Fixed-Order Controller Design for Systems with Polytopic Uncertainty via Convex Optimization

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Abstract

Convex parameterization of fixed-order robust stabilizing controllers for systems with polytopic uncertainty is represented as an LMI using KYP Lemma. This parameterization is a convex inner-approximation of the whole non-convex set of stabilizing controllers and depends on the choice of a central polynomial. It is shown that with an appropriate choice of the central polynomial, the set of all stabilizing fixed-order controllers that place the closed-loop poles of a polytopic system in a disk centered on the real axis, can be outbounded with some LMIs. These LMIs can be used for robust pole placement of polytopic systems.

I. INTRODUCTION

Nowadays, many control design problems are formulated as convex optimization problems and solved efficiently using recently developed numerical algorithms. Yet, a challenging problem is the design of restricted-order controllers by convex optimization methods. The main problem stems from the fundamental algebraic property that the stability domain in the space of polynomial's parameters is non-convex for polynomials with order higher than two [1]. To overcome the non-convexity, there are different strategies, which are explained in [2]. One possibility is to consider an approximation of the non-convex domain with an outer-or-inner convex set. Although an inner approximation introduces some conservatism in the design method, it is preferred because the stability is ensured. Several convex inner approximations of the stability domain around a central polynomial have been proposed in the literature. However, the LMI

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approximations are more flexible since they can represent the other convex sets like polytopes, spheres and ellipsoids.

The problem becomes more complicated when a fixed-order controller should stabilize a model with structured polytopic uncertainty. This problem is usually studied in the state space representation of the system for the full-order controllers using Lyapunov equation. A conservative solution is to find one Lyapunov function to stabilize all models. The other solution which is less conservative is to design a parameter dependent Lyapunov function. However, it is not easy to find this Lyapunov function for polytopic systems. The stabilization problem can be converted to regional pole placement using the concept of D-stability. It is to define a subregion of the stability domain and to modify accordingly the structure of the Lyapunov equation and then design a stabilizing controller [3]. The desired regions are restricted to strips, circles, sectors and hyperbolas. In [4] a unified robust pole placement design method for both continuous and discrete-time systems is introduced. The controller meets the H_2 and/or H_∞ specifications for a nominal plant model and assigns the closed-loop poles in an LMI region which is introduced in [5] and covers many desired regions, using LMI constraints. This problem is extended to the case of systems with a specific type of unstructured uncertainty in [6]. Recently, design of a state feedback controller for a polytopic uncertain system which assigns the closed-loop poles in the same LMI regions is proposed using a non-convex optimization method [7]. However, the final controller does not even guarantee the stability of the system and a robust stability analysis should be carried out after the design is completed. In [8], a sufficient condition via a non-convex optimization is given to design a state feedback controller, which assigns the closed-loop poles of all the vertices of the system polytope in a sector. In [9], a state feedback controller which brings the closed-loop poles to the desired multi-constraints region via a non-convex optimization is designed. The only convex parameterization of fixed-order stabilizing controllers for polytopic systems is given in [10]. Using polynomial positivity, an LMI inner approximation of the stability domain in the polynomial parameter space is proposed. The design method relies on a central polynomial whose choice has not been really investigated.

In this paper, a similar approach is adopted based on the strict positive realness of transfer functions using the KYP Lemma. The derivation of the LMIs from the KYP Lemma is very straightforward and similar to those of [10] and [11]. On the other hand, it has been recently shown that the LMIs originated from the KYP Lemma can be solved very efficiently even with a large number of parameters [12], [13]. Furthermore, it will be shown that a particular choice of the central polynomial makes the LMI an outer approximation of all controllers which bring the closed-loop poles of a polytopic system to a desired disk centered on the real axis. As a result, the proposed LMI gives a sufficient constraint for the stability of the polytopic system and a necessary constraint for the robust regional pole placement.

It should be mentioned that a circle centered on the real axis has already been considered as the desired region for pole clustering [14], [15], [16]. However, state or output feedback controllers have been studied only for systems with unstructured uncertainty.

The paper is organized as follows. The preliminaries and problem formulation can be found in Section II. In Section III, a convex parameterization of fixed-order stabilizing controllers for a polytopic system is given via LMIs. Section IV proposes a choice of central polynomial to parameterize all controllers that cluster the closed-loop poles of the polytope into a desired circular region together with some simulation examples. The concluding remarks are given in Section V.

II. Preliminaries and Problem Formulation

A. Polytopic systems

In order to build up the background of the proposed method, some basics on the polytopes are recalled.

A polytope in an n-dimensional space is the convex hull of a set of points called generators in this space. The minimal set of generators is unique and constitute the vertex set of polytope. An exposed edge of a polytope is the line between two vertices of the polytope, such that the whole polytope lies on just one side of it [17]. If the generators are the coefficients of a polynomial, a polytope of polynomials is obtained. A discrete-time n-th order polytopic system can be defined by a set of transfer functions as follows :

$$G_i(z) = \frac{b_i(z)}{a_i(z)} = \frac{b_i^0 z^n + b_i^1 z^{n-1} + \cdots + b_i^n}{z^n + a_i^1 z^{n-1} + \cdots + a_i^n} \tag{1}$$

with $a_i^j, b_i^j \in \mathbb{R},\ i = 1, \cdots, q,\ j = 0, \cdots, n$, where $\mathbb{R}$ is the set of real numbers and q is the number of $2n+2$ dimensional polytope vertices $[1\ a_i^1\ \cdots\ a_i^n\ b_i^0\ b_i^1\ \cdots\ b_i^n]$. This polytopic system covers a wide variety of structured uncertainties, like multiple models and interval uncertainty.

