

On the Structure of the Capacity Region of Asynchronous Memoryless Multiple-Access Channels

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Abstract

The asynchronous capacity region of memoryless multiple-access channels is the union of certain polytopes. It is well-known that vertices of such polytopes may be approached via a technique called successive decoding. It is also known that an extension of successive decoding applies to the dominant face of such polytopes. The extension consists of forming groups of users in such a way that users within a group are decoded jointly whereas groups are decoded successively. This paper goes one step further. It is shown that successive decoding extends to every face of the above mentioned polytopes. The group composition as well as the decoding order for all rates on a face of interest are obtained from a label assigned to that face. From the label one can extract a number of structural properties, such as the dimension of the corresponding face and whether or not two faces intersect. Expressions for the the number of faces of any given dimension are also derived from the labels.

Index Terms— Multiple-access channel, polytope, polymatroid, face, group successive decoding.

1 Introduction

The asynchronous capacity region of an M -user memoryless multiple-access channel (MAC) is the union of certain M -dimensional polytopes. It is well known that if a desired rate tuple

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lies on the *vertex* of the so-called *dominant face* of such a polytope, one can decode *one user at a time successively*, using the codewords of already decoded users as side information [8, Section 14.3.2]. For example, for a 2-user code of blocklength n and rates R_1 and R_2 , respectively, decoding user 1 and 2 successively requires finding the first codeword within a codebook of size 2^{nR_1} and subsequently finding the second codeword in a codebook of size 2^{nR_2} . A decoder that makes a joint search does so in the bigger space of $2^{nR_1}2^{nR_2}$ pairs of alternatives. It should be said however that the probability of error is in general bigger for the successive decoder.

In [16] it is shown that successive decoding for dominant-face vertices extends to *group* successive decoding for rate tuples that are in the boundary of the dominant face. More specifically, each point on the boundary of the dominant face belongs to a face of some dimension $k \in \{0, 1, \dots, M - 2\}$. For a rate tuple on such a face of dimension k , successive decoding requires forming $M - k$ groups. For instance, for a vertex (a face of dimension 0) we need M groups, which means that each “group” contains a single user, implying, as it should, single user decoding of vertices. Alternatively, if the rate of interest is on a face of dimension 1, the number of groups is $M - 1$, i.e., all except two users can be decoded one at a time successively, and the group of two is decoded jointly. In [16] it is also shown that if the rate tuple of interest is on the dominant face but not on its boundary, then one can split a user and a channel input and make sure that the new rate tuple, which has an additional component, lies on the boundary of the dominant face of the newly created channel. By iterating this procedure one obtains *rate splitting multiple-access* [9, 10].

In this paper we focus on some structural and operational properties of the M -dimensional polytopes that form the capacity region. We extend the labeling technique of [12, 16] so as to have a label for every face. The label is unique if the polytope is non-degenerated. A degenerated polytope (to be properly defined later) is one for which certain faces collapse. To avoid complications due to the collapsing of faces we consider only non-degenerated cases. From the label, we can deduce structural properties such as which faces intersect and the dimensionality of a face. The label also specifies how to do successive decoding of groups, which is an operational property. In particular, we will see that group decoding applies to every face (not only the faces of the dominant face).

The paper is organized as follows. In Section 2 we define the relevant polytopes and characterize and label their faces. The main result of Section 2 is Proposition 5. It specifies which faces intersect and which do not. In Section 3 we make the link between the label and group successive decoding. In Section 4 we give expressions for the number of faces of any given dimension. Section 5 concludes the paper.

2 Labeling faces

Recall that an M -user discrete memoryless multiple-access channel is defined in terms of M discrete input-alphabets¹ \mathcal{X}_i , $i \in \{1, \dots, M\}$, an output alphabet \mathcal{Y} , and a stochastic matrix $W : \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_M \rightarrow \mathcal{Y}$ with entries $W_{Y|X_1, X_2, \dots, X_M}(y|x_1, x_2, \dots, x_M)$ describing the probability that the channel output is y when the inputs are x_1, x_2, \dots, x_M . For any input distribution in product form² P_{X_1}, \dots, P_{X_M} , define \mathcal{R} to be

$$\mathcal{R} = \{R \in \mathbb{R}_+^M : R(\mathcal{S}) \leq I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq [M]\},$$

where $R(\mathcal{S}) \triangleq \sum_{i \in \mathcal{S}} R_i$, $X_{\mathcal{S}} \triangleq (X_i)_{i \in \mathcal{S}}$, $\mathcal{S}^c \triangleq [M] \setminus \mathcal{S}$, $[M] = \{1, 2, \dots, M\}$, and $I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c})$ is the mutual information between $X_{\mathcal{S}}$ and Y given $X_{\mathcal{S}^c}$. \mathbb{R}_+ denotes the nonnegative reals. The capacity region depends on whether or not there is synchronism. A discrete-time channel is *synchronous* if the transmitters are able to index channel input sequences in such a way that all inputs with time index n enter the channel at the same time. If this is not the case, meaning that there is an unknown shift between time indices, then the channel is said to be *asynchronous*.

The capacity region for either the synchronous or asynchronous channel may be described in terms of the region

$$\mathcal{C}_{DMC} = \bigcup_{P_{X_1} P_{X_2} \dots P_{X_M}} \mathcal{R}[W; P_{X_1} P_{X_2} \dots P_{X_M}],$$

where the union is over all product input distributions. The capacity region of the asynchronous multiple-access channel with arbitrarily large shifts between time indices is \mathcal{C}_{DMC} [3, 4], whereas if shifts are bounded or the multiple-access channel is synchronous then its capacity region is the convex hull of \mathcal{C}_{DMC} [5, 6, 7].

Definition 1 A region \mathcal{R} is called non-degenerated if the following two conditions hold

- (a) $I(X_{\mathcal{S}}; Y) > 0$ for all non-empty sets $\mathcal{S} \subseteq [M]$,
- (b) $I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) < I(X_{\mathcal{S}}; Y|X_{\mathcal{B}})$ for all $\emptyset \subset \mathcal{S} \subset [M]$, $\mathcal{A} \subset \mathcal{B} \subset [M]$, and $\mathcal{S} \cap \mathcal{B} = \emptyset$.

The above definition is natural. Essentially it says that each input carries information and all inputs interfere with one another. Notice that for a non-degenerated channel it is also true that for all $\mathcal{A} \subset [M]$, $\emptyset \subset \mathcal{S} \subset \mathcal{T} \subseteq [M]$, and $\mathcal{A} \cap \mathcal{T} = \emptyset$,

$$I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) < I(X_{\mathcal{T}}; Y|X_{\mathcal{A}}). \tag{1}$$

¹All results presented in this paper carry over to the Gaussian multiple-access channel.

²Random variables and their sample values will be represented by capital and lowercase letters, respectively.

To see this, we first observe that the independence of the input random variables implies that $I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) \geq I(X_{\mathcal{S}}; Y)$ whenever \mathcal{S} and \mathcal{A} do not intersect. Thus, condition (a) implies $I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) > 0$ for every non-empty subset \mathcal{S} of $[M]$ and every subset \mathcal{A} of $[M]$ that does not intersect with \mathcal{S} . Now we can use the chain rule of mutual information to obtain $I(X_{\mathcal{T}}; Y|X_{\mathcal{A}}) = I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) + I(X_{\mathcal{T} \setminus \mathcal{S}}; Y|X_{\mathcal{A} \cup \mathcal{S}}) > I(X_{\mathcal{S}}; Y|X_{\mathcal{A}})$, where the inequality holds since the second term on its left must be positive.

An example of a channel that does not fulfill condition (a) above is the two-user binary adder channel when the sum is modulo 2 and the inputs are assigned uniform probability. In this case condition (a) is violated since $I(X_i; Y) = 0$ for $i = 1, 2$. Then \mathcal{R} is a triangle as opposed to a pentagon. An example for which condition (b) is not fulfilled is when we have two parallel channels. In this case condition (b) is violated since $I(X_1; Y|X_2) = I(X_1; Y) = 1$. The same is true if we swap X_1 and X_2 . In this case \mathcal{R} is a rectangle.

