\}$$
$$\hat{D} = D + u^{l(r+1)}\frac{\partial}{\partial u^{l(r)}}, \quad r = 0, \ldots, n+m$$

Construct the following filtration of $\hat{Z}$

$$\hat{Z}^{(0)} = \hat{Z} \qquad \hat{Z}^{(k+1)} = \{\zeta \in \hat{Z}^{(k)} \mid [\hat{D}, \zeta] \in \hat{Z}^{(k)}\}$$

for $k = 0, \ldots, n+m$. Define s as the number

$$s \;=\; \dim\big(\;([U, D] + U + [Z, D] + Z) \mod (U + Z)\;\big).$$

Then, the map ϕ induces a covering if and only if

$$\dim \hat{Z}^{(n+m-1)} - \dim \hat{Z}^{(n+m)} = s.$$

Proof. Define $Y \subset T\mathcal{U}^{n+m*}$ as $Y = \perp_{T\mathcal{U}^{n+m*}} \hat{Z}$. Compute the sequence

$$Y^{(0)} = Y \qquad Y^{(k+1)} = Y^{(k)} + DY^{(k)}.$$

By Lemma A.15, we have that $Y^{(k)} = \perp_{T\mathcal{U}^{n+m*}} \hat{Z}^{(k)}$. Hence $\dim \hat{Z}^{(n+m-1)} - \dim \hat{Z}^{(n+m)} = \dim Y^{(n+m)} - \dim Y^{(n+m-1)}$. The result then follows from Proposition 3.13, (Lemma 3.16) and Corollary 3.20. □

Example 3.23. *Consider $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ with*

$$\pi_{\mathcal{UM}} : (t, x^1, x^2, x^3, x^4, u^1, u^2) \mapsto (t, x^1, x^2, x^3, x^4)$$

the equations (3.1) *given by*

$$\dot{x}^1 = x^2 \qquad \dot{x}^2 = u^1 \qquad \dot{x}^3 = x^4 \qquad \dot{x}^4 = u^2 \tag{3.53}$$

and the surjective map

$$\phi : (t, x^1, x^2, x^3, x^4) \mapsto (t, y^1, y^2, y^3) = (t, x^1, x^3, \frac{x^4}{x^2}).$$

The distribution $Z \in T\mathcal{M}$, $Z = \ker \phi_*$ *is then given by* $Z = \{x^2 \frac{\partial}{\partial x2} + x^4 \frac{\partial}{\partial x^4}\}$ *and* $U \in T\mathcal{U}$ *by* $U = \{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\}$. *Computation leads to*

$$[U, D] + U = \{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^4}\}$$
$$[Z, D] + Z = \{x^2 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^4}, x^2 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^3} - u^1 \frac{\partial}{\partial x^1} - u^2 \frac{\partial}{\partial x^4}\}$$

From which we obtain that $s = \dim \big(([U, D] + U + [Z, D] + Z) \mod (U + Z) \big) = 2$. *Next*

$$\hat{Z} = \{x^2 \frac{\partial}{\partial x2} + x^4 \frac{\partial}{\partial x^4}, \frac{\partial}{\partial u^{l(0)}}, \ldots, \frac{\partial}{\partial u^{l(7)}}\} \qquad l = 1, 2$$
$$\hat{D} = x^2 \frac{\partial}{\partial x^1} + u^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} + u^2 \frac{\partial}{\partial x^4} + u^{l(r+1)} \frac{\partial}{\partial u^{l(r)}} \qquad l = 1, 2 \quad r = 0, \ldots, 6$$

From $[\hat{D}, \frac{\partial}{\partial u^{l(r)}}] = -\frac{\partial}{\partial u^{l(r-1)}}, r \geq 0$, $[\hat{D}, \frac{\partial}{\partial u^{1(0)}}] = -\frac{\partial}{\partial x^2}$, $[\hat{D}, \frac{\partial}{\partial u^{2(0)}}] = -\frac{\partial}{\partial x^4}$ *and from*

$$\begin{aligned} -[\hat{D}, x^2 \frac{\partial}{\partial u^{1(r)}} + x^4 \frac{\partial}{\partial u^{2(r)}}] &= x^2 \frac{\partial}{\partial u^{1(r-1)}} + x^4 \frac{\partial}{\partial u^{2(r-1)}} \\ &= -u^{1(0)} \frac{\partial}{\partial u^{1(r)}} - u^{2(0)} \frac{\partial}{\partial u^{2(r)}} \end{aligned}$$

we get, for $0 < k \leq 6$

$$\hat{Z}^{(k)} = \Big\{x^2 \frac{\partial}{\partial u^{1(k-1)}} + x^4 \frac{\partial}{\partial u^{2(k-1)}}, \frac{\partial}{\partial u^{l(k)}}, \ldots, \frac{\partial}{\partial u^{l(7)}}\Big\} \qquad l = 1, 2.$$

Therefore, $\dim \hat{Z}^{(6)} - \dim \hat{Z}^{(5)} = 2 = s$. *Hence, by Proposition 3.22, the map* ϕ *induces a covering. Since the system* (3.53) *is linear and controllable, by Proposition 3.7ii) the covered system is (locally) flat. Equations for the covered system are for instance given by*

$$\dot{y}^1 = v^1 \qquad \dot{y}^2 = y^3 v^1 \qquad \dot{y}^3 = v^2.$$

Note that these equations are locally static-feedback equivalent to the nonholonomic car equations.

3.3 Coverings of Constrained Systems

We now approach the following question. Given a control system and a set of state constraints, does there exists a covering of the constrained system by the unconstrained one? More precisely, consider the system described by (3.1), and a set of functions

$c^1(t,x),\dots,c^r(t,x)$ of the system state, i.e. $c^j \in \mathcal{C}^\infty(\mathcal{M})$. The constrained system is then described by the equations

$$\dot{x}^i = f^i(t,x,u) \tag{3.54a}$$

$$c^j(t,x) = 0 \qquad j = 1,\dots,r < m. \tag{3.54b}$$

Our aim is to give a sufficient condition for the system described by (3.54a) to cover the system described by (3.54). Loosely speaking, if the constraints (3.54b) are feasible, i.e. if there are solutions to the constrained system, then there locally exists a Lie-Bäcklund immersion from the constrained system (3.54) to the unconstrained system (3.54a), see [108]. However, this does not imply the existence of a covering, i.e. a map *in the other direction*. The discussion of this section bears many similarities with the notion of *relative flatness* and related results from [108].
We make the following simplifying assumption, let $\mathcal{U}^\infty$ be the infinite composite bundle on which the prolongation of equations (3.54) are defined and $\mathcal{A}^1$ the associated differential module. Then, around a point $p \in \mathcal{U}^\infty$ of interest, we assume that the 1-forms $dc^1,\dots,dc^r$ generate a free submodule of $\mathcal{A}^1$, i.e. that $dc^1,\dots,dc^r$ are differentially independent.
The next result provides a sufficient test to verify if a given surjective map π with domain $\mathcal{M}$, induces a control system, as in Section 3.2.2 respectively 3.2.4, that is covered not only by (3.54a) but also by (3.54).

Proposition 3.24. *Let $\pi_{\mathcal{U}\mathcal{M}} : \mathcal{U} \to \mathcal{M}$ be the bundle on which the unconstrained system* (3.54a) *is defined, let $D \in T\mathcal{U}$ be given by*

$$D = \frac{\partial}{\partial t} + f^i(t,x,u)\frac{\partial}{\partial x^i}$$

and $c^1,\dots,c^r \in \mathcal{C}^\infty(\mathcal{M})$. Let also $Z \subset T\mathcal{M}$ be an involutive ρ-dimensional distribution satisfying $Z \lrcorner\, dt = 0$ and $Y \subset T\mathcal{M}^*$ the integrable codistribution annihilating Z, i.e. $Y = \perp_{T\mathcal{M}} Z$. Choose $\mu^1,\dots,\mu^s \in T\mathcal{M}^*$, any independent representatives of the space $(Y + DY)/Y$. Moreover, assume that the $r + s \le m$ 1-forms

$$\mu^1,\dots,\mu^s, dc^1,\dots,dc^r$$

are differentially independent around some point $p \in \mathcal{U}^\infty$.
Then, $Y \cap \{dc^1,\dots,dc^r\} = \{0\}$. Also, there (locally) exist functions $y^1,\dots,y^{n-\rho} \in \mathcal{C}^\infty(\mathcal{M})$ such that $\{dt, dy^j\} = Y$, functions $v^1,\dots,v^s \in \mathcal{C}^\infty(\mathcal{U})$ and smooth real-valued functions $g^1(t,y,v),\dots,g^{n-\rho}(t,y,v)$ such that the system described by

$$\dot{y}^j = h^j(t,y,v) \qquad j = 1,\dots,n-\rho \qquad \operatorname{rank}\frac{\partial g}{\partial v} = s$$

is covered by the unconstrained system (3.54a)*. Moreover, the same system is also covered by the constrained system* (3.54)*.*

Proof. By assumption, $\{dt, dy^1,\dots,dy^{n-\rho}, dv^1,\dots,dv^s\} = Y + DY$, therefore by Lemma 3.16, $dv^1,\dots,dv^s$ are differentially independent if and only if $\mu^1,\dots,\mu^s$ are. This also implies that $dv^1,\dots,dv^s, dc^1,\dots,dc^r$ are differentially independent if and only if $\mu^1,\dots,\mu^s$,

$dc^1, \ldots, dc^r$ are. By Proposition 3.13, choosing $z^1, \ldots, z^\rho$ in such a way that (t, y, z) is a coordinate system on $\mathcal{M}$, the unconstrained system equations (3.54a) may be rewritten in the form

$$\begin{aligned} \dot{z}^k &= g^k(t, z, y, u) \\ \dot{y}^j &= h^j(t, v) \qquad v^q = v^q(t, z, y, u). \end{aligned}$$

We now show that $Y \cap \{dc^1, \ldots, dc^r\} = \{0\}$. Assume $\exists \omega \in Y \cap \{dc^1, \ldots, dc^r\}$ and $\omega \neq 0$. Since $\{dc\}$ is free, $\dim\{\omega, \dot{\omega}, \ldots, \omega^{(K)}\} = K + 1$ for all $K \geq 0$. Therefore, since Y has finite dimension and $\omega \in Y$, there must be a $P \geq 1$ such that $\omega^{(P)} \in Y + DY$ and $\omega^{(P)} \notin Y$. Hence $\omega^{(P)}$ is a representative of $(Y + DY)/Y$ that is not differentially independent from $\{dc\}$, a contradiction. The above implies that there is a subset $\tilde{z}^1, \ldots, \tilde{z}^{\rho - r}$ of the variables $z^1, \ldots, z^\rho$ such that $(t, y, \tilde{z}, c)$ is a local coordinate system on $\mathcal{M}$ and such that the unconstrained system equations (3.54a) take the new form

$$\begin{aligned} \dot{c}^i &= e^i(t, y, \tilde{z}, c, u) \\ \dot{\tilde{z}}^k &= \tilde{g}^k(t, y, \tilde{z}, c, u) \\ \dot{y}^j &= h^j(t, v) \qquad v^q = v^q(t, y, \tilde{z}, c). \end{aligned}$$

We already know that $dv^1, \ldots, dv^s, dc^1, \ldots, dc^r$ is a differentially independent set of 1-forms. Hence, we may apply the DEA a number of times and get the system

$$\begin{aligned} \dot{c}^{i(L_c)} &= \bar{u}^i_c \\ \dot{v}^{\iota(L_v)} &= \bar{u}^\iota_v \\ \dot{\tilde{z}}^k &= \tilde{g}^k(t, y, \tilde{z}, c, \ldots, c^{(L_c)}, v, \ldots, v^{(L_v)}, \tilde{u}) \\ \dot{y}^j &= h^j(t, v) \end{aligned} \tag{3.55}$$

with $\tilde{u}$ some subset of the original input variables u. The system (3.55) is equivalent to (3.54a) by endogenous feedback. Now the map

$$\begin{aligned} \pi_y : (t, y, \tilde{z}, c, \ldots, c^{(L_c)}, v, \ldots, v^{(L_v)}, \tilde{u}, \dot{\tilde{u}}, \ldots, \bar{u}_c, \dot{\bar{u}}_c, \ldots, \bar{u}_v, \dot{\bar{u}}_v, \ldots) \\ \mapsto (t, y, v, \ldots, v^{(L_v)}, \bar{u}_v, \dot{\bar{u}}_v, \ldots) \end{aligned}$$

is clearly a covering of the system

$$\dot{y}^j = h^j(t, v) \tag{3.56}$$

by the system (3.55), so that the same equations are covered by (3.54a).
Next, the constrained system (3.54) is equivalent by endogenous feedback to the system

$$\begin{aligned} \dot{v}^{\iota(L_v)} &= \bar{u}^\iota_v \\ \dot{\tilde{z}}^k &= \tilde{g}^k(t, y, \tilde{z}, 0, \ldots, 0, v, \ldots, v^{(L_v)}, \tilde{u}) \\ \dot{y}^j &= h^j(t, v) \end{aligned} \tag{3.57}$$

and the map

$$\begin{aligned} \bar{\pi}_y : (t, y, \tilde{z}, v, \ldots, v^{(L_v)}, \tilde{u}, \dot{\tilde{u}}, \ldots, \ldots, \bar{u}_v, \dot{\bar{u}}_v, \ldots) \\ \mapsto (t, y, v, \ldots, v^{(L_v)}, \bar{u}_v, \dot{\bar{u}}_v, \ldots) \end{aligned}$$

is a covering of the system (3.56) by the system (3.57). Hence, the constrained system (3.54) covers (3.56). □

The previous proposition allows one to verify if a given "subsystem" is simultaneously covered by the unconstrained and constrained systems (3.54a) and (3.54) respectively. However, it does not say if the unconstrained system covers the constrained one. Before stating a sufficient criterion for this problem, we use the assumption made on the state constraints. Indeed, since $\{dc^1, \ldots, dc^r\}$ are assumed differentially independent, we may apply the Dynamic Extension Algorithm and transform the system (3.54a) into the equivalent adapted problem

$$\dot{\tilde{x}}^i = F^i(t, \tilde{x}, c \ldots, c^{(L_c)}, \tilde{u}) \tag{3.58a}$$

$$\dot{c}^{j(L_c)} = \bar{u}_c^j \tag{3.58b}$$

$$\tilde{u} \subset u \quad \operatorname{card} \tilde{u} + \operatorname{card} c = m$$

$$\tilde{x} \subset x \quad \operatorname{card} \tilde{x} + \operatorname{card} c = n$$

for some $L_c \geq 0$. (Note that we may assume an extension of the same length L_c on each constraint c^i without loss of generality). The set of variables $\tilde{x}$ is any subset of x (or functions of x) such that $\{dt, d\tilde{x}, dc\} = \{dt, dx\}$, i.e. $(t, \tilde{x}, c)$ are local coordinates on $\mathcal{M}$. The constrained system (3.54) is then equivalent to

$$\dot{\tilde{x}}^i = F^i(t, \tilde{x}, 0 \ldots, 0, \tilde{u}). \tag{3.59}$$

Before stating the result, let us redefine the bundle $\mathcal{U}$ according to the extended system (3.58). The bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ shall have the local coordinate expressions

$$\pi_{\mathcal{UM}} : (t, \tilde{x}, c \ldots, c^{(L_c)}, \tilde{u}, \bar{u}_c^j) \mapsto (t, \tilde{x}, c \ldots, c^{(L_c)})$$

and out of the Cartan distribution defined on $\mathcal{U}$ by equations (3.58), we pick

$$D = \frac{\partial}{\partial t} + F^i(t, \tilde{x}, c \ldots, c^{(L_c)}, \tilde{u}) \frac{\partial}{\partial \tilde{x}^i} + c^{j(1)} \frac{\partial}{\partial c^{j(0)}} + \ldots + c^{j(L_c)} \frac{\partial}{\partial c^{j(L_c - 1)}} + \bar{u}_c^j \frac{\partial}{\partial c^{j(L_c)}}.$$

Corollary 3.25. *Let $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ be the bundle on which the unconstrained system* (3.58) *is defined, and let $D \in T\mathcal{U}$ be as above. Suppose that there exists an involutive distribution $Z \subset T\mathcal{M}$ satisfying $Z \lrcorner\, dt = 0$ and that*

$$\dim Z = (L_c + 1)r \qquad r = \operatorname{card} c.$$

Let $Y \subset T\mathcal{M}^*$ be the integrable codistribution annihilating Z, i.e. $Y = \perp_{T\mathcal{M}} Z$ and assume that

$$\dim \big((Y + DY)/Y\big) = \operatorname{card} \tilde{u} = m - r.$$

Choose $\mu^1, \ldots, \mu^{m-r} \in T\mathcal{M}^*$, any independent representatives of the space $(Y + DY)/Y$. Moreover, assume that the m 1-forms

$$\mu^1, \ldots, \mu^{m-r}, dc^1, \ldots, dc^r$$

are differentially independent around some point $p \in \mathcal{U}^\infty$. Then, there exists a covering of the constrained system (3.59) *by the unconstrained system* (3.58) *and a covering of* (3.54) *by* (3.54a).

Proof. By Proposition 3.24, there are functions $y^1, \ldots, y^{n-r} \in \mathcal{C}^\infty(\mathcal{M})$, $v^1, \ldots, v^{m-r} \in \mathcal{C}^\infty(\mathcal{U})$ and smooth functions $h^1, \ldots, h^{n-r}$ such that

$$\begin{aligned} Y = \{dt, dy\} \qquad & Y + DY = \{dt, dy, dv\} \\ \dot{y}^q = h^q(t, y, v) \qquad & q = 1, \ldots n-r. \end{aligned}$$

Also, $Y \cap \{dc\} = \{0\}$. Assume there is a $\omega \in \{dc, \ldots, dc^{(P)}\}$ for some $P \geq 0$ such that $\omega \in Y$. Then, either $\omega^{(Q)} \in Y$ $\forall Q \geq 0$, which implies a differential relation on dc (since Y is finite dimensional) and contradicts the differential independence of dc, or $\exists Q \geq 1$ such that $\omega^{(Q)}$ is a non-zero representant of $(Y + DY)/Y$, which contradicts the differential independence of μ, dc. Therefore, we conclude that $Y \cap \{dc, \ldots, dc^{(L_c)}\} = \{0\}$ which in turn, counting dimensions, implies that $\{t, y, c, \ldots, c^{(L_c)}\}$ form a local coordinate system on $\mathcal{M}$ and $\{t, y, c, \ldots, c^{(L_c)}, v, \bar{u}_c\}$ form a local coordinate system on $\mathcal{U}$. In these coordinates, the equations (3.58) take the (decoupled) form

$$\begin{aligned} \dot{y}^q &= h^q(t, y, v) \qquad q = 1, \ldots n-r \\ c^{j(L_c+1)} &= \bar{u}_c^j \end{aligned} \tag{3.60}$$

and the constrained system (3.59) is therefore equivalent to

$$\begin{aligned} \dot{y}^q &= h^q(t, y, v) \qquad q = 1, \ldots n-r \\ c^j &= 0, \ldots, c^{j(L_c)} = 0 \end{aligned}$$

which is clearly equivalent to $\dot{y}^q = h^q(t, y, v)$, since these relations are independent of c and its derivatives. But the system $\dot{y}^q = h^q(t, y, v)$ is covered by the unconstrained system (3.58) by Proposition 3.24. □

Remark 3.26. Assume that the unconstrained system (3.58) is linear (or static feedback linearizable, or flat) and that the conditions of Corollary 3.25 hold. Then the constrained system is flat. Also, in this case, the decoupled form (3.60) in the proof above shows that if $\mathfrak{y}^1(t, y, v, \ldots, v^{(L)}), \ldots, \mathfrak{y}^{m-r}(t, y, v, \ldots, v^{(L)})$ are the flat outputs of the constrained system, then $\mathfrak{y}^1, \ldots, \mathfrak{y}^{m-r}, dc^1, \ldots, dc^r$ are flat outputs for the unconstrained system. This shows the link of our result with Theorem 7.2 and Corollary 7.3 in [108] where it is shown for example, that given a flat system, if c is a subset of the flat outputs, then the constrained system satisfying $c = 0$ is flat.