It should be mentioned that in this paper only the discrete-time polytopic systems are considered. However, the continuous-time systems can also be treated in a very similar way.

B. Strictly positive real systems

A real rational transfer function $H(z)$ is strictly positive real (SPR) if and only if [18]

1) $H(z)$ is analytic in $|z| \geqslant 1$ and
2) $Re\ H(z) > 0 \quad \forall z,$ such that $|z| = 1$

Hence, for SPRness of a Schur stable real transfer function it is enough to check positivity of its real part on $|z| = 1$. With a simple application of Nyquist criterion, it can be easily shown that if $H(z) = c(z)/d(z)$ is SPR then $c(z)$ is Schur stable. This means that to test the Schur stability of a polynomial $c(z)$, it is sufficient to check that its ratio to another Schur polynomial $d(z)$ is SPR.

The SPR condition for a stable transfer function is closely related to the phase of its numerator and denominator as formulated in the following lemma :

Lemma 1 $H(z) = c(z)/d(z)$ *with Schur stable* $d(z)$ *is SPR if and only if* $\forall z$*, such that* $|z| = 1$

$$|\phi(c(z)) - \phi(d(z))| < \frac{\pi}{2} \tag{2}$$

where $\phi(\cdot)$ *denotes the phase.*

As a result, polynomial $c(z)$ is Schur stable if and only if there exists a Schur stable polynomial $d(z)$ such that Inequality (2) is satisfied (See also Lemma 1 in [10]).

The SPR condition can be given in the state space by the Kalman-Yakubovich-Popov Lemma :

Lemma 2 (KYP Lemma for discrete-time systems) *A transfer function* $H(z) = C(zI - A)^{-1} B + D$ *is SPR if and only if there exists a matrix* $P = P^T > 0$ *such that [19], [18] :*

$$\begin{bmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & -D - D^T + B^T PB \end{bmatrix} < 0 \tag{3}$$

Therefore, all SPR transfer functions with fixed denominator (which leads to a fixed A and B using controllable canonical form realization) can be parameterized by an LMI. Since the numerator of an SPR transfer function is stable, this LMI represents also a convex set of Schur stable polynomials. (See [10] for a similar set of LMIs for SPRness using positivity in polynomials.)

C. Problem formulation

Consider the discrete-time SISO LTI polytopic system in (1). The goal is to design a fixed-order controller

$$K\left(z\right)=\frac{y\left(z\right)}{x\left(z\right)}=\frac{y_0 z^m+y_1 z^{m-1}+\cdots+y_m}{z^m+x_1 z^{m-1}+\cdots+x_m} \tag{4}$$

which stabilizes and places the closed-loop poles of the whole polytope, in a disk centered on the real axis inside the unit circle.

The method proposed in this paper gives a convex set of fixed-order stabilizing controllers that contains (if there exists any) all the fixed-order controllers that place the closed-loop poles in the desired circular region.

III. Convex Parameterization of Fixed-Order Stabilizing Controllers

Consider the vertices of the system polytope given in (1) and the fixed-order controller in (4). Then,

$$c_i\left(z\right)=a_i\left(z\right)x\left(z\right)+b_i\left(z\right)y\left(z\right) \qquad i=1,\ldots,q \tag{5}$$

are the n_{cl}-th order ($n_{cl}=n+m$) characteristic polynomials of the vertices of the system polytope. Since characteristic polynomial is affine with respect to the system parameters, the whole characteristic polynomials of the system polytope develop a new polytope of closed-loop characteristic polynomials whose vertices are contained in (5) [17].

To proceed, a convex set of stabilizing controllers is obtained using the KYP lemma (see [11], [10] for similar results). Consider that a Schur stable polynomial $d(z)$ (central polynomial) is given. Then, by parameterizing all $c_i(z)$, which make $c_i(z)/d(z)$ SPR, a convex set of stable characteristic polynomials in terms of controller parameters can be given by a set of LMIs. This result is stated in the following proposition :

Proposition 1 *Consider the polytopic system in (1), the fixed-order controller in (4), the characteristic polynomials in (5) and a given Schur stable polynomial $d(z)=z^{n_{cl}}+d_1 z^{n_{cl}-1}+\cdots+d_{n_{cl}}$ of order n_{cl}. Suppose that the transfer functions $c_i(z)/d(z)$ are represented in the state space by the controllable canonical realization (A,B,C_i,D_i). Therefore, A and B are fixed, $D_i=1+y_0 b_i^0$ and $C_i=k^T S_i-d^T D_i$, which depend linearly on controller parameters and plant model parameters, where $k^T=[x_1,\ldots,x_m,y_0,y_1,\ldots,y_m]$ is the vector of the controller parameters,*