Fig. 1 shows an example of a non-degenerated \mathcal{R} (first subfigure) for $M = 2$ and all possible degenerated variations. Fig. 2 shows examples of degenerated cases for $M = 3$. All examples of Fig. 2 are for binary input channels and modulo 2 sums (when applicable). The first row depicts regions for the channel $Y = X_1 + X_2 + X_3$. If we denote by p_i the probability that $X_i = 1$, $i = 1, 2, 3$, then the first region (non degenerated) is obtained with $p_i \in [0, 1] \setminus \{0, 1/2, 1\}$, $i = 1, 2, 3$, the second region in the same row may be obtained with $p_1 = 0.5, p_2, p_3 \in [0, 1] \setminus \{0, 1/2, 1\}$, the third with $p_1 = p_2 = 0.5, p_3 \in [0, 1] \setminus \{0, 1/2, 1\}$, and the fourth with $p_1 = p_2 = p_3 = 0.5$. The first three subfigures of the second row correspond to the channel $Y = (Y_1, Y_2) = (X_1 + X_2, X_2 + X_3)$. The first region may be obtained with $p_i \in [0, 1] \setminus \{0, 1/2, 1\}$, $i = 1, 2, 3$, the second with $p_1 = p_2 = 0.5, p_3 \in [0, 1] \setminus \{0, 1/2, 1\}$, and the third with $p_1 = p_2 = p_3 = 0.5$. The last region in the second row may be obtained from the MAC $Y = (Y_1, Y_2, Y_3) = (X_1, X_2, X_3)$ with $p_i \in [0, 1] \setminus \{0, 1/2, 1\}$, $i = 1, 2, 3$. The first three subfigures in the third row correspond to the channel $Y = (Y_1, Y_2) = (X_1 + X_2, X_3)$, with the input distributions of the first one being $p_1, p_2 \in [0, 1] \setminus \{0, 1/2, 1\}, p_3 \in (0, 1)$, of the second being $p_1 \in [0, 1] \setminus \{0, 1/2, 1\}, p_2 = 0.5, p_3 \in (0, 1)$, and of the third being $p_1 = p_2 = 0.5, p_3 \in (0, 1)$. The last figure in the third row may be obtained with $p_1 \in \{0, 1\}$ and $p_i \in (0, 1)$, $i = 2, 3$.

An object of the form $\{R \in \mathbb{R}_+^M : R(\mathcal{S}) = c\}$, for some constant c , is an hyperplane of \mathbb{R}_+^M of dimension $M - 1$. The set $\{R \in \mathbb{R}_+^M : R(\mathcal{S}) \leq c\}$ is one of the two half-spaces bounded by such an hyperplane. \mathcal{R} is a finite intersection of such half-spaces. A linear inequality $Ra \leq a_0$, where R is a row vector, a a column vector and a_0 is a scalar, is valid for \mathcal{R} if it is satisfied for all points $R \in \mathcal{R}$. Following [11, Def. 2.1], a *face* of \mathcal{R} is defined as any set of the form

$$\mathcal{F} = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : Ra = a_0\},$$

where $Ra \leq a_0$ is a valid inequality for \mathcal{R} . The dimension of a face is the dimension of its affine hull, namely $\dim(\mathcal{F}) := \dim(\text{aff}(\mathcal{F}))$. In words, a face of \mathcal{R} is the intersection of \mathcal{R} with

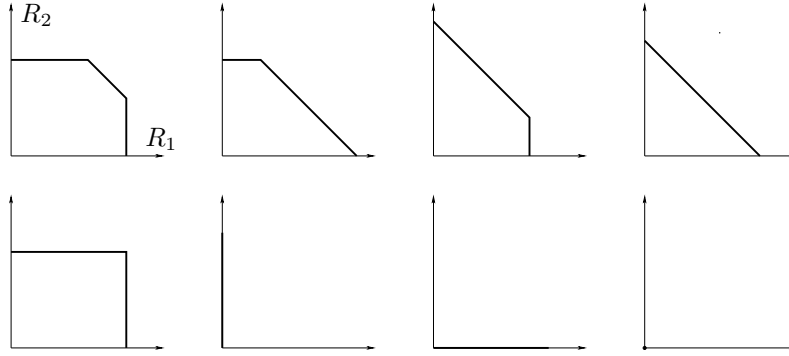


Figure 1: Shapes of \mathcal{R} for a 2-user channel. The first sub-figure is that of a non-degenerated case. In all pictures, the abscissa represents R_1 and the ordinate R_2 .

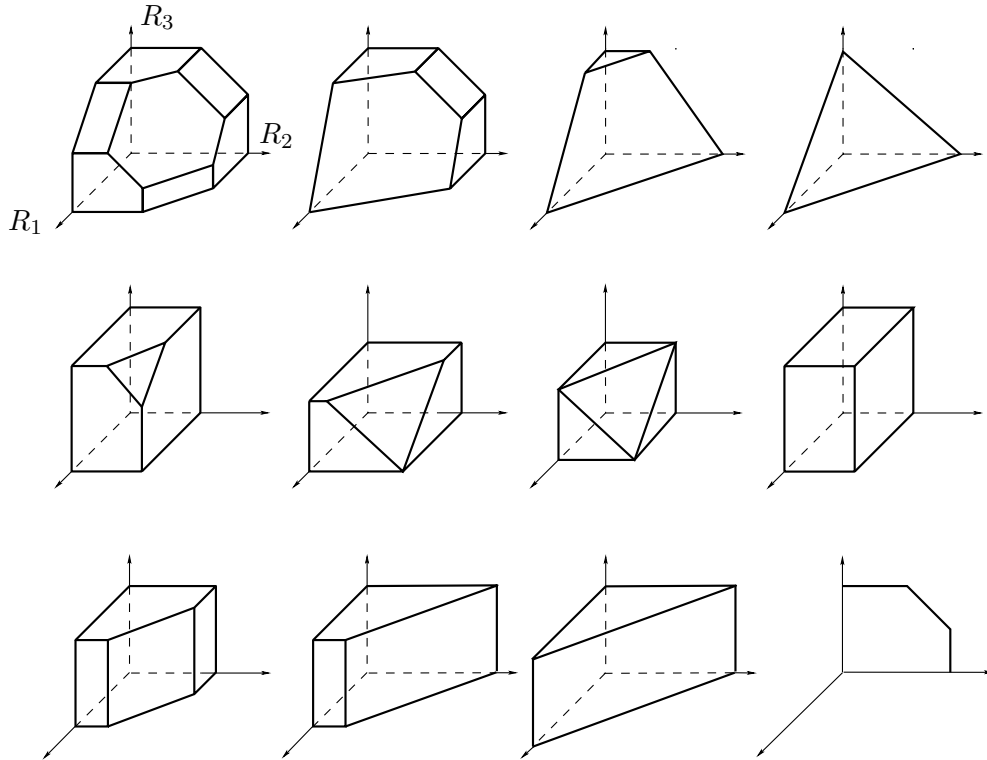


Figure 2: Some versions of \mathcal{R} for a 3-user channel. The first sub-figure is that of a non-degenerated case.

an $(M - 1)$ dimensional hyperplane that keeps \mathcal{R} on one side. Since the inequality $R\mathbf{0} \leq 0$ ($\mathbf{0}$ being all zero vector) is valid for \mathcal{R} , we observe that \mathcal{R} itself is a face. The empty set is also a face. These two faces are commonly referred to as *improper*. In this paper we are not concerned with improper faces. Hence by *face* we implicitly mean a proper face. Note also that the number of faces of any dimension is maximal in non-degenerated cases. We consider only such cases.

Faces of dimension 0, 1, $M - 2$, and $M - 1$ are called *vertices*, *edges*, *ridges*, and *facets*, respectively. In non-degenerated cases, for a single user channel \mathcal{R} has two vertices and one edge and for a 2-user channel it has five vertices and five edges. In Fig. 3 we see that there are 16 vertices, 24 edges and 10 facets for a non-degenerated region \mathcal{R} of a 3-user channel.

For every $i \in [M]$, there is a *back facet* of the form

$$\mathcal{B}_i = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R_i = 0\},$$

and for every $\mathcal{S} \subseteq [M]$, $\mathcal{S} \neq \emptyset$, there is a *front facet*

$$\mathcal{F}_{\mathcal{S}} = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R(\mathcal{S}) = I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c})\}.$$

There are M back facets and $2^M - 1$ front facets, one for each non-empty subset of $[M]$.