Example 3.27. *Consider the system described on the bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ with coordinates $\pi_{\mathcal{UM}} : (t, x^1, x^2, x^3, x^4, u^1, u^2, u^3) \mapsto (t, x^1, x^2, x^3, x^4)$ and satisfying the equations*

$$\begin{aligned} \dot{x}^1 &= u^1 + 2u^3 x^1 \\ \dot{x}^2 &= u^2 + 2u^3 x^2 \\ \dot{x}^3 &= -u^3 x^1 \\ \dot{x}^4 &= -u^3 x^2. \end{aligned} \tag{3.61}$$

It is easily verified that the system is flat with flat outputs x^3, x^4, u^3. Consider also the constraint

$$c = (x^1)^2 + (x^2)^2 - 1 = 0. \tag{3.62}$$

For the involutive distribution Z of Corollary 3.25, choose $Z = \{x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}\}$. Then

$$Y = \perp_{T\mathcal{M}} Z = \{dt, dx^3, dx^4, x^1 dx^2 - x^2 dx^2\}.$$

Computations show that $\dim\big((Y + DY)/Y\big) = 2$ *and two representatives are given e.g. by*

$$\mu^1 = u^3 dx^1 + x^1 du^3 \qquad \mu^2 = (x^2 u^1 - x^1 u^2) dx^1 - x^1 x^2 du^1 + (x^1)^2 du^2.$$

Further computations show that the matrix

$$\begin{pmatrix} \frac{\partial}{\partial u^1} \\ \frac{\partial}{\partial u^2} \\ \frac{\partial}{\partial u^3} \end{pmatrix} \lrcorner \begin{pmatrix} \mu^1 & \mu^2 & d\dot{c} \end{pmatrix}$$

has determinant $-2(x^1)^2(c+1)$ *which is non-zero if* $x^1 \neq 0$. *This shows that* μ^1, μ^2, dc *are (locally) differentially independent since the input differentials* du^1, du^2, du^3 *are supposed so. Hence, by Corollary 3.25, the system* (3.61) *under the constraint* (3.62) *is flat.*

We shall close this section by stressing the fact that the difficulty in the application of Corollary 3.25 is to actually find an appropriate involutive distribution Z (or an appropriate integrable codistribution Y).

3.4 A Non-Flat System Linearizable by Singular Static Feedback

By Proposition 3.7, any system linearizable by a non-singular dynamic feedback is flat. The same does no hold true for singular feedbacks. We close this chapter by giving an example of a system that is linearizable by singular static feedback but that is not flat. The example also shows that in general, this property is not invariant under endogenous feedback. The system in the example is linearizable, but the system obtained by prolonging one of the inputs once is not.

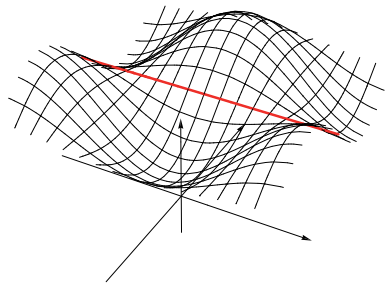
Example 3.28. *Consider the following system with three states and two inputs*

$$\dot{x}^1 = u^1 \qquad \dot{x}^2 = x^1 + u^2 \qquad \dot{x}^3 = x^2 + \cos(u^1)\sin(u^2). \tag{3.63}$$

Using the singular static feedback $u^1 = u$ *and* $u^2 = 0$, *the system transforms to a chain of three integrators*

$$\dot{x}^1 = u^1 \qquad \dot{x}^2 = x^1 \qquad \dot{x}^3 = x^2. \tag{3.64}$$

However, the system (3.63) *is not flat. Indeed, it is known that if a system* $\dot{x} = f(x, u)$ *is flat, then the submanifold of the first jet space given by the equations* $p = f(x, u)$, *parameterized by* u *and with* x *considered as fixed parameters is a ruled manifold, see [129, 117]. It is easily checked that the corresponding manifold is not ruled in the case of equations* (3.63). *The manifold is illustrated in the following figure, black and meshed for* (3.63) *and red for* (3.64).



Finally, adding u^2 to the state and adding the equation $\dot{u}^2 = \bar{u}^2$ to (3.63) *leads to a non-linearizable system. Indeed, the new state variable u^2 may be assigned an initial condition different from zero, making the reduction to* (3.64) *impossible.*

Remark 3.29. The construction of the example can be viewed as follows. In the proof of Theorem 2 in [117], it is shown that the equations of a given dynamically linearizable system may always be "perturbed" so as to destroy the linearizability property. In this view, one can consider that (3.63) is a perturbed version of the single-input linearizable system (3.64) and that u^2 is the perturbation parameter (η in the notations of [117]).

Regarding the previous example, one should note that the singular feedback is such that the input variables u^1 and u^2 are not only made differentially dependent, but also algebraically dependent; and it is exactly for this reason that the ruled manifold condition can be satisfied for the new, smaller set of inputs.
However, consider the situation of a linear controllable system (3.1) and a dynamic controlled invariant distribution Z defined on the system's state-space manifold. If $Z = \{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^p}\}$, we may rewrite the (linear controllable system) in the form of (3.43). Further assume that the differentials of the functions $\kappa^1, \ldots, \kappa^s$ are not differentially independent. Then the system (3.44) is linearizable by a singular dynamic feedback. However, by construction, the functions $\kappa^1, \ldots, \kappa^s$ are algebraically independent. Hence, one may not use the same trick as in Example 3.28 to devise an instance of a system (3.44), linearizable by singular dynamic feedback but non-flat.

3.5 Conclusion

In this chapter, we dealt with the question whether a given system covers another one which is specified or "induced" by some surjective map defined on the state space. We obtained both a necessary condition and a necessary and sufficient condition introducing a generalization of controlled invariance, coined dynamic controlled invariance and applying a variant of the dynamic extension algorithm. We then went on to give a sufficient condition for a constrained system to be covered by its unconstraint counterpart, thereby

also providing a sufficient test for the flatness of the constrained system, given the fact that the unconstrained one is flat as well. Finally, we gave an example of a system linearizable by singular static feedback that is not flat and discussed some implications.

Chapter 4

Bilinear Systems

In our previous discussions, we encountered two different kinds of filtrations of (co-)distributions or modules, related to the data of a given control problem. In the case of static feedback linearizability, the emphasis was put on the Pfaffian system given by the system's contact forms. This approach can be understood as rearranging infinitesimal versions of the equations. The procedure originates in the Goursat normal form for systems of codimension two [61, 13] and is known as the GS-algorithm [55, 58]. Since then, it has been generalized further in various ways, e.g. [143, 142, 147, 146, 148]. The other approach consists in finding 1-forms of highest relative degree which can be seen as sorting out the system variables differentials in an appropriate way. This procedure potentially requires some input prolongations. We presented it in Section 2.2 as the computation of a basis of the differential module associated with the system. In [5, 4], it is called the infinitesimal Brunovsky normal form. See also [69, 72] for earlier results. In the case of static feedback linearizability, both types of filtrations enjoy some respective integrability properties. In this situation, both computations also happen to result in equivalent objects. Concurrently, in the flatness problem, the integrability issue is the most difficult one, c.f. Chapter 2. The first object of the present chapter is to devise a class of control systems given by equations of a more general type than linear ones, but simple enough so as to make a general assertion regarding integrability of their filtration. Indeed, using a specific relative derived flag [108], we show that one can compute a filtration with guaranteed integrability. However, as we will see, this comes at a cost. In a second step, we apply the algorithm to a specific class of equations. These equations, when imposed to satisfy a quadratic constraint, are shown to be flat. Moreover, we observe that when some parameters are set in an appropriate way, two well known physical flat systems are obtained, the non-holonomic car and the pendulum.

In Section 4.1, the two types of filtrations are compared in the integrable case. In Section 4.2 we define the studied class of bilinear systems and show that upon adequate prolongation, the associated filtration always satisfies the integrability conditions. We then deduce a sufficient condition for the flatness of that class of systems. Section 4.3 defines a set of "generalized pendulum" equations and shows their flatness. Some trajectory planning simulations are also presented.

4.1 Two Filtrations in the Integrable Case

Let us briefly recall the two different filtrations defined in Chapters 1 and 2. A control system with n states and m inputs is defined on the bundle $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$ given by $\pi_{\mathcal{UM}} : (t, x, u) \mapsto (t, x)$ in local coordinates. The system satisfies the equation

$$\dot{x}^i = f^i(t, x, u) \qquad \operatorname{card} x = n \quad \operatorname{card} u = m.$$

The associated Cartan distribution and codistribution on $\mathcal{U}$ are generated by

$$\begin{aligned} \Omega &= \{dx^1 - f^1(t, x, u)dt, \ \ldots, \ dx^n - f^n(t, x, u)dt\} \\ V &= \{D, \frac{\partial}{\partial u^1} \ldots, \frac{\partial}{\partial u^m}\} \quad D = \frac{\partial}{\partial t} + f^i(t, x, u)\frac{\partial}{\partial x^i}. \end{aligned}$$

Next, the infinite input prolongation is defined on $\mathcal{U}^\infty$ which has the local coordinates $(t, x, u, u^{(1)}, u^{(2)}, \ldots)$. On $\mathcal{U}^\infty$, the Cartan distribution is spanned by the unique vector field

$$D_\infty = \frac{\partial}{\partial t} + f^i(t, x, u)\frac{\partial}{\partial x^i} + u^{l(p+1)}\frac{\partial}{\partial u^{l(P)}} \qquad \begin{aligned} &i = 1, \ldots, n \\ &l = 1, \ldots, m \quad p = 0, \ldots \end{aligned}$$

In Chapter 2, we defined the set of smooth functions on $\mathcal{U}^\infty$ and denoted it by $\mathcal{R}$. The $\mathcal{R}$-module of 1-forms in $T\mathcal{U}^{\infty *}$ generated by $\{dt, dx^1, \ldots, dx^n\}$ was denoted by H. In Sections 1.2.2 and 2.2 respectively, the two following filtrations were defined

$$H^{(0)} = H \qquad H^{(k+1)} = \{\omega \in H^{(k)} \mid D_\infty(\omega) \in H^{(k)}\} \tag{4.1}$$

$$\Omega^{(0)} = \Omega \qquad \Omega^{(k+1)} = \{\omega \in \Omega^{(k)} \mid d\omega \in \Omega^{(k)}\} \tag{4.2}$$

where $d\omega \in \Omega^{(k)}$ means that $d\omega$ lies inside the ideal generated by $\Omega^{(k)}$ in $\Lambda T\mathcal{U}^*$. The next result states that in the case where the codistributions $\Omega^{(k)} + \{dt\}$ are integrable for all k, these coincide with $H^{(k)}$. Recall that the integrability of $\Omega^{(k)} + \{dt\}$ is what characterizes static-feedback linearizable systems.

Lemma 4.1. *Let $\pi_{\mathcal{U},\infty 0} : \mathcal{U}^\infty \to \mathcal{U}$ be the projection given in coordinates by $\pi_{\mathcal{U},\infty 0} : (t, x, u, u^{(1)}, u^{(2)}, \ldots) \mapsto (t, x, u)$. Assume $\Omega^{(k)} + \{dt\}$ is integrable for $k \geq 0$. Then*

$$H^{(k)} = \pi^*_{\mathcal{U},\infty 0}\big(\Omega^{(k)} + \{dt\}\big). \tag{4.3}$$

Proof. From the definition, one clearly has $H^{(0)} = \pi^*_{\mathcal{U},\infty 0}\big(\Omega^{(0)} + \{dt\}\big)$. Assume $\Omega^{(k)} + \{dt\}$ is integrable for $k \geq 0$. Then, for some particular value of k, assume (4.3) holds. There exists functions $\zeta^i, z^i, \gamma^j, g^j \in C^\infty(\mathcal{U})$ such that

$$\Omega^{(k)} = \{d\zeta^i - z^i dt, \ d\gamma^j - g^j dt\} \qquad \Omega^{(k+1)} = \{d\zeta^i - z^i dt\}.$$

From (4.2) we deduce $dz^i \wedge dt \in \Omega^{(k)}$, which implies $dz^i \in \{d\zeta, d\gamma, dt\} = \Omega^{(k)} + \{dt\}$. Observe that $\Omega^{(0)} + \{dt\} = \{dx, dt\} = \pi^*_{\mathcal{UM}} T\mathcal{M}^*$. It follows that the z^i, ζ^i, γ^j are not only functions on $\mathcal{U}$ but are pulled-back from some functions on $\mathcal{M}$, i.e. $z^i = z^i(t, x)$,

$\zeta^i = \zeta^i(t,x)$ and $\gamma^j = \gamma^j(t,x)$.
Since D annihilates the elements of $\Omega^{(k)}$, it follows that $D(\zeta^i) = z^i$ and $D(\gamma^j) = g^j$. And since ζ^i and γ^j depend only on (t,x), we have that $\pi^*_{\mathcal{U},\infty 0} D(\zeta^i) = D_\infty(\pi^*_{\mathcal{U},\infty 0}\zeta^i) = \pi^*_{\mathcal{U},\infty 0}(z^i)$ and $\pi^*_{\mathcal{U},\infty 0} D(\gamma^j) = D_\infty(\pi^*_{\mathcal{U},\infty 0}\gamma^j) = \pi^*_{\mathcal{U},\infty 0}(g^j)$.
From $dz^i \in \{d\zeta, d\gamma, dt\}$ it follows that $D_\infty(\pi^*_{\mathcal{U},\infty 0}\zeta^i) = \pi^*_{\mathcal{U},\infty 0} z^i$ are functions of $(\pi^*_{\mathcal{U},\infty 0}\zeta^i, \pi^*_{\mathcal{U},\infty 0}\gamma^j, t)$. We have shown that $\pi^*_{\mathcal{U},\infty 0}\big(\Omega^{(k+1)} + \{dt\}\big) \subset H^{(k+1)}$. The result follows by comparing the dimensions of $H^{(k+1)}$ and $\Omega^{(k+1)} + \{dt\}$. □

4.2 Bilinear Systems

We now consider systems with n states x^i, m inputs u^k and L mutiplicators λ^l considered as inputs, of the form

$$\dot{\vec{x}} = \vec{k} + A\vec{x} + B\vec{u} + \sum_{l=1}^{L} \lambda^l (Q_l \vec{x} + \vec{c}_l) \tag{4.4}$$

where $\vec{k} \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, Q_l \in \mathbb{R}^{n\times n},\ \vec{c}_l \in \mathbb{R}^n$. To describe the same set of equations, we shall also use the following notation

$$\dot{x}^i = f^i = k^i + a^i_j x^j + b^i_k u^k + \lambda^l (Q^i_{lr} x^r + c^i_l) \tag{4.5}$$

where $k^i, a^i_j, b^i_k, Q^i_{lr}, c^i_l \in \mathbb{R}$ are the entries of the corresponding matrices and vectors in (4.4). The variables u^k are inputs to the system and the $\lambda^l = \lambda^{l(0)}$ are also seen as inputs on which one admits an arbitrary prolongation of length P. The system is therefore defined on a bundle $\mathcal{U}$ with coordinates $(t, x^i, u^k, \lambda^{(0)l}, \ldots, \lambda^{(P)l})$. The Cartan distribution on $\mathcal{U}$ reads

$$V = \{D, \frac{\partial}{\partial \lambda^{1(P)}}, \ldots, \frac{\partial}{\partial \lambda^{L(P)}}\} \qquad D = \frac{\partial}{\partial t} + f^i \frac{\partial}{\partial x^i} + \lambda^{(p+1)l} \frac{\partial}{\partial \lambda^{(p)l}}$$
$$p = 0, \ldots, P-1.$$

and the Cartan codistribution

$$\Omega = \{\ dx^i - f^i dt,\ d\lambda^{(p)l} - \lambda^{(p+1)l} dt\ \} \qquad i = 1, \ldots, n \quad p = 0, \ldots, P-1.$$

One may complete Ω and V so as to build two dual bases of $T\mathcal{U}^*$ and $T\mathcal{U}$ respectively:

$$\underbrace{\begin{array}{l} \Omega \left\{ \begin{array}{l} dx^i - f^i dt = \omega^i \\ d\lambda^{(p)l} - \lambda^{(p+1)l} dt \end{array} \right. \\ du^j \\ d\lambda^{(P)l} \\ dt \end{array}}_{T\mathcal{U}^*} \qquad \underbrace{\begin{array}{l} V \left\{ \begin{array}{l} \frac{\partial}{\partial x^i} \\ \frac{\partial}{\partial \lambda^{(p)l}} \quad p = 0, \ldots, P-1 \\ \frac{\partial}{\partial u^j} \\ \frac{\partial}{\partial \lambda^{(P)l}} \\ D \end{array} \right. \end{array}}_{T\mathcal{U}} \tag{4.6}$$

One easily checks that applying any form to any vector gives 1 if these are on the same row and if the indices agree and leads to 0 otherwise. Hence the two bases are indeed dual to each other. Concerning the derived flag (4.2) of Ω, we have the following result

Proposition 4.2. *If $P \geq \max(2K-1, K+1)$, then for $k = 1, \ldots, K$, the kth derived system $\Omega^{(k)}$ takes the form*

$$\Omega^{(k)} = \{\ n_i^j(dx^i - f^i dt),\ d\lambda^{l(p)} - \lambda^{l(p+1)}dt \mid p = 0, \ldots, P-k-1\ \}$$

where the matrix N_k with entries n_i^j depends only on $\lambda^{l(0)}, \ldots, \lambda^{l(k-2)}$. The codistribution

$$\Omega^{(k)} + \{dt\}$$

is completely integrable. Moreover, to compute N_k, choose N_1 as any (constant) matrix of maximal rank satisfying

$$N_1 B = 0.$$

The other elements are obtained by applying the following recursive computation for all k, $1 \leq k \leq K-1$:

- *find an M of max. rank s.t. $N_k M = 0$*
- *find an H of max. rank s.t. $H(\dot{N}_k + N_k(A + \sum_{l=1}^{L} \lambda^l Q_l))M = 0$*
- *$N_{k+1} = HN_k$*

In the second step, $\dot{N}_k = \sum_{p=0}^{k-2} \frac{\partial N_k}{\partial \lambda^{l(p)}} \lambda^{l(p+1)}$. The matrices N_k can be chosen with entries polynomial in $\lambda^{l(p)}$.

