Study of the maximal interpolation errors of the local polynomial method for frequency response function measurements

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***Abstract*—Frequency response function measurements take a central place in the instrumentation and measurement field because many measurement problems boil down to the characterisation of a linear dynamic behaviour. The major problems to be faced are leakage- and noise errors. The local polynomial method (LPM) was recently presented as a superior method to reduce the leakage errors with several orders of magnitude while the noise sensitivity remained the same as that of the classical windowing methods. At the resonance frequencies, where often most information about the system is to be retrieved, the dominating error is the interpolation error. In this paper it is shown that the interpolation error for sufficiently low damping is bounded by $(B_{LPM}/B_{3dB})^{R+2}$ with B_{LPM} the local bandwidth of the LPM, R the degree of the local polynomial that is selected to be even (user choices), and B_{3dB} the 3dB bandwidth of the resonance, which is a system property.**

***Index Terms*—frequency response function, nonparametric, error analysis, interpolation errors**

I. Introduction

The major challenge for the instrumentation and measurement society is to develop improved and new measurement techniques. Measuring the frequency response function (FRF) to characterise the dynamic behaviour of a system is an important sub-class among these problems. In this paper we focus on the local polynomial method (LPM) [1], [2], [3] that was recently presented as a superior alternative to the widely spread and popular windowing methods [4], [5] to solve that problem. All nonparametric methods suffer from leakage and noise errors. Leakage errors form a fundamental restriction for the standard methods and are present even in the absence of measurement or process noise. At a cost of an increase of the computation time, the LPM reduces the leakage errors with several orders of magnitude while the disturbing noise sensitivity remains the same as that of the standard procedures. Because the continuously increase of the available computer power removes the major drawback of the more calculations demanding LPM, the authors believe that it will become the standard method in many applications where data with high signal-to-noise ratios are available. For that reason it is extremely important to understand the errors of the LPM.

The basic idea of the LPM is to use local polynomial approximations of the transfer function and the transient behaviour of the system caused by initial condition effects, usually a polynomial of degree two is used. This finite order approximation will create systematic errors, and it is the goal of this paper to provide an upper bound on these errors. This allows the reader on the one hand to better understand the underlying error mechanism, and on the other hand it allows us to provide the user with a simple rule of thumb to choose the measurement conditions.

In Section II a brief introduction to the LPM is given, in Section III the upper bound on the polynomial approximation error for a lightly damped system is obtained and verified in Section IV, followed by the conclusions.

II. The local polynomial method: a brief introduction

In this section we first describe the system and measurement set-up in Section II-A, next we discuss very briefly the basic idea in Section II-B, and finally we formulate the LPM as a linear-least-squares problem that is solved frequency per frequency. We refer the reader to [1], [3] for more detailed information.

A. Set-up

In this paper we focus on a linear discrete time single-input-single-output (SISO) system $G_0(q)$. Since we focus on the interpolation errors, we do not consider here the impact of disturbing noise and assume that the input u_0 and the output y_0 are exactly known.

$$y_0(t) = G_0(q)\, u_0(t)\,, \tag{1}$$

with q^{-1} the backward shift operator. All results apply also to continuous time systems. For a finite record length $t = 0, \ldots, N-1$, as it is in practical applications, this equation has to be extended with the initial conditions (transient) effects of the dynamic plant t_G:

$$y(t) = G_0(q)\, u_0(t) + t_G(t) \tag{2}$$

Using the discrete Fourier transform (DFT)

$$X(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-j2\pi kt/N}, \tag{3}$$

an exact frequency domain formulation of (2) is obtained:

$$Y_0(k) = G_0(\Omega_k) U_0(k) + T_G(\Omega_k), \tag{4}$$

where the index k points to the frequency kf_s/N with f_s the sampling frequency, and $\Omega_k = e^{-j2\pi k f_s/N}$. The contributions U_0, Y_0 in (4) are an $O\left(N^0\right)$, the transient term T_G, is an $O\left(N^{-1/2}\right)$. In these expressions, $O(x)$ stands for ordo(x): a function that goes to zero at least as fast as x. It is most important for the rest of this paper to understand that (4) is an exact relation [9], [6], [7], [8]. The finite record length requires the use of a transient term in (2), and it turns out that the leakage errors of the DFT are modelled by very similar terms in the frequency domain. All these terms $t_G(t), T_G(\Omega_k)$ are described by rational forms in q^{-1} (time domain) or z^{-1} (frequency domain), hence it are smooth functions of the frequency.