$d^T = [d_1, d_2, \dots, d_{n_{cl}}]$ *contains the parameters of* $d(z)$ *and* S_i *is a Sylvester matrix of dimension* $(2m+1) \times n_{cl}$*, which is composed of the system parameters* a_i *and* b_i*. Then the set of all controller parameters which make* $c_i(z)/d(z)$ *SPR for* $i = 1, \dots, q$*, and hence, stabilizes the whole system polytope is given by the following LMIs :*

$$\begin{array}{l} P_i = P_i^T > 0, \\ \begin{bmatrix} A^T P_i A - P_i & A^T P_i B - C_i^T \\ B^T P_i A - C_i & -D_i - D_i^T + B^T P_i B \end{bmatrix} < 0 \end{array} \tag{6}$$

Remarks :

• A feasible point of the LMIs in (6) gives the parameters of a controller that stabilizes not only the q vertices (1), but also all the models in their convex hull (the polytope made by them). The reason is that this set of LMIs is affine with respect to the parameters of the plant model. (Notice that Edge theorem [17] is not used to prove the stability of the whole polytope.)

• Note that the set of all fixed-order stabilizing controllers is a non-convex set. However, for each fixed $d(z)$ the feasible set of inequality (6) gives an LMI inner approximation of this non-convex set. Thus, the choice of $d(z)$ (central polynomial) is crucial to have a logical approximation, to minimize the conservatism of the method and more important, to bring the closed-loop poles of the whole system polytope to a desired region.

Before discussing about the choice of the central polynomial, it is interesting to state that if there is a controller which stabilizes a polytopic system, there exists always a $d(z)$ that makes the LMIs in (6) feasible. The reason is that for any stable polytope (here, polytope of characteristic polynomials), there always exists a polynomial $d(z)$ (an SPR-maker) that produces SPR transfer functions when divided by any member of the polytope (See the proof of Theorem 2.1 in [20]). The existence of an SPR-maker gives enough motivation to investigate for finding a suitable central polynomial.

The following lemma is needed in the sequel :

Lemma 3 *[17], [21] Let* $p_1(z)$ *and* $p_2(z)$ *be two monic Schur stable polynomials of the same degree. The whole line between these polynomials :* $p_\lambda(z) = \lambda p_1(z) + (1-\lambda)p_2(z),\ \lambda \in [0,1]$ *is stable if and only if* $\forall z$*, such that* $|z| = 1$*,*

$$|\phi(p_1(z)) - \phi(p_2(z))| < \pi \tag{7}$$

It means that the phase difference between any two members of a polytope of polynomials, does not exceed π.

Now, a specific polytope of polynomials is considered. It can be proved that for a polynomial of order n_{cl}, the polytope made from the $n_{cl}+1$ vertices

$$v_i(z) = (z+1)^i (z-1)^{n_{cl}-i} \quad i = 0, \ldots, n_{cl} \tag{8}$$

is the smallest polytope outbounding the stability domain of polynomials of order n_{cl} in parameter space [22]. It can be easily verified that the phase difference between the vertices $v_i(z)$ and $v_j(z)$ becomes greater than π for $|i-j| > 2$, which happens when $n_{cl} > 2$. Therefore, the line between them is not even the boundary of stability domain which confirms the fact that the stability domain of the parameters of the polynomials with order greater than two is not convex. However, for the second-order polynomials, the mentioned phase difference reaches at most to π and hence, the lines between these vertices are the boundary of the stability domain in parameter space.

Next, consider the polytope in (8), with α and β ($|\alpha|, |\beta| < 1$) instead of ± 1 :

$$v_i(z) = (z-\alpha)^i (z-\beta)^{n_{cl}-i} \quad i = 0, \cdots, n_{cl} \tag{9}$$

This polytope has some interesting specifications :

1) Its stability analysis is very easy, using the following lemma :

   Lemma 4 *[21], [22] The polytope made by vertices (9) is Schur stable if and only if the line between $v_0(z)$ and $v_{n_{cl}}(z)$ is stable.*

   When $\alpha = -\beta$, then Lemma 4 leads to the following simple condition :

   Corollary 1 *([21], Corollary 1) The polytope made by vertices (9) with $\alpha = -\beta$ is Schur stable if and only if $\alpha < \tan(\pi/2n_{cl})$.*

2) For n_{cl} even, it is easy to find a $d(z)$, whose phase lies exactly in the midmost of the phase plot of the whole polytope :

   Lemma 5 *[22] For n_{cl} even, phase plot of*

$$d(z) = (z-\alpha)^{n_{cl}/2} (z-\beta)^{n_{cl}/2} \tag{10}$$

lies exactly in the midmost of the phase plot of the whole polytope made by vertices (9).

As a result, if the whole polytope defined by (9) is stable, the phase difference between each member of the polytope and (10) is less than $\pi/2$ (Lemma 3). Therefore, taking into account Lemma 1, $d(z)$ in (10) is an SPR-maker for this polytope.