It is convenient to extend the notation \mathcal{B}_i and $\mathcal{F}_{\mathcal{S}}$ as follows

$$\begin{aligned} \mathcal{B}_{\mathcal{A}} &= \bigcap_{i \in \mathcal{A}} \mathcal{B}_i, \text{ with } \mathcal{B}_{\emptyset} = \mathcal{R} \text{ by convention,} \\ \mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m} &= \bigcap_{j=1}^m \mathcal{F}_{\mathcal{S}_j}, \text{ with } \mathcal{F}_{\emptyset} = \mathcal{R} \text{ by convention,} \\ \mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}} &= \mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m} \cap \mathcal{B}_{\mathcal{A}}. \end{aligned}$$

Note that $\mathcal{F}_{\mathcal{S}|\emptyset} = \mathcal{F}_{\mathcal{S}}$, $\mathcal{F}_{\emptyset|\mathcal{A}} = \mathcal{B}_{\mathcal{A}}$, and $\mathcal{F}_{\emptyset|\emptyset} = \mathcal{R}$. Fig. 3 shows a non-degenerated \mathcal{R} for a 3-user channel and some of the labels.

Next we show that two front facets intersect if and only if the index set of one is a subset of the index set of the other (Lemma 2) and that a front and a back facet intersect if and only if the index of the back facet is not an element of the set that defines the front facet (Lemma 4).

Lemma 2 $\mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$ is not empty iff $\mathcal{S}_1 \subseteq \mathcal{S}_2$ or $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

Proof: The “if” direction is clearly true if $\mathcal{S}_1 = \mathcal{S}_2$. Assume without loss of generality that $\mathcal{S}_1 \subset \mathcal{S}_2$. We want to show the existence of an $R \in \mathcal{R}$ such that

$$\begin{aligned} R(\mathcal{S}_1) &= I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c}), \text{ and} \\ R(\mathcal{S}_2) &= I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}). \end{aligned}$$

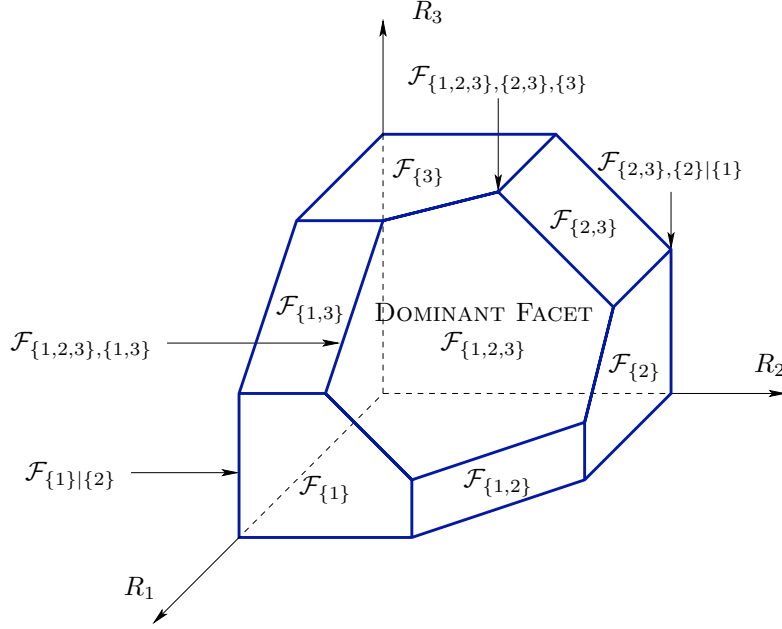


Figure 3: Region \mathcal{R} with labels for a 3-user MAC.

Without loss of generality, we re-index users so that $\mathcal{S}_1 = [k]$ and $\mathcal{S}_2 = [\ell]$, where $\ell > k$. Consider $R = (R_1, \dots, R_M)$ defined as follows

$$R_i = \begin{cases} I(X_i; Y | X_{i+1}, \dots, X_M), & i = 1, \dots, M-1, \\ I(X_M; Y) & i = M. \end{cases}$$

Observe that R is a vertex of the dominant face. Hence $R \in \mathcal{R}$. Furthermore, from the chain rule for mutual information

$$R([i]) = \sum_{j=1}^i R_j = \sum_{j=1}^i I(X_j; Y | X_{j+1}, \dots, X_M) = I(X_{[i]}; Y | X_{[i]^c}).$$

Thus, for $i = k$ we get $R([k]) = I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c})$ and for $i = \ell$, $R([\ell]) = I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c})$. Hence $R \in \mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$.

To prove the “only if” direction, let $R \in \mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$. Then

$$\begin{aligned}
I(X_{\mathcal{S}_1 \cup \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c}) &\stackrel{(a)}{\geq} R(\mathcal{S}_1 \cup \mathcal{S}_2) = R(\mathcal{S}_1) + R(\mathcal{S}_2) - R(\mathcal{S}_1 \cap \mathcal{S}_2) \\
&\stackrel{(b)}{=} I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) - R(\mathcal{S}_1 \cap \mathcal{S}_2) \\
&\stackrel{(c)}{\geq} I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) - I(X_{\mathcal{S}_1 \cap \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cap \mathcal{S}_2)^c}) \\
&\stackrel{(d)}{=} I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) \\
&\stackrel{(e)}{\geq} I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) \\
&= I(X_{\mathcal{S}_1 \cup \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c})
\end{aligned}$$

where (a) and (c) follow from the fact that $R \in \mathcal{R}$, (b) from the definition of $\mathcal{F}_{\mathcal{S}_i}$, $i = 1, 2$, (d) from the chain rule for mutual information, and (e) holds since the inputs are independent and conditioning on independent inputs can not decrease mutual information. By comparing the first and the last term of the above chain, we see that (a), (c), and (e) must be equalities. Equality in (e) means

$$I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{\mathcal{S}_1^c}) = I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c}).$$

Since \mathcal{R} is non-degenerated (by assumption), the above equality implies that either $\mathcal{S}_1 \setminus \mathcal{S}_2 = \emptyset$, i.e., $\mathcal{S}_1 \subseteq \mathcal{S}_2$ or $\mathcal{S}_1 = \mathcal{S}_1 \cup \mathcal{S}_2$, i.e., $\mathcal{S}_2 \subseteq \mathcal{S}_1$. This completes the proof. \square

The following Lemma is from [16].

Lemma 3 Assume $R \in \mathcal{F}_{\mathcal{S}}$. Then for every $\mathcal{L} \subseteq \mathcal{S}$

$$I(X_{\mathcal{L}}; Y | X_{\mathcal{S}^c}) \leq R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y | X_{\mathcal{L}^c}). \quad (2)$$

Proof: The second inequality is true for every $R \in \mathcal{R}$. To prove the first inequality observe that

$$\begin{aligned}
R(\mathcal{L}) &= R(\mathcal{S}) - R(\mathcal{S} \setminus \mathcal{L}) \\
&\stackrel{(a)}{=} I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c}) - R(\mathcal{S} \setminus \mathcal{L}) \\
&\stackrel{(b)}{\geq} I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c}) - I(X_{\mathcal{S} \setminus \mathcal{L}}; Y | X_{(\mathcal{S} \setminus \mathcal{L})^c}) \\
&\stackrel{(c)}{=} I(X_{\mathcal{L}}; Y | X_{\mathcal{S}^c}),
\end{aligned} \quad (3)$$

where (a) is true since $R \in \mathcal{F}_{\mathcal{S}}$, (b) since $R \in \mathcal{R}$ and (c) follows from the chain rule for mutual information. \square

Lemma 4 $\mathcal{F}_{\mathcal{S}} \cap \mathcal{B}_{\mathcal{A}} \neq \emptyset$ iff $\mathcal{A} \cap \mathcal{S} = \emptyset$.

Proof: If $\mathcal{A} = \emptyset$ then the Lemma is clearly true. Assume $\mathcal{A} \neq \emptyset$. To prove one direction, let and $R \in \mathcal{F}_{\mathcal{S}} \cap \mathcal{B}_{\mathcal{A}}$. Then $0 = R(\mathcal{A}) = R(\mathcal{S} \cap \mathcal{A}) \geq I(X_{\mathcal{S} \cap \mathcal{A}}; Y | X_{\mathcal{S}^c})$, where the inequality follows from Lemma 3. This implies that $I(X_{\mathcal{S} \cap \mathcal{A}}; Y | X_{\mathcal{S}^c}) = 0$. Since R is non-degenerated, it follows that $\mathcal{A} \cap \mathcal{S} = \emptyset$. To prove the other direction, assume $\mathcal{A} \cap \mathcal{S} = \emptyset$ and pick a rate \tilde{R} such that $\tilde{R} \in \mathcal{F}_{\mathcal{S}}$. Let R be obtained from \tilde{R} by setting to 0 all coordinates with index in \mathcal{A} . Clearly $R \in \mathcal{B}_{\mathcal{A}}$ but also $R \in \mathcal{F}_{\mathcal{S}}$ since $R(\mathcal{S}) = \tilde{R}(\mathcal{S})$.