B. LPM: the basic idea

In this section we give a very brief introduction to the polynomial method. A detailed description, together with a full analysis is given in [1], [2], a comparison with the classical spectral windowing methods is found in [3].

The basic idea of the local polynomial method is very simple: the transfer function G_0 and the transient term T_G are smooth functions of the frequency so that they can be approximated in a narrow frequency band around a user specified frequency k by a complex polynomial. The complex polynomial parameters are estimated from the experimental data. Next $G_0(\Omega_k)$, at the central frequency k, is retrieved from this local polynomial model as the measurement of the FRF at that frequency.

C. LPM: a local linear-least-squares estimate

We start from the full OE-expression (2), and consider again the equivalent relation for the DFT-spectra:

$$Y_0(k) = G_0(\Omega_k)U_0(k) + T_G(\Omega_k) \tag{5}$$

Making use of the smoothness of G_0 and T, the following polynomial representation holds for the frequency lines $k+r$, with $r = 0, \pm 1, \ldots, \pm n$.

$$\begin{aligned} &G_0(\Omega_{k+r}) = \\ &G_0(\Omega_k) + \textstyle\sum_{s=1}^{R} g_s(k) r^s + O\left(\left(\frac{r}{N}\right)^{(R+1)}\right) \end{aligned} \tag{6}$$

$$\begin{aligned} &T_G(\Omega_{k+r}) = \\ &T(\Omega_k) + \textstyle\sum_{s=1}^{R} t_s(k) r^s + N^{-\frac{1}{2}} O\left(\left(\frac{r}{N}\right)^{(R+1)}\right) \end{aligned} \tag{7}$$

Putting all parameters $G_0(\Omega_k), T_G(\Omega_k)$ and the parameters of the polynomial g_p, t_p, with $p = 1, \ldots, R$ in a column vector θ, and their respective coefficients in a row vector $K(k,r)$ allows (5) to be rewritten (neglecting the remainders) as:

$$Y(k+r) = K(k,r)\theta, \tag{8}$$

Collecting (8) for $r = -n, -n+1, \ldots, 0, \ldots, n$ finally gives

$$Y_n = K_n \theta, \tag{9}$$

with Y_n, K_n the values of $Y(k+r), K(k,r)$, stacked on top of each other. Observe that the matrix K_n depends upon U_0. Solving this equation in least squares sense eventually provides the polynomial least squares estimate for $\hat{G}_{\text{poly}}(\Omega_k)$. In order to get a full rank matrix K_n, enough spectral lines should be combined: $n \geq R+1$. The smallest interpolation error is obtained for $n = R+1$.

III. Upper bounding the interpolation error

A. Introduction

From the previous section, it turns out that the local polynomial approximation plays a central role in the FRF-estimation. The polynomial approximation errors will set directly the errors of the LPM in the noiseless case. These depend upon the polynomial degree R in (6) and (7), and on the selected bandwidth n in (9). For that reason it is important to bound the maximum approximation error. It will be shown that the approximation errors can be described by a single invariant of the system/setup, given by $(B_{LPM}/B_{3dB})^{R+2}$ with B_{LPM} the local bandwidth of the LPM, R the degree of the local polynomial that is selected to be even (user choices), and B_{3dB} the 3 dB bandwidth of the resonance ω_n (where $|G(\omega)|_{dB} \geq G_{\max\, dB} - 3$, see Figure 1), which is a system property. It can be shown that for a second order system $B_{3dB} = 2\varsigma\omega_n$, with ς the damping of the system.