3) It contains all the polynomials whose roots are located in the disk centered on the real axis on the midmost of α and β. The following lemma explains this important specification :

Lemma 6 *([22], Corollary 1) Consider a closed bounded region of complex plane which is symmetric with respect to the real axis and intersects the real axis at α and β. Then if this region is outbounded by a circle, which is centered on the real axis and passes through the real points α and β, all monic polynomials of degree n_{cl} whose roots lie inside this region are inside the polytope defined by the vertices (9).*

Now, suppose that there exists a controller, which leads to a characteristic polynomial polytope inside the polytope defined by (9). Such a controller is certainly a feasible point of LMIs (6), with (10) as its central polynomial. Therefore, for n_{cl} even, and with α and β chosen such that the polytope defined by (9) becomes as large as possible, i.e. it touches the stability boundary, then (10) is an SPR-maker of the same order, which can be chosen as the central polynomial.

The following example emphasizes the importance of the choice of the central polynomial.

Example: The objective is to find the set of all second-order stable polynomials using LMIs (6). The stability domain of the second-order polynomials in parameter space is the interior of the polytope (a triangle) with three vertices $(z-1)^2, (z-1)(z+1), (z+1)^2$, which is a convex set [1]. To exploit LMIs (6), we should first fix a Schur stable $d(z)$ to have fixed A and B. Let $d(z) = z^2$ be chosen as proposed in [10] for the same example. For this choice of central polynomial, the feasible set of LMIs (6) deos not cover the whole stability domain (See Fig. 1). However, for this convex stability domain, one can expect to find an LMI, whose feasibility domain covers the whole triangle. Indeed, choosing $v_1(z) = (z-1)(z+1)$ as the central polynomial, the feasible set of non-strict LMIs in (3) becomes exactly the stability triangle. After the first submission of this paper, we were informed that the same result about using $v_1(z)$ as an SPR-maker has been already observed in [23]. However, the uniqueness of this choice is proved in this paper.

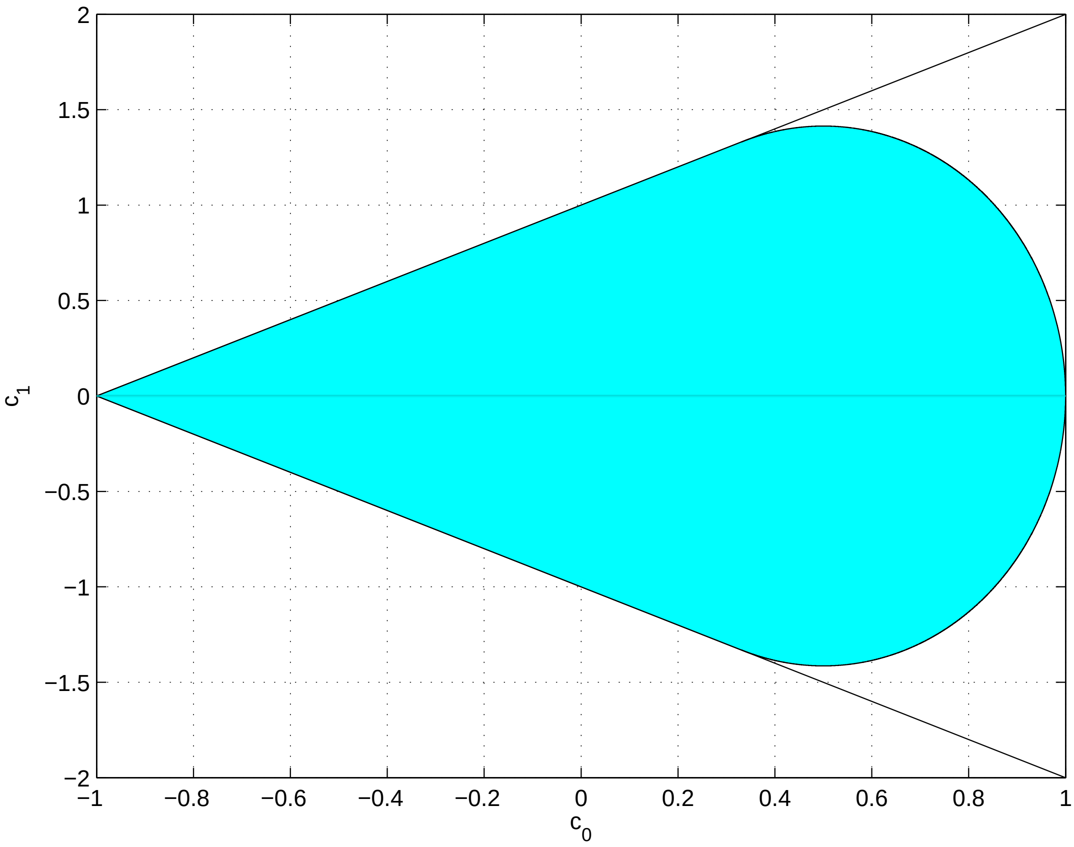

Fig. 1. Stability domain of $c\,(z) = z^2 + c_1 z + c_0$ in parameter space, which is almost the same as feasibility set of LMIs (6) with $d(z) = (z-1)(z+1)$ (Triangle) and their feasibility set with $d\,(z) = z^2$ (shaded)

IV. Controller parameterization for Robust Pole Placement

The main objective of this paper is to parameterize all controllers that place the closed-loop poles of a polytopic system in a specified region of the complex plane. Since the set of these controllers is not convex in the space of the controller parameters, an outer convex approximation of this set which is an inner approximation of all stabilizing controllers is given by a set of LMIs. This set will contain all controllers that place the closed-loop poles in the desired region and does not contain any destabilizing controller.