□

Proposition 5 The intersection $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}$ is not empty, if and only if the following two conditions are satisfied

- (i) The set sequence $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m$ is telescopic, i.e., there is a permutation π on the index set $[m]$ such that $\mathcal{S}_{\pi(1)} \supset \mathcal{S}_{\pi(2)} \supset \dots \supset \mathcal{S}_{\pi(m)}$, and
- (ii) $\mathcal{A} \cap \mathcal{S}_{\pi(1)} = \emptyset$.

Proof: Assume that, after re-indexing if necessary, $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{S}_m$ and $\mathcal{A} \cap \mathcal{S}_1 = \emptyset$. The construction in the “if” part of the proof of Lemma 2 leads to an \tilde{R} in $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}$. Let R be obtained from \tilde{R} by setting to 0 all coordinates with index in \mathcal{A} . This does not affect coordinates with index in \mathcal{S}_i . Hence $R \in \mathcal{F}_{\mathcal{S}_i}$, $i = 1, \dots, m$ and $R \in \mathcal{B}_{\mathcal{A}}$.

To prove the converse, we observe that if \mathcal{S}_i is not contained in \mathcal{S}_j or vice versa, then by Lemma 2, $\mathcal{F}_{\mathcal{S}_i} \cap \mathcal{F}_{\mathcal{S}_j} = \emptyset$. Similarly, if $\mathcal{S}_1 \cap \mathcal{A} \neq \emptyset$, then according to Lemma 4, $\mathcal{F}_{\mathcal{S}_1} \cap \mathcal{B}_{\mathcal{A}} = \emptyset$. This concludes the proof. □

A fundamental result of polytopes asserts that any proper face is the intersection of those facets that contain it (see e.g. [21, Theorem 3.7, p. 35].) Hence Proposition 5 uniquely labels all proper faces of \mathcal{R} . (For our purpose, it would have been more straightforward to define a proper face as a nonempty intersection of facets. For obvious reasons we have decided to use the established definition.)

There is one facet of \mathcal{R} that stands out from the others. It is the *dominant facet* (commonly called dominant face) $\mathcal{F}_{[M]}$. It is special since points in the dominant facet have maximal sum-rate. Observe that, from Lemma 4, the dominant facet is the only facet that does not intersect with any back facet. The structure of the dominant facet was presented in [12]. In the one-user case, the dominant facet is a vertex, in the two-user case it is an edge that has two vertices, in the three-user case a hexagon (Fig. 3). In Fig. 2, we see that there are 24 vertices, 36 edges and 14 two-dimensional faces in the dominant facet of a 4-user channel. The dominant facet is a geometrical object called *permutahedron* [11].

Notice that the notation for a vertex in Fig. 2, has been simplified. Instead of writing the

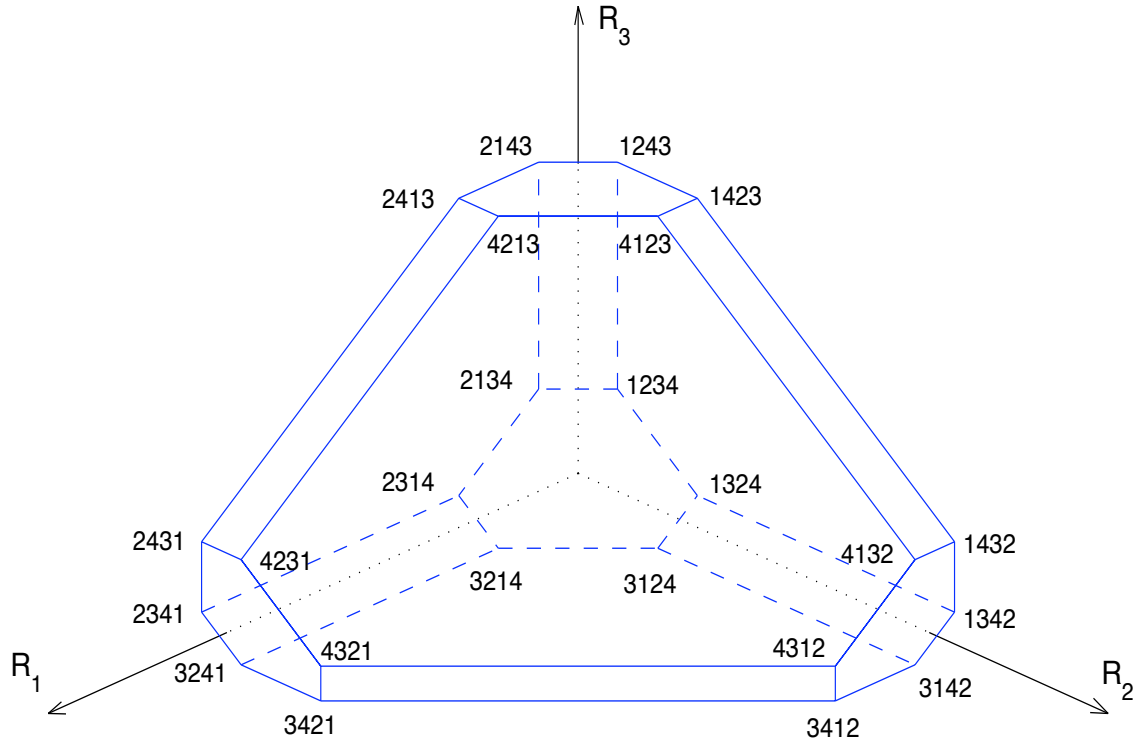


Figure 4: Dominant facet of a 4-user MAC. The 4th dimension, not shown here, has coordinate $R_4 = I(X_{\{1,2,3,4\}}; Y) - R_1 - R_2 - R_3$. Labels describe the decoding order used to approach the corresponding vertex via successive decoding.

telescopic sequence $\mathcal{F}_{\{1,2,3,4\},\{1,2,3\},\{1,3\},\{3\}}$, we have written the sequence of “decrements,” i.e., 4, 2, 1, 3 (commas are not shown in Fig. 2). Besides being more compact, the sequence of decrements gives the order in which users are decoded. It is also a convenient notation to count vertices. Since each permutation on the set $[M]$ is a vertex in the dominant facet, it is clear that there are $M!$ such vertices.

The reader familiar with polymatroids will recognize that Lemma 2 is a consequence of the fact that the function $f : S \rightarrow I(X_S; Y|X_{S^c})$ is submodular and nondecreasing, implying that $([M], f)$ is a polymatroid. In fact \mathcal{R} and \mathcal{D} are the associated *independence* and *base polyhedra*, respectively (see e.g. [20]). The structural properties of the *dominant face* derived in this section are implied by the fact that f is submodular and non-decreasing. In fact those properties could have been inferred from Theorems 3.22, 3.30 and 3.34 of [20]. However, for the structural properties of general faces we do need Lemma 3 in addition to the fact that f is submodular and non-decreasing. Specifically, Lemma 3 implies that all rates are strictly positive for all front faces. Without this Lemma some of the front faces could collapse into some of the back faces.

3 Structure, dimensionality, and group successive decoding

In this section we show that the faces of \mathcal{R} consist of the Cartesian product of regions and of dominant facets of channels that are obtained from the original channel W . To distinguish those channels, we use subscripts that indicate the channel inputs and outputs. The original channel W will be denoted by $W_{Y|X_{[M]}}$. Recall that the region \mathcal{R} is completely specified by the channel $W_{Y|X_{[M]}}$ and by the input distribution $P_{X_{[M]}}$.

For any two sets $\mathcal{U}, \mathcal{V} \subset [M]$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$, there is a channel with inputs $X_{\mathcal{U}}$ and outputs $(Y, X_{\mathcal{V}})$. Specifically,

$$\begin{aligned} W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}(y, x_{\mathcal{V}}|x_{\mathcal{U}}) &= P_{X_{\mathcal{V}}}(x_{\mathcal{V}})W_{Y|X_{\mathcal{U}}, X_{\mathcal{V}}}(y|x_{\mathcal{U}}, x_{\mathcal{V}}) \\ &= P_{X_{\mathcal{V}}}(x_{\mathcal{V}}) \sum_{x_{[M]\setminus\mathcal{U}\setminus\mathcal{V}}} W_{YX_{[M]\setminus\mathcal{U}\setminus\mathcal{V}}|X_{\mathcal{U}}, X_{\mathcal{V}}}(y, x_{[M]\setminus\mathcal{U}\setminus\mathcal{V}}|x_{\mathcal{U}}, x_{\mathcal{V}}) \\ &= P_{X_{\mathcal{V}}}(x_{\mathcal{V}}) \sum_{x_{[M]\setminus\mathcal{U}\setminus\mathcal{V}}} P_{X_{[M]\setminus\mathcal{U}\setminus\mathcal{V}}}(x_{[M]\setminus\mathcal{U}\setminus\mathcal{V}})W_{Y|X_{[M]}}(y|x_{[M]}), \end{aligned}$$

where by convention $P_{X_{\emptyset}}(x_{\emptyset}) = 1$.

