Algebraic Techniques for Linear Deterministic Networks

Javad Ebrahimi and Christina Fragouli
EPFL, Lausanne, Switzerland.
Email: {javad.ebrahimi,christina.fragouli}@epfl.ch

***Abstract*—We here summarize some recent advances in the study of linear deterministic networks, recently proposed as approximations for wireless channels. This work started by extending the algebraic framework developed for multicasting over graphs in [1] to include operations over matrices and to admit both graphs and linear deterministic networks as special cases. Our algorithms build on this generalized framework, and provide as special cases unicast and multicast algorithms for deterministic networks, as well as network code designs using structured matrices.**

I. Introduction

Deterministic network models have recently attracted a significant interest in the literature, in particular in the context of serving as approximate characterizations for wireless networks [9]–[11]. The work in [9], [10] established an information theoretical min-cut max-flow theorem for deterministic networks, showing that the unicast capacity equals the minimum rank of the transfer matrices of cuts separating the source from the destination. In the same work, the authors also extended to deterministic networks the multicasting theorem in network coding [7], and showed that in the presence of N receivers, if the mincut to each receiver equals at least h, it is possible to multicast information to all receivers at this rate. The proof uses information theoretical arguments and does not directly lead to polynomial time designs.

For deterministic networks, a polynomial time algorithm that allows unicast information transmission using block length of one over arbitrary networks was first developed in [24], for an extended version see [25]. This work was followed up by three other alternative designs for unicast algorithms [26]–[28]. Additional traffic scenaria over deterministic networks were investigated for example in [31], [32], but for specific network configurations, typically consisting from a small number of nodes.

Network coding solved the problem of multicasting at optimal rate the same information from a single source to multiple receivers [7], [8]. Several algorithms were subsequently developed that offer polynomial time designs for multicasting over graphs [1]–[3]. In particular, Koetter and Medard [1] translated the problem of network code design to an algebraic problem over graphs. This algebraic framework provided the theoretical foundation for the development of most of the finite length coding results and practical algorithms for network coding today.

In our work [29], [30] we have extended the algebraic framework developed for multicasting over graphs in [1] in two ways: (i) to include operations over matrices and (ii) to accept both graphs and linear deterministic networks as special cases. Independently from our work, the framework in [1] was also extended over deterministic networks in [16]. Building on this extended framework, we propose new algebraic code designs for scalar and vector network coding, that apply both over networks represented as graphs, as well as arbitrary linear deterministic networks. A key ingredient of our designs is the use of vector network codng.

In scalar network coding, when multicasting to N receivers at rate h, the source transmits h scalar values over some finite field. The size of the employed finite field is a design parameter, that for complexity considerations we typically desire to minimize. Intermediate network nodes linearly combine their received symbols by multiplying them with scalar coefficients, called coding coefficients in the literature. The network code design consists of selecting the coding coefficients, and the finite field of operation, so that each receiver has a full rank set of equations to solve and can thus retrieve the source symbols [1].

In vector network coding, the source transmits h vectors of length L, where the elements of the vectors are over a fixed finite field $\mathbb{F}_q$, for example, the binary field $\mathbb{F}_2$. Intermediate network nodes perform coding operations over vectors, namely, multiply their incoming vectors with $L \times L$ coding matrices and then add them to create the new vectors that they propagate towards the destinations. That is, intermediate nodes linearly combine their incoming vectors using coding matrices, where these matrices play a similar role as scalar coding coefficients in traditional algebraic network coding. The code design consists in selecting the length L and the $L \times L$ coding matrices so that each receiver receives information at rate h. Scalar network coding over a field of size $\mathbb{F}_q$ can be viewed as a special case of vector network coding with $L = 1$.

The contributions in our work [29], [30] include the following:

1) We provide a polynomial time algorithm for the design of coding matrices of vector network coding when multicasting to N receivers. Our metric of optimization is the smallest size L such that there exist $L \times L$ coding matrices that allow all N destinations to successfully decode the source information. The size of these matrices plays the same role as the size of the finite field in traditional network coding. Our algorithm reduces the problem of finding a small size L to the problem of finding a small

degree coprime factor of an algebraic polynomial, and leads to solutions not possible with using scalar network coding, as illustrated through examples in [29], [30].

2) Both for the case of graphs and deterministic networks, we show that $L \approxeq \log(N(\log N - 1)h\Lambda)$ is always sufficient, where Λ is a network parameter, and we can find such matrices in polynomial time. We also provide probabilistic guarantees, and show for example that in a fraction $\frac{1023}{1024}$ of polynomials derived from transfer functions, we will be able to find in polynomial time binary coding matrices of size at most 3×3 that lead to a valid code.
3) For the case of deterministic networks, our approach offers the first code design for multicasting over arbitrary linear deterministic networks. Moreover, as a special case, it gives an alternative algorithm for a unicast connection over a deterministic network, adding to the existing algorithms [24]–[28].
4) Our approach also gives a new algorithm for scalar network code design, that operates in polynomial time. This new algorithm jointly minimizes the employed field size while selecting the coding coefficients. In contrast, existing algorithms [1]–[5] first select a fixed finite field and then proceed to design the network codes over this predetermined field. As a consequence, these algorithms would operate over a field of size N, i.e., the worst case guarantee. However, our algorithm, due to jointly optimizing the effective field size as well as the coefficients, can result in a much smaler field while still being a polynomial time algorithm. A theoretical side-result of our work is establishing a connection between the problem of identifying the minimum field size required for network coding and finding the smallest coprime factor of algebraic polynomials.