Instead of focusing on the original problem in (5), we solve here the underlying and more generic problem of the local approximation of a transfer function in a given frequency band by a polynomial of degree R. Using a similar notation as in (9), we have that:

$$G_n = \tilde{K}_n \tilde{\theta}, \tag{10}$$

with $\tilde{\theta}^T = [G_0(\Omega_k), g_1, \ldots, g_R]$, and $\tilde{K}_n$ the corresponding frequency matrix that does no longer depend on U.

It is well known that the transfer function of a system can be written as the sum of a set of first order systems with complex or real poles. The complex conjugated poles can be grouped in 2nd order contributions. The approximation of the transfer function reduces to the approximation of a first order system if we can assume that the damping of the poles is low enough (e.g. $\varsigma < 0.25$). This is not a hard restriction, because the polynomial approximation errors grow for a decreasing damping, so this is the worst case situation. For that reason we can also assume without great loss of generality that we have only single poles with a low damping. The situation of coinciding poles is excluded from this study. In this section we will make for simplicity all the calculations on the continuous time representation of the system, but all results apply without loss of generality also for discrete time systems, as long as the damping is sufficiently small.

B. Normalised second order system

Consider the normalised 2^{nd}order system with resonance frequency ω_n and damping ς :

$$G(s) = \frac{G_{DC}\omega_n^2}{s^2 + 2s\varsigma\omega_n + \omega_n^2} = \frac{b}{s-p} + \frac{\bar{b}}{s-\bar{p}}, \tag{11}$$

with $s = j\omega$ the frequency variable, $b = -jG_{DC}\omega_n/(2\sqrt{1-\varsigma^2})$ and $p = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}$. The over-score denotes the complex conjugate. The maximum of the transfer function is at $\omega = \omega_n\sqrt{1-2\varsigma^2}$. Around that frequency, the first term

$$\tilde{G}(s) = \frac{b}{s-p} \text{ for } s \simeq j\omega_n \tag{12}$$

in (11) dominates. This term becomes maximum at $\omega = \omega_n\sqrt{1-\varsigma^2}$. In this paper we will focus on systems with $\varsigma \ll 1$, so that both maxima coincide almost completely with ω_n which we call the resonance frequency $s = j\omega_n$. So we can focus completely on (12) that can be rewritten as

$$\tilde{G}(s = j\omega) = \frac{-\Delta - j}{1+\Delta^2}\frac{G_{DC}}{2\varsigma}, \tag{13}$$

with $\Delta = \frac{\omega-\omega_n}{B_{3dB}/2}$. Observe that the real and imaginary part of $\tilde{G}(\Delta)$ are respectively an odd and even function of Δ.

C. Least squares approximation of degree R

We are interested in $\tilde{G}(s = j\omega) = \tilde{G}(\Delta = 0)$ estimated from a least squares fit in the frequency band $\tilde{\omega} \in \left[\omega_n - \frac{B}{2}, \omega_n - \frac{B}{2}\right]$, or $\Delta \in [-1, 1]$. For a symmetric grid around zero, the fit of the real part will also be anti-symmetric and hence equal zero for $\Delta = 0$, hence the error is completely set of the fit on $\tilde{G}_{imag}$. Since the latter is even, only even terms will appear in the polynomial least squares approximation. For that reason we set R to be even. Consider the Taylor expansion

$$\tilde{G}_{imag}(\Delta) = \frac{G_{DC}}{2\varsigma}\sum_{k=0}^{\infty}(-1)^{k+1}\Delta^{2k}, \tag{14}$$

for $|\Delta| < 1$. A fit of a polynomial of degree R will capture all the contributions up to degree R of the Taylor expansion, and the error will be dominated by the next contribution of the Taylor approximation of degree $R+2$ so that we find that the error E_G on the estimate of $\tilde{G}(\Delta = 0)$ is given by