A rate tuple for $W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}$ is an expression of the form $R_{\mathcal{U}} \triangleq (R_i)_{i \in \mathcal{U}}$. The corresponding region $\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}}$ is defined by

$$\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}} \triangleq \mathcal{R}[W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}; P_{X_{\mathcal{U}}}] = \{R \in \mathbb{R}_+^{|\mathcal{U}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{V} \cup (\mathcal{U} \setminus \mathcal{L})}), \quad \forall \mathcal{L} \subseteq \mathcal{U}\}. \quad (4)$$

The dimensionality of $\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}}$ is $|\mathcal{U}|$. Its dominant facet is the $(|\mathcal{U}| - 1)$ -dimensional sub-region obtained by adding the equality $R(\mathcal{U}) = I(X_{\mathcal{U}}; Y|X_{\mathcal{V}})$ i.e.,

$$\begin{aligned} \mathcal{D}_{YX_{\mathcal{V}}|X_{\mathcal{U}}} &\triangleq \mathcal{D}[W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}; P_{X_{\mathcal{U}}}] \\ &= \{R \in \mathbb{R}_+^{|\mathcal{U}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{V} \cup (\mathcal{U} \setminus \mathcal{L})}), \forall \mathcal{L} \subset \mathcal{U}, R(\mathcal{U}) = I(X_{\mathcal{U}}; Y|X_{\mathcal{V}})\}. \end{aligned}$$

The following special cases will be used frequently

$$\begin{aligned} \mathcal{R}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}} &\triangleq \mathcal{R}[W_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}; P_{X_{\mathcal{S}}}] = \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}), \quad \forall \mathcal{L} \subseteq \mathcal{S}\}, \\ \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}} &\triangleq \mathcal{D}[W_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}; P_{X_{\mathcal{S}}}] \\ &= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}), \forall \mathcal{L} \subset \mathcal{S}, R(\mathcal{S}) = I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c})\}, \\ \mathcal{R}_{Y|X_{\mathcal{S}}} &\triangleq \mathcal{R}[W_{Y|X_{\mathcal{S}}}; P_{X_{\mathcal{S}}}] = \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{S} \setminus \mathcal{L}}), \quad \forall \mathcal{L} \subseteq \mathcal{S}\}, \\ \mathcal{D}_{Y|X_{\mathcal{S}}} &\triangleq \mathcal{D}[W_{Y|X_{\mathcal{S}}}; P_{X_{\mathcal{S}}}] \\ &= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{S} \setminus \mathcal{L}}), \forall \mathcal{L} \subset \mathcal{S}, R(\mathcal{S}) = I(X_{\mathcal{S}}; Y)\}. \end{aligned}$$

The next lemma says that \mathcal{F}_S is the Cartesian product of a region \mathcal{R} and a dominant facet \mathcal{D} . One expects this to be the case by looking at the facets $\mathcal{F}_{\{1\}}$, $\mathcal{F}_{\{2\}}$, and $\mathcal{F}_{\{3\}}$ of Fig. 3. For instance $\mathcal{F}_{\{1\}}$ is the Cartesian product of a singleton and a pentagon. The singleton, the value of R_1 , is the dominant facet \mathcal{D} of a single-user channel. The pentagon, the region that contains $R_{\{2,3\}}$, is the region \mathcal{R} of a two-user multiple-access channel. A perhaps less evident example is $\mathcal{F}_{\{1,2\}}$. This is the Cartesian product of the dominant facet \mathcal{D} of a two user multiple-access channel and the region \mathcal{R} of a single user channel.

Lemma 6 $R \in \mathcal{F}_S$ iff $R_{S^c} \in \mathcal{R}_{Y|X_{S^c}}$ and $R_S \in \mathcal{D}_{YX_{S^c}|X_S}$.

Proof : Let $R \in \mathcal{F}_S$. From the definition of \mathcal{F}_S , $\forall \mathcal{L} \subset \mathcal{S} \subseteq [M]$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ and $R(\mathcal{S}) = I(X_S; Y|X_{S^c})$. Therefore $R_S \in \mathcal{D}_{YX_{S^c}|X_S}$. Moreover, $\forall \mathcal{T} \subset \mathcal{S}^c$ we may write $[M] = \mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$ as the union of disjoint sets. Then

$$\begin{aligned} R(\mathcal{T}) + R(\mathcal{S}) &\leq I(X_{\mathcal{T} \cup \mathcal{S}}; Y|X_{\mathcal{Q}}) \\ &= I(X_{\mathcal{T}}; Y|X_{\mathcal{Q}}) + I(X_S; Y|X_{\mathcal{Q} \cup \mathcal{T}}) \\ &= I(X_{\mathcal{T}}; Y|X_{\mathcal{Q}}) + I(X_S; Y|X_{S^c}) \\ &= I(X_{\mathcal{T}}; Y|X_{S^c \setminus \mathcal{T}}) + R(\mathcal{S}). \end{aligned}$$

Hence $R(\mathcal{T}) \leq I(X_{\mathcal{T}}; Y|X_{S^c \setminus \mathcal{T}})$ and from (4) it follows that $R_{S^c} \in \mathcal{R}_{Y|X_{S^c}}$.

To prove the converse, let $R_S \in \mathcal{D}_{YX_{S^c}|X_S}$ and $R_{S^c} \in \mathcal{R}_{Y|X_{S^c}}$. We have to prove that $R(\mathcal{S}) = I(X_S; Y|X_{S^c})$ and that for all $\mathcal{L} \subseteq [M]$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$. The former is true since $R_S \in \mathcal{D}_{YX_{S^c}|X_S}$. To prove the latter, let $\mathcal{T} = \mathcal{L} \cap \mathcal{S}$ and $\mathcal{Q} = \mathcal{L} \cap \mathcal{S}^c$. Since $R_S \in \mathcal{D}_{YX_{S^c}|X_S}$, $R(\mathcal{T}) \leq I(X_{\mathcal{T}}; Y|X_{S^c \cup (\mathcal{S} \setminus \mathcal{T})}) = I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c})$ for all $\mathcal{T} \subseteq \mathcal{S}$. Furthermore, since $R_{S^c} \in \mathcal{R}_{Y|X_{S^c}}$, $R(\mathcal{Q}) \leq I(X_{\mathcal{Q}}; Y|X_{S^c \setminus \mathcal{Q}})$ for all $\mathcal{Q} \subseteq \mathcal{S}^c$. Hence

$$\begin{aligned} R(\mathcal{L}) &= R(\mathcal{T} \cup \mathcal{Q}) = R(\mathcal{T}) + R(\mathcal{Q}) \\ &\leq I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c}) + I(X_{\mathcal{Q}}; Y|X_{S^c \setminus \mathcal{Q}}) \\ &\leq I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c}) + I(X_{\mathcal{Q}}; Y|X_{(S^c \setminus \mathcal{Q}) \cup (\mathcal{S} \setminus \mathcal{T})}) \\ &= I(X_{\mathcal{T} \cup \mathcal{Q}}; Y|X_{(\mathcal{T} \cup \mathcal{Q})^c}) \\ &= I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}) \end{aligned}$$

for all $\mathcal{L} \subseteq [M]$ and this completes the proof. □

From Lemma 6 we obtain the dimension of \mathcal{F}_S

$$\begin{aligned} \dim(\mathcal{F}_S) &= \dim(\mathcal{R}_{Y|X_{S^c}}) + \dim(\mathcal{D}_{YX_{S^c}|X_S}) \\ &= |S^c| + |S| - 1 = M - 1, \end{aligned}$$

which is to be expected for a facet.

Finally, Lemma 6 tells us that a rate point in $\mathcal{F}_{\mathcal{S}}$ may be approached via group successive decoding where groups are decoded in the $\text{ord}(\mathcal{S}^c, \mathcal{S})$. (For a rigorous proof of this fact we need to use codes that have independent and identically distributed components. This may be done using random coding arguments as in [16].)