We believe that vector network coding is a very promising research direction, as it offers a natural generalization of network coding, and thus offers a larger space of choices for optimizing cost parameters, such as the operational complexity, or the communication block length.

REFERENCES

[1] R. Koetter and M. Médard, "Beyond routing: an algebraic approach to network coding", *IEEE/ACM Transactions on Networking*, vol. 11, no. 5, pp. 782-796, October 2003.

[2] S. Jaggi, P. Sanders, P. Chou, M. Effros, S. Egner, K. Jain and L. Tolhuizen, "Polynomial time algorithms for multicast network code construction", *IEEE Transactions on Information Theory,* vol. 51, no. 6, pp. 1973–1982, 2005.

[3] N. Harvey, "Deterministic network coding by matrix completion", MS Thesis 2005.

[4] C. Fragouli and E. Soljanin, "Network coding: Fundamentals," *Foundations and Trends in Networking*, vol. 2, pp. 1–133, 2007.

[5] T. Ho, R. Koetter, M. Médard, M. Effros, J. Shi, and D. Karger, "A random linear network coding approach to multicast," *IEEE Transactions on Information Theory*, vol. 52, iss. 10, pp. 4413-4430, October 2006.

[6] C. Fragouli and E. Soljanin, "A connection between network coding and convolutional codes," *IEEE International Conference on Communications (ICC)*, pp. 661–666, vol.2, June 2004.

[7] R. Ahlswede, N. Cai, S-Y. R. Li, and R. W. Yeung, "Network information flow", *IEEE Trans. Inform. Theory,* vol. 46, 2000.

[8] S-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory,* vol. 49, pp. 371–381, Feb. 2003.

[9] S. Avestimehr, S N. Diggavi and D N C. Tse, "Wireless network information flow", *Proceedings of Allerton Conference on Communication, Control, and Computing*, Illinois, September 2007. See http://licos.epfl.ch/index.php?p=research_ projWNC

[10] S. Avestimehr, S N. Diggavi and D N C. Tse, "A deterministic approach to wireless relay networks", *Proceedings of Allerton Conference on Communication, Control, and Computing*, Illinois, September 2007.

[11] S. Avestimehr, S N. Diggavi and D N C. Tse, "Wireless netwokr information flow: a deterministic approach", $arXiv:0906.5394$, 2009.

[12] J. Ebrahimi and C. Fragouli, "Multicasting algorithms for deterministic networks", *IEEE ITW*, Cairo, January 2010.

[13] J. Ebrahimi and C. Fragouli, "Vector network coding", *EPFL Technical Report*, $http://infoscience.epfl.ch/record/144144$, 2010.

[14] S. Jaggi, Y. Cassuto, M. Effros, "Low complexity encoding for network codes," *ISIT* 2006.

[15] M. Khojastepour, A. Keshavarz-Haddad, "Rotational coding achieves multicast capacity of deterministic wireless networks", *IEEE Allerton*, September 2009.

[16] M. Kim, M. Medard "Algebraic network coding approach to deterministic wireless relay networks", $http://arxiv.org/pdf/1001.4431$.

[17] C. Fragouli and E. Soljanin, "Information flow decomposition for network coding," *IEEE Transactions on Information Theory*, vol. 52, iss. 3, pp. 829–848, March 2006.

[18] P. Morandi, "Field and Galois Theory", *Springer*, 1996.

[19] Istvan Kovacs, Daniel S. Silver and Susan G. Williams, "Determinants of commuting-block matrices", *The American Mathematical Monthly*, vol. 106, no. 10, pp. 950-952, December 1999.

[20] Horn and Johnson, "Linear Algebra", *Cambdrige University Press*.

[21] I. Chaparlinski, "On finding primitive roots in finite fields", *Theoretical computer science*, vol. 157, no 2, pp. 273-275, May 1996.

[22] J. T. Schwartz, "Fast probabilistic algorithms for verification of polynomial identities", *Journal of the ACM*, vol. 27, no. 4, 1980.

[23] P.J. Davis, "Circulant Matrices", *Chelsea Publishing*, New York 1994.

[24] A. Amaudruz and C. Fragouli, "Combinatorial algorithms for wireless information flow", *SODA* 2009.

[25] J. Ebrahimi and C. Fragouli, "Combinatorial algorithms for wireless information flow", arXiv:0909.4808, under submission.

[26] M. Goemans, I. Satoru and R. Zenklusen, "An algorithmic framework for wireless information flow", *Allerton* 2009.

[27] S. Tabatabaei Yazdi and S. Savari, "A combinatorial study of linear deterministic relay networks", arXiv:0904.2401, under submission.

[28] C. Shi and A. Ramamoorthy, "Fast algorithms for finding unicast capacity of linear deterministic wireless relay networks", arXiv:0909.5507v1, under submission.

[29] J. Ebrahimi and C. Fragouli, "On the benefits of vector network coding", IEEE International Symposium on Information Theory (ISIT), 2010.

[30] J. Ebrahimi and C. Fragouli, "Multicasting algorithms for deterministic networks", ITW 2010, Cairo.

[31] S. Mohajer, S. Diggavi, C. Fragouli and D. Tse, "Transmission techniques for relay-interference networks", *Allerton* 2008, arXiv:0812.1597.

[32] S. Mohajer, S. Diggavi, C. Fragouli and D. Tse, "Capacity of deterministic Z-chain relay-interference network", in *IEEE Information Theory Workshop (ITW)*, Volos, Greece, June 2009.