Remark 4.3. Applying the algorithm of Proposition 4.2 and increasing P at each step if necessary, so as to guarantee that $P \geq \max(2k-1, k+1)$, does not necessarily lead to a zero matrix N, even if the system is strongly accessible. See the forthcoming Example 4.8.

Remark 4.4. In the statement of Proposition 4.2, we made no mention of the regularity of the obtained $\Omega^{(k)}$. For simplicity and throughout this chapter, we shall implicitly consider only neighborhoods of points $p \in \mathcal{U}$ within which the ranks of $N_1, \ldots, N_K$ are constant and equal to their generic ranks. These generic ranks are well defined since the matrices N_k may all be chosen polynomial in their arguments.

Proof. It is easy to verify that N_1 can be chosen as stated. Hence, N_1 can be chosen constant. The formula in the proposition implies that if N_κ depends on $\lambda^{l(0)}, \ldots, \lambda^{l(\kappa-2)}$, then $N_{\kappa+1}$ depends at most on $\lambda^{(0)l}, \ldots, \lambda^{(\kappa-1)l}$. It follows by induction that N_k depends at most on $\lambda^{(0)l}, \ldots, \lambda^{(k-2)l}$.
We now show the correctness of the algorithm. Assume $P \geq \max(2K-1, K+1)$ and $1 \leq k \leq K-1$. Moreover assume that $\Omega^{(k)}$ and $V^{(k)} = \perp_{T\mathcal{U}^*} \Omega^{(k)}$ fit in the following two

dual bases

$$\Omega^{(k)} \left\{ \begin{array}{l} n_i^j(dx^i - f^i dt) = n_i^j \omega^i \\ d\lambda^{l(p)} - \lambda^{l(p+1)} dt \\ \bar{n}_i^j(dx^i - f^i dt) = \bar{n}_i^j \omega^i \\ d\lambda^{l(p)} - \lambda^{l(p+1)} dt \\ du^j \\ d\lambda^{l(P)} \\ dt \end{array} \right. \qquad V^{(k)} \left\{ \begin{array}{ll} \tilde{m}_j^i \frac{\partial}{\partial x^i} & j = 1, \ldots, n - n_k \\ \frac{\partial}{\partial \lambda^{l(p)}} & p = 0, \ldots, P - k - 1 \\ m_j^i \frac{\partial}{\partial x^i} & j = 1, \ldots, n_k \\ \frac{\partial}{\partial \lambda^{l(p)}} & p = P - k, \ldots, P - 1 \\ \frac{\partial}{\partial u^j} & \\ \frac{\partial}{\partial \lambda^{l(P)}} & \\ D & \end{array} \right.$$

$$\underbrace{\qquad\qquad\qquad}_{T\mathcal{U}^*} \qquad \underbrace{\qquad}_{T\mathcal{U}}$$

These bases are dual of each other if and only if the following relations are satisfied (owing to the form of the dual bases (4.6)).

$$\left(\tilde{m}_j^i \frac{\partial}{\partial x^i}\right) \lrcorner \left(n_l^k \omega^l\right) = \tilde{m}_j^i n_l^k \delta_i^l = n_i^k \tilde{m}_j^i = \delta_j^k \tag{4.7}$$

$$\left(m_j^i \frac{\partial}{\partial x^i}\right) \lrcorner \left(n_l^k \omega^l\right) = m_j^i n_l^k \delta_i^l = n_i^k m_j^i = 0 \tag{4.8}$$

$$\left(\tilde{m}_j^i \frac{\partial}{\partial x^i}\right) \lrcorner \left(\bar{n}_l^k \omega^l\right) = \tilde{m}_j^i \bar{n}_l^k \delta_i^l = \bar{n}_i^k \tilde{m}_j^i = 0 \tag{4.9}$$

$$\left(m_j^i \frac{\partial}{\partial x^i}\right) \lrcorner \left(\bar{n}_l^k \omega^l\right) = m_j^i \bar{n}_l^k \delta_i^l = \bar{n}_i^k m_j^i = \delta_j^k \tag{4.10}$$

We shall use the following shorthands for the relevant elements.

$$\begin{aligned} \Omega^{(k)} &= \{\ \mu^j = n_i^j \omega^i, \qquad \eta^{lp} = d\lambda^{l(p)} - \lambda^{l(p+1)} dt\ \} && p = 0, \ldots, P - k - 1 \\ V^{(k)} &= \{\ v_s = m_s^r \frac{\partial}{\partial x^r}, \quad w_{hq} = \frac{\partial}{\partial \lambda^{h(q)}}, \quad D\ \} && q = P - k, \ldots, P. \end{aligned} \tag{4.11}$$

Assuming $\Omega^{(k)} + \{dt\}$ is integrable, by Lemma 1.6 and Proposition 1.8, we may compute $\Omega^{(k+1)}$ as

$$\begin{aligned} \Omega^{(k+1)} &= \{\omega \in \Omega^{(k)} \mid D \lrcorner\, d\omega \in \Omega^{(k)}\} \\ &\overset{(4.11)}{=} \{\omega \in \Omega^{(k)} \mid v_s \lrcorner\, D \lrcorner\, d\omega = 0, \quad w_{hq} \lrcorner\, D \lrcorner\, d\omega = 0, \quad q = P - k, \ldots, P\} \end{aligned} \tag{4.12}$$

The exterior derivative of the generators of $\Omega^{(k)}$ are given by

$$d\mu^j = dn_i^j \wedge \omega^i + n_i^j d\omega^i \qquad d\eta^{lp} = -d\lambda^{l(p+1)} \wedge dt$$

so that the elements $D \lrcorner\, d\mu^j$ and $D \lrcorner\, d\eta^{lp}$ read

$$\begin{aligned} D \lrcorner\, d\mu^j &= (D \lrcorner\, dn_i^j)\omega^i - (\underbrace{D \lrcorner\, \omega^i}_{0}) dn_i^j + n_i^j D \lrcorner\, d\omega^j \\ &= \dot{n}_i^j \omega^i - n_i^j D \lrcorner\, (df^i \wedge dt) \qquad \text{for } P \geq k - 1 \\ &= \dot{n}_i^j \omega^i + n_i^j \big(df^i - D(f^i)\ dt\big) \end{aligned}$$

$$D \lrcorner\, d\eta^{lp} = d\lambda^{l(p+1)} - \lambda^{l(p+2)} dt.$$

Hence, the elements of the right-hand side of (4.12) are

$$\begin{aligned} v_s \lrcorner\, (D \lrcorner\, d\mu^j) &= m_s^r \dot{n}_i^j \frac{\partial}{\partial x^r} \lrcorner\, \omega^i + m_s^r n_i^j \big(\frac{\partial f^i}{\partial x^r} - D(f^i) \frac{\partial}{\partial x^r} \lrcorner\, dt \big) \\ &= m_s^r \dot{n}_r^j + m_s^r n_i^j \frac{\partial f^i}{\partial x^r} = m_s^r (\dot{n}_r^j + n_i^j \frac{\partial f^i}{\partial x^r}) \end{aligned}$$

$$\begin{aligned} w_{hq} \lrcorner\, (D \lrcorner\, d\mu^j) &= \dot{n}_i^j \frac{\partial}{\partial \lambda^{h(q)}} \lrcorner\, \omega^i + n_i^j \big(\frac{\partial f^i}{\partial \lambda^{h(q)}} - D(f^i) (\frac{\partial}{\partial \lambda^{h(q)}} \lrcorner\, dt) \big) \qquad q = P-k, \dots, P \\ &= 0 \qquad \text{for } P \geq k+1, \text{ since } f^i \text{ dependes on } \lambda^{h(0)} \end{aligned}$$

$$v_s \lrcorner\, (D \lrcorner\, d\eta^{lp}) = m_s^r \frac{\partial}{\partial x^r} \lrcorner\, (d\lambda^{l(p+1)} - \lambda^{l(p+2)} dt) = 0$$

$$\begin{aligned} w_{hq} \lrcorner\, (D \lrcorner\, d\eta^{lp}) &= \frac{\partial}{\partial \lambda^{h(q)}} \lrcorner\, (d\lambda^{l(p+1)} - \lambda^{l(p+2)} dt) \\ &= \delta_h^l\, \delta_q^{p+1} \quad p = 0, \dots, P-k-1 \quad q = P-k, \dots, P \\ &= \begin{cases} \delta_h^l & \text{if } \ p = P-k-1 \ \text{ and } \ q = P-k \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, if $P \geq k+1$, the only two terms that are not identically zero are

$$v_s \lrcorner\, (D \lrcorner\, d\mu^j) = m_s^r (\dot{n}_r^j + n_i^j \frac{\partial f^i}{\partial x^r}) \tag{4.13}$$

$$w_{hq} \lrcorner\, (D \lrcorner\, d\eta^{lp}) = \begin{cases} \delta_h^l & \text{if } \ p = P-k-1 \ \text{ and } \ q = P-k \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Since $\Omega^{(k+1)} \subset \Omega^{(k)}$, there must exist functions $h_j^\alpha,\ z_{lp}^\alpha$ such that

$$\Omega^{(k+1)} = \{ h_j^\alpha \mu^j + z_{lp}^\alpha \eta^{lp} \}$$

By (4.12), these functions satisfy

$$\begin{aligned} v_s \lrcorner\, D \lrcorner\, d(h_j^\alpha \mu^j + z_{lp}^\alpha \eta^{lp}) &= \\ v_s \lrcorner\, D \lrcorner\, (dh_j^\alpha \wedge \mu^j + dz_{lp}^\alpha \wedge \eta^{lp}) &+ h_j^\alpha v_s \lrcorner\, D \lrcorner\, d\mu^j + z_{lp}^\alpha v_s \lrcorner\, D \lrcorner\, d\eta^{lp} \\ &= h_j^\alpha v_s \lrcorner\, D \lrcorner\, d\mu^j + z_{lp}^\alpha v_s \lrcorner\, D \lrcorner\, d\eta^{lp} = 0 \end{aligned}$$

$$\begin{aligned} w_{hq} \lrcorner\, D \lrcorner\, d(h_j^\alpha \mu^j + z_{lp}^\alpha \eta^{lp}) &= \\ w_{hq} \lrcorner\, D \lrcorner\, (dh_j^\alpha \wedge \mu^j + dz_{lp}^\alpha \wedge \eta^{lp}) &+ h_j^\alpha w_{hq} \lrcorner\, D \lrcorner\, d\mu^j + z_{lp}^\alpha w_{hq} \lrcorner\, D \lrcorner\, d\eta^{lp} \\ &= h_j^\alpha w_{hq} \lrcorner\, D \lrcorner\, d\mu^j + z_{lp}^\alpha w_{hq} \lrcorner\, D \lrcorner\, d\eta^{lp} = 0 \end{aligned}$$

where we used the fact that v_s, w_{hq} and D all annihilate the forms μ^j and η^{lp}. These equations may be rewritten in matrix notation as

$$\begin{aligned}
\begin{pmatrix} h_j^\alpha & z_{lp}^\alpha \end{pmatrix} \begin{pmatrix} v_s \lrcorner\, D \lrcorner\, d\mu^j & w_{hq} \lrcorner\, D \lrcorner\, d\mu^j \\ v_s \lrcorner\, D \lrcorner\, d\eta^{lp} & w_{hq} \lrcorner\, D \lrcorner\, d\eta^{lp} \end{pmatrix} &= \\
\begin{pmatrix} h_j^\alpha & z_{lp}^\alpha \end{pmatrix} \begin{pmatrix} v_s \lrcorner\, D \lrcorner\, d\mu^j & 0 \\ 0 & w_{hq} \lrcorner\, D \lrcorner\, d\eta^{lp} \end{pmatrix} &= \begin{pmatrix} 0 & 0 \end{pmatrix}
\end{aligned}$$

where the second row follows in case $P \geq K+1$. The null-space of a block diagonal matrix may always be chosen block diagonal. Therefore, we split the index α into two and rewrite the equation as

$$\begin{pmatrix} h_j^\gamma & 0 \\ 0 & z_{lp}^\xi \end{pmatrix} \begin{pmatrix} v_s \lrcorner\, D \lrcorner\, d\mu^j & 0 \\ 0 & w_{hq} \lrcorner\, D \lrcorner\, d\eta^{lp} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\Omega^{(k+1)}$ is generated by $\{h_j^\gamma \mu^j,\ z_{lp}^\xi \eta^{lp}\}$. Using (4.13) and (4.14), the relations become

$$h_j^\xi (\dot{n}_r^j + n_i^j \frac{\partial f^i}{\partial x^r}) m_s^r = h_j^\xi (\dot{n}_r^j + n_i^j (a_r^i + \lambda^l Q_{lr}^i)) m_s^r = 0 \tag{4.15}$$

and

$$z_{l,P-k-1}^\xi = 0. \tag{4.16}$$

In matrix form, (4.8) and (4.15) lead to the equations

$$N_k M = 0 \qquad M \text{ of max. rank} \tag{4.17}$$

$$H(\dot{N} + N_k(A + \sum_{l=1}^{L} \lambda^l Q_l))M = 0 \qquad H \text{ of max. rank.} \tag{4.18}$$

Setting $N_{k+1} = HN_k$, $\vec{\omega} = (\omega^1 \ \cdots \ \omega^n)^T$ and additionally using (4.16), we obtain that

$$\begin{aligned}
\Omega^{(k)} &= \{N_k \vec{\omega}, \quad d\lambda^{l(p)} - \lambda^{l(p+1)} dt \mid p = 0, \ldots, P-k-1\} \\
\Rightarrow \quad \Omega^{(k+1)} &= \{N_{k+1} \vec{\omega},\ d\lambda^{l(p)} - \lambda^{l(p+1)} dt \mid p = 0, \ldots, P-k-2\}
\end{aligned}$$

which proves the correctness of the algorithm for $k = 1, \ldots, K$ (under the condition that $\Omega^{(k)} + \{dt\}$ is integrable). Clearly

$$\begin{aligned}
\Omega^{(k)} + \{dt\} &= \{N_k \vec{\omega},\ \ d\lambda^{l(p)},\ dt \mid p = 0, \ldots, P-k-1\} \\
&= \{N_k d\vec{x},\ d\lambda^{l(p)},\ dt \mid p = 0, \ldots, P-k-1\}
\end{aligned} \tag{4.19}$$

And since N_k depends at most on $\lambda^{l(0)}, \ldots, \lambda^{l(k-2)}$, (4.19) is integrable if $P \geq 2k-1$. Together with the requirement $P \geq k+1$ used throughout the proof one obtains the condition $P \geq \max(2K-1, K+1)$. □

4.2.1 Flatness of Bilinear Systems

Proposition 4.2 together with Lemma 4.1 provide a simple sufficient condition for flatness of bilinear control systems of the form (4.4). Indeed, assume that there is a pair of numbers P and K satisfying $P \geq \max(2K-1, K+1)$ and such that the algorithm of Proposition 4.2 yields a sequence of matrices $N_1, \ldots, N_K$ with $N_K = 0$. It then follows that the filtration (4.2) of the Cartan codistribution of (4.4) with P prolongations on λ^l is of the form

$$\begin{array}{lll}
\Omega^{(0)} & = \{\ \vec{\omega}, & d\lambda^{l(p)} - \lambda^{l(p+1)}dt \ \mid\ p = 0, \ldots, P-1\ \} \\
\Omega^{(1)} & = \{\ N_1\vec{\omega}, & d\lambda^{l(p)} - \lambda^{l(p+1)}dt \ \mid\ p = 0, \ldots, P-2\ \} \\
 & \vdots & \\
\Omega^{(K-1)} & = \{\ N_{K-1}\vec{\omega}, & d\lambda^{l(p)} - \lambda^{l(p+1)}dt \ \mid\ p = 0, \ldots, P-K\ \} \\
\Omega^{(K)} & = \{ & d\lambda^{l(p)} - \lambda^{l(p+1)}dt \ \mid\ p = 0, \ldots, P-K-1\ \} \\
 & \vdots & \\
\Omega^{(P-1)} & = \{ & d\lambda^{l(0)} - \lambda^{l(1)}dt\ \} \\
\Omega^{(P)} & = \{\ \ \}
\end{array}$$

Moreover the elements $\Omega^{(k)} + \{dt\}$ are all integrable and coincide with $H^{(k)}$ for $k \geq 0$ by Lemma 4.1. Therefore, the $\mathcal{R}$-modules $H^{(k)}$ are generated by

$$\begin{array}{lll}
H^{(0)} & = \{\ d\vec{x}, & d\lambda^{l(p)},\ dt \ \mid\ p = 0, \ldots, P-1\ \} \\
H^{(1)} & = \{\ N_1 d\vec{x}, & d\lambda^{l(p)},\ dt \ \mid\ p = 0, \ldots, P-2\ \} \\
 & \vdots & \\
H^{(K-1)} & = \{\ N_{K-1}d\vec{x}, & d\lambda^{l(p)},\ dt \ \mid\ p = 0, \ldots, P-K\ \} \\
H^{(K)} & = \{ & d\lambda^{l(p)},\ dt \ \mid\ p = 0, \ldots, P-K-1\ \} \\
 & \vdots & \\
H^{(P-1)} & = \{ & d\lambda^{l(0)},\ dt\ \} \\
H^{(P)} & = \{\ dt\ \}
\end{array}$$

Using Lemma 2.8, we deduce that

$$\lambda^{1(0)}, \ldots, \lambda^{L(0)}$$

are part of the flat outputs. The other flat outputs, m in number, are obtained as follows. One seeks integrable representatives of $H^{(0)}/(H^{(1)} + D_\infty H^{(1)})$ to $H^{(K-1)}/(H^{(K)} + D_\infty H^{(K)})$. Since $N_K = 0$, $H^{(K-1)}/(H^{(K)} + D_\infty H^{(K)})$ is represented by $N_{K-1}d\vec{x}$, which is integrable modulo $d\lambda^{l(0)}, \ldots, \lambda^{l(P-K)}$. For $k = 0, \ldots, K-2$, define the matrix

$$Q_k$$

as the matrix whose rows are the rows of

$$N_k$$

independent of the rows of the matrix

$$N_{k+1} + \dot{N}_{k+1} + N_{k+1}(A + \lambda^l Q_l).$$

The remaining flat outputs are then obtained as

$$Q_0\vec{x},\ \ldots, Q_{K-2}\vec{x},\ N_{K-1}\vec{x}.$$

Note that these flat outputs are polynomial in $\lambda^{l(s)}$ and linear in $x^1, \ldots, x^n$.