$$E_G = \alpha_{R+2}\Delta^{R+2} = \alpha_{R+2}\left(\frac{B}{B_{3dB}}\right)^{R+2}\frac{G_{DC}}{2\varsigma}. \tag{15}$$

with R even, and α_{R+2} a constant. Alternatively, (15) can be written as

$$E_G = \alpha_{R+2}\left(\frac{B}{B_{3dB}}\right)^{R+2} G_{max}. \tag{16}$$

The constant α_{R+2} can be determined explicitly making use of the explicit expression of the least squares solution of (10):

$$\tilde{\theta}_{LS} = (\tilde{K}_n^H\tilde{K}_n)^{-1}\tilde{K}_n^H. \tag{17}$$

All the coefficients in $\tilde{K}_n$ are a priory known, and the first error term contributing to the solution will be the term of degree $R+2$ in (14). Using integral approximations for the sums in the matrix- and vector products in (17) it can be shown that for $R = 2$, the error is given by:

$$E_G \simeq \begin{bmatrix} 1.125 & 0 & 1.875 \end{bmatrix}\begin{bmatrix} 2/5 \\ 0 \\ 2/7 \end{bmatrix}\left(\frac{B}{B_{3dB}}\right)^4 G_{max},$$

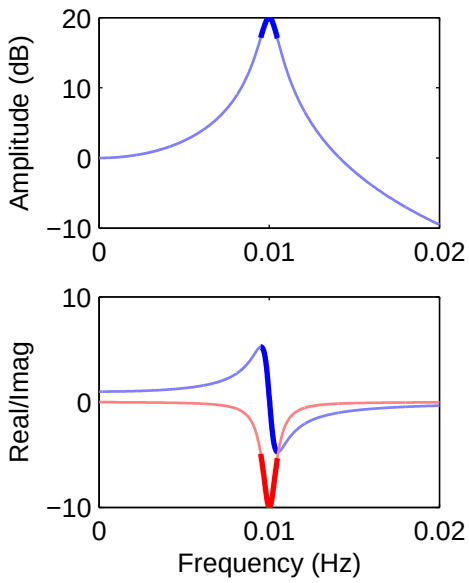

Figure 1. Behaviour of G_0 around its resonance frequency. The 3 dB bandwidth is emphasised in bold. The upper figure shows the amplitude in dB. The lower figure shows the real (blue) and imaginar (red) part of G_0.

or

$$E_G \simeq 0.857\left(\frac{B}{B_{3dB}}\right)^4 G_{max}. \tag{18}$$

It is possible to get more precise estimates for α_{R+2} by calculating the coefficient numerically for a given frequency grid. Calculating the coefficient for higher values of R can be done similarly. It is important here to observe that the invariants that are determining the maximum interpolation error are: the degree R of the polynomial, the ratio B/B_{3dB}, and G_{max}. In the next section we will show that with these 3 parameters it is possible to cover indeed the interpolation error for different resonance frequencies ω_n, dampings ς, and polynomial degrees R, as long as the damping $\varsigma < 0.25$.

IV. Verification of the upper bound

In this section we verify the error bound (16). First we verify the results on the normalised problem (13) for varying values of B/B_{3dB}, next we return to the original problem where we verify the results for varying dampings ς, and resonance frequencies ω_n.