Then next result is a generalization of Lemma 6. It says that when $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}$ is not empty it is the Cartesian product of a region and m dominant facets.

Theorem 7 Let $\mathcal{S}_1 \supset \mathcal{S}_2 \dots \supset \mathcal{S}_m$ form a telescopic sequence. $R \in \mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}$ iff $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$ and $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ for $i = 1, \dots, m$, where by way of convention we have defined $\mathcal{S}_{m+1} = \emptyset$.

Proof : Let $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m}$ and recall that $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m} = \bigcap_{i=1}^m \mathcal{F}_{\mathcal{S}_i}$. From Lemma 6 we have

$$R_{\mathcal{S}_i} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i}} \quad \text{and} \quad R_{\mathcal{S}_i^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_i^c}}, \quad i = 1, \dots, m. \quad (5)$$

This proves $R_{\mathcal{S}_i^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_i^c}}$. To complete the proof of the direct part it is sufficient to show that (5) implies

$$R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}, \quad \forall i = 1, \dots, m-1.$$

This is the case if $R(\mathcal{K}) \leq I(X_{\mathcal{K}}; Y|X_{\mathcal{S}_i^c \cup (\mathcal{S}_i \setminus \mathcal{S}_{i+1} \setminus \mathcal{K})}) = I(X_{\mathcal{K}}; Y|X_{(\mathcal{S}_{i+1} \cup \mathcal{K})^c})$ holds for all $\mathcal{K} \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1}$, $i = 1, \dots, m$, with equality if $\mathcal{K} = \mathcal{S}_i \setminus \mathcal{S}_{i+1}$. From (5), for any $\mathcal{K} \subseteq \mathcal{S}_i$, $R(\mathcal{K}) \leq I(X_{\mathcal{K}}; Y|X_{\mathcal{K}^c})$ with equality if $\mathcal{K} = \mathcal{S}_i$ and for any $\mathcal{L} \subseteq \mathcal{S}_{i+1}$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ with equality if $\mathcal{L} = \mathcal{S}_{i+1}$. Then for $\mathcal{K} \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1}$ we have

$$\begin{aligned} R(\mathcal{K}) &= R(\mathcal{S}_i) - R(\mathcal{S}_{i+1}) - R(\mathcal{S}_i \setminus \mathcal{S}_{i+1} \setminus \mathcal{K}) \\ &= I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}) - I(X_{\mathcal{S}_{i+1}}; Y|X_{\mathcal{S}_{i+1}^c}) - R(\mathcal{S}_i \setminus \mathcal{S}_{i+1} \setminus \mathcal{K}) \\ &\stackrel{(a)}{\leq} I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}) - I(X_{\mathcal{S}_{i+1}}; Y|X_{\mathcal{S}_{i+1}^c}) - I(X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1} \setminus \mathcal{K}}; Y|X_{\mathcal{S}_i^c}) \\ &= I(X_{\mathcal{K}}; Y|X_{(\mathcal{S}_{i+1} \cup \mathcal{K})^c}), \end{aligned}$$

where (a) follows from the fact that $\forall \mathcal{Q} \subset \mathcal{S}_i$, $R(\mathcal{Q}) \geq I(X_{\mathcal{Q}}; Y|X_{\mathcal{S}_i^c})$. The equality in (a) holds if $\mathcal{K} = \mathcal{S}_i \setminus \mathcal{S}_{i+1}$. This proves the direct part.

To prove the converse, let $R_{\mathcal{S}_i^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_i^c}}$, and $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$, for $i = 1, \dots, m$. We have to prove that $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ holds for all $\mathcal{L} \subseteq [M]$ with equality if $\mathcal{L} = \mathcal{S}_i$, $i = 1, \dots, m$. $R(\mathcal{S}_i) = I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c})$ is true since $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ and

$$R(\mathcal{S}_i) = \sum_{j=i}^m R(\mathcal{S}_j \setminus \mathcal{S}_{j+1}) = \sum_{j=1}^m I(X_{\mathcal{S}_j \setminus \mathcal{S}_{j+1}}; Y|X_{\mathcal{S}_j^c}) = I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}), \quad i = 1, \dots, m.$$

Now let $\mathcal{L} \subseteq [M]$ and define $\mathcal{L}_i = \mathcal{L} \cap \mathcal{S}_i \setminus \mathcal{S}_{i+1}$, $i = 0, 1, \dots, m$ with $\mathcal{S}_0 = [M]$ by convention. Then $\mathcal{L} = \bigcup_{i=0}^m \mathcal{L}_i$ is a disjoint partition. From $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$ and $\mathcal{L}_0 \subseteq \mathcal{S}_1^c$ it follows $R(\mathcal{L}_0) \leq I(X_{\mathcal{L}_0}; Y|X_{\mathcal{S}_1^c \setminus \mathcal{L}_0})$. Furthermore, since $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$, $R(\mathcal{L}_i) \leq I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_i^c \cup (\mathcal{S}_i \setminus \mathcal{S}_{i+1}) \setminus \mathcal{L}_i}) = I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_{i+1}^c \setminus \mathcal{L}_i})$ for all $\mathcal{L}_i \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1}$. Therefore,

$$\begin{aligned} I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}) &= \sum_{i=0}^m I(X_{\mathcal{L}_i}; Y|X_{\bigcup_{j=0}^{i-1} \mathcal{L}_j \cup \mathcal{L}^c}) \\ &\stackrel{(a)}{\geq} \sum_{i=0}^m I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_{i+1}^c \setminus \mathcal{L}_i}) \\ &\geq \sum_{i=0}^m R(\mathcal{L}_i) \\ &= R(\mathcal{L}), \end{aligned}$$

where (a) holds since

$$\bigcup_{j=0}^{i-1} \mathcal{L}_j \cup \mathcal{L}^c = [M] \setminus \bigcup_{j=i}^m \mathcal{L}_j \supseteq \mathcal{S}_{i+1}^c \setminus \bigcup_{j=i}^m \mathcal{L}_j = \mathcal{S}_{i+1}^c \setminus \mathcal{L}_i,$$

and (b) holds since for $j \geq i+1$, $\mathcal{L}_j \subseteq \mathcal{S}_{i+1}$ implies that \mathcal{L}_j does not intersect with \mathcal{S}_{i+1}^c . This completes the proof. \square

From the previous theorem, the dimension of $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}$ is

$$\begin{aligned} \dim(\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}) &= \dim(\mathcal{R}_{Y|X_{\mathcal{S}_1^c}}) + \sum_{i=1}^m \dim(\mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}) \\ &= M - |\mathcal{S}_1| + \sum_{i=1}^m (|\mathcal{S}_i| - |\mathcal{S}_{i+1}| - 1) \\ &= M - m. \end{aligned}$$

The theorem also implies that all points in $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}$ may be approached via group successive decoding with groups of users decoded according to the following order: $(\mathcal{S}_1^c, \mathcal{S}_1 \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \dots, \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \mathcal{S}_m)$.

Corollary 8 Let $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{S}_m$ be a telescopic sequence. $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}}$ iff $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$, $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ for $i = 1, \dots, m$, and $R_{\mathcal{A}} = \underline{0}$.

Proof : Recall that $\mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}} = \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m} \cap \mathcal{B}_{\mathcal{A}}$. Hence $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}}$ iff $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m}$ and $R_{\mathcal{A}} = \underline{0}$. The rest follows from Theorem 7.

□

From the above Corollary we conclude that

$$\begin{aligned} \dim(\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}) &= \dim(\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m}) - |\mathcal{A}| \\ &= M - |\mathcal{A}| - m. \end{aligned}$$

Furthermore, $R \in \mathcal{F}_{\mathcal{S}_1, \dots, \mathcal{S}_m | \mathcal{A}}$ may be approached by decoding groups of users in the order $([M] \setminus \mathcal{A} \setminus \mathcal{S}_1, \mathcal{S}_1 \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \dots, \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \mathcal{S}_m)$.

4 Number of faces of dimension D

Now we are ready to derive the number of D -dimensional faces in \mathcal{R} for any $D = 0, 1, \dots, M$. We start by describing the number of D -dimensional faces of the dominant facet.