Remark 4.5. It is not necessary to guess the values of P and K beforehand. The recursive algorithm of Proposition 4.2 can be applied without a priori knowledge of P and K, simply by adding appropriate derivatives of λ^l in each computation of $\dot{N}_k$ from N_k. As soon as the algorithm saturates, i.e. whenever some k such that $\operatorname{rank} N_{k+1} = \operatorname{rank} N_k$ is reached, then we set $K = k$. Next we may choose $P = \max(2K-1, K+1)$ so as to guarantee the integrability of $\Omega^{(0)} + \{dt\}, \ldots, \Omega^{(K)} + \{dt\}$

Remark 4.6. Consider the extended equations of system (4.5) given by

$$\dot{x}^i \quad = f^i = k^i + a^i_j x^j + b^i_k u^k + \lambda^l (Q^i_{lr} x^r + c^i_l) \tag{4.20a}$$

$$\dot{\lambda}^{l(p)} = \lambda^{l(p+1)} \qquad p = 0, \ldots, P-1. \tag{4.20b}$$

Since (4.20b) is independent of (4.20a), it can be considered as describing a subsystem of (4.20). Choosing P sufficiently large, the codistributions $\{d\vec{x}\}$, $\{N_1 d\vec{x}\}$, $\ldots$, $\{N_K d\vec{x}\}$ are in one-to-one correspondence with the result of the computation of the *relative derived flag* [108] of (4.20) with respect to (4.20b) given by

$$\begin{aligned} I^{(0)} &= \{dx^1 - f^1 dt, \ldots, dx^n - f^n dt\} \\ I^{(k+1)} &= \{\omega \in I^{(k)} \mid d\omega \in I^{(k)} + J\} \\ &\text{with} \quad J = \{d\lambda^{l(p)} - \lambda^{l(p+1)} dt \mid p \geq 0\}. \end{aligned}$$

In the case where $N_K = 0$ for some K and P large enough, the integrability of $\Omega^{(k)} + \{dt\}$ previously discussed is then equivalent to the integrability of $I^{(k)} + J + \{dt\}$ and $I^{(K)} = 0$. Hence, by Theorem 8.2 in [108], the system (4.20) is *relatively flat* with respect to the subsystem (4.20b) (and therefore relatively flat w.r.t. the *output subsystem* with λ^l as system outputs). Since the subsystem (4.20b) is clearly flat with flat outputs $\lambda^1, \ldots, \lambda^L$, we may use Proposition 5.2 in [108] to deduce once again that the system (4.5) is flat.

Let us give a simple example for the application of Proposition 4.2.

Example 4.7. *Consider the dimension* 4 *system with inputs* u, λ

$$\dot{x}_1 = u \qquad \dot{x}_2 = x_1 + \lambda x_3 \qquad \dot{x}_3 = \lambda x_2 \qquad \dot{x}_4 = x_3 + \lambda x_1$$

i.e.

$$\dot{\vec{x}} = A\vec{x} + Bu + \lambda Q\vec{x}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The algorithm produces:

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_2 = \begin{pmatrix} 0 & \lambda & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$N_3 = \begin{pmatrix} 0 & -\lambda^2 & \dot{\lambda} & \lambda \end{pmatrix} \qquad N_4 = 0.$$

leading to the flat outputs:

$$y_1 = \lambda \qquad \textit{and} \qquad y_2 = N_3\vec{x} = -\lambda^2 x_2 + \dot{\lambda} x_3 + \lambda x_4$$

The following example shows that Proposition 4.2 is only a sufficient condition for the flatness of bilinear control systems (4.4).

Example 4.8. *The system with state variables x^1, x^2 and inputs u, λ*

$$\dot{x}_1 = u \qquad \dot{x}_2 = x_2\lambda$$

is flat with flat outputs x^1, x^2 whenever $x_2 \neq 0$. However, setting

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

the algorithm yields

$$N_1 = N_2 = \ldots = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

There is no $K > 0$ such that $N_K = 0$.

4.3 Application: Pendulum-like Equations

This section introduces a very simple class of bilinear systems of the form (4.4) that are additionally required to satisfy a quadratic constraint. We verify flatness of all systems in this class and then show that the class contains the well known bi- and tri-dimensional pendulums (or VTOL) as well as the non-holonomic car.
Let $x, y \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ and let g_x, g_y be constant vectors in $\mathbb{R}^n$. For any integer $k > 0$ define the "generalized pendulum" equations of order k and dimension $2kn$ as

$$\begin{aligned} x^{(k)} &= g_x + u + \lambda(x - y) \\ y^{(k)} &= g_y + \lambda(y - x) \end{aligned} \qquad \text{s.t.} \quad C = (x - y)^T(x - y) - 1 = 0 \tag{4.21}$$

Corollary 4.9. *The system described by* (4.21) *is flat with flat output* $y = y^{(0)}$.

Proof. Define the vector of (first order) state variables as

$$z = (x^{(k-1)}\ y^{(k-1)}\ \cdots\ x\ y)^T$$

The unconstrained system equations (l.h.s. of (4.21)) are given in the form of (4.4) by the matrices

$$A = \begin{pmatrix} 0_{2n}^{2(k-1)n} & 0_{2n}^{2n} \\ I_{2(k-1)n} & 0_{2(k-1)n}^{2n} \end{pmatrix} \quad B = \begin{pmatrix} I_n \\ 0_{2(k-1)n}^{n} \end{pmatrix} \quad Q = \begin{pmatrix} 0_{2n}^{2(k-1)n} & \partial^2 C \\ 0_{2(k-1)n}^{2(k-1)n} & 0_{2n}^{2(k-1)n} \end{pmatrix}$$

the constant vector $k = (g_x^T\ g_y^T\ 0\ \cdots\ 0)^T$ and using the notation

$$\partial^2 C = \begin{pmatrix} \partial_{xx} C & \partial_{xy} C \\ \partial_{xy} C & \partial_{yy} C \end{pmatrix}.$$

Applying the algorithm of Proposition 4.2 leads to

$$N_{2k-1} = \begin{pmatrix} 0_n^{2(k-1)n} & I_n \end{pmatrix} \qquad N_{2k} = 0.$$

It should also be noted that the $N_1\ \ldots\ N_{2k-1}$ are all constant and independent of λ. This implies that $(N_{2k-1}z, \lambda) = (y, \lambda)$ is a set of linearizing outputs for the unconstrained system. Next, from (4.21)

$$y^{(k)} - g_y = \lambda(y - x) \quad \Rightarrow \quad (y^{(k)} - g_y)^T (y^{(k)} - g_y) = \lambda^2 (y - x)^T (y - x)$$

or

$$(y^{(k)} - g_y)^T (y^{(k)} - g_y) = \lambda^2 (C + 1). \tag{4.22}$$

This last relation shows that λ can be computed (on a suitable open set) from $y^{(k)}$ and C so that (y, C) is another set of flat outputs for the unconstrained system. This in turn implies that the constrained system with $C = 0$ is flat with flat output y by application of Corollary 3.25 or, more directly, by invoking Corollary 7.3 in [108]. Indeed, the mentioned result from [108] states that given a flat system with flat outputs $(y, \mathfrak{y})$, the system obtained by adding the constraints $\mathfrak{y}^j = 0$ is flat with flat outputs y. □

4.3.1 Special Cases

Second order, planar or 3D This is the case $k = 2$, $n = 2, 3$. If g_x and g_y are chosen so as to represent the effect of gravity, one obtains the physical planar respectively three-dimensional pendulum composed of two linked masses with forces acting on one of the two.

First order, planar, gravity-free This situation is given by $k = 1$, $n = 2$, $g_x = g_y = (0 \;\; 0)^T$ and reads

$$\begin{aligned} \dot{x}^1 &= u^1 + \lambda(x^1 - y^1) \\ \dot{x}^2 &= u^2 + \lambda(x^2 - y^2) \\ \dot{y}^1 &= \lambda(y^1 - x^1) \\ \dot{y}^2 &= \lambda(y^2 - x^2) \end{aligned} \qquad C = (x^1 - y^1)^2 + (x^2 - y^2)^2 - 1 = 0$$

These equations can be seen as describing the motion of two vertical pencils on a sheet of paper and linked by a thread of fixed length. The motion of one of the pencils is controlled. Using the constraint C, the system can be reparameterized by setting $x^1 - y^1 = \sin\theta$ and $x^2 - y^2 = \cos\theta$. In the coordinates y^1, y^2, θ, the system reads

$$\begin{aligned} \dot{y}^1 &= \lambda \sin\theta \\ \dot{y}^2 &= \lambda \cos\theta \\ \dot{\theta} &= \mu \end{aligned} \qquad \begin{aligned} \lambda &= -\tfrac{1}{2}(u^1 \sin\theta + u^2 \cos\theta) \\ \mu &= u^1 \cos\theta - u^2 \sin\theta \end{aligned}$$

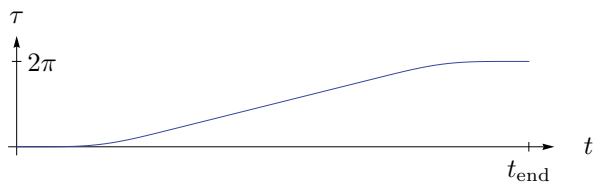
i.e. the non-holonomic car equations.

4.3.2 Trajectory Planning

To illustrate the behavior of system (4.21) as the order k and the "gravity" vectors g_x and g_y change, we present some simple trajectory planning simulations in the planar ($n = 2$) case. The flat outputs (y^1, y^2) are required to follow the path

$$y^1(t) = \sin(2\tau(t)) \qquad y^2(t) = \cos(\tau(t)).$$

The parameterization $\tau(t) \in \mathcal{C}^{2k}(\mathbb{R}, \mathbb{R})$ is chosen piecewise polynomial and such that $\tau(0) = 0$, $\tau(t_{\text{end}}) = 2\pi$ and such that the derivatives $\frac{d^s\tau}{dt^s}$ at $t = 0$ and $t = t_{\text{end}}$ are zero for $s = 1, \ldots, 2k+1$. Hence, the system is at rest at $t = 0$ and $t = t_{\text{end}}$. For some δ, $0 < \delta < \frac{t_{\text{end}}}{2}$, the parameterization $\tau(t)$ is an affine function on the interval $t \in [\delta, t_{\text{end}} - \delta]$. The following is a sketch of the parameterization $\tau(t)$.

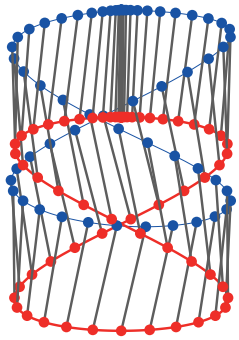


Figures 4.1 through 4.3 are qualitative representations of the obtained trajectories for order $k = 1, 2, 3, 4$ and different values of g_x and g_y. The red curve represents the path of the flat outputs y^1, y^2, which also happens to be the planar coordinates of one of the two "masses". The red and blue dots represent the position of both "masses" at regular time intervals. The blue "mass" is the actuated one with coordinates x^1, x^2, i.e the one on which the "forces" u^1, u^2 are applied and its trajectory is represented by the blue curve. Notice the symmetry in the trajectories for even orders k. Also, as k increases, the "speed"

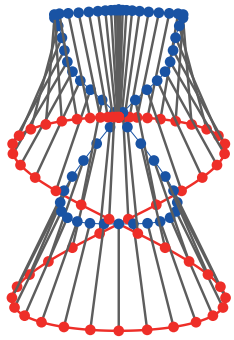
range of the actuated "mass" increases. This is due to the fact that the coordinates x^1, x^2 depend on higher time-derivatives of the reference trajectory $y^1(t), y^2(t)$ as k grows.
Finally, note that equation (4.22) has two solutions for λ in general. In the simulation, a simple heuristic was used to choose the right sign for $\lambda(t)$, so as to ensure a continuous solutions for $x^1(t), x^2(t)$.

4.4 Conclusion

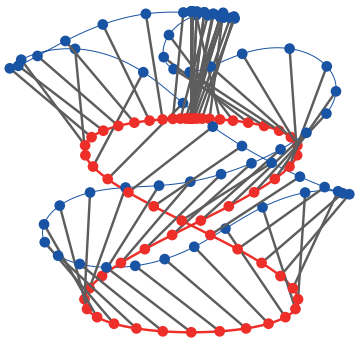
The approach in this chapter was to seek a class of systems simple enough to overcome the integrability issues one encounters when trying to assess flatness of a control system. Systems affine in the states and some inputs, bilinear in the state and some other inputs were shown to be of that kind. Indeed, an algorithm can be applied, and from its output, given it has saturated to an empty set, flatness can be deduce. The procedure was then applied to a set of equations. These equations, when imposed to satisfy a quadratic constraint were shown to describe a collection of flat systems. We observed that this collection contains the non-holonomic car and the planar respectively three-dimensional pendulums. Finally, some qualitative trajectory planning examples where presented with varying parameters; perhaps more for their relatively appealing esthetic qualities than their relevance.



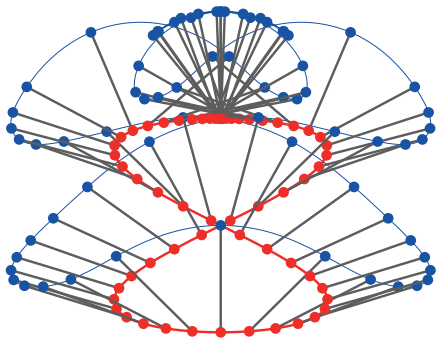
Order $k = 1$



Order $k = 2$
Planar pendulum.

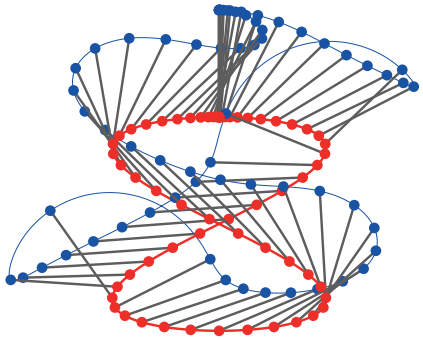


Order $k = 3$

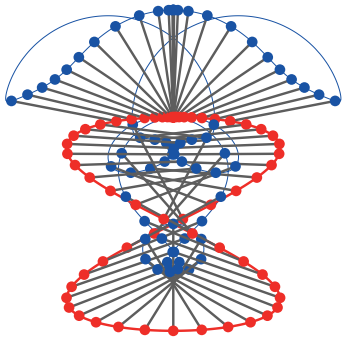


Order $k = 4$

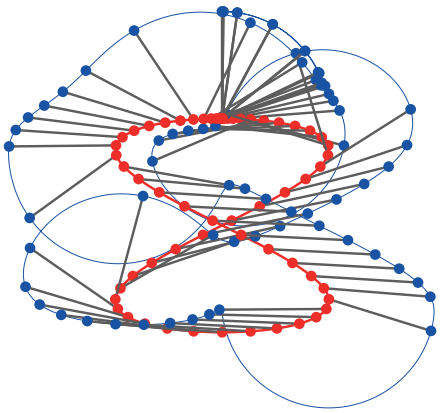
Figure 4.1: Trajectory planning with "strong gravity", $g_x = g_y = (0 \ \ -10)^T$.



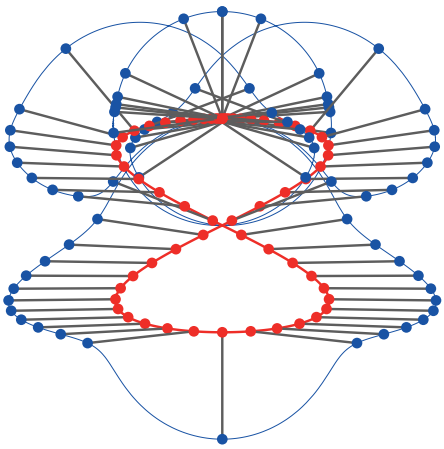
Order $k = 1$



Order $k = 2$
Planar pendulum.

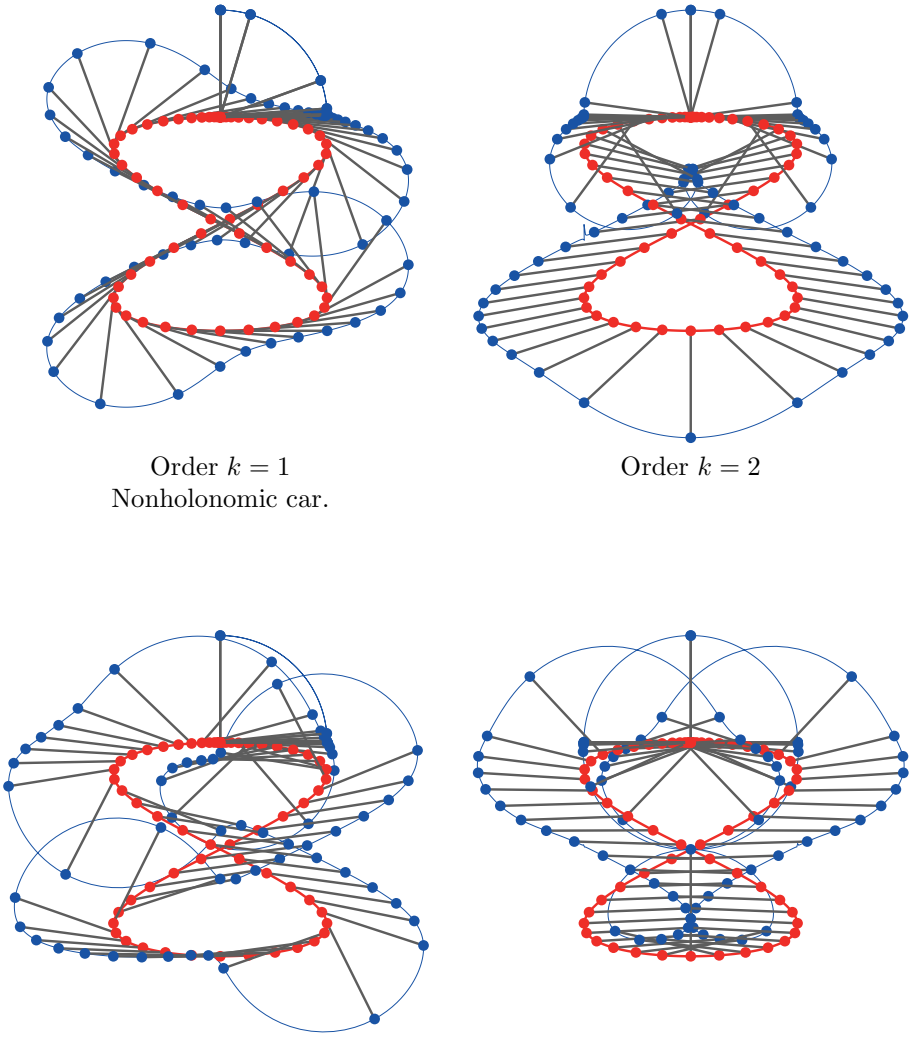


Order $k = 3$



Order $k = 4$

Figure 4.2: Trajectory planning with "weak gravity", $g_x = g_y = (0 \ \ -1)^T$.