Consider the n-th order polytopic system given in (1) and the m-th order controller in (4) such that $n_{cl} = n + m$ is even. The main results are presented in the following theorems :

Theorem 1 *Suppose that there exists an m-th order controller that clusters the closed-loop poles of the system polytope in a disk centered at the origin with radius $r = \tan(\pi/2n_{cl})$, then, this controller is in the feasible set of LMIs (6) by choosing the central polynomial as :*

$$d(z) = (z - r)^{n_{cl}/2}(z + r)^{n_{cl}/2} \tag{11}$$

Proof: Taking into account Lemma 6 and Corollary 1, all polynomials of degree n_{cl}, whose roots lie in the disk centered at the origin with radius $r = \tan(\pi/2n_{cl})$, are contained in the stable polytope defined by (9), with $\alpha = -\beta = r$. Then, according to Lemma 5, $d(z)$ in (11), is an SPR-maker of this polytope. Thus, taking into account Proposition 1, all controllers that place the closed-loop poles in the mentioned region are contained in the feasible set of (6). ■

This result can be extended to the case that the desired region is a disk centered on the real axis at $z = p$ and is formulated in the next theorem.

Theorem 2 *Consider a disk centered at $z = p$ with radius r defined as a solution to the following set of equations :*

$$\sin\left(\frac{r}{p}\cot\left(\frac{\pi}{n_{cl}}\right)\right) = \rho(r,\theta)\sin(\theta) \tag{12}$$

$$\cos\left(\frac{r}{p}\cot\left(\frac{\pi}{n_{cl}}\right)\right) = p + \rho(r,\theta)\cos(\theta) \tag{13}$$

$$\rho(r,\theta) = r\left[\sin(\theta)\cot\left(\frac{\pi}{n_{cl}}\right) + \sqrt{\sin^2(\theta)\cot^2\left(\frac{\pi}{n_{cl}}\right) + 1}\;\right] \tag{14}$$

where $z = p + \rho(r,\theta)e^{\pm j\theta},\ 0 \le \theta \le \pi$, is the boundary of the root locus of the polytope defined by the following vertices :

$$v_i(z) = (z - (p + r))^i(z - (p - r))^{n_{cl}-i}, \quad i = 0, \ldots, n_{cl} \tag{15}$$

Suppose that there exists an m-th order controller that places the closed-loop poles of the system polytope in the disk defined above, then, this controller is in the feasible set of LMIs (6) by choosing the central polynomial as :

$$d(z) = (z - (p + r))^{n_{cl}/2}(z - (p - r))^{n_{cl}/2} \tag{16}$$

Proof: Based on Lemmas 4 and 5, it is necessary to find r such that the whole polytope defined by (15) becomes stable. In this case, according to Lemma 3 the phase difference between each pair of its members becomes less than π and thus taking into account Lemmas 5 and 1, (16)

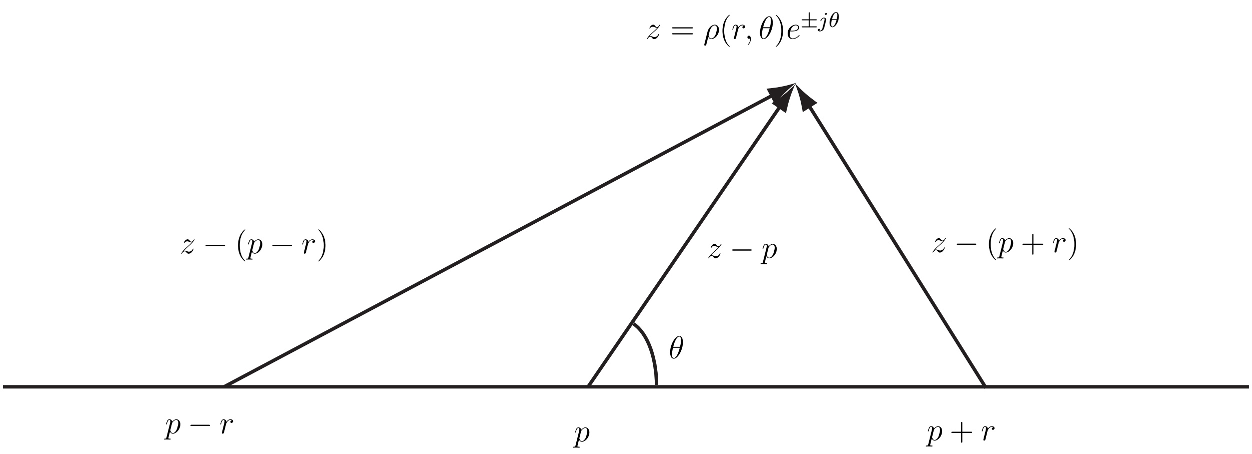

Fig. 2. Phasor of $z-(p-r)$ and $z-(p+r)$ and $z-p$, where z is a point on the boundary of the root locus of the edge between vertices $v_0(z)$ and $v_{n_{cl}}(z)$, p is the center of desired disk and r is its radius.