A. Study of the normalised problem

In this section we study the observed error for the normalised problem for $R = 2$ and a varying choice of B/B_{3dB}. We selected 21 frequencies equidistantly distributed in the interval [-1 1]. The numerically obtained value for $\alpha_4 = 0.1018$ to be compared to the integral expression in (18) that was 0.0857. In Figure 2 we show the ratio of the actual observed error and the error predicted from the theory. It can be seen that for $B/B_{3dB} < 0.2$, a very good agreement between the theory and the actual observations is found. For larger values, the theoretic bound is too conservative, and hence gives still a

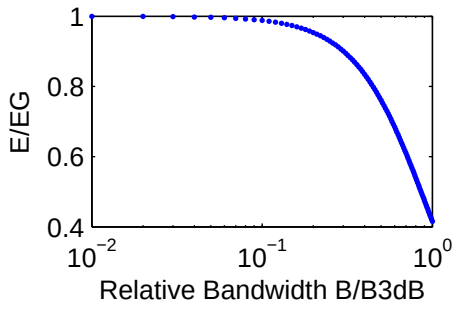

Figure 2. Evaluation of the upper bound on the normalized problem (13) as a function of the relative bandwidth.

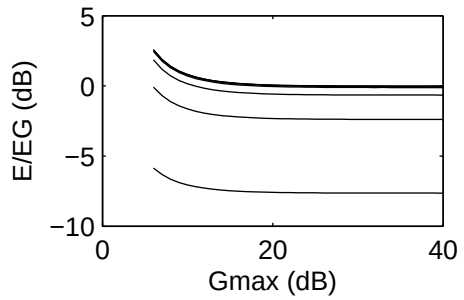

Figure 3. Evolution of the upper bound on the original problem (11) as a function of the maximal amplitude of the system.

safe bound. The deviations are due to the fact that higher order terms should be included for larger values of B/B_{3dB}, and for values close to 1 the Taylor series is even no longer converging.

B. Study of the original problem

In this section we compare again the observed and the theoretical error, but now on the original problem (11). The ratio B/B_{3dB} is varied between 0.01 and 1, the damping is varied so that G_{max} varied from 6 to 40 dB (with $G_{DC} = 1$). Again we can conclude from Figure 3 that the bound is precise. For low dampings (small values of G_{max}) the theoretical bound is too small. This error is due to the fact that the second term in the right side of (11), corresponding to the complex conjugated pole, becomes too important while it was left out in the theoretical analysis. We can also see that for larger bandwidths the theoretical bound is again too conservative.

C. Solving the LPM problem

When carrying over the results from Sections IV-A and IV-B, it should be kept in mind that all these results relied heavily on the symmetry/anti-symmetry of the real/imaginary part. In equation (9) this symmetry can be lost if the spectrum of the excitation signal is not symmetric around the centre frequency. In that case the convergence will be reduced with one order to $O\left((B/B_{3dB})^{R+1}\right)$. In practice we observe that with the LPM the gain obtained by moving from R_{even} to $R_{even}+1$ is much smaller than moving from R_{even} to $R_{even}+2$. This is still due to the previous explained mechanism. As a general conclusion we can state that the local bandwidth B that is used should be (significantly) smaller than B_{3dB}. This sets immediately an underlimit on the acceptable measurement time since B_{3dB} is directly linked to the measurement time:

$$\tau = 1/(\varsigma\omega_n) = 2/B_{3dB}.$$

Since we need at least $2R+3$ frequencies in the local interval B, and B should be choosen to be smaller than B_{3dB}, we find that the measurement time T_{meas} (which is the inverse of the frequency resolution in Hz) should be larger than

$$T_{meas} > 2\pi\frac{2R+3}{B_{3dB}} = (2R+3)\,\pi\tau$$

in order to be in the good operational conditions to use the LPM around the resonance frequency of the system. For example, for $R = 2$, the strict minimum will be $T_{meas} = 22\tau$ (corresponding to having 7 frequency points in B_{3dB}).

V. Conclusions

In this paper we analysed the polynomial approximation error of a 2nd order system. This is a generic problem because it appears in many modelling and measurement techniques. A theoretical bound on the interpolation error was derived that results in a set of normalised numbers (order and relative bandwidth of the fit) that give a lot of insight in the behaviour of the approximation. Using these results, it is for example possible to better understand the errors in the local polynomial method. We also could translate these results in very practical advices on the minimum required measurement time.

VI. Acknowledgement

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