Order $k = 1$
Nonholonomic car.

Order $k = 2$

Order $k = 3$

Order $k = 4$

Figure 4.3: Trajectory planning with zero gravity, $g_x = g_y = (0 \;\; 0)^T$.

Conclusion

Synthesis The topics and contributions presented in the thesis may be summarized by the following three points.

- We gave a variant of a known characterization of flatness. First a basis ω of the free summand of the differential module associated with a control system is computed. Given that the torsion part of the module is generated solely by dt, the time variable differential, the only obstacle in concluding flatness is the potential non-integrability of the codistribution spanned by the basis ω. To retrieve integrability, if possible, one is required to find an invertible matrix differential operator P transforming the set ω in a set of exact 1-forms $P\omega$. An alternative approach from [88] consists in defining a rectangular operator P_F based on the implicit system equations. The condition on the torsion part of the module then translates to hyper-regularity of P_F. A basis ω may then be obtained from P_F. Common to both approaches, one expands the exactness condition $d(P\omega) = 0$ and uses further differentiations so as to obtain a closed set of equations. We proposed an equivalent set of equations where one is asked to find a rectangular operator $\bar{P}$ such that the composite matrix $\begin{pmatrix} \bar{P} \\ P_F \end{pmatrix}$ is unimodular, i.e. square and invertible, and such that $d\bar{P}\ dx = 0$. These new conditions have a different structure than the one proposed so far. A part of the equations, akin to curvature is trivialized. It should however be stressed that our characterization does not formulate an easier problem. Indeed, the equations that have been trivialized always admit a solution and the set of solutions can be described in closed form, [21, 88].

- Given a control system and an involutive distribution Z defined on the state-space manifold, we considered the following problem: Does the integral manifold of Z locally corresponds to the state space of a control system related in some way to the original system? In the case where Z satisfies the classical controlled invariance, the answer is positive; there is a subsystem with codimension equal to $\dim Z$. The inputs to the subsystem can be made to coincide with a subset of the original inputs by an appropriate bundle preserving change of coordinates. In this situation, we showed that if the initial system is static feedback linearizable, so is the subsystem.
  We were led to the following observation. If one wishes to start with a linear controllable system and then devise some submanifold of its state space so as to obtain a flat "subsystem" that is *not* statically feedback linearizable, then the notion of

controlled invariance is of no interest. This motivated the definition of the notion of dynamic controlled invariance. To achieve the mentioned situation, the inputs of the "subsystem" are made to depend on the initial system state. Moreover, the dependency of the subsystem inputs in the full system inputs *must* be of defective rank. Unfortunately, this last requirement has important consequences. Indeed, the flatness of the subsystem is not anymore immediately implied by the linearity or flatness of the initial one.

A sufficient condition was obtained through the notion of a covering and related results from [21]. If the initial system is flat and if it can be shown to cover the subsystem "induced" by the distribution Z, then the subsystem is flat. A system covers another one if the map between the two corresponding diffieties satisfies certain conditions. This infinite dimensional criterion was translated to a finite dimensional one by means of a version of the dynamic extension algorithm.

The following more difficult problem was then considered. Given a system and a set of state constraints, is the constrained system covered by the unconstrained one? We approached the problem by first finding a sufficient condition to a related problem: Given a system, a set of state constraints and an involutive distribution Z on the state-space manifold, is there a system whose state-space is an integral manifold of Z and such that it is covered by the unconstrained system *and* by the constrained one as well? We showed that the answer is positive if the inputs of the subsystem v^j and the constraints c^k are such that dv^j, dc^k are a set of differentially independent 1-forms. This differential independence can be checked thanks to the dynamic extension algorithm. We then used this result to devise a sufficient condition to the initial problem.

In the same spirit, we went on by giving an instance of a system linearizable by singular static feedback that is not flat.

- In the characterization of flatness discussed in the second chapter, the general procedure can be decomposed in two steps. One starts by computing a certain filtration, thereby obtaining the basis of a module. An integrability problem then remains. In the case of statically linearizable systems, the mentioned filtration enjoys some important integrability properties at each stage of the computation. It is therefore tempting to seek a situation where the flag structure can be adapted at each step in a way that enforces the integrability condition to remain satisfied. We devised a simple class of bilinear systems for which this can be achieved. Concurrently, we proposed a efficient recursive algorithm form the computation of the adequate relative derived flag. The algorithm is such that flatness of the system may be concluded if it saturates to an empty set.

  We proposed a set of quadratically constrained bilinear equations and proved their generic flatness. The equations were shown to specialize to known physical flat systems when appropriate parameters are chosen.

Perspectives As a perspective, we would like to mention two aspects of the approached problems. These questions could not be answered in the course of our study and we believe they are of interest. To the best of our knowledge, these problems have received relatively

little attention in the literature, [16].

- Consider the situation of a linear controllable system and an involutive distribution $Z = \{\frac{\partial}{\partial z^k}\}$ defined on the system state-space manifold. Assume that the conditions of Proposition 3.13 lead to coordinates (t, y^i, z^k) such that the system may be rewritten in the form of equations (3.43), i.e.

$$\begin{aligned} \dot{z}^k &= g^k(t, z, y, u) \\ \dot{y}^i &= h^i(t, y, \kappa(t, z, y, u)) \qquad \operatorname{rank} \frac{\partial h}{\partial \kappa} = s. \end{aligned}$$

  Further assume that the conditions for the map $(t, y, z) \mapsto (t, y)$ to induce a covering are *not* satisfied. As we have seen, this means that the equations above define a singular dynamic feedback of the system

$$\dot{y}^i = h^i(t, y, \bar{\kappa}).$$

  By assumption, the system is hence linearizable by singular dynamic feedback. What additional conditions should be satisfied to conclude that the system $\dot{y}^i = h^i(t, y, \bar{\kappa})$ is flat, i.e. linearizable by a non-singular (regular) dynamic feedback? Our discussion of the dynamic extension algorithm leads us to propose that answering the following question would be a useful first step. Assume that the system $\dot{x} = f(t, x, u)$ with $\operatorname{card} x = n$ and $\operatorname{card} u = m$ is flat and that $h \in \mathbb{R}^n$ is a constant vector. What are the conditions (if any) for the system with $m + 1$ inputs given by

$$\dot{x} = f(t, x, u) + hv$$

  with v a new additional input to be flat? Note that since the new input enters the equations in an affine way, the ruled manifold condition cannot be broken, which is precisely the trick used in Example 3.28.

- In the proposed sufficient condition for the flatness of a class of bilinear systems, we have seen with Example 4.8 that even if a system is locally strongly accessible (only dt is torsion), the algorithm may saturate to a non-zero matrix corresponding to a codistribution larger than $\{dt\}$. This is in contrast with the filtration of the second chapter where $H^{(k^*)}$ is precisely spanned by the set of torsion elements of $\mathcal{A}^1$. Hence, one may say that we have traded termination for integrability and that the resulting conservatism leads to effective and easy computations. Can such a tradeoff be made for a larger, perhaps more interesting class of systems?

Appendix A

Tools

The aim of this appendix is to clarify the notations used throughout the text, to recall some basic notions and to derive a few technical results. Section A.1 very briefly reviews concepts from differential geometry. Some useful identities from exterior differential calculus are gathered for convenience. Section A.2 deals with some consequences of the Frobenius theorem, pullback of codistributions and projectability of distributions. In Section A.3, we discuss the computations of derived systems and filtrations.

A.1 Notations and Basic Notions

For convenience, some useful notions and identities are also discussed. One may refer to [98, 86, 132, 59] for more details on the subject of differential geometry. We use the Einstein summation convention whenever the notation is unambiguous.

Manifolds All discussions are local in nature, even when not explicitly stated. An n-dimensional manifold M may therefore always be identified with some open subset of $\mathbb{R}^n$. As open subsets of $\mathbb{R}^n$, all our manifolds will automatically be smooth (i.e. $\mathcal{C}^\infty$) manifolds. A manifold may be equipped with a set of (local) independent coordinates $x^i : M \to \mathbb{R}$ where $\operatorname{card} x = \dim M$ in the finite dimensional case and x is a countable set in the infinite dimensional case.

Maps A map between two manifolds $\phi : M \to N$ is an application sending any point $p \in M$ to a point $\phi(p) \in N$. All considered maps shall be smooth, that is, given two sets of coordinates x^i on M and y^j on N, the functions $y^j = \phi^j(x)$ are smooth functions in $\mathcal{C}^\infty(M)$. See Chapter 2 for the notion of a smooth function on an infinite dimensional manifold.
If the map ϕ is a surjection, there exists coordinates $x^i, \bar{x}^\iota$ on M and y^j on N such that ϕ takes the form $\phi : (x, \bar{x}) \mapsto (y = x)$. We will usually use the symbol π and call it a *projection*. The map describes a *submersion* of M to N. If ϕ is an injective map, i.e. there are coordinate x^i on M and $y^j, \bar{y}^k$ on N such that $\phi : (x) \mapsto (y = x, \bar{y} = 0)$, then ϕ is an

immersion from M to N. Since we work only locally, this also implies that $\phi(M) \equiv N$ and the map also describes an *embedding*.

Fiber Bundle and Bundle Maps By a *fiber bundle* or simply a *bundle* we shall mean a pair of manifolds M and B together with a projection map $\pi : M \to B$. The manifold B is called the base and for each point $b \in B$, the pre-image $\pi^{-1}(b)$ is the fiber over b. The whole bundle will be referred to as $\pi : M \to B$ or simply as M. Since we work locally, we may identify M with $\pi^{-1}(b) \times B$ for any fixed $b \in B$. In other words, we may find adapted coordinates x^i, y^j on M such that $\pi : (x, y) \mapsto (y)$. Let us insist on the fact that we rely on the local triviality of fibre bundles. The global properties are subtler. See [78, 131] for a general discussion and [59] for the local picture.
Let $\pi_{MB} : M \to B$ and $\pi_{NC} : N \to C$ be two bundles. A *bundle preserving map* or simply a *bundle map* from M to N is a map $\phi : M \to N$ such that there exists another map $\varphi : B \to C$ and such that the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi_{MB}"'] & N \arrow[d, "\pi_{NC}"] \\ B \arrow[r, "\varphi"] & C \end{tikzcd}$$

commutes. In any two systems of adapted coordinates such that $\pi_{MB} : (x, y) \mapsto (y)$ and $\pi_{NC} : (\nu, \gamma) \mapsto (\gamma)$, a bundle map takes the form $(x, y) \mapsto (\nu = \phi(x, y), \gamma = \varphi(y))$.

Tangent and Cotangent Bundles Let M be a manifold and $p \in M$ any point on M. We shall denote by T_pM the $\mathbb{R}$-linear space of tangent vectors to M at the point p and by T_pM^* the dual space of T_pM. The union of all tangent and cotangent space at all points in M, $TM = \cup_{p\in M} T_pM$ and $TM^* = \cup_{p\in M} T_pM^*$ are the tangent and cotangent bundles. A smooth section $v \in \Gamma TM$, i.e. a map $v : M \to TM$ assigning a vector in T_pM to each point $p \in M$ is called a *vector field* and we use the shorter notation $v \in TM$. Similarly, a smooth map $V : M \to \mathrm{Gr}(TM, s)$ assigning an s-dimensional sub-vectorspace of T_pM to each point $p \in M$ is called an *s-dimensional distribution* and we use the shorter notation $V \subset TM$.
A smooth section $\omega \in \Gamma TM^*$, i.e. a map $\omega : M \to TM^*$ assigning a covector in T_pM^* to each point $p \in M$ is called a 1-*form* and we use the shorter notation $\omega \in TM^*$. Similarly, a smooth map $\Omega : M \to \mathrm{Gr}(TM^*, s)$ assigning an s-dimensional sub-vectorspace of T_pM^* to each point $p \in M$ is called an *s-dimensional codistribution* and we use the shorter notation $\Omega \subset TM^*$.
At least locally, the constructions TM and TM^* are also $\mathcal{C}^\infty(M)$-modules. An s-dimensional distribution (respectively codistribution) is a $\mathcal{C}^\infty(M)$-submodule of TM (respectively TM^*) and may locally always be generated by s independent vector fields $v_1, \ldots, v_s \in TM$ (respectively s independent 1-forms $\omega^1, \ldots, \omega^s \in TM^*$). We shall use the notation $\{v_1, \ldots, v_s\}$ (resp. $\{\omega^1, \ldots, \omega^s\}$) to denote the (co-) distribution generated by such a basis. Assume $\dim M = n$ and let V be an s-dimensional distribution $V \subset TM$. We shall denote by $\Omega = \perp_{TM} V$, $\Omega \subset TM^*$, the $(n-s)$-dimensional codistribution of all forms annihilating

the vector fields belonging to V. Reciprocally, the meaning of $\perp_{TM^*} \Theta$ for $\Theta \subset TM^*$ should be clear.

Operations on Vector Fields and Forms The set of q-forms on a manifold M shall be denoted by $\Lambda^q TM^*$. A q-form ω is also a map

$$\omega : \underbrace{TM \times \ldots \times TM}_{q\text{-fold}} \to \mathcal{C}^\infty(M).$$

The interior product of a vector field $v \in TM$ and a q-form $\omega \in \Lambda^q TM^*$, denoted by $v \lrcorner\, \omega$ is the $(q-1)$-form satisfying

$$(v \lrcorner\, \omega)(v_1, \ldots, v_{q-1}) = q\omega(v, v_1, \ldots, v_{q-1}) \qquad \forall v_1, \ldots, v_{q-1} \in TM.$$

For $v \in TM$ and $\omega \in TM^*$ we simply have $v \lrcorner\, \omega = \omega(v)$.
We shall write $v(\omega)$ or sometimes simply $v\omega$ for the Lie derivative of ω along v; $v(\omega)$ is a form of the same degree than ω. Note the following very useful identities where d is the exterior derivative and $[.,.]$ is the Lie bracket:

Lemma A.1. *Let $v, w \in TM$, $\omega \in \Lambda^q TM^*$ and $\eta \in \Lambda^r TM^*$ then*

$$\begin{aligned}
v(\omega) &= v \lrcorner\, d\omega + d(v \lrcorner\, \omega) \\
v(w \lrcorner\, \omega) &= [v, w] \lrcorner\, \omega + w \lrcorner\, v(\omega) && \text{(Cartan formula)} \\
d(\omega \wedge \eta) &= d\omega \wedge \eta + (-1)^q \omega \wedge d\eta \\
v \lrcorner\, (\omega \wedge \eta) &= (v \lrcorner\, \omega) \wedge \eta + (-1)^q \omega \wedge (v \lrcorner\, \eta) \\
v(\omega \wedge \eta) &= v(\omega) \wedge \eta + \omega \wedge v(\eta) \\
v(d\omega) &= d(v(\omega)) \\
[v, w](\omega) &= v(w(\omega)) - w(v(\omega))
\end{aligned}$$

Note that since the Lie derivative along v of w is $[v, w]$, the Cartan formula can be seen as a Leibnitz (product) rule of the Lie derivative over the interior product.

Proof. All detailed proofs are found in Chapter 2 of [98]. □

Induced Tangent and Cotangent Maps Let M and N be two manifolds and ϕ a map $\phi : M \to N$. The map ϕ induces a tangent and a cotangent map

$$\begin{aligned}
\phi_* &: T_p M \to T_{\phi(p)} N & \qquad \phi_* &: TM \to TN \\
\phi^* &: \Lambda^q T_{\phi(p)} N^* \to \Lambda^q T_p M^* & \qquad \phi^* &: \Lambda^q TN^* \to \Lambda^q TM^*.
\end{aligned}$$

Assume that x^i and y^j are local coordinates on M and N and that $\phi : (x) \mapsto (y^j = \phi^j(x))$. The tangent map then acts as

$$\phi_* \frac{\partial}{\partial x^i} = \frac{\partial \phi^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

and the cotangent map as

$$\phi^* dy^j = \frac{\partial \phi^j}{\partial x^i} dx^i.$$

Given a vector field v, $\phi_* v$ is called the *pushforward* of v by ϕ and given a form ω, $\phi^*\omega$ is called the *pullback* of ω by ϕ.

Lemma A.2. *Let ϕ, M and N be as above. Let also $\omega \in \Lambda^q TN^*$, $\eta \in \Lambda^r TN^*$ and $v, v_i \in TM$. Then*

$$\begin{aligned} \phi^*(\omega \wedge \eta) &= (\phi^*\omega) \wedge (\phi^*\eta) \\ d(\phi^*\omega) &= \phi^* d\omega \\ (\phi^*\omega)(v_1, \ldots, v_q) &= \omega(\phi_* v_1, \ldots, \phi_* v_q) \end{aligned}$$

Proof. Again, see Chapter 2 of [98]. □

A.2 Frobenius and Friends

A.2.1 Cauchy Characteristic and Retracting Spaces

On a finite dimensional manifold M, consider a (locally constant dimensional) codistribution $\Omega \subset TM^*$ and the distribution $V \subset TM$ of vectors annihilating ω, i.e. $V = \perp_{TM^*} \Omega$. The distribution of vector fields in TM given by

$$\begin{aligned} \operatorname{Char}\Omega &= \{v \in TM \mid v \lrcorner\, \omega = 0,\ v \lrcorner\, d\omega \in \Omega,\ \forall \omega \in \Omega\} \\ &= \{v \in V \quad \mid \qquad\qquad v \lrcorner\, d\omega \in \Omega,\ \forall \omega \in \Omega\} \end{aligned}$$

is called the set of *Cauchy characteristic vector fields* of Ω. Note that $\operatorname{Char}\Omega$ is a subset of V. The dual of $\operatorname{Char}\Omega$

$$\operatorname{Retr}\Omega = \perp_{TM} \operatorname{Char}\Omega$$

defines the *retracting space* of Ω.

Lemma A.3.

$$\operatorname{Char}\Omega = \{v \in V \mid [v, V] \subset V\}$$

Proof. For all $\bar{v} \in V$ and $\omega \in \Omega$, and v such that $[v, \bar{v}] \in V$. The Cartan formula yields

$$\begin{aligned} 0 &= [v, \bar{v}] \lrcorner\, \omega = v(\bar{v} \lrcorner\, \omega) - \bar{v} \lrcorner\, v(\omega) = -\bar{v} \lrcorner\, v(\omega) \\ &= -\bar{v} \lrcorner\, (v \lrcorner\, d\omega + d(v \lrcorner\, \omega)) = -\bar{v} \lrcorner\, (v \lrcorner\, d\omega). \end{aligned}$$

□

The characteristic and retracting spaces have an important property.