is an SPR-maker of this polytope. Taking into account Lemma 4, to have a stable polytope, the root locus of the edge between $v_0(z)$ and $v_{n_{cl}}(z)$, which outbounds all the roots of the polytope, should be inside the unit circle. Now, according to Lemma 3, to compute the root locus of the mentioned edge, it is necesary to put : $\max|\phi(v_0(z)) - \phi(v_{n_{cl}}(z))| = \pi$, $\forall z$, such that $|z| = 1$. Noting Fig. 2, it is easy to show that :

$$|\phi(z-(p+r)) - \phi(z-(p-r))| = \\ \pi - \arctan\frac{\rho(r,\theta)\sin(\theta)}{r+\rho(r,\theta)\cos(\theta)} - \arctan\frac{\rho(r,\theta)\sin(\theta)}{-r+\rho(r,\theta)\cos(\theta)} = \frac{\pi}{n_{cl}} \quad (17)$$

where $0 \leqslant \arctan(\cdot) < \pi$. With some straightforward calculations over (17), it can be shown that the root locus of the mentioned edge is :

$$z = p + \rho(r,\theta)e^{\pm j\theta} \qquad 0 \leqslant \theta \leqslant \pi \quad (18)$$

with $\rho(r,\theta)$ defined in (14). Now, in order to force (18) to lie inside the unit circle, the distance from the origin to its farthest point should be equal to one. The distance ℓ of the root locus (18) to the origin can be easily computed as : $\ell^2 = (p+\rho(r,\theta)\cos(\theta))^2 + (\rho(r,\theta)\sin(\theta))^2$, where $0 \leqslant \theta < \pi$, and $\theta = \theta_{\max}$ corresponding to the farthest point, can be computed by maximizing the distance ℓ with respect to θ. After straightforward but tedious calculations the following result can be obtained :

$$\frac{\sin(\theta_{\max})}{p + cos(\theta_{\max})} = \tan(\gamma_{\max}) = \frac{r}{p}\cot\left(\frac{\pi}{n_{cl}}\right)$$

where $\gamma_{\max}$ is the phase of the farthest point of the root locus. Thus, to find the maximum stabilizing r, it is sufficient to solve the following equations:

$$\sin(\gamma) = \rho(r,\theta)\sin(\theta) \quad (19)$$

$$\cos(\gamma) = p + \rho(r,\theta)\cos(\theta) \quad (20)$$

The rest of the proof is similar to the proof of Theorem 1. ■

Remarks :

- It should be mentioned that r depends only on n_{cl} and p and can be easily computed by standard equation solvers from Eqs (12)-(14). It can be observed that r is a decreasing function of n_{cl} and p (Fig. 3).
- In the case that n_{cl} is not even, we can augment m by 1, or we can just accept more conservatism and solve the set of LMIs (6) for a strictly proper transfer function $c(z)/d(z)$, where the order of $d(z)$ is $n_{cl} + 1$.
- The root locus of feasible set of LMIs (6), depends on the place of the roots of central polynomial. Simulation results show that by moving p, for example to the right, the roots of feasible characteristic polynomials move also to the right. The reason is that the feasibility set of LMIs (6) moves according to the movement of the root locus of the moved polytope in (15). In the next example, this effect is shown for a polytopic system with 16 vertices.

A. Example

Consider the problem of robust controller design for a third-order system, which is affected by polytopic uncertainty. The vertices of the polytope are given in Table I , where

$$G = \frac{b_0 z^2 + b_1 z + b_2}{z^3 + a_1 z^2 + a_2 z + a_3},\ T_s = 1$$

Consider a controller of order three, which is supposed to place the closed-loop poles of these vertices inside the desired circle around $z = 0.5$. The radius of such a disk can be easily computed as $r = 0.1972$. Therefore, the proposed central polynomial is near to $d_1(z) = (z - 0.31)^3(z - 0.69)^3$. The stabilizing controller :