Proposition 9 The number of D -dimensional faces in the dominant facet of \mathcal{R} is

$$N_d(M, D) = \sum_{j=1}^{M-D} \binom{M-D}{j} (-1)^{M-D-j} j^M. \quad (6)$$

Proof: Any D -dimensional face on the dominant facet is labeled by $\mathcal{F}_{[M], \mathcal{S}_2, \dots, \mathcal{S}_{M-D}}$. The difference sets $[M] \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \dots, \mathcal{S}_i \setminus \mathcal{S}_{i+1}, \dots, \mathcal{S}_{M-D} \setminus \emptyset$ form an $(M-D)$ partition of $[M]$. There is a one-to-one correspondence between a D -dimensional face and such a partition. The number of such ordered partitions is

$$N_d(M, D) = \sum_{\substack{m_1, m_2, \dots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = M}} \binom{M}{m_1, m_2, \dots, m_{M-D}} = \sum_{\substack{m_1, m_2, \dots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = M}} \frac{M!}{\prod_i m_i!}. \quad (7)$$

To go further, we expand the following polynomial

$$\left(\frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^M}{M!} \right)^{M-D} = \sum_{k=M-D}^{M(M-D)} x^k \sum_{\substack{m_1, \dots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = k}} \frac{1}{m_1! m_2! \dots m_{M-D}!},$$

and note that the coefficient in front of x^M multiplied by $M!$ gives (7). Therefore,

$$\begin{aligned}
N_d(M, D) &= M! \operatorname{coeff} \left(\left(\sum_{i=1}^M \frac{x^i}{i!} \right)^{M-D}, x^M \right) \\
&\stackrel{(a)}{=} M! \operatorname{coeff} \left(\left(\sum_{i=1}^{\infty} \frac{x^i}{i!} \right)^{M-D}, x^M \right) \\
&\stackrel{(b)}{=} M! \operatorname{coeff} \left((e^x - 1)^{M-D}, x^M \right) \\
&\stackrel{(c)}{=} \left. \frac{d^M}{dx^M} (e^x - 1)^{M-D} \right|_{x=0},
\end{aligned}$$

where $\operatorname{coeff}(f(x), x^i)$ is the coefficient of x^i in the Taylor series expansion around zero of the function $f(x)$, (a) is true since taking all the terms up to M or up to infinity will not change the coefficient in front of x^M , (b) follows from the Taylor expansion of e^x , and (c) follows from the definition of the Taylor expansion.

To prove (6), we use the Binomial formula to expand $(e^x - 1)^{M-D}$, namely

$$(e^x - 1)^{M-D} = \sum_{j=0}^{M-D} \binom{M-D}{j} e^{jx} (-1)^{M-D-j}.$$

Taking the M -th derivative,

$$\frac{d^M}{dx^M} (e^x - 1)^{M-D} = \sum_{j=1}^{M-D} \binom{M-D}{j} e^{jx} (-1)^{M-D-j} j^M,$$

and setting $x = 0$ we obtain (6). □

Observe that by letting $D = 0$, using the fact that there are $M!$ vertices in the dominant facet, from (6) we obtain an alternative expression for $M!$ that is

$$M! = \sum_{j=1}^M \binom{M}{j} (-1)^{M-j} j^M.$$

The number of faces in the dominant facet is directly connected to the *Stirling number of the second kind* [15, 18], denoted by $\left\{ \begin{smallmatrix} M \\ n \end{smallmatrix} \right\}$. This is known as *Karamata notation* [14]. The Stirling number of the second kind is the number of ways we can partition a set of M elements into

n nonempty subsets. In calculating the number of D -dimensional faces, all permutations of such partitions have to be counted. That is,

$$N_d(M, D) = (M - D)! \left\{ \begin{matrix} M \\ M - D \end{matrix} \right\}.$$

Like for facets, it is useful to distinguish between front and back faces. Hence we say that a face $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}$, is called a *front face* if $\mathcal{A} = \emptyset$ and a *back face* if $\mathcal{A} \neq \emptyset$.

Proposition 10 The total number of front faces of dimension D , denoted by $N_f(M, D)$, equals

$$N_f(M, D) = N_d(M, D) + N_d(M, D - 1). \quad (8)$$

Proof: Any D -dimensional front face has a label $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{M-D} | \emptyset}$ for some $\mathcal{S}_1 \subseteq [M]$. If $\mathcal{S}_1 = [M]$, the front face is in the dominant facet and there are $N_d(M, D)$ such faces. If $\mathcal{S}_1 \subset [M]$ the front face is not in the dominant facet. Since there is a one-to-one relationship between the subscripts of $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{M-D} | \emptyset}$ and those of $\mathcal{F}_{[M], \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{M-D} | \emptyset}$ when $\mathcal{S}_1 \subset [M]$, it follows that the total number of front faces not in the dominant facet is exactly $N_d(M, D - 1)$. To obtain the total number of front D -faces we have to add this number and the number $N_d(M, D)$ of D -faces in the dominant facet. □

We now have an expression for $N_d(M, D)$ (Proposition 9) and an expression for $N_f(M, D)$ (Proposition 10). Next we derive an expression for the number of back faces $N_b(M, D)$.

Proposition 11 The total number of D -dimensional back faces in \mathcal{R} is given by

$$N_b(M, D) = \sum_{i=D}^{M-1} \binom{M}{i} N_f(i, D). \quad (9)$$

Proof: To derive (9), we observe that all back faces are front faces for some other channel with fewer users. This can be seen from the label $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}}$ of a back face, where $\mathcal{A} \neq \emptyset$. The dimension of this face is $M - m - |\mathcal{A}|$. Recall that $\mathcal{A} \cap \mathcal{S}_1 = \emptyset$. If we remove all users with index in \mathcal{A} , we obtain the front face $\mathcal{F}_{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \emptyset}$ of an $(M - |\mathcal{A}|)$ -user MAC. The dimensionality of this face is also $M - |\mathcal{A}| - m$. Running over all pertinent subsets $\mathcal{A} \subset [M]$ yields

$$N_b(M, D) = \sum_{\substack{\mathcal{A} \subset [M] \\ 0 < |\mathcal{A}| \leq M - D}} N_f(M - |\mathcal{A}|, D).$$

Since there are $\binom{M}{|\mathcal{A}|}$ subsets of cardinality $|\mathcal{A}|$,

$$\begin{aligned} N_b(M, D) &= \sum_{|\mathcal{A}|=1}^{M-D} \binom{M}{|\mathcal{A}|} N_f(M - |\mathcal{A}|, D) = \sum_{i=1}^{M-D} \binom{M}{i} N_f(M - i, D) \\ &= \sum_{i=1}^{M-D} \binom{M}{M-i} N_f(M - i, D) = \sum_{i=D}^{M-1} \binom{M}{i} N_f(i, D). \end{aligned}$$

□

Now we are ready to derive an expression for the total number of D -dimensional faces in \mathcal{R} .

Theorem 12 The total number of D -dimensional faces in \mathcal{R} , $0 \leq D \leq M$, is

$$N(M, D) = \sum_{i=D}^M \binom{M}{i} \left[(i+1-D)^i - \sum_{j=1}^{i-D} \binom{i-D}{j-1} (-1)^{i-D-j} j^i \right]. \quad (10)$$

Proof: First we observe that

$$N(M, D) = N_f(M, D) + N_b(M, D) = \sum_{i=D}^M \binom{M}{i} N_f(i, D).$$

Using (8) we obtain

$$N(M, D) = \sum_{i=D}^M \binom{M}{i} [N_d(i, D) + N_d(i, D-1)], \quad (11)$$

where $N_d(D, D) = 0$, $N_d(D, D-1) = 1$ and, by convention, $N_d(i, -1) = 0$ (the latter is needed for the case $D = 0$). Furthermore, from (6) we obtain

$$\begin{aligned} &N_d(i, D) + N_d(i, D-1) \\ &= (i-D+1)^i + \sum_{j=0}^{i-D} j^i (-1)^{i-D-j+1} \left[\binom{i-D+1}{j} - \binom{i-D}{j} \right] \\ &= (i-D+1)^i - \sum_{j=0}^{i-D} \binom{i-D}{j} \frac{j^{i+1} (-1)^{i-D-j}}{i-D+1-j} \\ &= (i-D+1)^i - \sum_{j=1}^{i-D} \binom{i-D}{j-1} (-1)^{i-D-j} j^i. \end{aligned} \quad (12)$$

Inserting (12) into (11) yields (10) and completes the proof.

□

Next we determine a closed form expression for the total number $N(M, 0)$ of vertices and the total number $N(M, 1)$ of edges.

Lemma 13 The total number of vertices in \mathcal{R} is $\lfloor eM! \rfloor$.