Proposition A.4. *With the notations as above, assume that* $\operatorname{Char}\Omega$ *has a constant dimension. Then* $\operatorname{Char}\Omega$ *is involutive and* $\operatorname{Retr}\Omega$ *is integrable. Moreover, setting* $n = \dim M$, $s = \dim \operatorname{Retr}\Omega$ *and* $r = \dim \Omega$, *there are local coordinates* $(y^1, \ldots, y^s, z^1, \ldots, z^{n-s})$ *on* M *such that* Ω *has a set of generators involving* $\{y^1, \ldots, y^s\}$ *only, i.e. there locally exists functions* $\alpha_i^j(y)$ *such that*

$$\Omega = \{\alpha_i^1(y)\ dy^i,\ \ldots\ ,\ \alpha_i^r(y)\ dy^i\}.$$

The number s *is minimal.*

Proof. See [13], Theorem 2.2, p. 31. □

A.2.2 Pulled-back Codistributions and Projectable Distributions

Given a map between two manifolds $\phi : M \to N$ and a codistribution $J \subset TN^*$, we shall write $I = \phi^* J \subset TM^*$ for the codistribution on M generated by all forms $\phi^*\mu$, $\mu \in J$. In other words, if $\{\mu^1, \ldots, \mu^p\}$ is a basis of $J \subset TN$, then $\phi^* J$ is the well-defined codistribution

$$I = \phi^* J \subset TM^* \quad \text{with basis} \quad \{\phi^*\mu^1, \ldots, \phi^*\mu^p\}. \tag{A.1}$$

With these notations, Proposition A.4 admits the following interpretation.

Corollary A.5. *Let* $\Omega \subset M$ *be an* r*-dimensional codistribution on an* n*-dimensional manifold* M. *Let* $s = \dim \operatorname{Retr}\Omega$. *There exists an* s*-dimensional manifold* N, *a (local) surjective map* $\pi : M \to N$ *and an* r*-dimensional codistribution* $\hat{\Omega} \subset TN^*$ *such that*

$$\Omega = \pi^* \hat{\Omega}.$$

Moreover

$$\ker \pi_* = \operatorname{Char}\Omega.$$

Proof. Use Proposition A.4 and set $\pi : (y^1, \ldots, y^s, z^1, \ldots, z^{n-s}) \mapsto (y^1, \ldots, y^s)$. □

Similarly, given a distribution $F \subset TM$, a surjective map $\pi : M \to N$ and a distribution $H \subset TN$, we will write

$$\pi_* F = H \quad \text{if} \quad \forall x \in M\ \pi_*(F|_x) = H|_{\pi(x)} \tag{A.2}$$

where $F|_x$ is the $\mathbb{R}$-vector space spanned by the vectors of F at the point $x \in M$. Note that $\pi_* F$ is not always well-defined. Moreover, even if $\pi_* F$ is well-defined, $\pi_* f$ for $f \in F$ is not necessarily a well-defined vector field on TN, as the next example shows.

Example A.6. *Consider* M *and* N *with two local sets of coordinates* (x, y, z) *and* (z) *respectively. Let the map* π *be given by* $\pi : (x, y, z) \mapsto (z)$. *Then with* $F = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\}$ *and* $H = \{\frac{\partial}{\partial z}\}$ *we have* $\pi_* F = H$. *But the vector field* $f = x\frac{\partial}{\partial z} \in F$ *is such that* $\pi_* f$ *is not a well-defined vector field in* TN.

One can link the two notions of (A.1) and (A.2) with the following lemma.

Lemma A.7. *Consider two manifolds M, N and a surjective projection $\pi : M \to N$. Let $J \subset TN^*$ be a codistribution on N and $I \subset TM^*$ such that $I = \pi^* J$, as in* (A.1)*. Consider $H \subset TN$ and $F \subset TM$ such that $H = \perp_{TN^*} J$ and $F = \perp_{TM^*} I$. Then $\pi_* F = H$ as in* (A.2)*.*

Proof. Since π is surjective, $\ker \pi^* = 0$ so that $\dim I = \dim J$, and $\dim M = \dim N + \dim \ker \pi_*$. By construction, $\dim H = \dim N - \dim J$ and $\dim F = \dim M - \dim I$. Also, for any $X \in \ker \pi_*$ and any $\mu \in J$, $X \lrcorner\, \pi^* \mu = \pi_* X \lrcorner\, \mu = 0$ so that $\ker \pi_* \subset F$. Using the above relations, we deduce that

$$\dim \pi_* F|_x = \dim N - \dim J = \dim H|_{\pi(x)}.$$

Hence, if we can show that $\pi_* F|_x \subset H|_{\pi(x)}$, we are done. Indeed, $\forall f \in F$ and $\forall \mu \in J$, $\pi_* f \lrcorner\, \mu = f \lrcorner\, \pi^* \mu = 0$ since $\pi^* \mu \in I = \perp_{TM} F$, implying $\pi_* F|_x \subset H|_{\pi(x)}$. □

Although (A.2) is a pointwise relation, there is a relation between vector fields in H and vector fields in F.

Lemma A.8. *Let M, N and π be as in* (A.2) *and let* (A.2) *be satisfied. Then, for any vector field $h \in H$, there exists a vector filed $f \in F$ such that $h = \pi_* f$. The two vector fields f and h are π-related.*

Proof. A vector field $h \in H \subset TN$ may always be lifted to a vector field $\hat{h} \in TM$ in such a way that $\pi_* \hat{h} = h$. This can be done by choosing a connection on the fibered manifold $\pi : M \to N$, see e.g. [59] p. 58 or [98]. Next define the codistribution $J \in TN^*$, $J = \perp_{TN} H$. For any $\mu \in J$ we have $\pi_* \hat{h} \lrcorner\, \mu = \hat{h} \lrcorner\, \pi^* \mu = 0$, therefore $\hat{h} \in F$. Set $f = \hat{h}$. □

The previous lemma has an immediate useful consequence regarding integrability.

Corollary A.9. *Let M, N be two manifolds and $\pi : M \to N$ a surjective projection map.*

i) Let $\hat{\Omega} \subset TN^*$ and $\Omega \in TM^*$ be such that $\Omega = \pi^* \hat{\Omega}$. Then Ω is integrable if and only if $\hat{\Omega}$ is.

ii) Let $F \subset TM$ and $H \subset TN$ be such that $\pi_* F = H$. Then $\pi_*(F + [F, F]) = H + [H, H]$.

Proof. To prove *i)*, define $F \subset TM$ and $H \subset TN$ such that $F = \perp_{TM^*} \Omega$ and $H = \perp_{TN^*} \hat{\Omega}$. If Ω is integrable, then F is involutive. Take any two vector fields $h_1, h_2 \in H$; by Lemma A.8, there are two vector fields $f_1, f_2 \in F$ such that $\pi_* f_i = h_i$. We have $\pi_* [f_1, f_2] = [h_1, h_2]$ and $[f_1, f_2] \in F$, hence $[h_1, h_2] \in H$ and H is involutive so that $\hat{\Omega}$ is integrable. The converse is clear.
To show *ii)*, choose a basis $\{h_1, \ldots, h_\rho\}$ of H. By Lemma A.8, there are $f_1, \ldots, f_\rho \in F$ such that $\pi_* f_i = h_i$. Since $\pi_* F = H$ and π is surjective, there exists a basis of F of the form $F = \{f_1, \ldots, f_\rho, k_1, \ldots, k_r\}$ with $k_j \in \ker \pi_*$. Then, $F + [F, F]$ is spanned by $\{f_i, k_j, [f_{i_1}, f_{i_2}], [f_i, k_j], [k_{j_1}, k_{j_2}]\}$. But by Lemma A.11, $[f_i, k_j] \in F + \ker \pi_*$ and clearly, $[k_{j_1}, k_{j_2}] \in \ker \pi_*$. □

We now consider the following question. One is given (finite dimensional) manifolds M, N and a (local) surjective projection map $\pi : M \to N$. Next, a codistribution $\Omega \subset TM^*$ is specified. Under what condition, does there exist a codistribution $\hat{\Omega} \subset TN^*$, such that

$$\Omega = \pi^* \hat{\Omega}$$

and $\dim \hat{\Omega} = \dim \Omega$? Similarly, given a distribution $V \subset TM$, is there a well defined distribution $\hat{V} \in TN$ such that

$$\pi_* V = \hat{V}$$

i.e. do the vectors $\pi_* v, \forall v \in V$ (locally) span a well defined distribution on N?
When the answer is negative, we will see that one can construct a largest codistribution contained in Ω satisfying the requirement, respectively a smallest distribution containing $V + \ker \pi_*$. The answer to the first question is clearly related to Corollary A.5.

Lemma A.10. *Given the surjective map $\pi : M \to N$ and a codistribution $\Omega \subset TM^*$ as above, there exists a codistribution $\hat{\Omega} \subset TN^*$ such that $\Omega = \pi^* \hat{\Omega}$ if and only if*

$$\forall \omega \in \Omega,\ \forall X \in \ker \pi_* : \qquad X \lrcorner\, \omega = 0, \quad X \lrcorner\, d\omega \in \Omega.$$

Proof. The condition shows that $\ker \pi_* \subset \operatorname{Char} \Omega$. By Corollary A.5, there exists a manifold $\tilde{N}$ and a surjective map $\tilde{\pi} : M \to \tilde{N}$, $\ker \tilde{\pi}_* = \operatorname{Char} \Omega$ and a codistribution $\tilde{\Omega} \in T\tilde{N}$ such that $\Omega = \tilde{\pi}^* \tilde{\Omega}$. Hence, locally, we may find coordinates (y^i, z^j, w^k) on M such that

$$\begin{aligned} \pi &: (y^i, z^j, w^k) \mapsto (y^i, z^j) \\ \tilde{\pi} &: (y^i, z^j, w^k) \mapsto (y^i) \end{aligned}$$

From the coordinate expressions above, we see that there locally exists a map $\hat{\pi} : N \to \tilde{N}$ given by

$$\hat{\pi} : (y^i, z^j) \mapsto (y^i)$$

and such that $\tilde{\pi} = \hat{\pi} \circ \pi$. Define $\hat{\Omega} \in TN^*$, $\hat{\Omega} = \hat{\pi}^* \tilde{\Omega}$. Clearly, $\Omega = \pi^* \hat{\Omega}$.
Conversely, Assume there is a codistribution $\hat{\Omega} \subset TN^*$ such that $\Omega = \pi^* \hat{\Omega}$. Take $\{\hat{\omega}^i\}$ a basis of $\hat{\Omega}$. Then any $\omega \in \Omega$ is of the form

$$\omega = \alpha_i\ \pi^* \hat{\omega}^i$$

for some functions α_i on M. It follows that for any $X \in \ker \pi_*$

$$\begin{aligned} X \lrcorner\, \omega &= X \lrcorner\, (\alpha_i\ \pi^* \hat{\omega}^i) = \alpha_i\ X \lrcorner\, \pi^* \hat{\omega}^i \\ &= \alpha_i (\pi_* X) \lrcorner\, \hat{\omega}^i = 0 \\ X \lrcorner\, d\omega &= X \lrcorner\, d(\alpha_i\ \pi^* \hat{\omega}^i) = X \lrcorner\, (d\alpha_i \wedge \pi^* \hat{\omega}^i + \alpha_i\ d(\pi^* \hat{\omega}^i)) \\ &= X(\alpha_i)\ \pi^* \hat{\omega}^i + \alpha_i X \lrcorner\, \pi^* d\hat{\omega}^i = X(\alpha_i)\ \pi^* \hat{\omega}^i + \alpha_i (\pi_* X) \lrcorner\, d\hat{\omega}^i \\ &= X(\alpha_i)\ \pi^* \hat{\omega}^i \in \Omega. \end{aligned}$$

□

Let us now answer the second question.

Lemma A.11. *Given the surjective map $\pi : M \to N$ and a distribution $V \subset TM$, the set of vectors $\pi_*(v|_x) \in TN|_{\pi(x)}$ for all $v \in V$ generates a well defined distribution on TN, denoted by $\pi_* V$ if and only if*

$$[V, \ker \pi_*] \subset V + \ker \pi_*.$$

Proof. The augmented distribution $V + \ker \pi_*$ clearly projects to the same set than V under π_*. Hence we may assume that $\ker \pi_* \subset V$ and replace the condition by $[V, \ker \pi_*] \subset V$. Next, $\forall \omega \in \Omega = \perp_{TM} V$, $\forall v \in V$ and $\forall X \in \ker \pi_*$ we have $v \lrcorner\, \omega = X \lrcorner\, \omega = 0$ and from the (modified) assumption it follows that

$$\begin{aligned} 0 &= [v, X] \lrcorner\, \omega = v(X \lrcorner\, \omega) - X \lrcorner\, v(\omega) = -X \lrcorner\, v(\omega) \\ &= -X \lrcorner\, (v \lrcorner\, d\omega + d(v \lrcorner\, d\omega)) = v \lrcorner\, X \lrcorner\, d\omega \\ \Rightarrow\ & X \lrcorner\, d\omega \in \Omega. \end{aligned}$$

Hence, by Lemma A.10, $\Omega = \pi^* \hat{\Omega}$ for some $\hat{\Omega} \subset TN^*$. The distribution $\pi_* V$ is then given by $\perp_{TN^*} \hat{\Omega}$. □

Assume the conditions of Lemma A.10 are not satisfied. The following result produces the largest codistribution $\Omega^{(r^*)} \subset \Omega$ for which Lemma A.10 applies.

Lemma A.12. *Consider the surjective map $\pi : M \to N$ and a codistribution $\Omega \subset TM^*$. Compute the following sequence of nested codistributions*

$$\Omega^{(0)} = \Omega \cap \perp_{TM} \ker \pi_* \qquad \Omega^{(r+1)} = \{\omega \in \Omega^{(r)} \mid X \lrcorner\, d\omega \in \Omega^{(r)}, \forall X \in \ker \pi_*\}.$$

There is an integer r^*, $0 \le r^* \le \dim \Omega$ such that $\dim \Omega^{(r)}$ is a strictly decreasing sequence for $r = 0, \ldots, r^*$ and $\dim \Omega^{(r)}$ is constant for $r \ge r^*$. Moreover, $\Omega^{(r^*)} \subset \Omega$ is the largest codistribution in Ω satisfying Lemma A.10.

Proof. Since $\Omega^{(r)} \subset \Omega^{(r+1)}$, if $\dim \Omega^{(r+1)} = \dim \Omega^{(r)}$ for some r, then $\Omega^{(r+1)} = \Omega^{(r)}$ and $\Omega^{(r+k)} = \Omega^{(r)}$ for all $k \ge 0$. Let r^* be the smallest such number, then we see that $\Omega^{(r^*)}$ satisfies Lemma A.10.
Take $\bar{\Omega} \subset \Omega$ an arbitrary codistribution in Ω satisfying Lemma A.10. Assume that $\bar{\Omega} \subset \Omega^{(r)}$ also holds for some r. Then, $\forall X \in \ker \pi_*$ and $\forall \omega \in \bar{\Omega}$, $X \lrcorner\, d\omega \in \bar{\Omega} \subset \Omega^{(r)}$. Therefore $\bar{\Omega} \subset \Omega^{(r+1)}$ and by induction $\bar{\Omega} \subset \Omega^{(r^*)}$. Hence, $\Omega^{(r^*)}$ is the largest codistribution in $\Omega = \Omega^{(0)}$ satisfying Lemma A.10.

□

A.3 Derived Systems and Filtrations

Given a distribution (or codistribution) E on some manifold M, a derived system $E^{(1)}$ computed from E is an other distribution (or codistribution) contained in or containing E. By duality, to the derived system of some distribution always corresponds a derived

system of the codistribution annihilating it and vice versa.
Suppose that the rule for computing a derived system $E^{(1)}$ of E has been specified. One may define a recursion

$$E^{(0)} = E \qquad\qquad E^{(k+1)} = (E^{(k)})^{(1)} \quad \forall k \geq 0$$

thereby producing an ascending or descending sequence of nested vector spaces or modules

$$E = E^{(0)} \supset E^{(1)} \supset E^{(2)} \supset \dots$$

or

$$\dots \subset E^{(2)} \subset E^{(1)} \subset E = E^{(0)}.$$

Such a sequence is called a *filtration* of E. Note that in the finite dimensional case, a filtration always saturates after a finite number of steps.

A.3.1 Dual Derived Systems

In this section, we give instances of useful derived systems of distributions and codistributions on finite dimensional manifolds. In each case, we state a duality result that allows one to perform the computation either on the distribution or on its annihilating codistribution.
Consider the distributions V, W and a codistribution Ω on some manifold M, we shall use the following notations where $\{v_k\}$, $\{w_r\}$ and $\{\omega^j\}$ are given bases of V, W and Ω :

$$[V, V] = \{[v_{k_1}, v_{k_2}]\} \qquad [V, W] = \{[v_k, w_r]\}$$

$$V\Omega = \{v_k(\omega^j)\}.$$

Importantly, note that $[V, V]$, $[V, W]$ and $V\Omega$ are *not* independent of the chosen bases, however $V + [V, V]$, $V + W + [V, W]$ and $\Omega + V\Omega$ are. By

$$\Omega \wedge \Omega$$

we shall either mean the ideal in ΛTM^* generated by the elements $\omega^{j_1} \wedge \omega^{j_2}$ or the submodule of $\Lambda^2 TM^*$ generated by the same elements.

Lemma A.13. *Let M be a finite dimensional manifold and $V \subset TM$ a distribution. Let $I \subset \Lambda TM^*$ be the ideal such that $I \cap \Lambda^1 TM^* \equiv \perp_{TM} V$. That is, I is the ideal generated by the 1-forms annihilating V. Then*

$$I^{(1)} := \{\omega \in I \mid d\omega \in I\} \qquad \textit{and} \qquad V^{(1)} := V + [V, V]$$

are such that

$$I^{(1)} \cap \Lambda^1 TM^* \ \equiv \perp_{TM} V^{(1)}$$

Proof. Let ω be any 1-form of I, i.e. $\omega \in I^{(1)} \cap \Lambda^1 TM^*$ and v_1, v_2 any two elements of V. Hence $v_1 \lrcorner\, \omega = v_2 \lrcorner\, \omega = 0$. The Cartan formula then shows that

$$\begin{aligned}[v_1, v_2] \lrcorner\, \omega &= v_1(v_2 \lrcorner\, \omega) - v_1 \lrcorner\, v_2(\omega) = -v_1 \lrcorner\, v_2(\omega) \\ &= -v_1 \lrcorner\, (v_2 \lrcorner\, d\omega + d(v_2 \lrcorner\, \omega)) = -v_1 \lrcorner\, (v_2 \lrcorner\, d\omega).\end{aligned}$$

But for the 2-form $d\omega$, $-v_1 \lrcorner\, (v_2 \lrcorner\, d\omega) = d\omega(v_2, v_1) = 0$ if and only $d\omega \in I$. □

Lemma A.14. *Let M be a finite dimensional manifold and $V, W \subset TM$ distributions. Let also $\Omega, \Theta \subset TM^*$ be codistributions $\Omega = \perp_{TM} V$ and $\Theta = \perp_{TM} W$. Then*

$$\begin{aligned}\Phi :&= \{\mu \in \Omega \cap \Theta \ \mid v \lrcorner\, w \lrcorner\, d\mu = 0, \forall v \in V, \forall w \in W\} \qquad Z = V + W + [V, W] \\ &= \{\mu \in \Omega \cap \Theta \ \mid w \lrcorner\, d\mu \in \Omega, \quad \forall w \in W\} \\ &= \{\mu \in \Omega \cap \Theta \ \mid v \lrcorner\, d\mu \in \Theta, \quad \forall v \in V\}\end{aligned}$$

are such that $\Phi = \perp_{TM} Z$.