$$K(z) = \frac{2z^3 - 1.8z^2 + .16z}{z^3 - 2.1z^2 + 1.28z - .18}$$

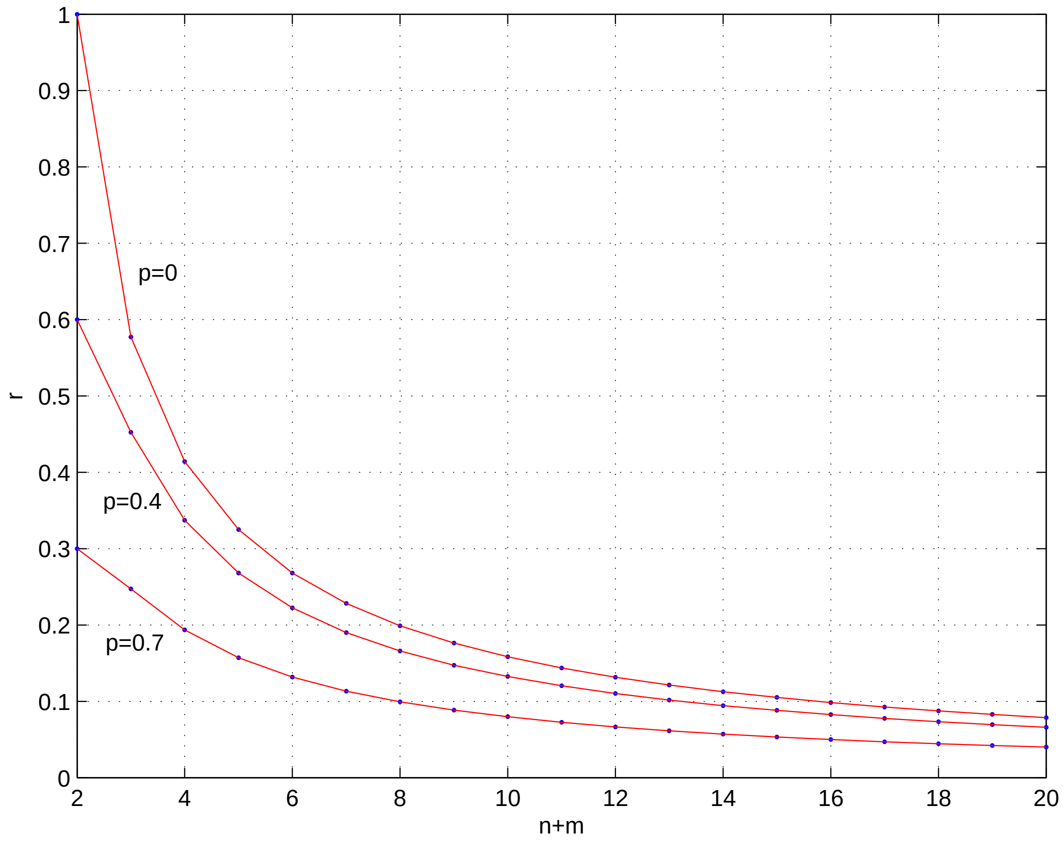

Fig. 3. *r* versus n_{cl} for different *p*

clusters the poles of these two vertices at $z = 0.31$ and $z = 0.69$ respectively. Taking $d_2(z) = (z - 0.5)^6$ or $d_3(z) = z^6$ as the central polynomial, this controller is not a feasible point of LMIs 6, whereas with $d_1(z)$ this controller is a feasible point of LMIs (6).

V. Conclusion

An LMI parameterization of all controllers that put the closed-loop pole of a polytopic system in a disk centered on the real axis is given. The proposed LMI gives a sufficient stability condition for the polytopic system and a necessary condition for the robust regional pole placement. It is shown that the radius of this disk decreases when the closed-loop order increases or the distance between the origin and the disk center is augmented. The capability of the proposed method is illustrated via some simulation examples.

TABLE I

PARAMETERS OF TWO VERTICES OF EXAMPLE IV-A

$a_1^1 = 1.115100244722316$	$a_1^2 = -0.024899755277851$	$b_0^1 = -0.437550122361158$	$b_0^2 = -1.007550122361074$
$a_2^1 = -0.0841162256667$	$a_2^2 = 0.12953602988889$	$b_1^1 = 0.89986825966674$	$b_1^2 = 1.933042131888844$
$a_3^1 = -0.004930576005557$	$a_3^2 = -0.59954535045$	$b_2^1 = -0.16254546208058$	$b_2^2 = -0.923026721524995$