Proof: From (11),

$$\begin{aligned}
N(M, 0) &= \sum_{i=0}^M \binom{M}{i} N_d(i, 0) \\
&\stackrel{(a)}{=} \sum_{i=0}^M \binom{M}{i} i! = \sum_{i=0}^M \frac{M!}{(M-i)!} = \sum_{i=0}^M \frac{M!}{i!} \\
&= M! \sum_{i=0}^{\infty} \frac{1}{i!} - M! \sum_{i=M+1}^{\infty} \frac{1}{i!} \\
&\stackrel{(b)}{=} eM! - M! \sum_{i=M+1}^{\infty} \frac{1}{i!},
\end{aligned}$$

where in (a) we have used the well known fact that the number of vertices $N_d(i, 0)$ of the dominant facet of an i -user region is $i!$ and (b) follows from the Taylor series expansion of e . Since $eM! - M! \sum_{i=M+1}^{\infty} \frac{1}{i!}$ is an integer, and

$$\begin{aligned}
\sum_{i=M+1}^{\infty} \frac{M!}{i!} &= \sum_{i=1}^{\infty} \frac{M!}{(M+i)!} = \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^i (M+j)} \\
&< \sum_{i=1}^{\infty} \prod_{j=1}^i \frac{1}{M+1} = \sum_{i=1}^{\infty} \left(\frac{1}{M+1} \right)^i \\
&= \frac{1/(M+1)}{1 - 1/(M+1)} = \frac{1}{M} \leq 1,
\end{aligned}$$

it follows that

$$N(M, 0) = \sum_{i=0}^M \frac{M!}{i!} = \left\lfloor N(M, 0) + \sum_{i=M+1}^{\infty} \frac{M!}{i!} \right\rfloor = \lfloor eM! \rfloor. \tag{13}$$

□

Lemma 14 The total number of edges in \mathcal{R} is $\frac{M}{2} \lfloor eM! \rfloor$.

Proof: From (11) we have

$$N(M, 1) = \sum_{i=1}^M \binom{M}{i} (N_d(i, 1) + N_d(i, 0)).$$

Furthermore, since $N_d(i, 0) = i!$, from (7),

$$N_d(i, 1) = \sum_{\substack{m_1, m_2, \dots, m_{i-1} \\ m_j \geq 1, \forall j \\ \sum_j m_j = i}} \binom{i}{m_1, \dots, m_{i-1}} = (i-1) \binom{i}{2, 1, \dots, 1} = \frac{i!(i-1)}{2}.$$

Therefore,

$$\begin{aligned} N(M, 1) &= \sum_{i=1}^M \binom{M}{i} \left(i! + \frac{i-1}{2} i! \right) = \frac{1}{2} \sum_{i=1}^M \binom{M}{i} i! (i+1) \\ &= \frac{1}{2} \sum_{i=1}^M \frac{M!}{(M-i)!} (i+1) = \frac{1}{2} \sum_{j=0}^{M-1} \frac{M!}{j!} (M-j+1) \\ &= \frac{M+1}{2} \sum_{j=0}^{M-1} \frac{M!}{j!} - \frac{1}{2} \sum_{k=0}^{M-2} \frac{M!}{k!} \\ &\stackrel{(a)}{=} \frac{1}{2} [(M+1)(\lfloor eM! \rfloor - 1) - (\lfloor eM! \rfloor - M - 1)] = \frac{M}{2} \lfloor eM! \rfloor, \end{aligned}$$

where in (a) we use (13) to obtain $\sum_{j=0}^{M-1} M!/j! = \lfloor eM! \rfloor - 1$ and $\sum_{j=0}^{M-2} M!/j! = \lfloor eM! \rfloor - M - 1$.

□

5 Summary

The capacity region of an asynchronous memoryless multiple-access channel is the union of certain polytopes. The points in those polytopes are exactly the rate tuples that can be approached at an arbitrarily small error probability. In this paper we have developed operational and structural properties that apply to those polytopes. The centerpiece of our developments are the labels that we use to tag their faces. For non-degenerated cases (the only kind considered in this paper), the set of labels is the set of expressions of the form $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A})$, where $\mathcal{A} \subseteq [M]$ and $[M] \setminus \mathcal{A} \supset \mathcal{S}_1 \supset \mathcal{S}_2, \dots, \supset \mathcal{S}_m$. This extends the labeling introduced in [12]. Each label of the above form tags one face and each face has a unique such tag. We have shown that the label $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}$ tags a face of dimension

$M - m - |\mathcal{A}|$. By counting the number of such expressions for a fixed k , we find the number of faces of a given dimension.

We have also shown that a rate tuple on the face with label $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m | \mathcal{A}$ may be approached via successive decoding, as follows: the users with index in $([M] \setminus \mathcal{A} \setminus \mathcal{S}_1)$ are decoded first, followed by the users with index in $\mathcal{S}_1 \setminus \mathcal{S}_2$, followed by those with index in $\mathcal{S}_2 \setminus \mathcal{S}_3$ etc. The users with index in \mathcal{S}_m are decoded last. The users with index in \mathcal{A} do not need to be decoded since they have vanishing rate. The decoding order $([M] \setminus \mathcal{A} \setminus \mathcal{S}_1, \mathcal{S}_1 \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \dots, \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \mathcal{S}_m)$ is an equivalent alternative way to label faces.

Table 1 summarizes the expressions for the number of faces of a given dimension, where $f_n^{(i)}(0) = \frac{d}{dx}(e^x - 1)^n \Big|_{x=0}$. The logarithm of the total number of D -dimensional faces as a function of D , for $M = 1, 2, \dots, 20$ is shown in Fig. 5.

Table 1: Number of vertices, edges, facets and D -dimensional faces for an M -user MAC.

Objects	In \mathcal{R}	In the dominant facet
Vertices	$\lfloor eM! \rfloor$	$M!$
Edges	$\frac{M}{2} \lfloor eM! \rfloor$	$M!(M-1)/2$
Facets	$M + 2^M - 1$	$2^M - 2$
D -faces	$\sum_{i=D}^M \binom{M}{i} \left(f_{i-D}^{(i)}(0) + f_{i-D+1}^{(i)}(0) \right)$	$f_{M-D}^{(M)}(0)$

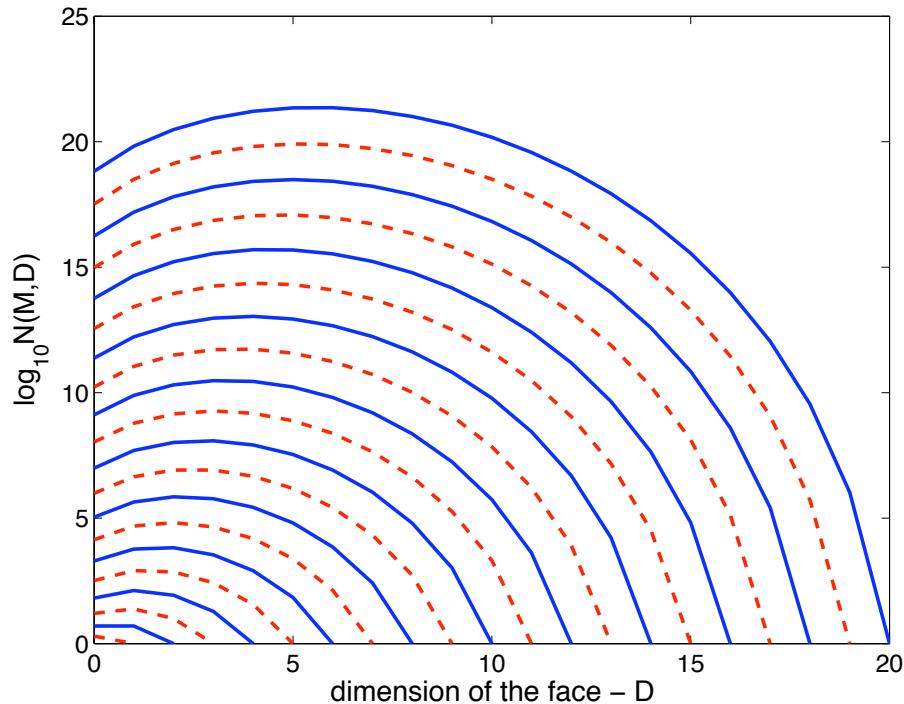


Figure 5: Total number of D -dimensional faces (expressed in logarithmic form) as a function of D . Each curve corresponds to a value of M . The curve that corresponds to $M = m$, $m = 1, 2, \dots, 20$, is the one that hits the abscissa at $D = m$.

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