Proof. Use the Cartan formula to show that

$$\begin{aligned}[v, w] \lrcorner\, \mu &= v(w \lrcorner\, \mu) - w \lrcorner\, v(\mu) = -w \lrcorner\, v(\mu) \\ &= -w \lrcorner\, (v \lrcorner\, d\mu + d(v \lrcorner\, \mu)) = v \lrcorner\, w \lrcorner\, d\mu\end{aligned}$$

whenever $v \lrcorner\, \mu = w \lrcorner\, \mu = 0$. □

Lemma A.15. *Let M be a finite dimensional manifold, $V \subset TM$ a distribution and $\Omega \subset TM^*$ the codistribution satisfying $\Omega = \perp_{TM} V$. Let also $W \subset V$ be another distribution. Then*

$$\Omega^{(1)} := \Omega + W\Omega \qquad \textit{and} \qquad V^{(1)} := \{v \in V \mid [w, v] \in V, \ \forall w \in W\}$$

are such that

$$\Omega^{(1)} = \perp_{TM} V^{(1)}.$$

Proof. The inclusion $W \subset V$ ensures that the definition of $V^{(1)}$ is independent of the choices of bases for W and V. Let ω, v and w be any three elements such that $\omega \in \Omega$, $v \in V$ and $w \in W$. Hence $v \lrcorner\, \omega = 0$. The Cartan formula then implies

$$v \lrcorner\, w(\omega) = w(v \lrcorner\, \omega) - [v, w] \lrcorner\, \omega = [w, v] \lrcorner\, \omega.$$

Under these conditions, $[w, v] \lrcorner\, \omega = 0$ if and only if $v \lrcorner\, w(\omega) = 0$. □

Lemma A.16. *Let M be a finite dimensional manifold and $V \subset TM$ a distribution. Let $\Omega \subset TM^*$ be the codistribution such that $\Omega = \perp_{TM} V$. Let also $W \subset V$ be another distribution. Then*

i)

$$\Omega^{(1)} := \{\omega \in \Omega \mid w(\omega) \in \Omega, \ \forall w \in W\} \qquad \textit{and} \qquad V^{(1)} := V + [W, V]$$

are such that

$$\Omega^{(1)} = \perp_{TM} V^{(1)}.$$

ii) Moreover, set $\Phi \subset TM^*$, $\Phi = \perp_{TM} W$ *such that* $\Omega \subset \Phi$ *and denote by* $\{\Omega, \Phi \wedge \Phi\}$ *the ideal in* ΛTM^* *generated by the* 1*-forms* $\omega \in \Omega$ *and the* 2*-forms* $\phi^1 \wedge \phi^2$*, with* $\phi^1, \phi^2 \in \Phi$*. Then* $\Omega^{(1)}$ *may also be computed as*

$$\Omega^{(1)} = \{\omega \in \Omega \mid d\omega \in \{\Omega, \Phi \wedge \Phi\}\}$$

iii) Additionally assuming that W *is involutive, i.e.* $[W, W] \subset W$ *and choosing any* $\bar{V} \subset V$ *such that* $V = \bar{V} + W$*, we also obtain the same systems as in i) by computing*

$$\Omega^{(1)} := \{\omega \in \Omega \mid \bar{v}(\omega) \in \Omega,\ \forall \bar{v} \in \bar{V}\} \qquad \textit{and} \qquad V^{(1)} := V + \left[\bar{V}, W\right]$$

Proof. i) Since $W \subset V$, $V + [W, V]$ is independent of the choice of bases. The equivalence follows from the same formula as in the proof of Lemma A.15.
ii) Again as in Lemma A.15, for all $w \in W, v \in V, \omega \in \Omega$, $[w, v] \lrcorner\, \omega = 0$ if and only if $v \lrcorner\, w(\omega) = 0$. Moreover,

$$0 = v \lrcorner\, w(\omega) = v \lrcorner\, (w \lrcorner\, d\omega + d(w \lrcorner\, \omega)) = v \lrcorner\, w \lrcorner\, d\omega.$$

In particular and since $W \subset V$,

$$w_1 \lrcorner\, w_2 \lrcorner\, d\omega = 0 \quad \forall w_1, w_2 \in W\ \omega \in \Omega^{(1)}$$

so that $d\omega \in \Phi$, with Φ taken as an ideal. Therefore, assuming that $\Omega = \{\omega^j\}$ and $\Phi = \{\omega^j, \phi^s\}$, there must exist $\alpha_j, \beta_s \in TM^*$ such that

$$d\omega = \alpha_j \wedge \omega^j + \beta_s \wedge \phi^s.$$

Hence, for all $w \in W$, $v \in V$, $\omega \in \Omega^{(1)}$

$$\begin{aligned} 0 &= v \lrcorner\, w \lrcorner\, d\omega \\ &= v \lrcorner\, ((w \lrcorner\, \alpha_j)\omega^j + (w \lrcorner\, \beta_s)\phi^s) \\ &= (w \lrcorner\, \beta_s)(v \lrcorner\, \phi^s). \end{aligned}$$

A rank argument shows that this implies that $w \lrcorner\, \beta_s = 0$, therefore $\beta_s \in \Phi$ which in turn implies $d\omega \in \{\Omega, \Phi \wedge \Phi\}$. Conversely, assume $\omega \in \Omega$ and $d\omega \in \{\Omega, \Phi \wedge \Phi\}$. Then there are 1-forms α_j and functions γ_{kr} s.t.

$$d\omega = \alpha_j \wedge \omega^j + \gamma_{kr}\phi^k \wedge \phi^r$$

Hence, $\forall v \in V$

$$v \lrcorner\, d\omega = (v \lrcorner\, \alpha_j)\omega^j + \gamma_{kr}((v \lrcorner\, \phi^k)\phi^r - (v \lrcorner\, \phi^r)\phi^k) \in \Phi$$

so that $\forall w \in W$

$$\begin{aligned} 0 &= w \lrcorner\, v \lrcorner\, d\omega = w \lrcorner\, (v(\omega) - d(v \lrcorner\, \omega)) = w \lrcorner\, v(\omega) \overset{\text{Cartan}}{=} v(w \lrcorner\, \omega) - [v, w] \lrcorner\, \omega \\ &= -[v, w] \lrcorner\, \omega \end{aligned}$$

which shows that $\omega \in \perp_{TM} (V + [W, V]) = \Omega^{(1)}$.
To show *iii)* notice that $V + \left[\bar{V}, W\right]$ is indeed independent of the choice of generators for $\bar{V}$ and

$$V + [W, V] = V + \left[W, \bar{V} + W\right] = V + \left[W, \bar{V}\right] + [W, W] = V + \left[W, \bar{V}\right].$$

On the other hand, if $\omega \in \Omega$ and $w_1, w_2 \in W$, then $[w_1, w_2] \lrcorner\, \omega = 0$, so that it suffices to satisfy the relations $[\bar{v}, v] \lrcorner\, \omega = 0$ for all $v \in V$ and $\bar{v} \in V, \bar{v} \notin W$. □

In Lemma A.15 and A.16 *i)* above, the requirement that W is a subset of V may be relaxed. The cost is that the result of the computed derived system is no more independent of the choice of basis of W (but remains independent of the choice of basis of V). Hence, the distribution W must be replaced by a fixed set of elements. Therefore, in the following we replace W by a single vector field $D \in TM$. Note that the result generalizes without effort to the case of a set of vectors $(D_1, \ldots, D_\delta)$.

Lemma A.17. *Let M be a finite dimensional manifold and $V \subset TM$ a distribution. Let $\Omega \subset TM^*$ be the codistribution such that $\Omega = \perp_{TM} V$. Let also $D \in TM$ be any vector field in TM. Define*

i)
$$\Omega^{(1)} := \{\omega \in \Omega \mid D(\omega) \in \Omega\} \qquad \textit{and} \qquad V^{(1)} := V + [D, V]$$

ii)
$$\Omega^{(1)} := \Omega + D\Omega \qquad \textit{and} \qquad V^{(1)} := \{v \in V \mid [D, v] \in V\}$$

Then, in both cases i) and ii) the derived systems verify

$$\Omega^{(1)} = \perp_{TM} V^{(1)}.$$

Proof. From the Cartan formula, we have that for $v \in V$ and $\omega \in \Omega$, $[D, v] \lrcorner\, \omega = 0$ if and only if $v \lrcorner\, D\omega = 0$. □

Bibliography

[1] R. L. Anderson and N. H. Ibragimov. *Lie-Bäcklund Transformations in Applications.* SIAM, 1979.

[2] F. Antritter and J. Lévine. Toward a computer algebraic algorithm for flat output determination. *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 7–14, 2008.

[3] F. Antritter and J. Lévine. Flatness characterization: Two approaches. In J. Lévine and Ph. Mullhaupt, editors, *Advances in the Theory of Control, Signal and Systems with Physical Modeling*, volume 407, pages 127–139, 2010.

[4] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet. Infinitesimal Brunovský normal form for nonlinear system with application to dynamic linearization. In B. Jakubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pages 19–33. Banach Center publications, 1995.

[5] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet. A linear algebraic framework for dynamic feedback linearization. *IEEE Transactions on Automatic Control*, 40(1):127–132, January 1995.

[6] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer, 2nd edition, 1988.

[7] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer, 2nd edition, 1989.

[8] D. Avanessoff. *Linéarisation dynamique des systèmes non linéaire et paramétrage de l'ensemble des solutions.* PhD thesis, Université de Nice Sophia Antipolis, 2005.

[9] A. Baker. *Matrix Groups, An Introduction to Lie Group Theory.* Springer, 2002.

[10] N. Berline and C. Sabbah, editors. *Aspects de la théorie du contrôle.* Les Éditions de l'École Polytechnique, 2008.

[11] R. M. Bianchini and G. Stefani. Graded approximations and controllability along a trajectory. *SIAM J. Control And Optimization*, 28(4):903–924, 1990.

[12] G. E. Bredon. *Topology and Geometry.* Springer, 1993.

[13] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior Differential Systems.* Springer Verlag, 1991.

[14] R. L. Bryant, Ph. Griffiths, and D. Grossman. *Exterior Differential Systems and Euler-Lagrange Partial Differential Equations.* The University of Chicago Press, 2003.

[15] C. I. Byrnes and A. J. Krener. On the existence of globally (f, g)-invariant distributions. In R. W. Brockett, R. S. Millman, and H. J. Sussmann, editors, *Differential Geometric Control Theory*, pages 209–225, 1983.

[16] B. Charlet, J. Lévine, and R. Marino. On dynamic feedback linearization. *Systems & Control Letters*, 13:143–151, 1989.

[17] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optimization*, 29(1):38–57, 1991.

[18] C.-T. Chen. *Linear System Theory and Design.* Oxford University Press, 3rd edition, 1999.

[19] V. N. Chetverikov. On the structure of integrable C-fields on infinitely extended equations. *Soviet. Math. Dokl.*, 33(1):45–48, 1986.

[20] V .N. Chetverikov. On the structure of integrable C-fields. *Differential Geometry and its Applications*, 1:309–325, 1991.

[21] V .N. Chetverikov. Flatness conditions for control systems. *DIPS-01 http://diffiety.ac.ru*, 2002.

[22] V .N. Chetverikov. Higher symmetries and the brunovskii infinitesimal form of controlled systems. *Differential Equations*, 38(11):1619–1627, 2002.

[23] V. N. Chetverikov. Flat control systems and deformations of structures on diffieties. *Forum Math.*, 16:903–923, 2004.

[24] V .N. Chetverikov. Flatness of dynamic feedback linearizable systems. *Differential Equations*, 40(12):1665–1674, 2004.

[25] V. N. Chetverikov. A nonlinear spencer complex for the group of invertible differential operators and its applications. *Acta Applicandae Mathematicae*, 83:1–23, 2004.

[26] V .N. Chetverikov. Linearization method for the flatness problem and for finding the compatibility operator. *Differential Equations*, 42(10):1479–1489, 2006.

[27] V.N. Chetverikov, A.N. Kanatnikov, and A.P. Krishchenko. Classical and higher symmetries of control systems. In *Preprints of the 15th World Congress IFAC 2002, Barcelona, Spain*, 2002.

[28] P. M. Cohn. *Free Rings and their Relations.* Academic Press, London, 1985.

[29] G. Conte, A. M. Perdon, and C. H. Moog. The differential field associated to a general analytic nonlinear dynamical system. *IEEE Transactions on Automatic Control*, 38(7):1120–1124, 1993.

[30] J.-M. Coron. Linearized control systems and applications to smooth stabilization. *SIAM J. Control And Optimization*, 32(2):358–386, 1994.

[31] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms.* Springer-Verlag, 1992.

[32] D. A. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Springer, 2nd edition, 2005.

[33] E. Delaleau and P. S. Pereira da Silva. Filtrations in feedback synthesis: Part I – Systems and feedbacks. *Forum Math.*, 10:147–174, 1998.

[34] E. Delaleau and P. S. Pereira da Silva. Filtrations in feedback synthesis: Part II – Input-output decoupling and disturbance decoupling. *Forum Math.*, 10:259–275, 1998.

[35] E. Delaleau and W. Respondek. Lowering the orders of derivatives of controls in generalized state space systems. *Journal of Mathematical Systems, Estimation, and Control*, 5(3):1–27, 1995.

[36] E. Delaleau and J. Rudolph. Control of flat systems by quasi-static feedback of generalized states. *International Journal of Control*, 71(5):745–765, 1998.

[37] M. D. Di Benedetto, Grizzle J. W., and C. H. Moog. Rank invariants of nonlinear systems. *SIAM J. Control And Optimization*, 27(3):658–672, 1989.

[38] P. T. Eendebak. *Contact Structures of Partial Differential Equations.* PhD thesis, Universiteit Utrecht, 2007.

[39] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, 1995.

[40] M. Fliess. Automatique et corps différentiels. *Forum Mathematicum*, 1:227–238, 1989.

[41] M. Fliess. Generalized controller canonical forms for linear and nonlinear dynamics. *IEEE Transactions on Automatic Control*, 35(9):994–1001, 1990.

[42] M. Fliess. Some basic structural properties of generalized linear systems. *Systems & Control Letters*, 15:391–396, 1990.

[43] M. Fliess, J. Lévine, Ph. Martin, F. Ollivier, and P. Rouchon. A remark on nonlinear accessibility conditions and infinite prolongations. *Systems & Control Letters*, pages 77–83, 1997.

[44] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *Comptes rendus de l'Académie des sciences, Paris*, 1(315):619–624, 1992.

[45] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. *Comptes rendus de l'Académie des sciences, Paris*, 1(317):981–986, 1993.

[46] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Difffential flatness and defect: An overview. In B. Jakubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pages 209–225. Banach Center publications, 1995.

[47] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of non-linear systems: Introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.

[48] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Deux applications de la géométrie locale des diffiétés. *Annales de l'I. H. P., section A*, 66(3):275–292, 1997.

[49] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on Automatic Control*, 44:922–937, 1999.

[50] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. *Open Problems in Mathematical Systems and Control Theory*, chapter "Some open questions related to flat nonlinear systems". Springer Verlag, 1999.

[51] M. Fliess, Ph. Martin, F. Ollivier, and P. Rouchon. Flatness and dynamic feedback linearizability: Two approaches. *ECC'95*, 1995.

[52] A. R. Forsyth. *Theory of Differential Equations, Part I. Exact Equations and Pfaff's Problem.* Dover Publications, New York, 1959.

[53] A. Frölicher and A. Nijenhuis. Theory of vector valued differential forms. part I. *Indagationes Mathematicae*, 18:338–360, 1956.

[54] G. Fu, M. Halás, Ü. Kotta, and Z. Li. Some remarks on Kähler differentials and ordinary differentials in nonlinear control theory. *Systems & Control Letters*, 60:699–703, 2011.

[55] R. B. Gardner and W. F. Shadwick. Feedback equivalence for general control systems. *Systems & Control Letters*, 15:15–23, 1990.

[56] R. B. Gardner and W. F. Shadwick. Symmetry and the implementation of feedback linearization. *Systems & Control Letters*, 15:25–33, 1990.

[57] R. B. Gardner and W. F. Shadwick. An algorithm for feedback linearization. *Differential Geometry and its Applications*, 1(2):153–158, 1991.

[58] R. B. Gardner and W. F. Shadwick. The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Transactions on Automatic Control*, 37(2):224–230, February 1992.

[59] G. Giachetta, L. Mangiarotti, and G. Sardanashvily. *New Lagrangian and Hamiltonian Methods in Field Theory*. World Scientific, 1997.

[60] R. Godement. *Introduction à la théorie des groupe de Lie*. Springer, 2003.

[61] E. J. B. Goursat. *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*. Hermann, Paris, 1891.

[62] K. A. Grasse. On controlled invariance for fully nonlinear systems. *International Journal of Control*, 56(5):1121–1137, 1992.

[63] L. Guo, Ph. J. Cassidy, W. F. Keigher, and W. Y. Sit, editors. *Differential Algebra and Related Topics*. World Scientific, 2002.

[64] D. Hartley and R. W. Tucker. A constructive implementation of the Cartan-Kähler theory of exterior differential systems. *J. Symbolic Computation*, 12:655–667, 1991.

[65] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

[66] R. Hermann and A. J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, 1977.

[67] R. M. Hirschorn. Invertibility of multivariable nonlinear systems. *IEEE Transactions on Automatic Control*, 24:855–865, 1979.

[68] R. M. Hirschorn. Invertibility of nonlinear control systems. *SIAM J. Control And Optimization*, 17:289–297, 1979.

[69] L. Hunt, R. Su, and G. Meyer. Global transformations of nonlinear systems. *IEEE Transactions on Automatic Control*, 28:24–31, 1983.

[70] A. Isidori. *Nonlinear Control Systems*. Springer Verlag, Berlin, Heidelberg, New York, second edition, 1989.