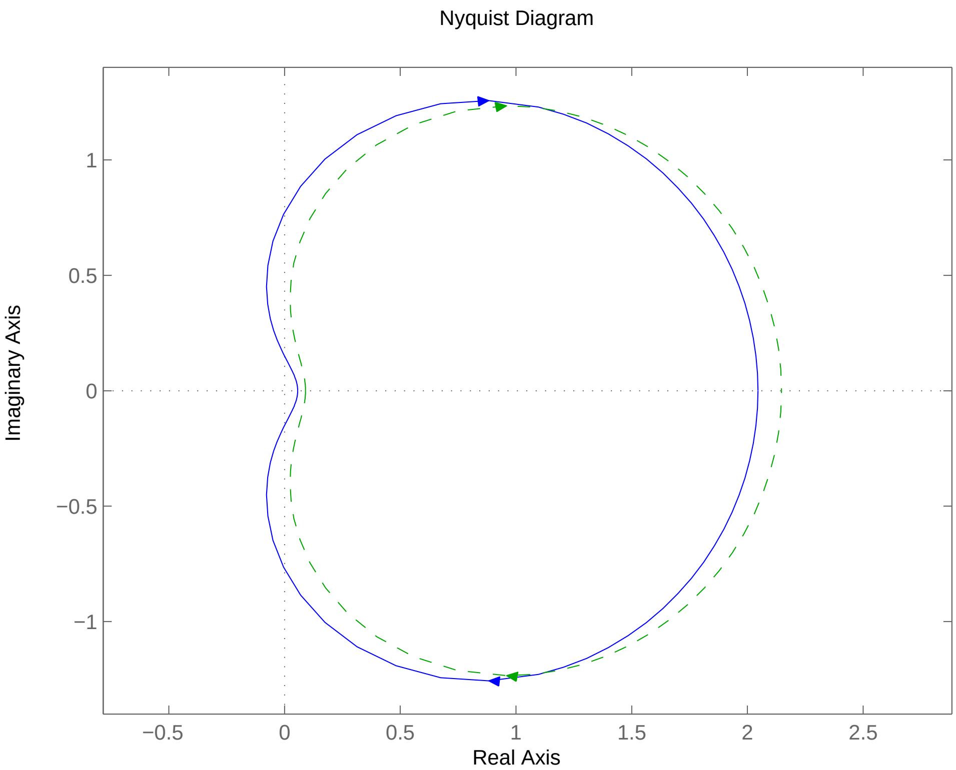

Fig. 4. Nyquist diagrams for Example IV-A, $c_2(z)/d_1(z)$ (dashed) and $c_2(z)/d_2(z)$ (solid) that is not SPR.

REFERENCES

[1] J. Ackermann, *Robust Control: Systems with Uncertain Physical Parameters*. New York, NY: Springer Verlag, 1993.

[2] D. Henrion and M. Sebek, "Overcoming non-convexity in polynomial robust control design," in *Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Leuven, Belgium, 2004.

[3] W. M. Haddad and D. S. Bernstein, "Controller design with regional pole constraints," *IEEE Transactions on Automatic Control*, vol. 37, no. 1, pp. 54–69, January 1992.

[4] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, July 1997.

[5] M. Chilali and P. Gahinet, “H_∞ design with pole placement constraints : An LMI approach,” *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 358–367, March 1996.

[6] M. Chilali, P. Gahinet, and P. Apkarian, “Robust pole placement in LMI regions,” *IEEE Transactions on Automatic Control*, vol. 44, no. 12, pp. 2257–2270, December 1999.

[7] R. P. E.N. Goncalves and R. Takahashi, “Improved optimisation approach to the robust H_2/H_∞ control problem for linear systems,” in *IEE Proceedings Control Theory and Applications*, vol. 152, no. 2, March 2005, pp. 171–176.

[8] D. Arzelier, J. Bernussou, and G. Garcia, “Pole assignment of linear uncertain systems in a sector via a Lyapunov-type approach,” *IEEE Transactions on Automatic Control*, vol. 38, no. 7, pp. 1128–1132, July 1993.

[9] J. L. Wu and T. T. Lee, “A new approach to optimal pole placement,” *Automatica*, vol. 33, no. 10, pp. 1917–1921, 1997.

[10] D. Henrion, M. Sebek, and V. Kucera, “Positive polynomials and robust stabilization with fixed-order controllers,” *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1178–1186, 2003.

[11] B. Dumitrescu, “Parameterization of positive-real transfer functions with fixed poles,” *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 49, no. 4, pp. 523–526, April 2002.

[12] P. A. Parrilo, “Structured semi-definite programs and semi-algebraic geometry methods in robustness and optimization,” Ph.D. dissertation, California Institute of Thechnology, Pasadena, California, 2000.

[13] R. Wallin, A. Hansson, and L. Vandenberghe, *Efficiently solving semidefinite programs originating from the Kalman-Yakubovich-Popov lemma using general purpose SDP solvers*. http://www.control.isy.liu.se/publications: Technical report from the Control and Communication group in Linkoping, Report number: LiTH-ISY-R-2503, 2003.

[14] G. Garcia and J. Bernussou, “Pole assignment for uncertain systems in a specified disk by state feedback,” *IEEE Transactions on Automatic Control*, vol. 40, no. 1, pp. 184–190, 1995.

[15] G. Garcia, J. Daafouz, and J. Bernussou, “Output feedback disk pole assignment for systems with positive real uncertainty,” *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1385–1391, September 1996.

[16] B. S. Kim, H. S. Han, and J. G. Lee, “Pole placement of uncertain discrete systems in the smallest disk by state feedback,” in *35th IEEE Conference on Decision and Control*, Kobe, Japan, 1996.

[17] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach*. NJ, USA: Prentice Hall Information and System Sciences Series, 1995.

[18] I. D. Landau, R. Lozano, and M. M’Saad, *Adaptive Control*. London: Springer-Verlag, 1997.

[19] L. Lee and J. L. Chen, “Strictly positive real lemma for discrete-time descriptor systems,” in *39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000.

[20] B. D. O. Anderson, S. Dasgupta, F. J. Khargonekar, P. P. Kraus, and M. Mansour, “Robust strict positive realness : Characterization and construction,” *IEEE Transactions on Circuits and Systems*, vol. 37, pp. 869–876, 1990.

[21] L. Jetto, “Strong stabilization over polytopes,” *IEEE Transactions on Automatic Control*, vol. 44, no. 6, pp. 1211–1216, 1999.

[22] A. Tesi, A. Vicino, and G. Zappa, “Convexity properties of polynomials with assigned root location,” *IEEE Transactions on Automatic Control*, vol. 39, no. 3, pp. 668–672, 1994.

[23] B. Dumitrescu and R. Niemistö, “Multistage IIR filter design using convex stability domains defined by positive realness,” *IEEE Transactions on Signal Processing*, vol. 52, no. 4, pp. 962– 974, 2004.