[71] A. Isidori, C. H. Moog, and A. De Luca. A sufficient condition for full linearization via dynamic feedback. In *Proceedings of 25th conference on decision and control*, pages 203–208, 1986.

[72] B. Jakubczyk and W. Respondek. On linearisation of control systems. *Bulletin de l'Académie Polonaise des Sciences*, 28(9-10):517–522, 1989.

[73] J. Johnson. Kähler differentials and differential algebra. *Annals of Mathematics*, 89(1):92–98, 1969.

[74] J.-P. Jouanolou. *Equations de Pfaff algébriques*. Springer, 1979.

[75] A.N. Kanatnikov and A.P. Krishchenko. Symmetries and decomposition of nonlinear systems. *Differential Equations*, 30(11):1735–1745, 1994.

[76] H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, Englewood Cliffs, N.J., 3rd edition, 2002.

[77] B. Kiss, J. Lévine, and Ph. Mullhaupt. Modelling, flatness and simulation of a class of cranes. *Periodica Polytechnica Ser. El. Eng.*, 43(3):215–225, 1999.

[78] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume I.* Wiley, 1996.

[79] E.R. Kolchin. *Differential algebra and algebraic groups.* Pure and Applied Mathematics, 54. New York-London: Academic Press. XVII, 446 p. $ 24.50 , 1973.

[80] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis.* Dover, 1975.

[81] A. J. Krener. A decomposition theory for differentiable systems. *SIAM J. Control And Optimization*, 15(5):813–829, 1977.

[82] A. J. Krener. Approximate linearization by state feedback and coordinate change. *Systems & Control Letters*, 5:181–185, 1984.

[83] A. Kushner, V. Lychagin, and V. Rubtsov. *Contact Geometry and Nonlinear Differential Equations.* Cambridge University Press, 2007.

[84] P. Lancaster. *The Theory of Matrices, Second Edition, With Applications.* Academic Press, San Diego, CA, 1985.

[85] S. Lang. *Algebra.* Springer-Verlag, 3rd edition, 2002.

[86] J. M. Lee. *Introduction to Smooth Manifolds.* Springer, 2006.

[87] J. Lévine. *Analysis and Control of Nonlinear Systems.* Springer, 2009.

[88] J. Lévine. On necessary and sufficient conditions for differential flatness. *Applicable Algebra in Engineering, Communication and Computing*, 22(1):47–90, 2011.

[89] J. Lévine and Ph. Mullhaupt, editors. *Advances in the Theory of Control, Signal and Systems with Physical Modeling.* Springer, 2010.

[90] J. Lévine and D. V. Nguyen. Flat output characterization for linear systems using polynomial matrices. *Systems & Control Letters*, 48(1):69–75, 2003.

[91] C. Lobry. Controlabilité des systèmes non linéaires. *SIAM J. Control*, 8(4):573–605, 1970.

[92] C. Lobry. Erratum: Controlabilité des systèmes non linéaires. *SIAM J. Control And Optimization*, 15(2), 1976.

[93] S. Lovett. *Differential Geometry of Manifolds.* AK Peters, 2010.

[94] E. L. Mansield. *Differential Gröbner Bases.* PhD thesis, University of Sydney, 1991.

[95] Ph. Martin. *Contribution à l'Étude des Systèmes Différentiellement Plats.* PhD thesis, École des Mines de Paris, 1992.

[96] Ph. Martin, R.M. Murray, and P. Rouchon. *Advances in the control of nonlinear systems*, chapter "Flat systems, equivalence and feedback". Springer, 2001.

[97] C. H. Moog. Nonlinear decoupling and structure at infinity. *Math. Control Signals Systems*, 1:257–268, 1988.

[98] S. Morita. *Geometry of Differential Forms.* Iwanami Series in Modern Mathematics, vol. 201. American Mathematical Society, 1998.

[99] H. Nijmeijer. Right-invertibility for a class of nonlinear control system: A geometric approach. *Systems & Control Letters*, 7:125–132, 1986.

[100] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer, 1990.

[101] P. J. Olver. Evolution equations possessing infinitely many symmetries. *J. Math. Phys.*, 18:1212–1215, 1977.

[102] P. J. Olver. *Applications of Lie Groups to Differential Equations.* Springer Verlag, New York, 1986.

[103] W. Pasillas-Lépine and W. Respondek. Contact systems and corank one involutive subdistributions. *Acta Applicandae Mathematicae*, 69:105–128, 2001.

[104] P. S. Pereira da Silva. Flatness of nonlinear control systems: a Cartan-Kähler approach. In *Mathematical Theory of Networks and Systems (MTNS'2000)*, pages 1–10, 2000.

[105] P. S. Pereira da Silva. Geometric properties of the dynamic extension algorithm. Technical report, Escola Politécnica da USP - BT/PTC, 2000.

[106] P. S. Pereira da Silva. Flatness of nonlinear control systems and exterior differential systems. In *Nonlinear control in the year 2000 volume 2*, pages 205–227, 2001.

[107] P. S. Pereira da Silva and C. C. Filho. Flatness based position/force control of constrained under-actuated robots. In *CDROM Anais Do XIII Congresso Brasileiro de Automática*, 2000.

[108] P. S. Pereira da Silva and C. C. Filho. Relative flatness and flatness of implicit systems. *SIAM Journal Control and Optimization*, 39(6):1929–1951, 2001.

[109] P. S. Pereira da Silva, H. B. Silveira, C. C. Filho, and S. Batista. An infinite dimensional differential-geometric approach for nonlinear systems: Part II - system. *In preparation.*

[110] P. S. Pereira da Silva, H. B. Silveira, C. C. Filho, and S. Batista. An infinite dimensional differential-geometric approach for nonlinear systems: Part I - $\mathbb{R}^A$-manifolds, diffieties and systems. *Submitted*, 2008.

[111] J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In B. Jakubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pages 319–339. Banach Center publications, 1995.

[112] M. Quincampoix. Singular perturbations for systems of differential inclusions. In B. Jakubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, volume 32, pages 341–349. Banach Center publications, 1995.

[113] M. Rathinam. *Differentially Flat Nonlinear Control Systems.* PhD thesis, California Institute of Technology, 1997.

[114] W. Respondek. On decomposition of nonlinear control systems. *Systems & Control Letters*, 1(5):301–308, 1982.

[115] W. Respondek. *New Trends in Nonlinear Dynamics and Control, LNCIS 295*, chapter "Symmetries and Minimal Flat Outputs of Nonlinear Control Systems", pages 65–86. Springer-Verlag Berlin Heidelberg, 2003.

[116] J.F. Ritt. *Differential Algebra.* New York: Dover Publications, Inc. VIII, 184 p. , 1955.

[117] P. Rouchon. Necessary conditions and genericity of dynamic feedback linearization. *Journal of Mathematical Systems, Estimation, and Control*, 4(2):1–14, 1994.

[118] M. K. Sain and J. L. Massey. Invertibility of linear time-invariant dynamical systems. *IEEE Transactions on Automatic Control*, 14:141–149, 1969.

[119] D. J. Saunders. *The Geometry of Jet Bundles.* Cambridge University Press, 1989.

[120] H. Schenck. *Computational Algebraic Geometry.* London Mathematical Society, 2003.

[121] K. Schlacher and M. Schöberl. Construction of flat outputs by reduction and elimination. *CD Proceedings IFAC Symposium on Nonlinear Control Systems NOLCOS 2007*, pages 666–671, 2007.

[122] M. Schöberl and K. Schlacher. On parametrizations for a special class of nonlinear systems. In *8th IFAC Symposium on Nonlinear Control Systems*, pages 1261–1266, 2010.

[123] M. Schöberl and K. Schlacher. Some remarks on affine derivative systems concerning flatness. In *Proc. Appl. Math. Mech.*, volume 10, pages 631–632, 2010.

[124] M. Schöberl and K. Schlacher. On calculating flat outputs for pfaffian systems by a reduction procedure - demonstrated by means of the vtol example. In *9th IEEE International Conference on Control and Automation*, pages 477–482, 2011.

[125] W. M. Seiler. *Involution*. Springer, 2010.

[126] W. F. Shadwick. Absolute equivalence and dynamic feedback linearization. *Systems and Control Letters*, 15:35–39, 1990.

[127] S. N. Singh. Decoupling of invertible nonlinear systems with state feedback and precompensation. *IEEE Transactions on Automatic Control*, 25(6):1237–1239, 1980.

[128] S. N. Singh. A modified algorithm for invertibility in nonlinear systems. *IEEE Transactions on Automatic Control*, 26(2):595–598, 1981.

[129] W. M. Sluis. A necessary condition for dynamic feedback linearization. *Systems and Control Letters*, 21:277–283, 1993.

[130] E. D. Sontag. *Mathematical Control Theory*. Springer Verlag, second edition, 1998.

[131] N Steenrod. *The Topology of Fiber Bundles*. Princeton University Press, 1951.

[132] S. Sternberg. *Lecture Notes on Differential Geometry*. AMS Chelsea, 2nd edition, 1983.

[133] O. Stormark. *Lie's Structural Approach to PDE Systems*. Cambridge University Press, 2000.

[134] A. A. Strubelkopf. *Œvres Complètes*. Plomb, 1928.

[135] H. J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Transactions of the American Mathematical Society*, 120:171–188, June 1973.

[136] H. J. Sussmann. On the number of directions needed to achieve controllability. *SIAM J. Control*, 13(2):414–419, 1975.

[137] H. J. Sussmann. A sufficient condition for local controllability. *SIAM J. Control And Optimization*, 16(5):790–802, 1978.

[138] H. J. Sussmann. Lie brackets and local controllability: A sufficient condition for scalar-input systems. *SIAM J. Control And Optimization*, 21(5):686–713, 1983.

[139] H. J. Sussmann. A product expansion for the Chen series. In C. I. Byrnes and A. Lindquist, editors, *Theory and Applications of Nonlinear Control Systems*, 1986.

[140] H. J. Sussmann and V. Jurdjevic. Controllability of nonlinear systems. *Journal of Differential Equations*, 12:95–116, 1972.

[141] P. Tabuada and G. J. Pappas. Quotients of fully nonlinear control systems. *SIAM Journal on Control and Optimization*, 43(5):1844–1866, 2005.

[142] D. Tilbury and S. Sastry. The multi-steering n-trailer system: A case study of goursat normal forms and prolongations. *International Journal of Robust and Nonlinear Control*, 5(4):343–364, 1995.

[143] D. Tilbury and S. S. Sastry. On Goursat normal forms, prolongations, and control systems. *Proceedings of the 33rd IEEE Conference on Decision and Control*, 11(40):1797–1802, December 1994.

[144] J. Tsinias. Smoothly global stabilizability by dynamic feedback and generalizations of Artstein's theorem. *SIAM J. Control And Optimization*, 33(4):1071–1085, 1995.

[145] I. Vaisman. Basics of Lagrangian foliations. *Publicacions Matemàtiques*, 33:559–575, 1989.

[146] P. J Vassiliou. A constructive generalised goursat normal form. *Differential Geometry and its Applications*, 12:332–350, 2006.

[147] P. J Vassiliou. Efficient construction of contact coordinates for partial prolongations. *Foundations of Computational Mathematics*, 6:269–308, 2006.

[148] P. J Vassiliou. Contact Geometry and its Applications to Control. In J. Lévine and Ph. Mullhaupt, editors, *Advances in the Theory of Control, Signal and Systems with Physical Modeling*, volume 407 of *Lecture Notes in Control and Information Sciences*, pages 225–237, 2010.

[149] F. Verhulst. *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics*. Springer, New York, NY, 2005.

[150] A. M. Vinogradov. Geometry of nonlinear differential equations. *Journal of Soviet Mathematics*, 17(1):1624–1649, 1981.

[151] Grizzle J. W. and Marcus S. I. The structure of nonlinear control systems possessing symmetries. *IEEE Transactions on Automatic Control*, 30(3):248–258, 1985.

[152] D. C. Youla and P. F. Pickel. The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices. *IEEE transactions on circuits and systems*, 31(6):513–518, 1984.

[153] V. V. Zharinov. *Geometrical Aspects of Partial Differential Equations*. World Scientific, 1992.

Glossary of Symbols

$[.,.]$	Lie bracket, commutator
$\lrcorner$	interior product
$\perp$	for $B \subset A$, $\perp_A B$ are the elements in the dual of A, annihilating B
$\wedge$	wedge product
$\mathcal{A}^p$	$\mathcal{R}[D]$-module of p-forms on $\mathcal{U}^\infty$
$\mathcal{A}^{p,s}$	$\mathcal{M}^0_{s,s}[D]$-module of s-length columns of p-forms
$\mathcal{A}^{*,s}$	$\mathcal{M}_{s,s}[D]$-module of s-length columns of forms of any degree
$\mathcal{B}$	time manifold with coordinate (t)
$\mathfrak{C}$	Cartan distribution
card	cardinality
$\mathcal{C}^\infty(M)$	smooth real valued functions on the finite dimensional manifold M
D	vector field of the Cartan distribution satisfying $D\lrcorner\, dt = 1$
d	exterior derivative
$\mathcal{E}$	submanifold of $J^1\mathcal{M}$ defined by the equations $F(t,x,p) = 0$
Γ	ΓM, sections of the fibered manifold M
$\mathcal{H}_{p,q}[D]$	hyper-regular $p \times q$ matrix differential operators
j, j^k	$j(\sigma), j^k(\sigma)$; first, respectively k^{th} jet of σ
ker	kernel
$\Lambda^p\mathcal{R}[D]$	graded (scalar) differential operators with p-form coefficients
$\Lambda^p T\mathcal{U}^{\infty *}$	$\mathcal{R}$-module of p-forms on $\mathcal{U}^\infty$
$\mathcal{M},\ \mathcal{N}$	time and state manifold with coordinates (t,x)
$\mathcal{M}^l_{v,s}[D]$	$v \times s$ matrices of differential operators with l-form coefficients

ω^i	elements of the Cartan codistribution on $\mathcal{U}$ (Chapters 1 and 4) or representatives of a basis of $\mathcal{A}^1/\mathcal{T}$ (Chapter 2)
P_F	$(n-m)\times n$ matrix operator constructed from the system implicit equations
ϕ_*	push-forward; for $\phi : M \to N$, denotes the induced map $\phi_* : TM \to TN$
ϕ^*	pull-back; for $\phi : M \to N$, denotes the induced map $\phi^* : TN^* \to TM^*$
π	surjective smooth map
π_{10}	projection on jet manifolds $\pi_{10} : J^1M \to M$
π_{kl}	projection on jet manifolds $\pi_{kl} : J^kM \to J^lM$ for $k > l$
$\pi_{\mathcal{MB}}$	projection $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$, $\pi_{\mathcal{MB}} : (t,x) \mapsto (t)$
$\pi_{\mathcal{UM}}$	projection $\pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{M}$, $\pi_{\mathcal{UM}} : (t,x,u) \mapsto (t,x)$
$\pi_{\mathcal{U},kl}$	projection between input prologantions $\pi_{\mathcal{U},kl} : \mathcal{U}^k \to \mathcal{U}^l$ for $k > l$
$\mathbb{R}$	field of real numbers
$\mathcal{R}$	ring ($\mathbb{R}$-algebra) of smooth real-valued functions on $\mathcal{U}^\infty$
$\mathcal{R}[D]$	ring of (scalar) differential operators with coefficients in $\mathcal{R}$
σ	section on the bundle $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$, $\sigma : (t) \mapsto (t, x(t))$
$\hat{\sigma}$	lift of a section σ on $\pi_{\mathcal{MB}} : \mathcal{M} \to \mathcal{B}$ to a section on $\pi_{\mathcal{MB}} \circ \pi_{\mathcal{UM}} : \mathcal{U} \to \mathcal{B}$, $\hat{\sigma} : (t) \mapsto (t, x(t), u(t))$
TM	tangent bundle of the manifold M
TM^*	cotangent bundle of the manifold M
$\mathcal{U},\ \mathcal{V}$	time, state and input manifold with coordinates (t,x,u)
$\mathcal{U}^k$	k^{th} input prolongation manifold with coordinates $(t,x,u,\ldots,u^{(k)})$
$\mathcal{U}_s[D]$	unimodular (invertible) $s\times s$ matrix differential operators

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EDUCATION

PhD École Polytechnique Fédérale de Lausanne (EPFL), Laboratoire d'Automatique, Switzerland 2007-present

Master of Science MSc *en Génie électrique et électronique*, EPFL, School of Electrical Engineering 2005-2007

- Master Thesis : *SwissCube attitude determination and control*, for the SwissCube micro-satellite

Bachelor Last year at EPFL 2005

Two first years at ETHZ, Zürich 2002-2004

WORK EXPERIENCE

2007-2012 Teaching and research duties at the Laboratoire d'Automatique

- Supervised 6 master theses
  - *Linéarisation par rétroaction du robot Delta* by Patrick Clerc, 2012
  - *Flying the Skybotix Coax* by Sandro Montanari, 2012
  - *Analysis and Control of Instabilities in Railway Power Systems* by Hu Shuai, 2010
  - *Optimal racing line calculation for a Formula 1 car using a quasi-static lap time simulation* by Constantin Niemeyer, 2009
  - *SwissCube Attitude Determination and Control* by Martin Ehrensprenger, 2008
  - *SwissCube Attitude Determination and Control* by Jordi Martin-Benet, 2007
- Supervised 11 semester projects
- Teaching assistant for various courses

2006 Programmed a computer simulator for the hydrofoil sailboat "L'Hydroptère" at the Laboratoire d'Automatique, EPFL

2004 Four months internship at Phillips AG, Zürich at the Business Line "Display Drivers"

2003 Teaching assistant for the course "Digitaltechnik" at ETHZ

TECHNICAL SKILLS

- Modeling and analysis of dynamical systems
- Design of control strategies
- Electronics, circuit design
- Proficient in various programming languages: Mathematica, Matlab, Simulink, C++, Java

PUBLICATIONS

- *Application of Legendrian Foliations in Differential Flatness Problems.* Basile Graf and Philippe Müllhaupt. IEEE Conference on Decision and Control, 2012, Hawaii.
- *Bilinear Control Systems with Forced Integrable Filtration.* Basile Graf and Philippe Müllhaupt. ICNAAM 2012: International Conference of Numerical Analysis and Applied Mathematics. Kos, Greece, 2012.
- *Feedback Linearizability and Flatness in Restricted Control Systems.* Basile Graf and Philippe Müllhaupt. 18th IFAC World Congress, Milano, Italy, 2011.
- *Nonlinear Analysis of a Coaxial Micro-Helicopter with Bell Stabilizer Bar.* Basile Graf and Philippe Müllhaupt. To appear in the Journal of the American Helicopter Society.
- *Modeling and Flatness of Rigid and Flexible Cable Suspended Underactuated Robots.* Philippe Müllhaupt and Basile Graf. 2010 IEEE International Conference on Control, Yokohama, Japan, 2010.

LANGUAGES

- French
- English
- German and Swiss-German